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Number of SOSOFS(n, s) For n a Product of Prime Powers

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Abstract

Formulas for the number of F-squares with p symbols, q symbols, and r symbols are presented, For n = pqr, p < q < r, and p, q, and r, prime numbers, the numbers of the various F-squares are determined. It is shown how to generalize the results to powers of primes and how to construct sum-of-squares orthogonal arrays from the complete sets of F-squares. A complete set of sum-of-squares orthogonal F-squares attains the maximum number, the upper bound, that can be constructed.

Key words: Upper bound, complete set, factorial arrangement, degrees of freedom.

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Introduction

A geometry for all values of n, n a product of prime powers, has been developed by Federer (2003b). Related to this is a construction of complete sets of F-squares. An F-square, FS(n, s), is an n × n row-column array of s symbols with the s symbols appearing equally frequent in rows and equally frequent in columns. This has been called a regular F-square to distinguish it from a semi-F-square which has the s symbols occurring equally frequent in rows (columns) but not in columns (rows). Pesotan *et al.* (2003) denote the semi-F-square as a row (column) frequency square, RFS(n, s) (CFS(n, s)). These authors investigated the maximum number of RFS(n, s) (CFS(n, s)) and reviewed the history of upper bounds for a set of FS(n, s). Here we determine the number and the upper bound on the number of sum-of-squares orthogonal F-squares with s symbols, SOSOFS(n, s). Statistical ideas and concepts are used for this investigation.

Number of SOSOFS

Ideas from factorial treatment design and associated degrees of freedom provide the basis for determining the number of SOSOFS(n, s) in a complete set of sum-of-orthogonal F-squares for a particular value of n. Partitioning degrees of freedom and sums of squares for the factorial effects into main effects and interactions in an analysis of variance results in the well-known orthogonal decomposition of the total sums of squares and total degrees of freedom. Federer (2003b) presented a method for constructing F-squares from the main effect and interactions sums of squares and degrees of freedom. The construction results in regular F-squares and semi-F-squares. When the sum of squares for the F-squares constructed from a main effect or an interaction account for all of the degrees of freedom and sums of squares of the main effects or interactions, the set of F-squares is said to be sum-of-squares orthogonal, SOSO. The set is said to be complete if all degrees of freedom and sums of square s have been accounted for leaving nothing left to construct additional F-squares. Thus the upper bound is reached with a complete set.

For $p < q < r < \ldots$, p, q, and r, .. prime numbers, and $n = p^a q^b r^c \ldots$, F-squares with p, q, r, .. symbols will result from the construction method given by Federer (2003b). The question arises as to the number of FS(n, p), FS(n, q), FS(n, r), etc. squares in a complete set of SOSOFSs. To determine this, degree of freedom concepts are used.

First, consider the case where $n = pq$, $p < q$, p and q prime numbers. Let factor A with p levels and factor B with q levels be associated with rows and let factor C with p levels and factor D with q levels be associated with columns. This results in a four factor factorial treatment design. The row × column interaction is associated with $(n - 1)^2 = (pq - 1)^2$ degrees of freedom and is composed of interactions of the four factors. A partitioning of the degrees of freedom in an analysis of variance is:

Source of variation	Degrees of freedom p=2,q=3		p-3,q=5		p, q	
Total	36		225		$n^2 = p^2q^2$	
Mean	1		1		1	
Row	5		14		$n - 1 = pq - 1$	
A		1		2	$p - 1$	
B		2		4	$q - 1$	
A × B		2		8	$(p - 1)(q - 1)$	
Column	5		14		$n - 1 = pq - 1$	
C		1		2	$p - 1$	
D		2		4	$q - 1$	
C × D		2		8	$(p - 1)(q - 1)$	
Row × column	25		196		$(n - 1)^2 = (pq - 1)^2$	
A × C		1		4		$(p - 1)^2$
A × D		2		8		$(p - 1)(q - 1)$
A × C × D		2		16		$(p - 1)^2 (q - 1)$
B × C		2		8		$(q - 1)(p - 1)$
B × D		4		16		$(q - 1)^2$
B × C × D		4		32		$(p - 1)(q - 1)^2$
A × B × C		2		16		$(q - 1)(p - 1)^2$
A × B × D		4		32		$(p - 1)(q - 1)^2$
A × B × C × D		4		64		$(p - 1)^2(q - 1)^2$

For n = 2(3), there is only one FS(6, 2); it is formed from the A × C interaction. There are 12 FS(5, 3)s that are formed from the remaining interactions using the method of construction given by Federer (2003b). The two symbols in the FS(6, 2)s have one degree of freedom for its sum of squares. Each of the FS(6, 3)s are associated with two degrees of freedom. These 12 F-squares account for all the degrees of freedom and likewise the sums of squares as this is a SOSO set of F-squares. The upper bound is attained as there are no degrees of freedom left to construct additional F-squares.

For n = 3(5), there are two FS(15, 3)s. There are 43 FS(15, 5)s formed from the remaining interactions. These 45 F-squares form a complete set of SOSOFSs. In general, there are p - 1 FS(n, p)s and $p^2(q + 1) - 2p$ FS(n, q)s. The interaction of the two factors with p levels is associated with $(p - 1)^2$ degrees of freedom. The p - 1 FS(n, p)s constructed from this interaction each have p - 1 degrees of freedom.

If $n = p^a q^b$, then the number of FS(n, p)s is $(p^a - 1)^2/(p - 1)$. The number of FS(n, q)s is $[(q^b - 1)/(q - 1)][p^{2a}(q^b + 1) - 2p^a]$. For example, let $n = 18 = 2(3^2)$. There is one FS(18, 2) and $[(3^2 - 1)/(3 - 1)][2^2(3^2 + 1) - 2p] = 144$ FS(18, 3)s. Federer (2003a) has presented a computer program for constructing this set of F-squares as well as other sets. For $n = 12 = 2^2(3)$, there are nine FS(12, 2)s and 56 FS(12, 3)s.

Let the row numbers be represented by a three factor factorial of factors A at p levels, B at q levels, and C at r levels. Likewise, use the three factors D at p levels, E at q levels, and F at r levels, to represent the column numbers. This results in a six factor factorial arrangement. Let $p < q < r$. Using the interactions of these factors to construct F-squares as described by Federer (2003a, b), there are p - 1 FS(n, p)s, $p^2(q + 1) - 2p$ FS(n, q)s, and $pq[p^2(q + 1) - 2p]$ FS(n, r)s. For n = 30 = 2(3)(5), there is one FS(30, 2), 12 FS(30, 3)s, and 204 FS(30, 5)s to form a complete set.

The above leads to the following theorem:

Theorem:For $n = pqr$, $p < q < r$, p, q, and r prime numbers, the construction method of Federer (2003b) produces p - 1 F(n, p) squares, $p^2(q + 1) - 2p$ F(n, q) squares, and $pq[p^2(q + 1) - 2p]$ F(n, r) squares to form a complete set of sum-of-squares orthogonal F-squares. This set is the maximum number of F-squares.

As demonstrated above, this theorem is easily generalized to the case where $n = p^a q^b r^c$ and to the extension of more factors. The above set of F-squares may be used to construct a sum-of-squares orthogonal array. Also, F-squares may be formed from the rows and column categories. One FS(n, p), p FS(n, q)s, and pq FS(n, r)s may be constructed from rows and the same number from columns. Adding these F-squares to the set in the theorem, we have p + 1 FS(n, p)s, $p^2(q + 1)$ FS(n, q)s, and $pq[p^2(q + 1) -2p] + 2pq$ FS(n, r)s. Thus the following corollary:

Corollory:A sum-of-squares orthogonal array with n^2 rums and with p + 1 rows of p symbols, $p^2(q + 1)$ rows of q symbols, and $pq[p^2(q + 1) -2p] + 2pq$ rows of r symbols is produced from the set of F -squares in the theorem and the row and column F-squares.

Comments

The sum-of-squares orthogonal arrays produced by the methods described herein greatly adds to the number of standard orthogonal arrays. The availability of such arrays adds to the diversity and flexibility for using arrays as codes. Since complete sets of SOSOFSs can be generated using a computer (Federer, 2003a), these arrays are easily available to the users of codes.

Using the method for constructing F-squares given by Federer (2003a, b) for n = 8 = 2(4) and n = 12 = 2(6) did not result in a complete set of SOSOFSs. For n = 2(4) = 8, seven degrees of freedom remain unaccounted for. This means that additional F-squares could be constructed using the method for adding to a given set of F-squares as described by Federer (2003a, b). Thus, this number would exceed the maximum number of SOSOFSs but the set would not be SOSO. For n = 12 = 2(6), 40 degrees of freedom were unaccounted for by this method. These 40 degrees of freedom would allow for an at least an additional eight FS(12, 6)s. Here again the upper bound for SOSOFSs would be exceeded but the set would not be a set of SOSOFSs.

Literature Cited

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