ON CONJECTURES RELATED TO CHARACTER VARIETIES OF KNOTS AND JONES POLYNOMIALS

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It is well known that the Kauffman Bracket Skein Module of a knot complement $K_q(S^3 \setminus K)$ is canonically a module over the $\mathbb{Z}_2$-invariants of the quantum torus, $A_q^{\mathbb{Z}_2}$, and this module determines the colored Jones polynomials $J_n(K;q)$ of the knot K. Berest and Samuelson identified a conjecture for knots under which a close variant of $K_q(S^3 \setminus K)$ canonically becomes a module over a certain Double Affine Hecke Algebra, from which they defined a family of polynomials $J_n(K;q,t_1,t_2)$ generalizing the classical polynomials of Jones.

In this thesis an analogue of Habiro's cyclotomic equation for the $J_n(K;q)$ is discovered for $J_n(K;q,t_1,t_2)$. An integrality result for the coefficients in this equation is found as a corollary, offering evidence for the conjecture of Berest and Samuelson for all knots.

Separately, the conjecture of Berest and Samuelson is studied at the particular value $q = -1$ where it is known to relate to properties of $SL_2(\mathbb{C})$-character varieties of knots. Computational methods are used to establish that the conjecture holds for some non-invertible knots, which was not previously known.

BIOGRAPHICAL SKETCH

Joe was born in 1990 to Mike and Trish Gallagher in Orlando, Florida. Always supported by his parents, he spent most of his years growing up either mesmerized by the natural beauty of the Floridan landscape or immersed in music. Thanks to one high-school teacher that allowed Joe to enter his calculus class a year early and another who transformed a physics classroom into a bonding experience, Joe left town in 2008 to study math and physics at the University of Virginia.

In college, he benefited greatly from long, frustrating nights with good friends and became convinced that he would want to pursue math in graduate school. Special thanks are due all of his patient teachers, who at least tried to beat out bad habits and conferred in him a true sense of wonder about math.

At Cornell, he discovered a community like no other. He made great friends and received tireless advising and support from Yuri. Perhaps most importantly, he had the good fortune to meet a certain other graduate student the semester before she ran over the border to Toronto. (She left a note in her dedication too). While his future is far from certain, he will be joining her in Toronto to continue what they started in Ithaca.

This thesis is dedicated to my family, my friends, and everyone who helped me see the good in the work.

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I would also like to thank a few close friends in the math department who provided an excellent atmosphere in which to do math. I count among this number Drew Zemke, Ahmad Rafiqi, Sasha Patotski, Jeffrey Bergfalk, Amin Saeid and Valente Ramirez. You have provided the soul to the discipline as I've known it.

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INTRODUCTION

1.1 Background

The broad thrust of this thesis is the exploration of the deformation theory of algebras and modules arising in low dimensional topology.

To introduce topology, let us fix an oriented knot $K \subset S^3$. If we delete a tubular neighborhood of this knot, there is a natural choice of generators for the fundamental group of $\partial(S^3 \setminus K) \cong S^1 \times S^1$ - a meridian/longitude pair (m, l) - which determines the peripheral map

$$\alpha_K : \pi_1(S^1 \times S^1) \to \pi_1(S^3 \setminus K).$$

This map is known, due to a combination of theorems of Waldhausen [33] and Gordon and Luecke [9] to be a complete knot invariant. However, the map above can be quite complicated, and so it is a common theme to instead take a group G and study how α_K relates representations of $\pi_1(S^1 \times S^1)$ and $\pi_1(S^3 \setminus K)$ into G.

Let us begin with the example $G = \mathbb{C}^*$. As $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$, a representation of this into G is determined by a point of $(\mathbb{C}^*)^2$, which we can give coordinates (m, l). Since $\pi_1(S^3 \setminus K)$ abelianizes to $\mathbb{Z}$ with generator m, any G-rep of it is determined by the single element of $\mathbb{C}^*$ with coordinate m. Thus the restriction map can be viewed as the morphism of varieties

$$\iota : \mathbb{C}^* \to (\mathbb{C}^*)^2$$

$$m \mapsto (m, 1)$$

Note that the map above on representation spaces for $G = \mathbb{C}^*$ is identical for all knots, since all knot groups have the same abelianization. Thus from the perspective of a topologist this choice of G gives no distinguishing information about our knot K. However, if we replace $\mathbb{C}^*$ with $SL_2(\mathbb{C})$, the group homomorphism α_K induces functorially a morphism of $SL_2(\mathbb{C})$-character schemes

$$\mathcal{X}(\alpha_K) : \mathcal{X}(S^3 \setminus K) \to \mathcal{X}(S^1 \times S^1)$$

or dually, a map of commutative rings

$$\mathcal{X}(\alpha_K)^* : \mathcal{O}(\mathcal{X}(S^1 \times S^1)) \to \mathcal{O}(\mathcal{X}(S^3 \setminus K)). \tag{1.1}$$

In this case the ring $\mathcal{O}(\mathcal{X}(S^1 \times S^1))$ admits a simple description: it is the ring of invariants $\mathbb{C}[X^{\pm 1}, Y^{\pm 1}]^{\mathbb{Z}_2}$, where $\mathbb{Z}_2$ acts by simultaneously inverting X and Y. However, the target ring $\mathcal{O}(\mathcal{X}(S^3 \setminus K))$ depends on K, and the map $\mathcal{X}(\alpha_K)^*$ remains quite interesting. For example, the kernel of $\mathcal{X}(\alpha_K)^*$ determines the A-polynomial of the knot K and thus by work of [23] already distinguishes K from the unknot. It is this map of rings and its deformations we are interested in.

Let us explain precisely what we mean by "deformations" of a ring map. The main idea of what follows can be seen in very simple geometric example(s). First, we will revisit our first topological case from a different perspective. Consider the case of the affine line $\mathbb{C}$ including into its cotangent bundle $T^*\mathbb{C}$ as the zero section:

$$\iota : \mathbb{C} \to T^*\mathbb{C}$$

$$x \mapsto (x, 0)$$

dual to this is a ring map

$$\iota^* : \mathbb{C}[x, y] \to \mathbb{C}[x]$$

$$x \mapsto x, \quad y \mapsto 0.$$

As $T^*\mathbb{C}$ is a symplectic manifold, its ring of functions carries a Poisson bracket. Thus one may ask about the existence of a noncommutative deformation of $\mathbb{C}[x,y]$ in the direction of this bracket. It is well-known that such a deformation can be realized in the form of the Weyl algebra

$$A_\hbar(\mathbb{C}) = \frac{\mathbb{C}\langle x,y\rangle}{yx - xy = \hbar}$$

and in fact more is true: the algebra $A_\hbar(\mathbb{C})$ acts on $\mathbb{C}[x]$ via its standard representation as differential operators $x \mapsto x$, $y \mapsto \hbar\frac{\partial}{\partial x}$. This action can be described as a ring map

$$\iota_\hbar^* : A_\hbar(\mathbb{C}) \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}[x]). \tag{1.2}$$

Let us now look at the "multiplicative" verson of this story. The inclusion

$$\iota : \mathbb{C}^* \to (\mathbb{C}^*)^2$$

$$X \mapsto (X, 1)$$

is dual to the map of rings

$$\iota^* : \mathbb{C}[X^{\pm 1}, Y^{\pm 1}] \to \mathbb{C}[X^{\pm 1}]$$

$$X \mapsto X, \quad Y \mapsto 1$$

which is exactly the restriction map of character rings for $G = \mathbb{C}^*$ discussed above.

While not quite a cotangent bundle, $(\mathbb{C}^*)^2$ nonetheless carries a natural Poisson structure corresponding to the symplectic form $\omega = \frac{dx}{x} \wedge \frac{dy}{y}$. In this case as well there is a noncommutative algebra which realizes a deformation of the algebra structure on $\mathbb{C}[X^{\pm 1}, Y^{\pm 1}]$, namely the *quantum Weyl algebra*:

$$A_q = \frac{\mathbb{C}\langle X^{\pm 1}, Y^{\pm 1}\rangle}{\langle XY - q^2 YX\rangle}$$

Of course, more is true in this case as well: the algebra A_q also acts on $\mathbb{C}[X^{\pm 1}]$ via difference-operators, $Y.X^n = q^{-2n}X^n$ and $X.X^n = X^{n+1}$. This action also can be described as a map of rings

$$\iota_q^* : A_q \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}[X^{\pm 1}]). \tag{1.3}$$

It is now important to make two observations about the examples ending in equations (1.2) and (1.3). First: we were not considering ι^* in either case as a map of Poisson algebras. Second: even if we had been, the process of deformation is in no way functorial so we could not have expected ι^* to deform into a map of algebras. However, in both cases the same phenomenon occurs: we start with

$$f : A_0 \to B$$

a ring map of commutative algebras with A_0 Poisson, and A_t some family of associative algebras depending on t which deform A_0. We then find for general t ring maps

$$f_t : A_t \to \mathrm{End}_{\mathbb{C}}(B)$$

which give B a module structure over A_t.

Deforming the *particular* map of interest (1.1) is not a novel concept. Work of many authors (see [6] for a summary) has established that there is a functor from the category of 3-manifolds with morphisms open embeddings to $\mathbb{C}$-vector spaces (depending on a complex parameter q) called the Kauffman Bracket Skein Module

$$K_q : \text{3-Mfld} \to \text{Vect}$$

such that $K_q(S^1 \times S^1 \times [0,1])$ carries an algebra structure, and the "thickened" version of the inclusion

$$\alpha_{[0,1]} : S^1 \times S^1 \times [0,1] \to S^3 \setminus K \tag{1.4}$$

induces a vector space map

$$K_q(\alpha_{[0,1]}) : K_q(S^1 \times S^1 \times [0,1]) \to K_q(S^3 \setminus K) \tag{1.5}$$

which can be shown to define a left $K_q(S^1 \times S^1 \times [0,1])$-module structure on $K_q(S^3 \setminus K)$. When $q = -1$, every vector space $K_{-1}(M)$ gains the structure of a commutative $\mathbb{C}$-algebra, and the maps induced functorially by K_{-1} become $\mathbb{C}$-algebra maps. Additionally, when $q = -1$, there is a natural isomorphism of functors

$$\eta : K_{-1}(-) \xrightarrow{\sim} \mathcal{O}(\mathcal{X}(-))$$

which identifies $K_{-1}(S^1 \times S^1 \times [0,1])$ with $\mathcal{O}(\mathcal{X}(S^1 \times S^1))$ and takes $K_{-1}(\alpha_{[0,1]})$ to $\mathcal{X}(\alpha_K)^*$. Thus (1.5) represents a 1-parameter deformation of (1.1). Just as in the $q = -1$ case, we may also describe explicitly the algebra $K_q(S^1 \times S^1 \times [0,1])$. A theorem of Frohman and Gelca [3] shows $K_q \cong A_q^{\mathbb{Z}_2}$, where $\mathbb{Z}_2$ acts by simultaneously inverting X and Y.

The functor K_q also is related to the colored Jones polynomials, central objects in quantum topology. It is essentially a theorem of Kauffman [15] that $K_q(S^3) \cong \mathbb{C}$. In the case of a knot $K \subset S^3$, if $D^2 \times S^1$ represents a tubular neighborhood of the knot, the inclusion

$$D^2 \times S^1 \sqcup S^3 \setminus K \to S^3 \tag{1.6}$$

determines a *topological pairing*

$$\langle\ ,\ \rangle : K_q(D^2 \times S^1) \otimes_{A_q^{\mathbb{Z}_2}} K_q(S^3 \setminus K) \to \mathbb{C} \tag{1.7}$$

of $A_q^{\mathbb{Z}_2}$-modules. It is a theorem of Kirby and Melvin [27] that this pairing determines the sequence of Laurent polynomials in $\mathbb{Z}[q^{\pm 1}]$ associated to a knot K known as the colored Jones polynomials $J_n(K;q)$. Specifically, we have

$$J_n(K;q) = (-1)^{n-1} \langle \varnothing, S_{n-1}(Y + Y^{-1}).\varnothing \rangle \tag{1.8}$$

where $S_{n-1}(X+X^{-1}) = \frac{X^n - X^{-n}}{X - X^{-1}}$ are Chebyshev polynomials and $\varnothing$ represents the empty skein.

In [2], Berest and Samuelson sought to further deform (1.5) by applying the representation theory of *Double Affine Hecke Algebras* (or DAHA). They begin by noting that the DAHA of type $C^\vee C_1$, denoted $\mathcal{H}_{q,\underline{t}}$, is an algebra depending on 5 deformation parameters $(q,(t_1,t_2,t_3,t_4))$ which is closely related to the algebra $A_q \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by simultaneously inverting X and Y. In particular, if we let D_q be the localization of $A_q \rtimes \mathbb{Z}_2$ at all nonzero polynomials in X, there is an embedding due to Sahi [29]

$$\phi : \mathcal{H}_{q,\underline{t}} \to D_q$$

whose image consists of the subalgebra generated by X, X^{-1} and the operators

$$T_0 = t_1 sY - \frac{q\overline{t_1}X + \overline{t_2}}{q^{-1}X^{-1} - qX}(1 - sY), \qquad T_1 = t_3 s + \frac{\overline{t_3}X^{-1} + \overline{t_4}}{X^{-1} - X}(1 - s)$$

where s is the generator of $\mathbb{Z}_2$ and $\overline{t_i} = t_i - t_i^{-1}$. Note that when $\underline{t} = (1,1,1,1)$, the subalgebra is isomorphic to $A_q \rtimes \mathbb{Z}_2$.

Berest and Samuelson then define the *nonsymmetric skein module* of a knot K to be the left $A_q \rtimes \mathbb{Z}_2$-module given by

$$\widehat{K_q}(S^3 \setminus K) := A_q \otimes_{A_q^{\mathbb{Z}_2}} K_q(S^3 \setminus K) \tag{1.9}$$

Let $\widehat{K_q}^{\mathrm{loc}}(S^3 \setminus K)$ denote the localization of the nonsymmetric skein module at nonzero polynomials in X, and $\eta : \widehat{K_q}(S^3 \setminus K) \to \widehat{K_q}^{\mathrm{loc}}(S^3 \setminus K)$ be the natural localization map. The embedding ϕ gives $\widehat{K_q}^{\mathrm{loc}}(S^3 \setminus K)$ the structure of a $\mathcal{H}_{q,\underline{t}}$-module.

The authors then compute these nonsymmetric skein modules for the unknot, the figure-eight knot, and the $(2, 2p+1)$-torus knots. They show in every case, $\widehat{K_q}(S^3\backslash K)$ is free and finitely generated over the subalgebra $\mathbb{C}[X^{\pm1}]$ of $A_q \rtimes \mathbb{Z}_2$. Thus in all these cases, the localization map η is injective. Moreover, they show that the action of $\mathcal{H}_{q,(t_1,t_2,1,1)}$ on $\widehat{K_q}^{\mathrm{loc}}(S^3 \setminus K)$ preserves the image of η; taken together, this gives their

Theorem 1. *(Berest, Samuelson, 2016) For K the unknot, figure-eight knot or any $(2, 2p + 1)$-torus knot,*

1. *The localization map $\eta : \widehat{K_q}(S^3 \setminus K) \to \widehat{K_q}^{\mathrm{loc}}(S^3 \setminus K)$ is injective.*
2. *The action of $\mathcal{H}_{q,(t_1,t_2,1,1)}$ preserves the image of η*

and thus $\widehat{K_q}(S^3 \setminus K)$ carries the natural structure of a $\mathcal{H}_{q,(t_1,t_2,1,1)}$-module.

That half of the parameters (t_3, t_4) cannot be realized as deformation parameters in this way is interesting and is studied in the original paper, but will not concern us in what follows.

It has been shown by Le [17] in the case of two-bridge knots and Marche [20] in the case of torus knots that the skein modules $K_q(S^3 \setminus K)$ are free and finitely generated over the subalgebra $\mathbb{C}[X + X^{-1}]$ of $A_q^{\mathbb{Z}_2}$ corresponding to the skein of the meridian. Thus in all these cases one would imagine $\widehat{K_q}(S^3\backslash K)$ would be free and finitely generated over $\mathbb{C}[X^{\pm1}]$, and (1) would hold. Berest and Samuelson then posed the main conjecture of their paper: that the above theorem could be extended to all knots.

Conjecture 1. *(BS Conjecture) Let K be any knot, and $\eta : \widehat{K_q}(S^3 \setminus K) \to \widehat{K_q}^{\mathrm{loc}}(S^3 \setminus K)$ the natural localization of the nonsymmetric skein module of K at all nonzero polynomials in X. Then*

1. *The map* η *is an injection.*

2. *The natural action of* $\mathcal{H}_{q,(t_1,t_2,1,1)}$ *on* $\widehat{K_q}^{loc}(S^3 \setminus K)$ *preserves the image of* η

and thus $\widehat{K_q}(S^3 \setminus K)$ *carries the natural structure of a* $\mathcal{H}_{q,(t_1,t_2,1,1)}$*-module.*

The operator $L_{t_1,t_2} = Y_{t_1,t_2} + Y_{t_1,t_2}^{-1} := T_1T_0 + T_0^{-1}T_1^{-1}$ acting on $\mathbb{C}[X^{\pm 1}]$ occupies a central place in the theory of DAHA; it is known as the *Askey-Wilson operator*. It can be shown that L_{t_1,t_2} preserves the subspace of symmetric functions $\mathbb{C}[X+X^{-1}]$ and for generic parameters is diagonalizable in this basis. The eigenvectors of this operator are the famous *Askey-Wilson polynomials* [24]. Note that when $t_1 = t_2 = 1$, $Y_{1,1} = Y$ in D_q and thus L_{t_1,t_2} is a natural $\underline{t}$-deformation of $Y + Y^{-1}$.

As an application of their conjecture, Berest and Samuelson use this operator to define a two-parameter family of knot invariants which deform the colored Jones polynomials, which we will call the *generalized Jones polynomials*. A key fact which they use is if $q^4 \neq 1$, then a pairing between right (resp. left) $A_q^{\mathbb{Z}_2}$-modules M, N

$$M \otimes_{A_q^{\mathbb{Z}_2}} N \to \mathbb{C} \tag{1.10}$$

naturally induces a pairing between right (resp. left) $A_q \rtimes \mathbb{Z}_2$-modules $\widehat{M}, \widehat{N}$

$$\widehat{M} \otimes_{A_q \rtimes \mathbb{Z}_2} \widehat{N} \to \mathbb{C} \tag{1.11}$$

This, along with the Askey-Wilson operator $L_{t_1,t_2} = Y_{t_1,t_2} + Y_{t_1,t_2}^{-1}$, allows the definition

Definition 1. *Let* K *be a knot for which the BS conjecture holds. Let* $\widehat{K_q}(D^2 \times S^1)$ *represent the nonsymmetric skein module of the solid torus and*

$$\langle , \rangle : \widehat{K_q}(D^2 \times S^1) \otimes_{A_q \rtimes \mathbb{Z}_2} \widehat{K_q}(S^3 \setminus K) \to \mathbb{C}$$

the pairing naturally induced by the topological pairing. Then define the generalized Jones polynomials $J_n(K; q, t_1, t_2)$ *associated to* K *by*

$$J_n(K; q, t_1, t_2) = (-1)^{n-1}\langle \varnothing, S_{n-1}(Y_{t_1,t_2} + Y^{-1}_{t_1,t_2}).\varnothing\rangle \tag{1.12}$$

where $\varnothing$ *in a nonsymmetric skein module* $\widehat{K_q}(M)$ *is an element uniquely determined by the empty skein in the symmetric skein module* $K_q(M)$.

Note that by construction, when $(t_1, t_2) = (1, 1)$, (1.12) becomes (1.8), and thus $J_n(K; q, 1, 1) = J_n(K; q)$. It can also be shown that $J_n(K; q, t_1, t_2) \in \mathbb{C}[q^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}]$.

The subcase of the BS conjecture where $q = -1$ is of independent interest and was studied in [34]. Let π denote a finitely generated group, and introduce the *Brumfiel-Hilden algebra* associated to π

$$H\pi := \frac{\mathbb{C}[\pi]}{\{[g, h + h^{-1}]\ \forall g, h \in \pi\}}, \qquad H^+\pi := \{a \in H\pi \mid a = \sigma(a)\}$$

where $\sigma : H\pi \to H\pi$ is the anti-automorphism given by reduction of the map $g \mapsto g^{-1}$ on $\mathbb{C}[\pi]$. Let us return to the case $\pi = \pi_1(S^3 \setminus K)$, and let $(m, l) \subset \pi$ be a peripheral system. In the original work of Brumfiel and Hilden [10] is the following conjecture

Conjecture 2. *(BH Conjecture) Let* $H^+\pi[m^{\pm 1}]$ *be the subalgebra of* $H\pi$ *generated by* $H^+\pi$ *and* $m^{\pm 1}$. *Then we have*

$$l \in H^+\pi[m^{\pm 1}] \tag{1.13}$$

One of the central results of [34] establishes the relation of the BS Conjecture at $q = -1$ to the BH Conjecture

Theorem 2. *(Berest, Samuelson, 2018) Assume that the map*

$$(m - m^{-1}) : H^+\pi \to H^+\pi[m^{\pm 1}]$$

given by multiplication by $m - m^{-1}$ *is injective. Then, the BS Conjecture at* $q = -1$ *is equivalent to the BH Conjecture.*

Thus, the BS Conjecture for arbitrary q may be viewed as a quantization of the classical BH conjecture.

1.2 Main Results

Part I of this thesis is focused on exploring the BS conjecture in the special case $q = -1$, specifically the BH conjecture. Commenting on results of Berest and Samuelson, Francis Bonahon suggested that "if the BH conjecture fails, it should fail for a non-invertible knot". All previously known examples for which the BH conjecture was proven to hold were invertible knots, i.e. knots K for which there is a group automorphism

$$\rho : \pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus K)$$

such that if (m, l) is the peripheral system, $\rho(m) = m$, $\rho(l) = l^{-1}$.

The first contribution of this thesis can now simply be described as a further investigation of the BS conjecture in the new case of a non-invertible knot, in the specific case $q = -1$. (Recall that work of Trotter [32] establishes that the (p, q, r)-pretzel knot is non-invertible provided p, q, r are all distinct odd numbers)

Theorem 3. *For* $(p, q, r) = (3, 5, 7)$ *and* $(3, 5, 9)$*, the pretzel knots* $K_{p,q,r}$ *satisfy the Brumfiel-Hilden conjecture.*

The proof of Theorem 3 is reduced to a computer calculation. The resulting code has the following virtue:

Remark 1. *The code developed to establish Theorem 3 also allows for (relatively) efficient determination of* $SL_2(\mathbb{C})$ *character schemes for 3-generator groups.*

This, in turn, leads to the possibility of computing more efficiently A-polynomials of knots which admit 3-generator presentations of their knot groups. To the best of the author's knowledge, current technology for computing A-polynomials either depends on a 2-generator presentation of the knot group or, alternately, an efficient decomposition of the complement into tetrahedra ([7], [4]).

Additionally at the $q = -1$ level, where everything can be viewed purely group-theoretically, our deformation problem can also be posed for virtual knots. The result of this is the following theorem.

Theorem 4. *For every 2-bridge knot* K*, there exists an infinite family of virtual knots* $\{K_n\}$*,* $n \geq 1$ *such that (the analogue of) the Brumfiel-Hilden conjecture holds.*

Part II of this thesis is devoted to the study of the generalized Jones polynomials as defined by Berest and Samuelson. It can also be viewed as new evidence in support of the BS conjecture for arbitrary knots K and arbitrary q. To explain the result, we must first recall a classical fact from quantum topology.

Habiro [11] found a remarkable formula - the so-called cyclotomic expansion for the ordinary colored Jones polynomials $\{J_n(K;q)\}$ for any knot K:

Theorem 5. *(Habiro, 2001)*

$$J_n(K) = \frac{1}{q^2 - q^{-2}} \sum_{i=0}^{n-1} \prod_{j=-i}^{i} (q^{2(n+j)} - q^{-2(n+j)}) H_i(K)$$

where $H_i(K)$ *are integral Laurent polynomials in* q*, depending on* K*.*

The terms $H_i(K)$ are known as the *Habiro cyclotomic polynomials* of K. The coefficients of Habiro's cyclotomic expansion will appear repeatedly; we introduce the notation

$$c_{n,i} := \frac{1}{q^2 - q^{-2}} \prod_{j=-i}^{i} (q^{2(n+j)} - q^{-2(n+j)}), \quad i = 0, 1, \ldots, n-1$$

We also must recall the inductive definition of the *Macdonald polynomials* in terms of the Pieri relations. For a parameter $\beta \in \mathbb{C}^*$, define $p_n(X, \beta|q) \in \mathbb{C}_q[X + X^{-1}]$, $\mathbb{C}_q = \mathbb{C}(q)$ by

$$p_0 = 1, \qquad p_1 = X + X^{-1} \tag{1.14}$$

$$p_{n+1}(X, \beta|q) = (X + X^{-1})p_n(X; \beta|q) + \frac{(1-q^n)(1-\beta^2 q^{n-1})}{(1-\beta q^{n-1})(1-\beta q^n)} p_{n-1}(X, \beta|q) \tag{1.15}$$

The main theorem of this thesis is an explicit (t_1, t_2)-deformation of Habiro's cyclotomic formula.

Theorem 6. *If* K *satisfies the BS conjecture, then* $J_n(K; q, t_1, t_2)$ *is given by*

$$J_n(K; q, t) = \sum_{k=0}^{n-1} \widetilde{c_{n,k}}(q, t_1, t_2) H_k(K) \tag{1.16}$$

where $H_i(K) \in \mathbb{Z}[q^{\pm 1}]$ *are the Habiro cyclotomic polynomials of* K*. In the particular case* $t_2 = 1$*, the coefficients* $\widetilde{c_{n,k}}$ *are given by the formula*

$$\widetilde{c_{n,k}}(q, t, 1) = \frac{p_{n-k-1}(q^{2(k+1)}t^{-1}; q^{4(k+1)}|q^4)}{p_{n-k-1}(q^{2(k+1)}; q^{4(k+1)}|q^4)} \left(\prod_{i=1}^{k} \frac{q^{2i+1}t^{-1} - q^{-2i-1}t}{q^{2i+1} - q^{-2i-1}} \right) c_{n,k} \tag{1.17}$$

As a result of the proof of Theorem 6, we have the following corollary.

Corollary 1. *The expressions* $\widetilde{c_{n,i}}$*, and thus* $J_n(K; q, t_1, t_2)$*, lie in* $\mathbb{Z}[q^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}]$*.*

Contrast this with the original cyclotomic expansion: in that case, the most surprising part of the theorem is that the $H_i(K)$ were integral Laurent polynomials in q.

We offer two proofs of the formula in Theorem 6 when $t_2 = 1$. The first is based on an inductive argument, using the Pieri relations defining the Macdonald polynomials. The second uses a change of variable argument to deduce the same result from a generating function perspective. Of course, the two proofs are related.

1.3 Future Work

An important conjecture in the study of Jones polynomials is the Volume Conjecture (see [21])

Conjecture 3. *(Murakami, 2007) Let* $J_n(K; q)$ *be the* n*th colored Jones polynomial, normalized so that its value on the unknot is* 1*. Then*

$$\lim_{n\to\infty} \frac{1}{n} \log |J_n(K; e^{2\pi i/n})| = \frac{1}{2\pi} \mathit{Vol}(S^3 \setminus K)$$

where $\mathit{Vol}(S^3 \setminus K)$ *is the simplicial volume of the manifold* $S^3 \setminus K$*; when* $S^3 \setminus K$ *is hyperbolic, this coincides with its volume.*

The simplest case for which the Volume Conjecture is known is the figure-eight knot E. In fact, in the case of the figure-eight knot, a deformation of the limit is known, as the following theorem [22] of Murakami illustrates.

Theorem 7. *(Murakami, Yokota, 2012) Let* E *be the figure-eight knot. There exists a neighborhood* U *of* 0 *in* $\mathbb{C}$ *such that for any* $u \in (U \setminus \pi i \mathbb{Q}) \cup \{0\}$*, the following limit* *exists*

$$\lim_{n\to\infty} \frac{1}{n} \log J_n(K; \exp((u + 2\pi i)/n)) \tag{1.18}$$

and this limit determines the $SL_2(\mathbb{C})$ *Chern-Simons invariant associated with an irreducible representation of* $\pi_1(S^3 \setminus E)$ *to* $SL_2(\mathbb{C})$ *determined by the parameter* u.

The proofs of both of these statements are organized around Habiro's cyclotomic formula for the figure-eight. It would be very interesting to see how the parameters t_1, t_2 enter into this limit; that is, it would be very interesting to compute

$$\lim_{n\to\infty} \frac{1}{n} \log \left| \frac{J_n(E)(e^{2\pi i/n}, t_1, t_2)}{J_n(U)(e^{2\pi i/n}, t_1, t_2)} \right|. \tag{1.19}$$

as our deformation parameters t_1, t_2 are fundamentally different from Murakami's deformation parameter u in that they are independent of the variable q and do not represent simply a different specialization of it.

As in the $t_2 = 1$ case we have an explicit formula for our deformed Habiro coefficients $\widetilde{c_{n,k}}(q, t, 1)$, perhaps (1.19) could be computed in a manner similar to that of the limits in [21], [22].

1.4 Organization

The first three chapters after this one consist entirely of background material. Chapter 2 contains a digest of the theory of character schemes and all results from Brumfiel-Hilden theory which go into the proofs of Theorems 3 and 4. Chapter 3 reviews the Kauffman Bracket Skein Module, its relation to $SL_2(\mathbb{C})$-character schemes and the quantum torus. Chapter 4 assembles the few theorems about the A_1 and $C^\vee C_1$-DAHA that will be used in the proof of Theorem

6.

The longest chapter is Chapter 5; it contains a more lengthy restatement of the BS conjecture and the proof of Theorem 6 (Main Theorem), along with the derivation of Corollary 1. It depends only on Chapters 3 and 4 for background material. Finally, Chapter 6 contains the proofs of Theorems 3 and 4, and only depends on Chapter 2. It describes in pseudocode an algorithm which is documented in the appendix.

CHAPTER 2

CHARACTER VARIETIES

2.1 General Case

In this section we offer a basic treatment of the theory of representation schemes and character schemes, following Lubotzky and Magid [1]. We will try to strike a balance between the general and the specific, as each perspective can be valuable in its own right.

As a first concession to specificity, we fill fix our base field to be $\mathbb{C}$ throughout, and for us G will always mean GL_n or SL_n. Of course many results here can be extended to arbitrary algebraically closed base fields k of characteristic 0 and more general (e.g. matrix) groups. For us, we will consider G as functors from commutative $\mathbb{C}$-algebras to groups.

Fix a finitely generated group π and a group G, and let $\mathcal{H}om(\pi, G)$ be the functor from commutative $\mathbb{C}$-algebras to sets given by taking $\mathcal{H}om(\pi, G)(A) = \mathrm{Hom}_{\mathrm{Grp}}(\pi, G(A))$. Functoriality of $\mathcal{H}om(\pi, G)$ is a consequence of the functoriality of G. The first result about this functor is the following:

Proposition 1. *The functor* $\mathcal{H}om(\pi, G)$ *is representable. That is, there exists a commutative* $\mathbb{C}$*-algebra* $\mathcal{O}(\mathcal{H}om(\pi, G))$ *along with a representation* $\rho_u : \pi \to G(\mathcal{O}(\mathcal{H}om(\pi, G)))$ *such that for any commutative* $\mathbb{C}$*-algebra* A *and any representation* $\rho : \pi \to G(A)$*, there exists a unique* $f : \mathcal{O}(\mathcal{H}om(\pi, G)) \to A$ *such that* $\rho = \mathcal{H}om(\pi, G)(f) \circ \rho_u$.

Proof. We will focus on the GL_n case, and indicate as we proceed what are the modifications for the SL_n case.

As π is finitely generated, choose a presentation $\pi = \langle g_p \mid r_q(g_1, ..., g_n)\rangle$ with $1 \le p \le m$ and $q \in Q$. Call $B = \mathbb{C}[x_{i,j}^{(p)}]$, $1 \le i, j \le n$, $1 \le p \le m$, and denote by $X^{(p)}$ the $n \times n$ matrix with entries the variables $x_{i,j}^{(p)}$, so $\det(X^{(p)})$ is an element of B. Let C be the algebra formed from B by inverting all of these elements. (In the $G = SL_n$ case, C is instead the algebra formed by dividing B by the ideal generated by $\det(X^{(p)}) - 1$).

For each r_q consider the product $r_q(\overline{X}^1, ..., \overline{X}^m) \in GL_n(C)$. Letting $[r^q]_{i,j}$ denote the (i, j)-th entry of this matrix, we can define the ideal J of C to be that generated by

$$\{[r^q]_{i,j} - \delta_{i,j} \mid 1 \le i, j \le n, q \in Q\}$$

Finally, set $\mathcal{O}(\mathcal{H}om(\pi, G))) = C/J$. If we denote the image of $x_{i,j}^{(p)}$ and $X^{(p)}$ in the quotient by $\overline{x}_{i,j}^{(p)}$ and $\overline{X}^{(p)}$, respectively, we can define the representation ρ_u by sending $\rho_u(g_p) = \overline{X}^{(p)}$.

For any other $\mathbb{C}$-algebra A and any other representation $\rho : \pi \to GL_n(A)$, the equation

$$\rho = \mathcal{H}om(\pi, G)(f) \circ \rho_u$$

requires that $f(\overline{x}_{i,j}^{(p)}) = [\rho(g_p)]_{i,j}$. As the $\overline{x}_{i,j}^{(p)}$ generate $\mathcal{O}(\mathcal{H}om(\pi, G))$, f is therefore unique if it exists. Of course it does exist, as the relations defining $\mathcal{O}(\mathcal{H}om(\pi, G))$ are precisely those that are satisfied by any G-representation of π.

□

Note that while demonstrating the existence of $\mathcal{O}(\mathcal{H}om(\pi, G))$ required a presentation of π, general properties of representing objects implies that $\mathcal{O}(\mathcal{H}om(\pi, G))$ is unique up to unique isomorphism. We will also denote by $\mathcal{H}om(\pi, G)) = \mathrm{Spec}(\mathcal{O}(\mathcal{H}om(\pi, G)))$ (which the reader probably saw coming) and call this the n*th representation scheme* of π in G. When the functor of points and not the space is meant, we will make a note of this.

By construction, $\mathcal{H}om(\pi, G)$ parametrizes representations of π into G, which are often best thought of up to conjugation. The following proposition tells us how this equivalence looks scheme-theoretically.

Proposition 2. *The map* $\alpha : G \times \mathcal{H}om(\pi, G) \to \mathcal{H}om(\pi, G)$ *is a morphism of schemes, where* $\alpha(T, \rho) = T\rho T^{-1}$. *More is true:* α *is a group scheme action in the sense that*

1) $I \cdot \rho = \rho$ *for all* ρ.

2) $T_1 \cdot (T_2 \cdot \rho) = (T_1 T_2) \cdot \rho$.

Proof. What is meant above is simply for every $\mathbb{C}$-algebra A, α is the map on sets

$$G(A) \times \mathcal{H}om(\pi, G)(A) \to \mathcal{H}om(\pi, G)(A)$$

taking $(T, g \mapsto \rho(g))$ to $(g \mapsto T\rho(g)T^{-1})$. In this sense, properties (1) and (2) are evident. That this is a natural transformation is readily checked, from which it follows again by representability that this is given by a scheme map. □

This gives the important corollary that $G(\mathbb{C})$ acts on $\mathcal{O}(\mathcal{H}om(\pi, G))$ by ring maps. Unraveling the functorial language shows that this action is precisely the anticipated one. Let $T \in G(\mathbb{C})$, and let $Y^{(p)} = T\overline{X}^{(p)}T^{-1}$ in the language of before. Then the assignment $\overline{x}_{i,j}^{(p)} \mapsto [Y^{(p)}]_{i,j}$ defines a ring map of $\mathcal{O}(\mathcal{H}om(\pi, G))$ as the entries of the $Y^{(p)}$ satisfy the same relations as the $\overline{X}^{(p)}$:

$$\alpha_T : \mathcal{O}(\mathcal{H}om(\pi, G)) \to \mathcal{O}(\mathcal{H}om(\pi, G))$$

which is the one produced by general theory above.

Let us denote by $\mathcal{O}(\mathcal{X}(\pi, G)) = \mathcal{O}(\mathcal{H}om(\pi, G))^{G(\mathbb{C})}$. Our next objective will be to demonstrate that for a fixed G both the assignments

$$\pi \to \mathcal{O}(\mathcal{H}om(\pi, G)), \qquad \pi \to \mathcal{O}(\mathcal{X}(\pi, G))$$

are functorial in the variable π.

Proposition 3. *If* $\phi : \pi_1 \to \pi_2$ *is a group homomorphism, then the map* $\phi^* : \mathcal{H}om(\pi, G) \to \mathcal{H}om(\pi, G)$ *given by* $\rho \mapsto \rho \circ \phi$ *is a map of schemes, and thus dually induces a map* $\phi_* : \mathcal{O}(\mathcal{H}om(\pi_1, G)) \to \mathcal{O}(\mathcal{H}om(\pi_2, G))$. *This map restricts to a map* $\phi_* : \mathcal{O}(\mathcal{X}(\pi_1, G)) \to \mathcal{O}(\mathcal{X}(\pi_2, G))$.

Proof. As in the previous cases, it is easy to check that ϕ is a natural transformation. That the map ϕ_* restricts to a map from $\mathcal{O}(\mathcal{X}(\pi_1, G)) \to \mathcal{O}(\mathcal{X}(\pi_2, G))$ is a consequence of the fact that the natural transformations on $\mathcal{H}om(\pi, G)$ defining the $G(\mathbb{C})$-action and ϕ^* commute, as can be seen:

$$(T\rho T^{-1}) \circ \phi = T(\rho \circ \phi)T^{-1}$$

□

Let us summarize what has been shown: for G either GL_n or SL_n and π any finitely generated group, there exists a $\mathbb{C}$-algebra $\mathcal{O}(\mathcal{H}om(\pi, G))$ parametrizing representations of π into G in the sense that there are bijections (natural in A)

$$\mathrm{Hom}_{\mathrm{Alg}_{/\mathbb{C}}}(\mathcal{O}(\mathcal{H}om(\pi, G)), A) \to \mathrm{Hom}_{\mathrm{Grp}}(\pi, G(A))$$

Moreover, $G(\mathbb{C})$ acts on $\mathcal{O}(\mathcal{H}om(\pi, G))$ by ring automorphisms and thus there is a well-defined invariant ring $\mathcal{O}(\mathcal{X}(\pi, G))$. Finally, the assignment $\pi \to \mathcal{O}(\mathcal{H}om(\pi, G))$ is functorial and all morphisms are $G(\mathbb{C})$-equivariant, so the assignment $\pi \to \mathcal{O}(\mathcal{X}(\pi, G))$ is functorial as well. It is a nontrivial result that for π a finitely presented group, the ring $\mathcal{O}(\mathcal{X}(\pi, G))$ is always finitely generated as a $\mathbb{C}$-algebra. Thus we define the *character scheme* of π in G as $\mathcal{X}(\pi, G) := \mathrm{Spec}(\mathcal{O}(\mathcal{X}(\pi, G)))$.

Also appearing in the literature are the terms "representation variety" and "character variety" of a pair (π, G). The former, denoted $\mathrm{Hom}(\pi, G)$, is the zero locus of the ideal J defining $\mathcal{O}(\mathcal{H}om(\pi, G))$. Thus, strictly speaking, it may not be irreducible. The ring of regular functions on $\mathrm{Hom}(\pi, G)$ is then given by $\mathcal{O}(\mathrm{Hom}(\pi, G)) = \mathcal{O}(\mathcal{H}om(\pi, G))/\sqrt{0}$.

The "character variety" of a pair (π, G), denoted $X(\pi, G)$ is defined in terms of $\mathcal{O}(\mathcal{X}(\pi, G))$. While it is not evident from the definition that $\mathcal{O}(\mathcal{X}(\pi, G)))$ is always a finitely generated algebra provided π is also finitely generated, it is true that this is the case. In fact, fixing a presentation of π gives a realization

$$\mathcal{O}(\mathcal{X}(\pi, G))) \cong \mathbb{C}[y_1, ..., y_m]/I$$

for some ideal I, defining an affine algebraic set, the vanishing locus of I. Define $X(\pi, G)$ as the vanishing locus of I, and thus $\mathcal{O}(X(\pi, G)) = \mathcal{O}(\mathcal{X}(\pi, G))/\sqrt{0}$.

2.2 $SL_2(\mathbb{C})$ character varieties

One of the natural appearances of G-character varieties in nature comes when $G = SL_2(\mathbb{C})$ and π is the fundamental group of a three-dimensional manifold.

Work of Thurston [31] showed that in some sense "many" three-manifolds M carry a hyperbolic structure, determined by a discrete faithful representation

$$\rho : \pi_1(M) \to PSL_2(\mathbb{C})$$

and that this representation always admits a lift to $SL_2(\mathbb{C})$. Thus, the $SL_2(\mathbb{C})$ character variety for $\pi = \pi_1(M)$ represents in some sense the moduli space of hyperbolic structures on M. In what follows we will implicitly specialize to the $SL_2(\mathbb{C})$ case and use the shorthand

$$\mathcal{O}(\mathcal{H}om(\pi)), \quad \mathcal{H}om(\pi), \quad Hom(\pi)$$

$$\mathcal{O}(\mathcal{X}(\pi)), \quad \mathcal{X}(\pi), \quad X(\pi)$$

to denote the objects

$$\mathcal{O}(\mathcal{H}om(\pi, SL_2(\mathbb{C}))), \quad \mathcal{H}om(\pi, SL_2(\mathbb{C})), \quad Hom(\pi, SL_2(\mathbb{C}))$$

$$\mathcal{O}(\mathcal{X}(\pi, SL_2(\mathbb{C}))), \quad \mathcal{X}(\pi, SL_2(\mathbb{C})), \quad X(\pi, SL_2(\mathbb{C})).$$

A reader that is familiar primarily with character varieties in the setting of 3-dimensional manifolds is most likely familiar with the exposition given by Culler and Shalen in their seminal work [18] on splittings of 3-manifolds. Thus to make the notation above more accessible, we present their development of character varieties here in our language.

Fix a finitely generated group π. For any element $g \in \pi$, we have via the universal representation $\rho_u(g)$ an element of $SL_2(\mathcal{O}(\mathcal{H}om(\pi)))$. Thus taking $tr_g = tr(\rho_u(g)) \in \mathcal{O}(\mathcal{H}om(\pi))$ gives an element which lies in $\mathcal{O}(\mathcal{X}(\pi))$. Denoting this map in aggregate $Tr : \pi \to \mathcal{O}(\mathcal{X}(\pi))$, Culler and Shalen define their "ring of characters" R_{CS} to be the image of Tr inside $\mathcal{O}(\mathcal{X}(\pi))$. To pass to an algebraic set, they give the following argument that R_{CS} is finitely generated provided π is.

Theorem 8. *(Culler, Shalen, 1983) Let π be a finitely generated group with generating set $g_1, ..., g_n$. Then the set*

$$\{tr_g \mid g = g_{i_1} g_{i_2} ... g_{i_k},\}$$

with all indicies $i_1, ..., i_k$ distinct generates R_{CS}.

Proof. The proof is based on the trace identity

$$\mathrm{tr}(x)\mathrm{tr}(y) = \mathrm{tr}(xy) + \mathrm{tr}(xy^{-1}) \tag{2.1}$$

which holds for all $x, y \in SL_2(\mathbb{C})$ due to the Cayley-Hamilton theorem for $SL_2(\mathbb{C})$, $y + y^{-1} = \mathrm{tr}(y) \cdot I$. Let R^0_{CS} be the subring of R_{CS} generated by the traces as in the statement of the theorem. We first show that for all $g \in \pi$ of the form

$$g = g_{i_1}^{m_1} g_{i_2}^{m_2} ... g_{i_k}^{m_k}, \quad m_i \in \mathbb{Z}$$

where all the i_j are distinct we have $\mathrm{tr}_g \in R^0_{CS}$. The proof will be by induction on the integer

$$\nu = \sum_{i=1}^{k} K_i$$

where $K_i = -m_i$ if $m_i \leq 0$ and $m_i - 1$ if $m_i > 0$. Note that if $\nu = 0$, there is nothing to prove. Else, we may assume (by conjugating g if necessary) that $m_r \neq 0, 1$. In the case $m_r < 0$, we have

$$g = gg_{i_r} \cdot g_{i_r}^{-1}$$

and thus by applying (2.1) we find

$$\mathrm{tr}_g = \mathrm{tr}_{gg_{i_r}} \mathrm{tr}_{g_{i_r}^{-1}} - \mathrm{tr}_{gg_{i_r}^2}$$

where each trace term on the right hand side corresponds to an element of π with strictly smaller ν. A similar reduction applies when $m_r > 0$, using instead $g = gg_{i_r}^{-1} \cdot g_{i_r}$.

Now consider the general case, $g = g_{i_1}^{m_1}...g_{i_k}^{m_k}$ with not all i_j distinct. Our argument will now be via induction on the number of repeated indicies. We may assume there exists at least one repeated index by virtue of what we have already shown. We may also assume (replacing g with a conjugate if necessary) that there exists some $j < k$ with $i_j = i_k$, and critically we may choose this conjugate in a way which does not increase the count of repeats. Writing

$$V = g_{i_1}^{m_1}...g_{i_j}^{m_j} \quad W = g_{i_{j+1}}^{m_{j+1}}...g_{i_k}^{m_k}$$

we have, again applying (2.1)

$$\mathrm{tr}_g = \mathrm{tr}_{VW} = \mathrm{tr}_V \mathrm{tr}_W - \mathrm{tr}_{VW^{-1}}$$

and note that each of the words V, W and VW^{-1} have at least one fewer repeat than g. This completes the proof. □

At this point, it is unclear how R_{CS} and $\mathcal{O}(\mathcal{X}(\pi))$ are related. As we will see in the next section, R_{CS} is actually all of $\mathcal{O}(\mathcal{X}(\pi))$.

2.3 Brumfiel Hilden Theory

In this section, we will keep the specialization of the previous section, namely: all representation and character varieties are with respect to $SL_2(\mathbb{C})$.

To study concretely the structure of $\mathcal{O}(\mathcal{X}(\pi))$, we would like a method of building a presentation for $\mathcal{O}(\mathcal{X}(\pi))$ from a presentation of π. This is handled in an exhaustive manner in the text [10] of Brumfiel and Hilden. As we will make central use of the theory and language they develop in their text, we give here a summary of both.

To any $SL_2(\mathbb{C})$-representation ρ of π we have a corresponding algebra map

$$\mathbb{C}\rho : \mathbb{C}\pi \to M_{2\times 2}(\mathbb{C}) \tag{2.2}$$

given by extending $\mathbb{C}$-linearly. We can say slightly more, however; since for any $g \in \pi$ we must have $\rho(g) + \rho(g^{-1}) = \mathrm{tr}(\rho(g))I$ - the Cayley-Hamilton identity in $SL_2(\mathbb{C})$ again - we can form the algebra

$$H\pi := \frac{\mathbb{C}\pi}{\langle [h, g + g^{-1}] \quad \forall h, g \in \pi \rangle}$$

and by construction the map $\mathbb{C}\rho$ always factors to give a map

$$H\rho : H\pi \to M_{2\times 2}(\mathbb{C}). \tag{2.3}$$

Note that the anti-involution $\iota : \mathbb{C}\pi \to \mathbb{C}\pi$ taking $\iota(g) = g^{-1}$ descends to a well-defined map on $H\pi$. Let us call the $\mathbb{C}$-linear projections onto the $1, -1$-eigenspaces of this map $+, -$ respectively, so that on group elements

$$g^+ = \frac{g + g^{-1}}{2} \quad g^- = \frac{g - g^{-1}}{2}$$

As the elements of the $+1$ eigenspace are central by the defining relations of $H\pi$, they form a commutative subalgebra $H^+\pi$ which should be thought of as "all the traces" of representations of π

$$H^+\pi := \langle g + g^{-1} \mid g \in \pi \rangle \subset H\pi.$$

The elements of the -1-eigenspace do not form an algebra, though they are a module over $H^+\pi$. Calling them $H^-\pi$, we have a splitting of $H\pi$ as $H^+\pi$-modules:

$$H\pi = H^+\pi \oplus H^-\pi$$

Note that the relation between representations of π and maps from $H\pi$ is precisely the same if we replace $SL_2(\mathbb{C})$ and $M_{2\times 2}(\mathbb{C})$ with $SL_2(A)$ and $M_{2\times 2}(A)$ for any $\mathbb{C}$-algebra A. Thus, associated to the universal representation ρ_u of the previous section, we have an algebra map

$$H\rho_u : H\pi \to M_{2\times 2}(\mathcal{O}(\mathcal{H}om(\pi))) \tag{2.4}$$

The first major theorem established in Brumfiel and Hilden's work is the following:

Theorem 9. *(Brumfiel, Hilden 1995) The map* $H\rho_u$ *as described above is injective. More precisely, if* $M_{2\times 2}(\mathcal{O}(\mathcal{H}om(\pi)))^{SL_2(\mathbb{C})}$ *is the ring of* $SL_2(\mathbb{C})$*-equivariant matrix-valued functions on* $\mathcal{H}om(\pi)$*, then we can identify the images*

$$H\rho_u(H\pi) = M_{2\times 2}(\mathcal{O}(\mathcal{H}om(\pi)))^{SL_2(\mathbb{C})}$$

$$H\rho_u(H^+\pi) = \mathcal{O}(\mathcal{X}(\pi)).$$

On its own, this theorem would be a curiosity. However, almost the entirety of the rest of the work is devoted to giving compact presentations of $H^+\pi$ as a ring and $H\pi$ as a module over $H^+\pi$ when a presentation for π is known. Let us give a few examples of these structural theorems, which we will lean on heavily.

The first proposition shows that for a finitely-generated group π, $H^+\pi$ is generated as a ring by finitely many explicit elements. Additionally, it shows that $H\pi$ is finitely generated as an $H^+\pi$-module.

Proposition 4. *(Brumfiel, Hilden 1995) Suppose* π *is generated by* $\{g_i\}$*,* $1 \le i \le n$*. Let* $S \subset H^+\pi$ *be the subring given by* $S = \mathbb{C}[g_i^+, (g_j^- g_k^-)^+]$*,* $j < k$*. Then* $H^+\pi$ *is spanned as a module over* S *by* $\{1, (g_r^- g_s^- g_t^-)^+\}$*,* $r < s < t$*, and* $H^-\pi$ *is spanned as a module over* S *by* $\{g_i^-, (g_j^- g_k^-)^-\}$*,* $j < k$*. Thus,* $H\pi$ *is spanned as a module over* S *by* $\{1, g_i^-, (g_j^- g_k^-)^-, (g_r^- g_s^- g_t^-)^-\}$*,* $j < k$*,* $r < s < t$*.*

While this can be useful in some particular cases, it serves mostly to instill confidence that at the end of the day, any particular question about $H\pi$ for a finitely-generated group will be (at least theoretically) computationally accessible. To construct these algebras directly from a presentation of π requires first determining the structure of HF_n for F_n a free group on n generators and then establishing how a surjection $F_n \to \pi$ relates HF_n and $H\pi$.

2.3.1 Free Groups

Most of our interest will lie in studying HF_n for $n = 2, 3$, so we will begin with describing these algebras explicitly. For completeness, we will indicate how the story extends to general n, though we will not need these results exactly.

For $F_2 = \langle a, b \rangle$ the free group on two generators a and b, we introduce the notation

$$x := a^+, \quad y := b^+, \quad z := (a^- b^-)^+$$
$$|a| := a^-, \quad |b| := b^-, \quad |ab| := (a^- b^-)^-$$

with this, we have the following

Proposition 5. *(Brumfiel, Hilden, 1995) The commutative ring* H^+F_2 *is given by* $k[x, y, z]$*, which is a free polynomial algebra on the given three generators. As a module over* H^+F_2*,* H^-F_2 *is spanned by the elements* $\{|a|, |b|, |ab|\}$*. As an associative algebra,* HF_2 *is determined by the following multiplication table*

	$\lvert a\rvert$	$\lvert b\rvert$	$\lvert ab\rvert$
$\lvert a\rvert$	x^2-1	$z+\lvert ab\rvert$	$-z\lvert a\rvert+(x^2-1)\lvert b\rvert$
$\lvert b\rvert$	$z-\lvert ab\rvert$	y^2-1	$-(y^2-1)\lvert a\rvert+z\lvert b\rvert$
$\lvert ab\rvert$	$z\lvert a\rvert-(x^2-1)\lvert b\rvert$	$(y^2-1)\lvert a\rvert-z\lvert b\rvert$	$z^2-(x^2-1)(y^2-1)$

For $F_3 = \langle a, b, c\rangle$ the free group on three generators, already the structure of H^+F_3 is complicated. Unlike in the $n = 2$ case, HF_3 fails to be a free module over H^+F_3. Instead of looking for a presentation of HF_3 as a H^+F_3-module, we will therefore take the approach of looking for a subring $S \subset H^+F_3$ such that H^+F_3 and HF_3 are free as modules over S. We will now need the additional notation:

$$|abc| = (a^-b^-c^-)^+,\quad \begin{vmatrix} a \\ b \end{vmatrix} = (a^-b^-)^+,\quad \begin{vmatrix} a \\ c \end{vmatrix} = (a^-c^-)^+,\quad \begin{vmatrix} b \\ c \end{vmatrix} = (b^-c^-)^+,$$

Proposition 6. *(Prop 6.5 and 7.4 in [10]) Let $S \subset H^+F_3$ be the subring given by*

$$S = \mathbb{C}[a^+, b^+, c^+, \begin{vmatrix} a \\ b \end{vmatrix}, \begin{vmatrix} a \\ c \end{vmatrix}, \begin{vmatrix} b \\ c \end{vmatrix}]$$

then

1. *The ring S is a free polynomial ring on its generators.*
2. *As an S-module, H^+F_3 is free of rank 2 with basis $\{1, |abc|\}$.*
3. *As an S-module, HF_3 is free of rank 8 with basis $\{1, |abc|, |a|, |b|, |c|, |ab|, |ac|, |bc|\}$.*

We note that this has the corollary that H^-F_3 is also free as a S-module, since the S-module basis for HF_3 is an extension of that for H^+F_3.

To then use this to determine the structure of HF_3 as an algebra in its totality would require recording an 8×8 multiplication table. While this will eventually be necessary to do in the code, for now we will simply note the proposition which would give us all the information we need.

Proposition 7. *(Prop. 5.7 in [10]) With respect to the decomposition* $H\pi = H^+\pi \oplus H^-\pi$, *we have for all* $u, v, w, x, y, z \in H\pi$

1. $|u||x| = \begin{vmatrix} u \\ v \end{vmatrix} \oplus |ux|$

2. $|u||xy| = |uxy| \oplus \left(\begin{vmatrix} x \\ u \end{vmatrix} |y| - \begin{vmatrix} y \\ u \end{vmatrix} |x| \right)$

3. $|xy||u| = |uxy| \oplus \left(\begin{vmatrix} y \\ u \end{vmatrix} |x| - \begin{vmatrix} x \\ u \end{vmatrix} |y| \right)$

4. $|xyz||u| = \begin{vmatrix} u \\ x \end{vmatrix} |yz| - \begin{vmatrix} u \\ y \end{vmatrix} |xz| + \begin{vmatrix} u \\ z \end{vmatrix} |xy|$

5. $|uv||xy| = \begin{vmatrix} u & v \\ y & x \end{vmatrix} \oplus \left(\begin{vmatrix} u \\ y \end{vmatrix} |vx| + \begin{vmatrix} v \\ x \end{vmatrix} |uy| - \begin{vmatrix} u \\ x \end{vmatrix} |vy| - \begin{vmatrix} v \\ y \end{vmatrix} |ux| \right)$

6. $|uvw||xy| = \begin{vmatrix} v & w \\ y & x \end{vmatrix} |u| - \begin{vmatrix} u & w \\ y & x \end{vmatrix} |v| + \begin{vmatrix} u & v \\ y & x \end{vmatrix} |w|$

where

$$\begin{vmatrix} u & v \\ y & x \end{vmatrix} := \begin{vmatrix} u \\ y \end{vmatrix} \begin{vmatrix} v \\ x \end{vmatrix} - \begin{vmatrix} u \\ x \end{vmatrix} \begin{vmatrix} v \\ y \end{vmatrix}$$

To continue in this way for general HF_n, one may hope to repeat the general picture: produce a subring $S \subset H^+F_n$ over which both H^+F_n and HF_n are relatively simple - hopefully free - modules over S. While the natural extension of the S used for $n = 2, 3$ and its variants fail to be polynomial algebras for $n \geq 4$, we do have the following result.

Proposition 8. *(Brumfiel, Hilden 1995) If F_n is a free group, H^+F_n is an integral domain and HF_n is a torsion-free H^+F_n-module.*

2.3.2 Quotient Groups

Proposition 9. *(Brumfiel, Hilden, 1995) If π is obtained from $\hat{\pi}$ by dividing out by the normal subgroup generated by a family of words $\{w_j(g_i^{\pm 1})\}$ where $\{g_i\}$ are elements of $\hat{\pi}$, then $H\pi$ is obtained from $H\hat{\pi}$ by dividing by the two-sided ideal generated by $\{w_j - 1\}$.*

$$H\pi \cong H\hat{\pi}/((w_j - 1))$$

Proof. The surjection $\hat{\pi} \to \pi$ induces a surjection on the level of group algebras and thus a surjection from $H\hat{\pi}$ to $H\pi$. Since all w_j are sent to the 1 in the group algebra, they are also killed by this surjection, and we have a well-defined map

$$H\hat{\pi}/((w_j - 1)) \to H\pi$$

To obtain an inverse, let's call $B = H\hat{\pi}/((w_j - 1))$. The natural map

$$\hat{\pi} \to U(B)$$

to the group of units of B factors through π. Additionally, for all $g \in \hat{\pi}$ (and thus $g \in \pi$), the image of $g + g^{-1}$ is central in B since $g + g^{-1}$ is central in $H\hat{\pi}$ and this surjects onto B. As such, there is an induced map

$$H\pi \to B$$

which can be shown to be the inverse of the above map by following elements of $\hat{\pi}$. $\square$

We note that the surjection above maps $H^+\hat{\pi}$ onto $H^+\pi$. This proposition admits the following refinement, which makes it computationally effective:

Proposition 10. *(Brumfiel, Hilden 1995) Using the language of the previous proposition, with* $\{x_j\} = \{w_j(g_i^{\pm 1}) - 1\} \subset H\hat{\pi}$*, define* $I = ((x_i))$ *and additionally* $I^+ = \{x^+ \mid x \in I\}$, $I^- = \{x^- \mid x \in I\}$. *Then we have*

1) $H^+\pi = H^+\hat{\pi}/I^+$

2) I^+ *is the ideal generated by* $(x_i^+, (x_i^- g_i^-)^+, (x_i^- g_i^- g_j^-)^+)$.

3) I^- *is spanned as an* $H^+\hat{\pi}$*-module by the elements*

$$\{x_i^+ g_j^-, x_i^+ (g_j^- g_k^-)^-, x_i^-, (x_i^- g_j^-)^-, (x_i^- g_j^-)^+ g_k^-, (x_i^- g_j^-)^+ (g_k^- g_l^-)^-\}$$

Proof. (1) Note first that for the generators x_j of I, we also have $\iota(x_j) \in I$ since $-w_j^{-1}(w_j - 1) = w_j^{-1} - 1$. Since ι is an antiautomorphism, by extension we thus have $\iota(I) = I$ and therefore $I^+ \subset I$ as $x^+ = \frac{x+\iota(x)}{2}$. This in turn guarantees $I^+ = I \cap H^+\hat{\pi}$. Since

$$H^+\pi \cong H^+\hat{\pi}/(I \cap H^+\hat{\pi})$$

we have the result. The proofs of (2) and (3) follow from longer computations. □

2.3.3 Examples

With the results of the two previous sections we can begin to compute a few examples of $H\pi$ and $H^+\pi$ for two-generator groups.

Example: $H\mathbb{Z}^2$

Note that since $\mathbb{Z}^2$ is abelian, the algebra $H\mathbb{Z}^2$ is simply the group algebra $\mathbb{C}[a^{\pm 1}, b^{\pm 1}]$. Thus the only interesting part of this example is giving the structure of $H^+\mathbb{Z}^2$ and describing $H\mathbb{Z}^2$ as a module over $H^+\mathbb{Z}^2$. Since the free abelian group on two generators has the presentation $\langle a, b \mid ab = ba \rangle$, by the above propositions we consider the two-sided ideal I in $H\pi_2$ generated by

$$ab - ba = (x + |a|)(y + |b|) - (y + |b|)(x + |a|) = 2|ab|$$

For $H^+\mathbb{Z}^2$, since $(|ab|)^+ = (|ab||a|)^+ = (|ab||b|)^+ = 0$, I^+ must be generated by $(|ab||a||b|)^+ = (|ab|(z + |ab|)^+ = z^2 - (x^2 - 1)(y^2 - 1)$. Thus

$$H^+\mathbb{Z}^2 \cong \mathbb{C}[x, y, z]/(z^2 - (x^2 - 1)(y^2 - 1))$$

As for the module structure of $H^-\mathbb{Z}^2$: since $H^-\mathbb{Z}^2$ is given as an $H^+\pi_2$-module as $H^-\pi_2/I^-$, we use the previous proposition to produce generators of I^- as an $H^+\pi_2$-module:

$$\{|ab|,\ |ab||a|\ |ab||b|\}$$

$$\{|ab|,\ z|a| - (x^2 - 1)|b|,\ (y^2 - 1)|a| - z|b|\}$$

So if we define $M := \mathbb{C}[x, y] \otimes \{|a|, |b|\}$ and give M a $H^+\mathbb{Z}^2$-module structure via

$$x.(f(x, y)|a| + g(x, y)|b|) = xf(x, y)|a| + xg(x, y)|b|$$

$$y.(f(x, y)|a| + g(x, y)|b|) = yf(x, y)|a| + yg(x, y)|b|$$

$$z.(f(x, y)|a| + g(x, y)|b|) = (y^2 - 1)g(x, y)|a| + (x^2 - 1)f(x, y)|b|$$

The obvious map $M \to H^-\mathbb{Z}^2$ is an isomorphism of $H^+\mathbb{Z}^2$-modules.

Example: $H^+\pi$**,** π **the knot group of the** (p, q)**-torus knot**

We know (see [8]) that the (p, q)-torus knot for $2 \leq p < q$, $\gcd(p, q) = 1$ admits the following two-generator presentation

$$\pi = \langle a, b \mid a^p = b^q \rangle$$

We will use the following simple observation: if $c \in H\pi$, then $c^n = T_n(c^+) + U_{n-1}(c^+)c^-$ for all n, where $T_n(x)$ and $U_n(x)$ are Chebyshev polynomials of the first and second kind, respectively, given by the recurrences

$$T_0 = 1, \quad T_1 = x, \quad T_{n+1} = 2xT_n - T_{n-1}$$

$$U_0 = 1, \quad U_1 = 2x, \quad U_{n+1} = 2xU_n - U_{n-1}.$$

In this case, we are looking for I^+ where $I = ((T_p(x) - T_q(y) + U_{p-1}(x)|a| - U_{q-1}(y)|b|))$. This is generated by

$$I^+ = (T_p(x) - T_q(y), (x^2-1)U_{p-1}(x) - zU_{q-1}(y), zU_{p-1}(x) - (y^2-1)U_{q-1}(y))$$

CHAPTER 3

QUANTIZATION: THE KBSM CONSTRUCTION

3.1 Kauffman Bracket for S^3

Here we recall some elementary knot theory and give the Kauffman-bracket construction for the Jones polynomial. Refer here to the original paper of Kauffman [15].

For us, a *knot* (resp. *link*) will be a smooth embedding of S^1 (resp. $\sqcup_{i=1}^n S^1$) into S^3, considered only up to ambient isotopy of S^3. Thus a representative of a knot will simply be a particular choice of embedding. More precisely: two embeddings

$$\iota_1, \iota_2 : S^1 \to S^3$$

are said to be equivalent if there exists a continuous map

$$F : S^3 \times [0, 1] \to S^3$$

such that F_t is a homeomorphism for all t, F_0 is the identity and $F_1 \circ \iota_1 = \iota_2$. Note that our knots and links carry an orientation naturally induced by the orientation of S^1; composing a knot K with the involution $x \to -x$ of S^1 gives the reverse of a knot, denoted $-K$. We say that two knots K_1, K_2 are equivalent as unoriented knots if either $K_1 = K_2$ or $K_1 = -K_2$. Viewing S^3 as $\mathbb{R}^{3+}$, the one-point compactification of $\mathbb{R}^3$, a representative of a knot can be chosen such that the projection to $\mathbb{R}^2$ is transverse, and moreover has at worst double points. Such a projection of a knot is called a *knot diagram*. An example is given below:

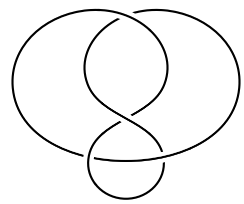
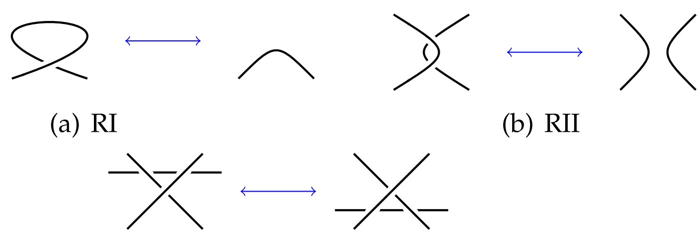


Figure 3.1: Fig-8 diagram

It is a classical theorem of Reidemeister that any two diagrams representing the same knot are related by the following collection of elementary local moves.



(a) RI

(b) RII

(c) RIII

Figure 3.2: Reidemeister Moves

By local it is meant that outside of the picture given, the diagrams remain the same.

Kauffman, in his study [15] of the skein relation defining the Jones polynomial, found it useful to consider the notion of "regular isotopy" of links. His definition was simply that two links were regularly isotopic if they arose from diagrams related by only Type II and III moves. This definition can be motivated geometrically by introducing the notion of a *framed knot*. A *framed knot* is an embedding

$$S^1 \times [0,1] \to S^3$$

considered up to ambient isotopy, with a framed link being defined similarly.

Restricting the embedding defining a framed knot to $S^1 \times \{1/2\}$ defines a knot, which then gives rise to a diagram. It can be shown that K_1, K_2 are equivalent as framed knots if and only if the diagrams D_1, D_2 resulting from this procedure are related by Type II and III moves.

Define a function on knot diagrams, the *writhe* $w(D)$ as follows: to each crossing p in the diagram D, assign the value $\epsilon(p)$ following the rule

(a) $+1$ (b) -1

Figure 3.3: Writhe

and set $w(D) = \sum_p \epsilon(p)$. It can be readily computed that $w(D)$ is not invariant under all three Reidemeister moves, but is an invariant of regular isotopy. The main discovery of Kauffman was the following

Theorem 10. *(Kauffman, 1987) The function*

$$\langle . \rangle : \textit{Unoriented diagrams} \to \mathbb{Z}[A, A^{-1}]$$

defined on knot diagrams inductively by the rules:

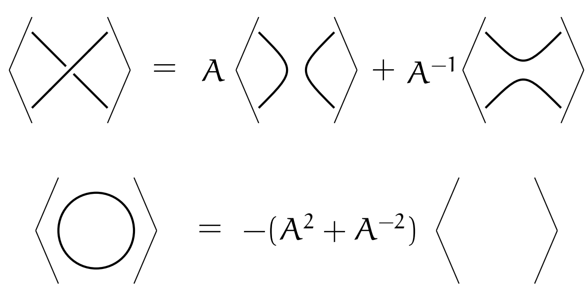


Figure 3.4: Skein Relations

is invariant under regular isotopy. (As in the Reidemeister moves, the pictures represent a local portion of a diagram which is otherwise unchanged.) The normalization of the empty diagram is taken to be 1.

It is clear that the first "skein" relation allows one to reduce a diagram to one with a series of disjoint circles, upon which the normalization relation can be used repeatedly to compute a value. The theorem of Kaufmann is that this value does not depend on the choice of diagram in the unoriented regular isotopy class one starts with. Moreover, it can be readily shown that while not invariant under the Type I move, the Kauffman bracket enjoys a fairly simple transformation:

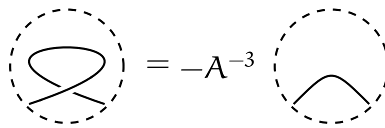


Figure 3.5: Type I Behavior

This equation coupled with the regular isotopy invariance of the bracket polynomial immediately implies that the following Laurent polynomial is an invariant of oriented links:

$$J(L; A) := (-A)^{-3w(D_L)} \langle D_L \rangle$$

where D_L is any diagram representing L.

It was observed by Kauffman that, properly normalized, this coincided with the definition of the then new polynomial invariant of oriented knots given by Jones in his seminal paper [14].

Theorem 11. *(Kauffman, 1987) Let $V_K(t)$ be the invariant of an oriented knot K as given by Jones. Then we have the equality*

$$J(K; t^{-1/4}) = V_K(t).$$

3.2 Skein Module for arbitrary M^3

In an attempt to organize a large body of work arising after the discovery of the Kauffman bracket polynomial and the HOMFLY-PT polynomial, Przytycki in [25] defined the Kauffman Bracket Skein Module (KBSM) of an oriented 3-manifold.

Here we give a definition of the KBSM as agrees with Przytycki. Fix a commutative ring R along with a choice of $r \in R^{\times}$. For M an oriented 3-manifold, denote by $\mathcal{L}$ the set of ambient isotopy classes of framed, unoriented links in M (including the empty link). Let $\mathcal{L}'$ denote the R-submodule of $R\mathcal{L}$ generated by all terms of the form

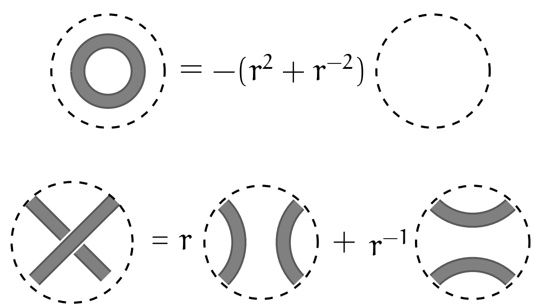


Figure 3.6: Framed skein relation

where outside of these oriented 3-balls the framed links can be any common embedding. Then define

$$S_{2,\infty}(M; R, r) := R\mathcal{L}/\mathcal{L}'$$

as the (R, r)-skein module of M. In the case $R = \mathbb{C}$ and $r = q$, we will denote this $\mathbb{C}$-vector space simply by $K_q(M)$. A few comments are in order: we will state them for $K_q(M)$, which will be in the continuation all we will consider, though the statements hold for the more general (R, r) case as well.

Remark 2. *(Functoriality) If* $\iota : M_1 \to M_2$ *is an oriented embedding, then it induces a map* $K_q(\iota) : K_q(M_1) \to K_q(M_2)$.

Remark 3. *(Monoidal) There is a natural isomorphism* $K_q(M_1 \sqcup M_2) \to K_q(M_1) \otimes K_q(M_2)$

Remark 4. *(Surfaces = Algebras) If* $M = \Sigma \times [0,1]$ *where* Σ *is a surface, then* $K_q(M)$ *carries the structure of an associative, unital algebra.*

Remark 5. *((*$\partial M \neq \varnothing$*) = Modules) If* $\partial M = \Sigma$*, then* $K_q(M)$ *carries the structure of a left module over the algebra* $K_q(\Sigma \times [0,1])$.

Proof. (1) is evident from the definition of the equivalence classes defining $K_q(M)$, since embeddings will preserve skein relations.

(2) Since the union is disjoint, any embedding $f : \sqcup_{i=1}^{n} S^1 \times [0,1] \to M_1 \sqcup M_2$ must split as $f_1 \sqcup f_2$, where $f_i : \sqcup_{i=1}^{n_i} S^1 \times [0,1] \to M_i$, $i \in 1,2$. Regular isotopies similarly split, and so the map

$$[f] \to [f_1] \otimes [f_2]$$

is the isomorphism we seek.

(3) This is a consequence of Remarks (1) and (2), as the embeddings $f_i : [0,1] \to [0,1]$, $i = 0,1$ given by $f_i(x) = 1/3x + 2/3i$ give an embedding $f_1 \sqcup f_0 : [0,1] \sqcup [0,1] \to [0,1]$ which in turn induces

$$\Sigma \times [0,1] \sqcup \Sigma \times [0,1] \to \Sigma \times [0,1]$$

and thus gives a multiplication $\mu : K_q(M) \otimes K_q(M) \to K_q(M)$. Note that in this multiplication, elements on the left are stacked "on top of" elements on the right.

(4) This is also a consequence of Remarks (1) and (2), along with the fact that in the smooth category if $N(\partial M)$ is a regular neighborhood of the boundary of M, there is a retraction which is isotopic to the identity $r : M \to M \setminus N(\partial M)$ that can be used along with the inclusion of $N(\partial M) \cong \partial M \times [0,1]$ to define an embedding

$$\iota_{N(\partial M)} \sqcup r : \partial M \times [0,1] \sqcup M \to M$$

and thus a function $A : K_q(\partial M \times [0,1]) \times K_q(M) \to K_q(M)$. It is apparent (with the outward pointing normal convention) that this function A satisfies the necessary relations with μ established before to define a left action of $K_q(\partial M \times [0,1])$ on $K_q(M)$. □

Note: In what follows, the skein algebras of three manifolds $\Sigma \times [0,1]$ - in particular, the three manifold $S^1 \times S^1 \times [0,1]$ - will be important. As such, the notation $K_q(\Sigma)$ will be used as shorthand to signify $K_q(\Sigma \times [0,1])$.

3.3 Relation to $\mathcal{O}(\mathcal{X}(M))$

Historically, what we now call "skein theory" - the study of the algebraic structure of skein algebras and modules associated to 3-manifolds - was advanced with little more tying it to "classical" topological invariants (e.g. the fundamental group, homology) than the Jones polynomial. Connections between evaluation of the Jones (or colored Jones) polynomial(s) at certain values and classical invariants were explored in [14], [26], and most impressively in [27]. In contrast, not much before 1998 was widely known about entire skein modules.

The work of Przytycki and Sikora in their two papers [12], [13] changed this.

It is their main result which we relate in this section.

At the special value $q = -1$, the skein relation

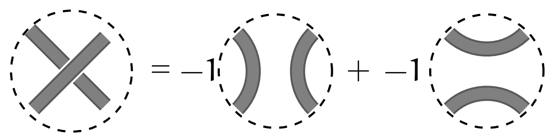


Figure 3.7: Skein at -1

implies that strands of skeins may be passed through themselves at will in $K_{-1}(M)$, and thus their framings can also be changed at will. In particular, this means that a skein consisting of a single loop at $q = -1$ is identifiable with its core circle up to free homotopy. Further, this means that if $L(M)$ denotes the $\mathbb{C}$-vector space of all formal linear combinations of (not framed) links in M, $K_{-1}(M)$ is a quotient of $L(M)$.

Observe as well that at $q = -1$ the vector space $K_{-1}(M)$ becomes a commutative algebra with the multiplication $L_1 \cdot L_2 = L_1 \cup L_2$. Strictly speaking, this does not produce another skein - however, a general position argument can be made to choose representatives of both L_1 and L_2 so that their union is disjoint, and crucially the ability to pass strands through each other gives that this is well-defined, independent of this choice. It is natural to then ask: is there an algebraic way to view this geometrically defined multiplication on $K_{-1}(M)$?

Recall from Proposition 1 the definition of the universal representation ρ_u associated to a group $\pi_1(M)$ and its $SL_2(\mathbb{C})$-representation scheme $\mathcal{O}(\mathcal{H}om(M))$. If composed with taking the trace, these maps together give a composite map

$$\mathrm{Tr} : \pi_1(M) \to \mathcal{O}(\mathcal{X}(M)) \tag{3.1}$$

such that $\mathrm{Tr}(\gamma)$ only depends on the conjugacy class of γ in $\pi_1(M)$.

We now define a function T from the vector space of all formal linear combinations of links in M, denoted $L(M)$, to $\mathcal{O}(\mathcal{X}(M))$. Fix a basepoint p of M and for any knot K, choose a path connecting p to K, producing an element $\gamma_K \in \pi_1(M)$. Set

$$T(K) = \mathrm{Tr}(\gamma_K)$$

since two different choices of path lead to conjugate γ_K, this is well defined. For links $L = K_1 \cup K_2 \cdots \cup K_n$, set

$$T(L) = T(K_1)T(K_2)\ldots T(K_n).$$

and finally extend T by linearity to define it on the vector space of all links. Now it is possible to state the theorem:

Theorem 12. *(Przytycki, Sikora, 1998) The map $T : L(M) \to \mathcal{O}(\mathcal{X}(M))$ descends to give an isomorphism of commutative algebras*

$$T : K_{-1}(M) \to \mathcal{O}(\mathcal{X}(M))$$

The subtlety in the proof comes in establishing the skein relation, which under T becomes the fundamental $SL_2(\mathbb{C})$ trace relation $\mathrm{tr}(A)\mathrm{tr}(B) = \mathrm{tr}(AB) + \mathrm{tr}(A^{-1}B)$.

3.4 $K_q(S^1 \times S^1)$ and the colored Jones polynomials

The link between the noncommutative algebra of the DAHA and skein theory is established by a beautiful theorem of Frohman and Gelca. For the original text, refer to [3]. Because this theorem will play such an essential role in what follows, we reproduce it here.

For any surface Σ, it is not difficult to see that $K_q(\Sigma)$ is spanned as a vector space by simple closed multicurves on $\Sigma \times \frac{1}{2}$ via projecting onto this level surface and resolving crossings. It can be shown that these multicurves in fact form a $\mathbb{C}$-basis of $K_q(\Sigma)$. In the case $\Sigma = S^1 \times S^1$, multicurves are of the form $(m, l)^n$ where (m, l) is the curve that wraps around m times in the meridian direction and l times in the longitudinal direction, where m and l are relatively prime, and n counts the number of parallel copies of this curve.

A different basis was taken by Frohman and Gelca: for any polynomial $p(x) \in \mathbb{C}[x]$ and any simple closed curve C on $S^1 \times S^1$, denote by $p(C) \in K_q(S^1 \times S^1)$ the image of the linear combination of C^n's given by the coefficients of p. In particular, take the Chebyshev polynomials of the first kind T_n, defined by

$$T_0 = 2, T_1 = x, T_{n+1} = xT_n - T_{n-1}$$

and consider the basis

$$(m, l)_T = T_{\gcd(m,l)}\left(\frac{m}{\gcd(m, l)}, \frac{l}{\gcd(m, l)}\right)$$

The observation to be made here is the

Lemma 1. *(Product Formula) The basis $\{(m, l)_T\}$ above satisfies the following product identity*

$$(m, l)_T * (r, s)_T = q^{ms-lr}(m + r, l + s)_T + q^{-ms+lr}(m - r, l - s)_T$$

Independently, consider the quantum torus

$$A_q := \frac{\mathbb{C}\langle X^{\pm 1}, Y^{\pm 1}\rangle}{\langle XY - q^2 YX\rangle}$$

where q is a complex parameter. The group $\mathbb{Z}_2$ acts on A_q by simultaneously inverting X and Y, and so the invariant elements form a subalgebra $A_q^{\mathbb{Z}_2}$. This subalgebra is spanned by elements of the form $X^m Y^l + X^{-m} Y^{-l}$, and if we take the shifted basis

$$\{f_{m,l} = q^{-ml} X^m Y^l + q^{-ml} X^{-m} Y^{-l}\}$$

it is then a calculation to show that the $f_{m,l}$ satisfy the same multiplication rule as the $(m,l)_T$. Thus we arrive at the

Theorem 13. *(Frohman, Gelca, 2000) The assignment* $(m,l)_T \to f_{m,l}$ *extends to an algebra isomorphism*

$$K_q(S^1 \times S^1) \to A_q^{\mathbb{Z}_2}.$$

Let us now relate this theorem to the colored Jones polynomials. Associated to the decomposition

$$S^3 = (S^1 \times D^2) \sqcup (S^3 \setminus K)$$

we have a map on skein modules - the *topological pairing*

$$\langle , \rangle : K_q(S^1 \times D^2) \otimes_{K_q(S^1 \times S^1)} K_q(S^3 \setminus K) \to K_q(S^3) \cong \mathbb{C}.$$

Note that this is $K_q(S^1 \times S^1)$-balanced since a skein in the intersection of $S^1 \times D^2$ and $S^3 \setminus K$ can just as readily be considered as lying in either space. Using the above theorem, identify $K_q(S^1 \times S^1)$ with $A_q^{\mathbb{Z}_2}$. The following theorem tells us that the data recorded by the colored Jones polynomials $J_n(K;q)$ lies entirely in this topological pairing.

Theorem 14. *(Kirby, Melvin, 1991) The colored Jones polynomials* $J_n(K;q)$ *are given by the formula*

$$J_n(K;q) = (-1)^{n-1} \langle \varnothing . S_{n-1}(Y + Y^{-1}), \varnothing \rangle$$

where $\varnothing$ *represents the empty skein. (With this convention, we have the normalizations*

$$J_n(O;q) = \frac{q^{2n} - q^{-2n}}{q^2 - q^{-2}}, \quad J_1(K;q) = 1$$

for O *the unknot and* K *any knot, respectively.)*

CHAPTER 4

DOUBLE AFFINE HECKE ALGEBRAS

4.1 Type A_1

Our goal in this section will be to give a definition of the *Macdonald polynomials*, $p_n(x)$, a family of polynomials in one variable depending on two complex parameters (q,t). These polynomials are orthogonal with respect to a certain inner product on the line and are intimitely related to the *Double Affine Hecke Algebra* or *DAHA* of type A_1. We will take a roundabout way to do this, essentially reversing the historical story. First, we will define the A_1-DAHA and investigate its polynomial representation. Subsequently, we will define the Macdonald polynomials as eigenvectors of a particular element of the DAHA acting in the polynomial representation. We will conclude by explicating some properties of the Macdonald polynomials which follow from this presentation and will be useful in what follows. The principal reference throughout will be the book of Cherednik, [5].

4.1.1 Definition and PBW Theorem

The DAHA of type A_1 is an associative, unital $\mathbb{C}$-algebra depending on 2 invertible complex parameters $q^{1/2}, t^{1/2}$, defined by

$$\mathcal{H}_{q,t} = \frac{\mathbb{C}\langle X^{\pm 1}, Y^{\pm 1}, T\rangle}{I}$$

where I is the two-sided ideal generated by the relations

$$TXT = X^{-1}, \quad TY^{-1}T = Y$$

$$Y^{-1}X^{-1}YXT^2q^{1/2} = 1$$

$$(T - t^{1/2})(T + t^{-1/2}) = 0$$

Alternately, note that the final relation implies that T is invertible, with inverse $T - (t^{1/2} - t^{-1/2})$; we can alternately choose a generator $\pi = YT^{-1}$ and define the same algebra via

$$\frac{\mathbb{C}\langle T, X^{\pm 1}, \pi\rangle}{J}$$

where J is the two-sided ideal generated by

$$TXT = X^{-1}, \quad \pi^2 = 1, \quad \pi X \pi^{-1} = q^{1/2}X^{-1}$$

$$(T - t^{1/2})(T + t^{1/2})$$

We want to realize this algebra as representing some family of operators on $\mathbb{C}[X^{\pm 1}]$, Laurent polynomials in X. The relevant result is the following

Theorem 15. *(Cherednik, 2005)*

(i) (PBW property) The elements $X^n T^\epsilon Y^m$, $n, m \in \mathbb{Z}$, $\epsilon = 0, 1$ *form a basis of* $\mathcal{H}_{q,t}$

(ii) Using $s(f(X)) = f(X^{-1})$ *and* $\pi(f(x)) = f(q^{1/2}X^{-1})$, *the formulas*

$$T \to t^{1/2}s + \frac{t^{1/2} - t^{-1/2}}{X^2 - 1}(s - 1), \quad X \to X, \quad Y \to \pi T$$

define a representation of $\mathcal{H}_{q,t}$ *in the space* $\mathbb{C}[X^{\pm 1}]$. *It is faithful for* q *apart from roots of unity.*

Proof. It is easier to check this defines a representation in the second presentation, where the abstract generator π acts as the operator of same name. The relations only involving π and X are quickly verified; the only two remaining are

$$TXT = X^{-1}, \quad (T - t^{1/2})(T - t^{-1/2})$$

We check the first, where it is sufficient to check on the monomials X^j:

$$TXT.X^j = TX.(t^{1/2}X^{-j} - (t^{1/2} - t^{-1/2})X^{-j}\sum_{k=0}^{j-1} X^{2k})$$

$$= (t^{1/2}s + \frac{t^{1/2} - t^{-1/2}}{X^2 - 1}(s-1)).(t^{1/2}X^{-j+1} - (t^{1/2} - t^{-1/2})X^{-j+1}\sum_{k=0}^{j-1} X^{2k})$$

since the second term is even, it is killed by the $s - 1$ term in T and preserved by the s operator, giving

$$= tX^{j-1} + (t-1)\frac{X^{j-1} - X^{-j+1}}{X^2 - 1} - (t-1)X^{-j+1}\sum_{k=0}^{j-1} X^{2k}$$

$$= tX^{j-1} + (t-1)X^{-j+1}\sum_{k=0}^{j-2} X^{2k} - (t-1)X^{-j+1}\sum_{k=0}^{j-1} X^{2k}$$

$$= X^{j-1} = X^{-1}.X^j$$

The computation in the case of the Hecke-type relation is similar, which establishes the first part of (ii). For (i), note that it is immediate from the relations that every element in $\mathcal{H}_{q,t}$ lies in the $\mathbb{C}$-span of the listed monomials. To demonstrate these are linearly independent, consider their action first on 1:

$$X^n T^\epsilon Y^m.1 = t^{(n+\epsilon)/2} X^n$$

which tells us that any dependence must split over dependences where n is fixed. Further, we compute the action of the monomials on X, using the fact that $Y.X = q^{-1/2}t^{-1/2}X$ and $T.X = t^{-1/2}X^{-1}$:

$$X^n T^\epsilon Y^m.X = \begin{cases} q^{-m/2}t^{-m/2}X^{n+1} & \epsilon = 0 \\ q^{-m/2}t^{-(m+1)/2}X^{n-1} & \epsilon = 1 \end{cases}$$

which tells us that for a fixed n, any dependence between monomials must further split as dependences where ϵ is fixed. Since the operators X and T are invertible, it now will suffice to show that the Y^i are independent. Note that $Y.X^j = t^{-1/2}q^{-j/2}X^j +$ lower order terms, and therefore any dependence

$$a_n Y^n + a_{n-1}Y^{n-1} + ... + a_0 = 0$$

induces for all j

$$a_n t^{-n/2}q^{jn/2} + a_{n-1}t^{-(n-1)/2}q^{j(n-1)/2} + ... + a_0 = 0 \tag{4.1}$$

Let us choose n distinct j, call them $j_1, j_2, ...j_n$. If we introduce $a'_k = t^{-k/2}a_k$, we see assembling the various (4.1) together gives the statement

$$\begin{pmatrix} q^{nj_1/2} & q^{(n-1)j_1/2} & \cdots & q^{j_1/2} & 1 \\ q^{nj_2/2} & q^{(n-1)j_2/2} & \cdots & q^{j_2/2} & 1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ q^{nj_n/2} & q^{(n-1)j_n/2} & \cdots & q^{j_n/2} & 1 \end{pmatrix} \begin{pmatrix} a'_n \\ a'_{n-1} \\ \vdots \\ a'_0 \end{pmatrix} = 0$$

But the matrix above is a Vandermonde matrix, and therefore as long as q is not a root of unity, the fact that all j_i are distinct guarantees that this matrix is invertible. Thus the Y^n are linearly independent and we are done with (i) and (ii) simultaneously. $\square$

4.1.2 Macdonald Polynomials

Let us continue examining the polynomial representation of the DAHA, restricting attention to the action of the Y operator. Of interest will be the following theorem, which appeared originally in slightly different language in [19].

Theorem 16. *(Macdonald, 1988) The operator $Y+Y^{-1}$ preserves the space $\mathbb{C}[X+X^{-1}]$ of symmetric polynomials. For pairs (q,t) such that the sequence $\{q^{n/2}t^{1/2}+q^{-n/2}t^{-1/2}\}$ consists of distinct values, there exist polynomials $p_n \in \mathbb{C}[X+X^{-1}]$ such that*

$$p_n = X^n + X^{-n} + \textit{lower terms}$$

$$(Y+Y^{-1})p_n = (q^{n/2}t^{1/2} + q^{-n/2}t^{-1/2})p_n$$

Proof. For the first statement, note that the relations in the first presentation of $\mathcal{H}_{q,t}$ tell us that T commutes with $Y+Y^{-1}$. We can now characterize symmetric polynomials in terms of the operator T: a polynomial $p(X)$ is symmetric if and only if $Tp(X) = t^{1/2}p(X)$. The first statement then follows.

For the second statement, it will suffice to show that $(Y+Y^{-1}).(X^n+X^{-n}) = (q^{n/2}t^{1/2}+q^{-n/2}t^{-1/2})(X^n+X^{-n}) + \text{lower terms}$. Since $Y=\pi T$, it is readily computed that

$$Y.(X^n+X^{-n}) = t^{1/2}(q^{-n/2}X^n + q^{n/2}X^{-n})$$

$$Y^{-1}(X^n+X^{-n}) = (t^{1/2}q^{n/2} - q^{-n/2}(t^{1/2}-t^{-1/2}))X^n + (t^{-1/2}q^{-n/2})X^{-n} + \text{lower order}$$

from which the equation follows. □

We will take as a definition that $p_n(X)$ are the Macdonald polynomials of type A_1. The first few of these polynomials are

$$p_0 = 1, \quad p_1 = X + X^{-1}, \quad p_2 = X^2 + X^{-2} + (t^{1/2} - t^{-1/2})\frac{q^{1/2} + q^{-1/2}}{t^{1/2}q^{1/2} - t^{-1/2}q^{-1/2}}.$$

It follows from this description of the p_n and some symmetry properties of the algebra $\mathcal{H}_{q,t}$ that much is known about these polynomials. For instance, we have the following identities - see, e.g., [5]

Theorem 17. *The Macdonald polynomials* $p_n(X)$ *satisfy the following properties:*

(a) Duality: $p_n(t^{1/2}q^{m/2})p_m(t^{1/2}) = p_m(t^{1/2}q^{n/2})p_n(t^{1/2})$ *for all* $m, n \in 0, 1, 2, \ldots.$

(b) Pieri Rule:

$$p_{n+1} = (X + X^{-1})p_n - \left(\frac{q^{n/2} - q^{-n/2}}{t^{1/2}q^{n/2} - t^{-1/2}q^{-n/2}}\right)\left(\frac{tq^{(n-1)/2} - t^{-1}q^{(1-n)/2}}{t^{1/2}q^{(n-1)/2} - t^{-1/2}q^{(1-n)/2}}\right)p_{n-1}$$

(c) Evaluation Formula:

$$p_n(t^{1/2}) = \prod_{k=0}^{n-1} \frac{tq^{k/2} - t^{-1}q^{-k/2}}{t^{1/2}q^{k/2} - t^{-1/2}q^{-k/2}}, \quad n \geq 1$$

Other normalizations of these $\{p_n\}$ are common; for instance, if we take $\pi_n(X) := p_n(X)/p_n(t^{1/2})$, the Pieri relation becomes

$$(X + X^{-1})\pi_n = \frac{tq^{n/2} - t^{-1}q^{-n/2}}{t^{1/2}q^{n/2} - t^{-1/2}q^{-n/2}}\pi_{n+1} + \frac{q^{n/2} - q^{-n/2}}{t^{1/2}q^{n/2} - t^{-1/2}q^{-n/2}}\pi_{n-1} \qquad (4.2)$$

4.2 Type $C^\vee C_1$

This section should be viewed as an extension of the previous section. The goal of this section is to summarize the properties of the DAHA of type $C^\vee C_1$, namely to describe its polynomial representation and the embedding of the DAHA into $A_q \rtimes \mathbb{Z}_2$ of Sahi.

4.2.1 Definition and PBW Theorem

The DAHA of type $C^\vee C_1$, denoted here by $\mathcal{H}_{q,\underline{t}}$ is an associative, unital $\mathbb{C}$-algebra depending on 5 complex parameters, q and $\underline{t} = (t_1, t_2, t_3, t_4)$ given by the presentation

$$\mathcal{H}_{q,\underline{t}} = \frac{\mathbb{C}\langle T_1, T_2, T_3, T_4\rangle}{I}$$

where I is the two-sided ideal generated by the relations

$$(T_1 - t_1)(T_1 + t_1^{-1}) \qquad (T_2 - t_2)(T_2 + t_2^{-1})$$

$$(T_3 - t_3)(T_3 + t_3^{-1}) \qquad (T_4 - t_4)(T_4 + t_4^{-1})$$

$$T_4T_3T_1T_2 = q$$

Analogously to the $\mathcal{H}_{q,t}$ case, this algebra comes equipped with a representation on the Laurent polynomial ring $\mathbb{C}[X^{\pm 1}]$.

Theorem 18. *(Sahi, 1997) Let s_0, s_1 denote the linear operators on $\mathbb{C}[X^{\pm 1}]$ defined by $s_0(X^m) = q^{-2m}X^{-m}$ and $s_1(X^m) = X^{-m}$. Then if we set*

$$S_1 = t_1 s_0 + \frac{(t_1 - t_1^{-1}) + (t_2 - t_2^{-1})q^{-1}X^{-1}}{1 - q^{-2}X^{-2}}(1 - s_0)$$

$$S_3 \to t_3 s_1 + \frac{(t_3 - t_3^{-1}) + (t_4 - t_4^{-1})X}{1 - X^2}(1 - s_1)$$

then the assignments $T_i \to S_i$ $i = 1, 3$, $T_2 \to q(S_0 - (t_1 - t_1^{-1}))X$ and $T_4 \to X^{-1}(S_1 - (t_3 - t_3^{-1}))$ define a faithful representation of $\mathcal{H}_{q,\underline{t}}$ into $\mathbb{C}[X^{\pm 1}]$.

A proof of this statement in this exact language can be found in [24], but the original proof of this fact dates back to [29]. That the linear operators S_i satisfy the Hecke-type relations of the T_i allows us to rewrite the last two assignments as $T_2 \to qS_0^{-1}X$ and $T_4 \to X^{-1}S_1^{-1}$. That the assignments give a representation is easy; showing that the resulting representation is faithful occupies the bulk of the work.

Remark 6. *Assuming this theorem, we could alternately take as generators for $\mathcal{H}_{q,\underline{t}}$ the elements $T := T_3, Y := T_3T_1$ and $X^{\pm 1}$, identifying $\mathcal{H}_{q,\underline{t}}$ with its image under the representation. With these generators, our relations then become*

$$XT = T^{-1}X^{-1} - \overline{t_4}$$

$$T^{-1}Y = Y^{-1}T + \overline{t_1}$$

$$T^2 = 1 + \overline{t_3}T$$

$$TXY = q^2T^{-1}YX - q^2\overline{t_1}X - q\overline{t_2} - \overline{t_4}Y.$$

Recalling the quantum torus $A_q = \frac{\mathbb{C}\langle X^{\pm 1}, Y^{\pm 1}}{XY = q^2YX}$ along with the action of $\mathbb{Z}_2$, define $A_q \rtimes \mathbb{Z}_2$ as

$$A_q \rtimes \mathbb{Z}_2 = \frac{\mathbb{C}\langle X^{\pm 1}, Y^{\pm 1}, s\rangle}{XY = q^2YX, sX = X^{-1}s, sY = Y^{-1}s, s^2 = 1}.$$

We then have the following observation, which will be essential in what follows.

Remark 7. *The algebra $\mathcal{H}_{q,\underline{t}}$ for $q = q$, $\underline{t} = (1, 1, 1, 1)$ is isomorphic to $A_q \rtimes \mathbb{Z}_2$ via the assignments*

$$T \mapsto s, \quad X \mapsto X, \quad Y \mapsto Y$$

in the language of the generators above.

This observation can be unified with the existence of the defining polynomial representation by considering $A_q \rtimes \mathbb{Z}_2$ as a subring of a larger algebra which naturally acts on $\mathbb{C}(X^{\pm 1})$: Let D_q denote the localization of the algebra $A_q \rtimes \mathbb{Z}_2$ at all nonzero polynomials in X. The result would be the following:

Theorem 19. *(Sahi, 1999) The defining representation of $\mathcal{H}_{q,\underline{t}}$ is equivalent to the existence of an algebra embedding*

$$\phi : \mathcal{H}_{q,\underline{t}} \to D_q.$$

and the image of ϕ preserves the subspace $\mathbb{C}[X^{\pm 1}] \subset \mathbb{C}(X)$. Moreover, the natural localization map

$$\eta : A_q \rtimes \mathbb{Z}_2 \to D_q$$

is injective, and when $\underline{t} = \underline{1}$, the images of ϕ and η coincide, with $\eta^{-1} \circ \phi$ defining an algebra isomorphism between $\mathcal{H}_{q,\underline{t}} \to A_q \rtimes \mathbb{Z}_2$

CHAPTER 5

BS CONJECTURE

5.1 Recap of Conjecture

In this section we will recount the main developments of [2], omitting all proofs.

At this point, we know two enticingly related facts: the first of which is for every oriented knot K, the vector space $K_q(S^3 \setminus K)$ is an $A_q^{\mathbb{Z}_2}$-module (the result of Frohman and Gelca). The second is that $\mathcal{H}_{q,\underline{t}}$, the DAHA of type $C^\vee C_1$, is an algebra deforming $A_q \rtimes \mathbb{Z}_2$. If $K_q(S^3 \setminus K)$ were instead a $A_q \rtimes \mathbb{Z}_2$-module, we could follow the general philosophy of the Introduction and ask when this module structure is a restriction of a more general $\mathcal{H}_{q,\underline{t}}$-module structure on the same space. However, it is easy to produce a $A_q \rtimes \mathbb{Z}_2$-module from an $A_q^{\mathbb{Z}_2}$-module via extension by scalars.

Definition 2. *The* nonsymmetric skein module $\widehat{K_q}(S^3 \setminus K)$ *is defined by*

$$\widehat{K_q}(S^3 \setminus K) = A_q \otimes_{A_q^{\mathbb{Z}_2}} K_q(S^3 \setminus K)$$

Since A_q is a left $A_q \rtimes \mathbb{Z}_2$-module, $\widehat{K_q}(S^3 \setminus K)$ carries the structure of a left $A_q \rtimes \mathbb{Z}_2$-module. Recall that we denoted by D_q the localization of $A_q \rtimes \mathbb{Z}_2$ at all nonzero polynomials in X. Denoting the corresponding localized module as $\widehat{K_q}(S^3 \setminus K)^{\mathrm{loc}}$, by the theorem of Sahi we have that $\widehat{K_q}(S^3 \setminus K)^{\mathrm{loc}}$ is a $\mathcal{H}_{q,\underline{t}}$-module. We can now state the BS conjecture in its entirety:

Conjecture 4. *(BS Conjecture) The following conditions hold, for any knot* K*:*

1. *The map* $\eta : \widehat{K_q}(S^3 \setminus K) \to \widehat{K_q}(S^3 \setminus K)^{\mathrm{loc}}$ *is injective.*

2. *If $\phi : \mathcal{H}_{q,\underline{t}} \to D_q$ is the embedding of Sahi, then the image of ϕ preserves the image of η.*

and thus $\widehat{K_q}(S^3 \setminus K)$ carries the structure of a $\mathcal{H}_{q,\underline{t}}$-module by the action

$$a.m = \eta^{-1}(\phi(a).\eta(m))$$

A computation in the case of the unknot reveals that this conjecture is overly optimistic. While (1) holds, (2) fails to hold if either t_3 or t_4 is not 1. However, by modifying the conjecture to fix $t_3 = t_4 = 1$, the authors show that this does in fact hold for $(2, 2p+1)$ torus knots, the trefoil and the figure eight knot.

More important to us, however, are the *generalized Jones polynomials* $\{J_n(K; q, t_1, t_2)\}$ which the authors produce for a knot K, assuming this conjecture holds. Recall the formula of Kirby and Melvin [27], relating the colored Jones polynomials to the topological pairing:

$$J_n(K; q) = (-1)^{n-1}\langle \varnothing.S_{n-1}(Y + Y^{-1}), \varnothing\rangle$$

To relate this to the nonsymmetric setting, we need to understand a little more about the relationship between $A_q^{\mathbb{Z}_2}$ and $A_q \rtimes \mathbb{Z}_2$. In particular, if we denote by $\mathbf{e} = \frac{1+s}{2}$ in $A_q \rtimes \mathbb{Z}_2$ where s is the generator of $\mathbb{Z}_2$ we have the following lemma.

Lemma 2. *(Satake isomorphism) The algebras $A_q^{\mathbb{Z}_2}$ and $\boldsymbol{e}(A_q \rtimes \mathbb{Z}_2)\boldsymbol{e}$ are isomorphic via the map $a \to a\boldsymbol{e}$.*

Thus for any $A_q^{\mathbb{Z}_2}$-module M, we must have M is isomorphic to $\mathbf{e}\widehat{M}$ as $A_q^{\mathbb{Z}_2}$-modules, where in truth it is $\mathbf{e}(A_q \rtimes \mathbb{Z}_2)\mathbf{e}$ acting on $\mathbf{e}\widehat{M}$, but we use the Satake isomorphism to identify the two. In particular, the $\mathbb{C}$-vector spaces

$$M \otimes_{A_q^{\mathbb{Z}_2}} N \qquad \widehat{M}\mathbf{e} \otimes_{\mathbf{e}(A_q \rtimes \mathbb{Z}_2)\mathbf{e}} \mathbf{e}\widehat{N}$$

are isomorphic via the map $(m, n) \to (m\mathbf{e}, \mathbf{e}n)$. Since the latter is clearly a subspace of $\widehat{M} \otimes_{A_q \rtimes \mathbb{Z}_2} \widehat{N}$, we can use this identification to then make sense of the inclusion

$$\iota : M \otimes_{A_q^{\mathbb{Z}_2}} N \to \widehat{M} \otimes_{A_q \rtimes \mathbb{Z}_2} \widehat{N}. \tag{5.1}$$

We will want to know when linear maps out of the domain of (5.1) can be extended to the target; this is the content of the following proposition.

Proposition 11. *Suppose* $q^4 \neq 1$*. To every pairing of* $A_q^{\mathbb{Z}_2}$*-modules*

$$\langle , \rangle : M \otimes_{A_q^{\mathbb{Z}_2}} N \to \mathbb{C}$$

there exists a unique pairing

$$\langle , \rangle : \widehat{M} \otimes_{A_q \rtimes \mathbb{Z}_2} \widehat{N} \to \mathbb{C}$$

lifting the former.

Applying this to the topological pairing $K_q(S^1 \times D^2) \otimes_{K_q(S^1 \times S^1)} K_q(S^3 \setminus K) \to \mathbb{C}$, we can then introduce

Definition 3. *The generalized Jones polynomials associated to the DAHA action are given by*

$$J_n(K; q, t_1, t_2) = (-1)^{n-1} \langle \varnothing . S_{n-1}(Y_{t_1,t_2} + Y_{t_1,t_2}^{-1}), \varnothing \rangle \tag{5.2}$$

where $Y_{t_1,t_2} = sT_0$ *is the Cherednik-Dunkl operator in* $\mathcal{H}_{q,\underline{t}}$*.*

Note that this Y_{t_1,t_2} is precisely the Y defined in our alternate presentation of the DAHA; we simply stress here its $\underline{t}$-dependence to reflect how it deforms the original formula of Kirby and Melvin. Since $Y_{1,1} = Y$, we see that automatically $J_n(K; q, 1, 1) = J_n(K; q)$.

5.2 Nonsymmetric skein modules and pairings

First, we recall the standard (symmetric) description of the skein module of $D^2 \times S^1$ and the (right) action of $A_q^{\mathbb{Z}_2}$ induced by the inclusion of the boundary torus, where the torus is oriented to be the boundary of the knot complement.

The generators of $A_q^{\mathbb{Z}_2}$ are $X + X^{-1}, Y + Y^{-1}$ and $XY + X^{-1}Y^{-1}$, which we will denote x, y and z, respectively. Under the isomorphism identifying the skein module of the oriented torus with $A_q^{\mathbb{Z}_2}$, the meridian and longitude are sent to x and y, respectively, and the $(1, 1)$ curve is sent to $q^{-1}z$. The skein module of the solid torus is identified with $\mathbb{C}[u]$, where u^n is identified with n parallel copies of the (0-framed) longitude. Under this identification, the empty skein becomes the element 1. Thus we have:

Lemma 3. *The right action of* $A_q^{\mathbb{Z}_2}$ *on* $K_q(D^2 \times S^1) \cong \mathbb{C}[u]$ *is determined by*

$$1.p(y) = p(u), \quad 1.x = -(q^2 + q^{-2}), \quad 1.q^{-1}z = -q^{-3}u$$

Proof. The first formula holds by definition of the isomorphism of $K_q(D^2 \times S^1)$ with $\mathbb{C}[u]$. The second holds as the meridian is contractible inside $D^2 \times S^1$ and the third as the $(1, 1)$ curve is simply the longitude with a positive framing twist.

That these formulas are sufficient to describe the action of a general element follows from the fact that the following relations hold in $A_q^{\mathbb{Z}_2}$

$$qxy - q^{-1}yx = (q^2 - q^{-2})z$$

$$qzx - q^{-1}xz = (q^2 - q^{-2})y$$

$$qyz - q^{-1}zy = (q^2 - q^{-2})x.$$

Repeated application of these commutator relations allows us to write any element of $A_q^{\mathbb{Z}_2}$ as a sum of monomials, with powers of y appearing only on the left. This, combined with the fact that z acts as $-q^{-2}y$, is sufficient to express the action of a general element of $A_q^{\mathbb{Z}_2}$ in terms of the action of elements of the form

$$\sum a_{i,j} x^i y^j \qquad a_i \in \mathbb{C}.$$

As x acts by a scalar, $\mathbb{C}$-linearity of the action tells us that our initial information will be sufficient to determine how this element acts. $\square$

To produce the non-symmetric version of this skein module, we need the following "effective" version of producing $\widehat{M}$ from M for M either a left or right $A_q^{\mathbb{Z}_2}$-module.

Proposition 12. *Suppose* $q^4 \neq 1$. *Let* M *be a right* $A_q^{\mathbb{Z}_2}$*-module. If* M' *is a right* $A_q \rtimes \mathbb{Z}_2$*-module such that* $M'\boldsymbol{e}$ *is isomorphic to* M *as* $A_q^{\mathbb{Z}_2}$*-modules (using the Satake isomorphism), then* M' *is isomorphic to* $\widehat{M}$ *as* $A_q \rtimes \mathbb{Z}_2$*-modules. The identical statement is true with "left" replacing "right" and* $\boldsymbol{e}M'$ *replacing* $M'\boldsymbol{e}$.

This is a consequence of the Morita equivalence of the algebras $A_q^{\mathbb{Z}_2}$ and $A_q \rtimes \mathbb{Z}_2$ when $q^4 \neq 1$. Using this, it is easy to check the following.

Proposition 13. *Let* $V = \mathbb{C}[U^{\pm 1}]$ *and give* V *a right* $A_q \rtimes \mathbb{Z}_2$*-module structure via the formulas*

$$f(U).g(Y) = f(U)g(U^{-1}), \quad f(U).X = -f(q^2U), \quad f(U).s = -f(U^{-1})$$

then V *is isomorphic to* $\widehat{K_q}(D^2 \times S^1)$ *as* $A_q \rtimes \mathbb{Z}_2$*-modules.*

Proof. To employ the previous proposition, we need to produce an isomorphism of $V\mathbf{e}$ with $K_q(D^2 \times S^1)$. Consider

$$f : K_q(D^2 \times S^1) \to Ve$$

$$f(u) \to (U - U^{-1})f(U + U^{-1})$$

By computing in Ve, we find:

$$(U - U^{-1}).f(Y + Y^{-1}) = (U - U^{-1})f(U + U^{-1})$$

$$(U - U^{-1}).(X + X^{-1}) = -(q^2 + q^{-2})(U - U^{-1})$$

$$(U - U^{-1}).q^{-1}(XY - Y^{-1}X^{-1}) = q^{-3}(U - U^{-1}).(Y + Y^{-1})$$

Note that under the isomorphism these computations become precisely the module-defining equations of Lemma 1, and therefore the module structures are identical. □

At this stage, a remark is in order:

Remark 8. *Lifting the formula of Kirby and Melvin to the nonsymmetric pairing gives us the identity:*

$$\begin{aligned} J_n(K, q) &= (-1)^{n-1}\langle S_{n-1}(u), 1\rangle \\ &= (-1)^{n-1}\langle (U - U^{-1})S_{n-1}(U + U^{-1}), 1\rangle = (-1)^{n-1}\langle U^n - U^{-n}, 1\rangle \end{aligned} \tag{5.3}$$

One other non-symmetric skein module will be of importance to us, namely that of the complement of the unknot, or $S^1 \times D^2$. The skein module of the unknot is identified with $\mathbb{C}[x]$, where by x^n is meant n parallel copies of the (0-framed) meridian. As before, we have a structure observation for the symmetric action:

Lemma 4. *The left action of $A_q^{\mathbb{Z}_2}$ on $K_q(S^1 \times D^2) \cong \mathbb{C}[x]$ is determined by*

$$x.f(x) = xf(x), \quad y.1 = -(q^2 + q^{-2}), \quad z.1 = -q^{-3}$$

And we have the following desymmetrization

Proposition 14. *Let* $V = \mathbb{C}[X^{\pm 1}]$ *and give* V *a left* $A_q \rtimes \mathbb{Z}_2$*-module structure via the formulas*

$$X.f(X) = Xf(X), \quad Y.f(X) = -f(q^{-2}X), \quad s.f(X) = -f(X^{-1})$$

then V *is isomorphic to* $\widehat{K_q}(S^1 \times D^2)$*.*

The proofs of both of these are identical to the case of the solid torus. Moreover, the identification of $\mathbf{e}V$ with $K_q(S^1 \times D^2)$ is explicitly given by

$$f : K_q(S^1 \times D^2) \to \mathbf{e}V$$

$$f(x) \to (X - X^{-1})f(X + X^{-1}).$$

We pause here to make a very simple remark, which will appear as an important fact in later proofs in this chapter:

Remark 9. *If* $\langle , \rangle : \widehat{K_q}(D^2 \times S^1) \otimes_{A_q \rtimes \mathbb{Z}_2} \widehat{K_q}(S^1 \times D^2) \to \mathbb{C}$ *denotes the pairing between the nonsymmetric skein modules of the solid torus and the unknot, then* $\langle 1, 1 \rangle \neq 0$ *provided* $q^4 \neq 1$.

This follows from the fact that $\langle U - U^{-1}, X - X^{-1} \rangle = 1$, as this represents the pairing of the two empty skeins.

5.3 First formula for $\widetilde{c_{n,i}}$

In this section we will give a formula for the coefficients $\widetilde{c_{n,i}}$ of the Main Theorem (Theorem 6) which will be used later to establish specifically the nice formula in the $t_2 = 1$ case. To this end we will establish the

Theorem 20. *For all knots K, there exist universal expressions $a_{n,p} \in \mathbb{C}(q)[t_1^{\pm 1}, t_2^{\pm 1}]$ such that*

$$J_n(K; q, t_1, t_2) = \sum_{p=1}^{n} (-1)^{n+p} a_{n,p} J_p(K; q)$$

whenever the left hand side is defined. Moreover, the $a_{n,k}$ are given recursively with boundary conditions

$$a_{1,1} = 1, \qquad a_{n,0} = 0, \qquad a_{n,k} = 0 \quad (k > n)$$

and recursion

$$a_{n+1,k} = A_k a_{n,k-1} + (A_k - A_{k+1}) a_{n,k} + A_{-k} a_{n,k+1} - a_{n-1,k}$$

where

$$A_k := \frac{q^{2k-1} t_1^{-1} - q^{-2k+1} t_1 + \overline{t_2}}{q^{2k-1} - q^{-2k+1}}$$

.

and by combining this with Habiro's cyclotomic expansion get the formula we desire as a

Corollary 2. *Let $a_{n,p}$ be as in the previous theorem. Then*

$$J_n(K; q, t_1, t_2) = \sum_{i=0}^{n-1} \widetilde{c_{n,i}} H_i(K)$$

where

$$\widetilde{c_{n,i}} = \sum_{p=1}^{n} (-1)^{n+p} a_{n,p} c_{p,i} \tag{5.4}$$

and $c_{n,p}$ are the coefficients in Habiro's cyclotomic formula.

We stress here that while this corollary could be taken as the first statement in Theorem 6, it is not at all evident that the $\widetilde{c_{n,i}}$ as described here live in $\mathbb{Z}[q^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}]$. It is also not clear from this formula how the particularly nice case $t_2 = 1$ produces such a compact formula.

The following lemma regarding computing in $\widehat{K_q}(D^2 \times S^1) \cong V = \mathbb{C}[U]$ will be used

Lemma 5. *For any* $P(X) \in \mathbb{C}(X)$*, we have in* V

$$U^k.P(X) = P(-q^{2k}).U^k$$

$$U^k.P(X^{-1}) = P(-q^{-2k}).U^k$$

for all $k \in \mathbb{Z}$

Proof. The result for monomials is a direct result of the definition of the $A_q \rtimes \mathbb{Z}_2$ action. For polynomials and then rational expressions it follows by linearity. □

Now we can give the proof of Theorem 20.

Proof. We begin by noting that by the use of the Kirby-Melvin formula

$$J_p(K; q) = (-1)^{p-1} \langle U^p - U^{-p}, 1 \rangle$$

and the definition of the generalized Jones polynomials

$$J_n(K; q, t_1, t_2) = (-1)^{n-1} \langle U - U^{-1}.S_{n-1}(Y_t + Y_t^{-1}), 1 \rangle$$

The statement of the theorem is equivalent to

$$\langle U - U^{-1}.S_{n-1}(Y_t + Y_t^{-1}), 1 \rangle = \sum_{p=1}^{n} a_{n,p} \langle U^p - U^{-p}, 1 \rangle.$$

Note that since $S_0(x) = 1$, we have the first base case trivially

$$J_1(K; q, t_1, t_2) = \langle (U - U^{-1}).1, 1 \rangle = J_1(K; q)$$

which implies $a_{1,1} = 1$ and $a_{1,p} = 0$ for all $p > 1$.

In the second case, we have $S_1(x) = x$, and thus

$$J_2(K; q, t_1, t_2) = \langle (U - U^{-1}).(Y_t + Y_t^{-1}), 1 \rangle$$

Recall the definition of Y_t and Y_t^{-1}:

$$Y_t = t_1 Y + \frac{q\overline{t_1}X^{-1} + \overline{t_2}}{qX^{-1} - q^{-1}X}(s - Y)$$

$$Y_t^{-1} = t_1 Y^{-1} + \frac{q\overline{t_1}X + \overline{t_2}}{qX - q^{-1}X^{-1}}(s - Y^{-1}) - \overline{t_1}s$$

Defining the operator

$$a(X) = \frac{q\overline{t_1}X^{-1} + \overline{t_2}}{qX^{-1} - q^{-1}X}$$

we note that we may write Y_t and Y_t^{-1}, and thus their sum, more compactly

$$Y_t = t_1 Y + a(X)(s - Y)$$

$$Y_t^{-1} = t_1 Y^{-1} + a(X^{-1})(s - Y^{-1}) - \overline{t_1}s$$

$$Y_t + Y_t^{-1} = t_1(Y + Y^{-1}) + (a(X) + a(X^{-1}))s - (a(X)Y + a(X^{-1})Y^{-1}) - \overline{t_1}s \quad (5.5)$$

Using the Lemma, we can calculate

$$(U - U^{-1}).(Y_t + Y_t^{-1}) = a_{2,2}(U^2 - U^{-2}) + a_{2,1}(U - U^{-1})$$

where

$$a_{2,2} = (t_1 - a(-q^{-2})) \qquad a_{2,1} = \overline{t_1} - a(-q^2) - a(-q^{-2}) \qquad a_{2,p} = 0 \quad (p > 2)$$

Since A_k can be shown to coincide with $t_1 - a(-q^{-2(k-1)})$, we see that $a_{2,2} = A_2$, and a similar computation shows $a_{2,1} = A_1 - A_2$. It also follows from this that $a_{2,p} = a_{2,0} = 0$ for all $p > 2$.

Thus the statement of the theorem holds for $n = 1, 2$. To establish the inductive step, we will use the defining recursion for the Chebyshev polynomials $S_n(x)$

$$S_n(x) = xS_{n-1}(x) - S_{n-2}(x)$$

Thus we assume the statement of the theorem holds for some $n, n-1$, that is:

$$(U - U^{-1}).S_{n-1}(Y_t + Y_t^{-1}) = \sum_{p=1}^{n} a_{n,p}(U^p - U^{-p})$$

$$(U - U^{-1}).S_{n-2}(Y_t + Y_t^{-1}) = \sum_{p=1}^{n-2} a_{n-1,p}(U^p - U^{-p})$$

and we compute

$$(U-U^{-1}).S_n(Y_t+Y_t^{-1}) = (U-U^{-1}).S_{n-1}(Y_t+Y_t^{-1}).(Y_t+Y_t^{-1})-(U-U^{-1}).S_{n-2}(Y_t+Y_t^{-1})$$

$$\sum_{p=1}^{n+1} a_{n+1,p}(U^p - U^{-p}) = \sum_{p=1}^{n+1} a_{n,p}(U^p - U^{-p}).(Y_t + Y_t^{-1}) - \sum_{p=1}^{n+1} a_{n-1,p}(U^p - U^{-p})$$

where we can extend the sums to all be of the same range based on the boundary conditions mentioned above. Focusing attention on the first sum on the right hand side, an application of Lemma 5 gives

$$(U^k-U^{-k}).(Y_t+Y_t^{-1}) = (t_1-a(-q^{-2k}))(U^{k+1}-U^{-(k+1)})+(a(-q^{2k})+a(-q^{-2k})-\overline{t_1})(U^k-U^{-k})$$
$$+(t_1 - a(-q^{2k}))(U^{k-1} - U^{-(k-1)})$$

$$(U^k-U^{-k}).(Y_t+Y_t^{-1}) = A_{k+1}(U^{k+1}-U^{-(k+1)})+(A_k-A_{k+1})(U^k-U^{-k})+A_{-k+1}(U^{k-1}-U^{-(k-1)})$$

Inserting this back into the summations and comparing coefficients of $(U^k - U^{-k})$ on both sides gives the result

$$a_{n+1,k} = A_k a_{n,k-1} + (A_k - A_{k+1})a_{n,k} + A_{-(k+1)+1}a_{n,k+1} - a_{n-1,k}$$

$\square$

5.4 Proof of Theorem 6

In this section we will produce a second formula for the coefficients $\widetilde{c_{n,i}}$ which will finish the proof of Theorem 6. Along the way we will observe how Corollary 1 follows from the proof. Specifically, section 5.4.1 will introduce generating functions associated to the $\widetilde{c_{n,i}}$ and sketch how two lemmas (Lemma 7 and Lemma 8) allow us to give a new formula for the $\widetilde{c_{n,i}}$ from which Corollary 1 follows. Sections 5.4.2 and 5.4.3 then contain the proofs of these two lemmas.

The formula in the special case for $t_2 = 1$ in Theorem 6 will be addressed in 5.4.4. It will be alternately rederived using the method of generating functions. This is not strictly speaking a logically independent proof; however, it offers an important sanity check as the resulting formula itself carries some significant complexity.

5.4.1 Generating Function

We will want to in essence still use (5.4) as our starting point for calculating a nicer form for $\widetilde{c_{n,i}}$. Recall that the $a_{n,p}$ are implicitly defined by the equation

$$(U - U^{-1}).S_{n-1}(Y_t + Y_t^{-1}) = \sum_{p=1}^{n} a_{n,p}(U^k - U^{-k})$$

Instead of working with the element-wise recurrence for the $a_{n,p}$ within (5.4), it turns out to be much more convenient to use the defining recurrence for the $S_n(Y_t + Y_t^{-1})$ and employ generating functions. Let us extend the definition of the $a_{n,p}$ formally to negative p via $a_{n,-p} = -a_{n,p}$; then we can introduce the functions

$$F_n(U) = \sum_{p=-\infty}^{\infty} a_{n,p} U^p$$

between which we have the recursive statement which is equivalent to Theorem 20:

$$F_n(U).(Y_t + Y_t^{-1}) = F_{n+1}(U) + F_{n-1}(U). \tag{5.6}$$

We will also find it useful to use the extension of the Chebyshev polynomials to negative n, namely $S_{-n}(x) = -S_{n-2}(x)$, so that $F_n(U)$ is defined for all n.

This turns out to be fruitful because we have the following

Lemma 6. *Let* $P^{(i)}(X)$ *be the operator defined by*

$$P^{(i)}(X) = \prod_{j=1}^{i-1}(q^{2j}X - q^{-2j}X^{-1})(q^{-2j}X - q^{2j}X^{-1}) \qquad i = 1, 2, ... \tag{5.7}$$

where the empty product is 1. *If* $f(U) = \sum_{p=1}^{\infty} a_p(U^p - U^{-p})$, *then*

$$f(U).P^{(i)}(X) = \sum_{p=1}^{\infty} a_p \cdot \prod_{j=p-i+1}^{p-1}(q^{2(p+j)} - q^{-2(p+j)}) \prod_{j=p+1}^{p+i-1}(q^{2(p+j)} - q^{-2(p+j)})(U^p - U^{-p})$$

In particular, acting by $P^{(i)}(X)$ *and summarily evaluating the resulting function of* U *at* $U = -q^2$ *gives*

$$f(U).P^{(i)}(X)\Big|_{U=-q^2} = (q^2 - q^{-2}) \sum_{k=1}^{\infty} (-1)^k a_k c_{k,i-1} \tag{5.8}$$

Proof. By Lemma 5 in the previous section, we compute

$$U^k.P^{(i)}(X) = \prod_{j=1}^{i-1}(q^{2j}q^{2k} - q^{-2j}q^{-2k})(q^{-2j}q^{2k} - q^{2j}q^{-2k})$$

$$U^{-k}.P^{(i)}(X) = \prod_{j=1}^{i-1}(q^{2j}q^{-2k} - q^{-2j}q^{2k})(q^{-2j}q^{-2k} - q^{2j}q^{2k})$$

where both coincide with the product in the statement in the lemma, and thus we are done. □

Thus from the conclusion of the Lemma we recognize we have rewritten

$$\widetilde{c_{n,i}} = \frac{(-1)^n}{q^2 - q^{-2}} F_n(U).P^{(i+1)}(X)|_{U=-q^2}. \tag{5.9}$$

The key idea is to now introduce the formal parameter λ and define

$$F(U, \lambda) = \sum_{n=1}^{\infty} F_n(U)\lambda^n \tag{5.10}$$

and its variants,

$$F^{(i)}(U, \lambda) := F(U, \lambda).P^{(i)}(X) \tag{5.11}$$

We will seek to prove a determinantal expression for $F^{(i)}(-q^2, \lambda)$, from which will follow Theorem 6, and in particular will show Corollary 1. First it is necessary to prove the following two lemmas:

Lemma 7. *For all i, there exists a sequence $\{\alpha_k^i\}_{k=1}^i \in \mathbb{Z}[q^{\pm 1}]$ such that*

$$F^{(i)}(-q^2, \lambda) = \sum_{k=1}^{i} \alpha_k^i F(-q^{2(2k-1)}, \lambda)$$

Lemma 8. *For all i, there exist $b_{i,j} \in \mathbb{Z}[q^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}, \lambda^{\pm 1}]$ such that*

$$\sum_{j=1}^{n} b_{i,j} F(-q^{2j}, \lambda) = -(q^{2i} - q^{-2i})$$

Before we embark on the proofs of the lemmas, let us see how they allow us to conclude a determinantal formula for $F^{(i)}(-q^{2(2i-1)}, \lambda)$.

Lemma 8 tells us we have a triangular system of equations for the $F(-q^{2(2i-1)}, \lambda)$, i.e. for any i, we have

$$\begin{pmatrix} b_{1,1} & 0 & \cdots & 0 \\ b_{2,1} & b_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{2i-1,1} & b_{2i-1,2} & \cdots & b_{2i-1,2i-1} \end{pmatrix} \begin{pmatrix} F(-q^2, \lambda) \\ F(-q^4, \lambda) \\ \vdots \\ F(-q^{2(2i-1)}, \lambda) \end{pmatrix} = \begin{pmatrix} -(q^2 - q^{-2}) \\ -(q^4 - q^{-4}) \\ \vdots \\ -(q^{2(2i-1)} - q^{-2(2i-1)}) \end{pmatrix}$$

Writing this as $B_{2i-1}F_{2i-1} = -\beta_{2i-1}$ for short, where of course

$$F_{2i-1} = \begin{pmatrix} F(-q^2) \\ F(-q^4) \\ \vdots \\ F(-q^{2(2i-1)}) \end{pmatrix} \qquad \beta = \begin{pmatrix} q^2 - q^{-2} \\ q^4 - q^{-4} \\ \vdots \\ q^{2(2i-1)} - q^{2(2i-1)} \end{pmatrix}$$

we then have by Cramer's rule $F(-q^{2k}) = \det B^k_{2i-1} / \det B_{2i-1}$, where B^k_{2i-1} is B_{2i-1} with its kth column replaced by $-\beta_{2i-1}$.

Combining this with the Lemma 7, we conclude that if we consider the $2i \times 2i$ matrix

$$C_i = \begin{pmatrix} 0 & \alpha^i_1 & 0 & \alpha^i_2 & \cdots & \alpha^i_i \\ (q^2 - q^{-2}) & b_{1,1} & 0 & 0 & \cdots & 0 \\ (q^4 - q^{-4}) & b_{2,1} & b_{2,2} & 0 & \cdots & 0 \\ (q^6 - q^{-6}) & b_{3,1} & b_{3,2} & b_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (q^{2(2i-1)} - q^{-2(2i-1)}) & b_{2i-1,1} & b_{2i-1,2} & b_{2i-1,3} & \cdots & b_{2i-1,2i-1} \end{pmatrix}$$

then by cofactor expansion along the top row we have the equation

$$\begin{aligned} \det(C_i) &= -\sum_{k=1}^{i} \alpha^i_k(-\det(B^k_{2i-1})) \\ &= \det(B_{2i-1})F^{(i)}(-q^2, \lambda) \end{aligned}$$

and thus, realizing that $\det(B_{2i-1})$ is simply a product of diagonal entries, we obtain

$$F^{(i)}(-q^2, \lambda) = \frac{\det(C_i)}{\det(B_{2i-1})} = \frac{\det(C_i)}{\prod_{j=1}^{2i-1} b_{j,j}}. \tag{5.12}$$

Combining (5.12) and (5.9), we arrive at the following expression for the $\widetilde{c_{n,i}}$, where $[\lambda^n](f(\lambda))$ represents the coefficient of λ^n in a formal power series expansion of $f(\lambda)$

$$\widetilde{c_{n,i}} = \frac{(-1)^n}{q^2 - q^{-2}}[\lambda^n]\left(\frac{\det(C_{i+1})}{\prod_{j=1}^{2i+1} b_{j,j}}\right). \tag{5.13}$$

Corollary 1 results from this expression since the first column of C_{i+1} is divisible by $q^2 - q^{-2}$, and the determinant of C_{i+1} with the first column rescaled is a polynomial in λ with coefficients in $\mathbb{Z}[q^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}]$.

5.4.2 Proof of Lemma 7

The two lemmas in essence correspond to the two operators acting on $F(U, \lambda)$; the first one is related to the operator $P^i(X)$.

We begin by making the following observation about the nonsymmetric pairing:

Proposition 15. *For all* $F(U) \in \mathbb{C}[U^{\pm 1}]$ *such that* $F(U^{-1}) = -F(U)$, *and all* $k \in \mathbb{Z}$, *we have*

$$\langle F(U)\ ,\ X^k - X^{-k}\rangle = F(-q^{2k}) \cdot 2 \cdot \langle 1\ ,\ 1\rangle$$

Proof. It will suffice to show this for the basis $\{U^m - U^{-m}\}$; note that

$$\langle U^m - U^{-m}\ ,\ X^k - X^{-k}\rangle$$

$$= \langle 1.(Y^{-m} - Y^m)\ ,\ X^k - X^{-k}\rangle$$

$$= \langle 1\ ,\ (-1)^m(q^{2mk}X^k - q^{-2mk}X^k) - (-1)^m(q^{-2mk}X^{-k} - q^{2mk}X^{-k})\rangle$$

$$= ((-q^{2k})^m - (-q^{2k})^{-m}) \cdot \langle 1\ ,\ X^k + X^{-k}\rangle$$

$$= ((-q^{2k})^m - (-q^{2k})^{-m}) \cdot 2 \cdot \langle 1\ ,\ 1 \rangle$$

□

We will also use the structure of $P^i(X)$; recall that since

$$P^i(X) = \prod_{j=1}^{i-1} (q^{2j}X - q^{-2j}X^{-1})(q^{-2j}X - q^{2j}X^{-1})$$

is symmetric in X, X^{-1} of degree $2i - 2$, we must have

$$(X - X^{-1})P^i(X) = \sum_{k=1}^{i} \alpha_k^i (X^{2k-1} - X^{-2k+1})$$

for some α_k^i. The α_k^i lie in $\mathbb{Z}[q^{\pm 1}]$ since they are differences of coefficients of $P^i(X)$, which evidently lie in this same ring. Putting this together with the Proposition allows us to compute

$$F^i(-q^2, \lambda) \cdot 2 \cdot \langle 1\ ,\ 1 \rangle = \langle F(U, \lambda).P^i(X),\ X - X^{-1} \rangle$$

$$= \langle F(U, \lambda),\ P^i(X)(X - X^{-1}) \rangle$$

$$= \sum_{k=1}^{i} \alpha_k^i \langle F(U, \lambda), X^{2k-1} - X^{-2k+1} \rangle$$

$$= \sum_{k=1}^{i} \alpha_k^i F(-q^{2(2k-1)}, \lambda) \cdot 2 \cdot \langle 1\ ,\ 1 \rangle$$

and thus dividing through by the nonzero $2 \cdot \langle 1\ ,\ 1 \rangle$ gives the result.

5.4.3 Proof of Lemma 8

This lemma is based on examining the action of the other operator acting on $F(U, \lambda)$, namely the $Y_t + Y_t^{-1}$ operator. Recall that by definition we have the relations

$$F_n(U)(Y_t + Y_t^{-1}) = F_{n+1}(U) + F_{n-1}(U)$$

and thus, by multiplying through by λ^n and summing, we get

$$F(U,\lambda)(Y_t + Y_t^{-1}) = \lambda^{-1}F(U,\lambda) - \lambda^{-1}F_0(U) + \lambda F(U,\lambda) + F_{-1}(U)$$

$$F(U,\lambda)(Y_t + Y_t^{-1} - (\lambda + \lambda^{-1})) = -(U - U^{-1})$$

By evaluating this expression on both sides at $U = -q^{2i}$ for various i we will get our result. It is possible to do this without employing the pairing; however, to preserve a similar style with the proof of the previous lemma, we will use

$$\langle F(U,\lambda)(Y_t + Y_t^{-1} - (\lambda + \lambda^{-1})) + (U - U^{-1})\ ,\ X^i - X^{-i}\rangle = 0$$

and letting the operator $Y_t + Y_t^{-1}$ act on the other side of the bracket will give what we want.

The proof follows from the fact that the operator $Y_t + Y_t^{-1}$ preserves the the space of odd polynomials and their natural grading. Later, however, the exact value of these coefficients will be of use to us, so we compute them now. As

$$\frac{(s - Y)}{qX^{-1} - q^{-1}X}.X^i - X^{-i}$$

$$= -(q^i + q^{-i})\sum_{l=1}^{i}(q^{-1}X)^{i-(2l-1)}$$

we have

$$Y_t.(X^i - X^{-i}) = t_1(-q^{-2i}X^i + q^{2i}X^{-i}) - (q^i + q^{-i})\sum_{l=1}^{i}(q^{-1}X)^{i-2l}\overline{t_1}$$
$$- (q^i + q^{-i})\sum_{l=1}^{i}(q^{-1}X)^{i-(2l-1)}\overline{t_2} \quad (5.14)$$

and similarly, since

$$\frac{s - Y^{-1}}{qX - q^{-1}X^{-1}}.X^i - X^{-i}$$

$$= (q^i + q^{-i}) \sum_{l=1}^{i} (qX)^{i-(2l-1)}$$

we also have

$$Y_t^{-1}(X^i - X^{-i}) = t_1(-q^{2i}X^i + q^{-2i}X^{-i}) + (q^i + q^{-i}) \sum_{l=1}^{i} (qX)^{i-2(l-1)}\overline{t_1}$$
$$+ (q^i + q^{-i}) \sum_{l=1}^{i} (qX)^{i-(2l-1)}\overline{t_2} + (X^{-i} - X^i)\overline{t_1} \quad (5.15)$$

And thus, combining, we have:

$$(Y_t+Y_t^{-1}).(X^i-X^{-i}) = -(q^{2i}t_1^{-1}+q^{-2i}t_1)(X^i-X^{-i})+(q^i+q^{-i}) \sum_{l=1}^{i-1} (q^{i-2l}-q^{-i+2l})X^{i-2l}\overline{t_1}$$
$$+ (q^i + q^{-i}) \sum_{l=1}^{i} (q^{i-(2l-1)} - q^{-i+(2l-1)})X^{i-(2l-1)}\overline{t_2} \quad (5.16)$$

Since the RHS is evidently odd in X, we have as a result that

$$(Y_t + Y_t^{-1}).(X^i - X^{-i}) = \sum_{j=1}^{i} b'_{i,j}(X^j - X^{-j})$$

where, for future use, we record that the $b'_{i,j}$ are given by

$$b'_{i,j} = \begin{cases} -(q^{2i}t_1^{-1} + q^{-2i}t_1) & j = i \\ (q^i + q^{-i})(q^j - q^{-j})\overline{t_1} & j \neq i \quad\quad j \equiv i \mod 2 \\ (q^i + q^{-i})(q^j - q^{-j})\overline{t_2} & j \neq i \quad j+1 \equiv N \mod 2 \end{cases}$$

and thus, we may again use linearity of the pairing to conclude

$$0 = \langle F(U,\lambda)(Y_t + Y_t^{-1} - \lambda - \lambda^{-1}) + (U - U^{-1})\ ,\ X^i - X^{-i}\rangle$$

$$= \sum_{j=1}^{i} b'_{i,j}\langle F(U,\lambda)\ ,\ X^j - X^{-j}\rangle - (\lambda + \lambda^{-1})\langle F(U,\lambda), X^i - X^{-i}\rangle + \langle U - U^{-1}\ ,\ X^i - X^{-i}\rangle$$

$$= 2 \cdot \langle 1\ ,\ 1 \rangle \left[\sum_{j=1}^{i} b_{i,j} F(-q^{2j}, \lambda) - (\lambda + \lambda^{-1}) F(-q^{2i}, \lambda) + (-q^{2j} + q^{-2j}) \right]$$

And thus we note that by taking

$$b_{i,j} = \begin{cases} -b'_{i,j} + (\lambda + \lambda^{-1}) & i = j \\ -b'_{i,j} & i \neq j \end{cases}$$

we observe we have concluded the proof of the lemma.

5.4.4 Special case: $t_2 = 1$

In this section, we will examine in detail a specific subcase of Lemma 7, namely when the parameter $t_2 = 1$, and establish the explicit formula given in Theorem 6. In this case, a certain 'miraculous coincidence' will take place and the determinantal formula (5.13) in the general case will reduce to a product. Specifically, we will prove the following theorem:

Theorem 21. *The deformed Habiro coefficients* $\widetilde{c_{n,i}}(q, t_1, t_2)$ *admit the following closed-form expression when* $t_2 = 1$*:*

$$\widetilde{c_{n,i}}(q, t, 1) = \frac{(-1)^{n+i+1}}{q^2 - q^{-2}} [\lambda^n] \left(\frac{\left(\prod_{j=1}^{2i+1} (q^{2j} - q^{-2j})\right) \left(\prod_{k=2}^{i+1} A_k\right)}{\prod_{k=1}^{i+1} (q^{2(2k-1)} t^{-1} + q^{-2(2k-1)} t + \lambda + \lambda^{-1})} \right) \tag{5.17}$$

where, recalling from a previous section

$$A_p = \frac{q^{2p-1} t^{-1} - q^{-2p+1} t}{q^{2p-1} - q^{-2p+1}}$$

From which we will derive the equation that was promised in Theorem 6,

$$\widetilde{c_{n,k}}(q, t, 1) = \frac{p_{n-k-1}(q^{2(k+1)} t^{-1}; q^{4(k+1)} | q^4)}{p_{n-k-1}(q^{2(k+1)}; q^{4(k+1)} | q^4)} \left(\prod_{i=1}^{k} \frac{q^{2i+1} t^{-1} - q^{-2i-1} t}{q^{2i+1} - q^{-2i-1}} \right) c_{n,k}. \tag{5.18}$$

Notation: The proof of Theorem 21 will feature large expressions in the variable q, which will be dramatically compacted by the use of the following notation which we will adopt:

$$[n] = q^n - q^{-n} \qquad \langle n \rangle = q^n + q^{-n}$$

$$[n]! = [n][n-1]...[1] \qquad \binom{n}{k}_q = \frac{[n]!}{[k]![n-k]!}$$

Proof. (Theorem 21) We begin by observing that the conclusion of Lemma 7 can be significantly simplified in the case where $t_2 = 1$. If we restrict ourselves to the cases of odd i, we have:

Lemma 9. *(Lemma 7, $t_2 = 1$ case)*

For all $i \geq 1$, there exist $d_{i,j} \in \mathbb{Z}[q^{\pm 1}, t_1^{\pm 1}, \lambda^{\pm 1}]$ such that

$$\sum_{j=1}^{n} d_{i,j} F(-q^{2(2j-1)}, \lambda) = -[2(2i-1)]$$

Since we have $d_{i,j} = b_{2i-1,2j-1}$, we have explicit values for these coefficients:

$$d_{i,j} = \begin{cases} q^{2(2i-1)}t_1^{-1} + q^{-2(2i-1)}t_1 + \lambda + \lambda^{-1} & i = j \\ -\langle 2i-1 \rangle [2j-1] & i \neq j \end{cases}$$

□

So just as before, we can consider this as a matrix equation:

$$\begin{pmatrix} d_{1,1} & 0 & \cdots & 0 \\ d_{2,1} & d_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ d_{i,1} & d_{i,2} & \cdots & d_{i,i} \end{pmatrix} \begin{pmatrix} F(-q^2, \lambda) \\ F(-q^6, \lambda) \\ \vdots \\ F(-q^{2(2i-1)}, \lambda) \end{pmatrix} = \begin{pmatrix} -[2] \\ -[6] \\ \vdots \\ -[2(2i-1)] \end{pmatrix}$$

which yields a determinantal formula, analogous to the general case, but now with a matrix half the size; namely, if we consider the $(i+1)\times(i+1)$ matrix

$$E_i = \begin{pmatrix} 0 & \alpha_1^i & \alpha_2^i & \cdots & \alpha_{i-1}^i & \alpha_i^i \\ [2] & d_{1,1} & 0 & \cdots & 0 & 0 \\ [6] & d_{2,1} & d_{2,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [2(2i-3)] & d_{i-1,1} & d_{i-1,2} & \cdots & d_{i-1,i-1} & 0 \\ [2(2i-1)] & d_{i,1} & d_{i,2} & \cdots & d_{i,i-1} & d_{i,i} \end{pmatrix}$$

then in this particular case

$$F^i(-q^2,\lambda) = \frac{\det(E_i)}{\prod_{j=1}^{i} d_{j,j}}$$

We claim that the E_i are closely related to one another, and thus it is possible to give a simple inductive computation of $\det(E_i)$.

To compute $\det(E_{i+1})$, first use column operations to clear the top row; since $\alpha_{i+1}^{i+1} = 1$, the resulting matrix which we'll call $E_{i+1}^{(1)}$ is

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ [2] & d_{1,1} & 0 & \cdots & 0 & 0 \\ [6] & d_{2,1} & d_{2,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [2(2i-1)] & d_{i,1} & d_{i,2} & \cdots & d_{i,i} & 0 \\ [2(2i+1)] & d_{i+1,1} - \alpha_1^{i+1} d_{i+1,i+1} & d_{i+1,2} - \alpha_2^{i+1} d_{i+1,i+1} & \cdots & d_{i+1,i} - \alpha_i^{i+1} d_{i+1,i+1} & d_{i+1,i+1} \end{pmatrix}$$

Recall the defining equation for the α_k^{i+1}:

$$(X - X^{-1})P^{(i+1)}(X) = \sum_{k=1}^{i+1} \alpha_k^{i+1}(X^{2k-1} - X^{-2k+1})$$

Since $P^{(i)}(q^2) = 0$ provided $i \geq 2$, by evaluating both sides at $X = q^2$, we obtain

$$0 = \sum_{k=1}^{i+1} \alpha_k^{i+1}[2(2k-1)].$$

Therefore, if we denote $e_{i,j} = \sum_{k=1}^{i} \alpha_k^i d_{k,j}$, row operations adding multiples of all rows to the last results in the matrix $E_i^{(2)}$

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ [2] & d_{1,1} & 0 & \cdots & 0 & 0 \\ [6] & d_{2,1} & d_{2,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [2(2i-1)] & d_{i,1} & d_{i,2} & \cdots & d_{i,i} & 0 \\ 0 & e_{i+1,1} - \alpha_1^{i+1} d_{i+1,i+1} & e_{i+1,2} - \alpha_2^{i+1} d_{i+1,i+1} & \cdots & e_{i+1,i} - \alpha_i^{i+1} d_{i+1,i+1} & d_{i+1,i+1} \end{pmatrix}$$

By swapping the last row up into the second row and preserving the order of all the remaining rows, then dividing the second row by its penultimate entry, we obtain the last matrix:

$$E_i^{(3)} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & \frac{e_{i+1,1} - \alpha_1^{i+1} d_{i+1,i+1}}{e_{i+1,i} - \alpha_i^{i+1} d_{i+1,i+1}} & \frac{e_{i+1,2} - \alpha_2^{i+1} d_{i+1,i+1}}{e_{i+1,i} - \alpha_i^{i+1} d_{i+1,i+1}} & \cdots & 1 & \frac{d_{i+1,i+1}}{e_{i+1,i} - \alpha_i^{i+1} d_{i+1,i+1}} \\ [2] & d_{1,1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ [2(2i-3)] & d_{i-1,1} & d_{i-1,2} & \cdots & 0 & 0 \\ [2(2i-1)] & d_{i,1} & d_{i,2} & \cdots & d_{i,i} & 0 \end{pmatrix}$$

We now claim that the lower-left $i \times i$ matrix is *exactly* E_i, that is,

$$\frac{e_{i+1,j} - \alpha_j^{i+1} d_{i+1,i+1}}{e_{i+1,i} - \alpha_i^{i+1} d_{i+1,i+1}} = \alpha_j^i. \tag{5.19}$$

It is this statement which we single out as being the "miraculous" moment where the two operators $P^{(i)}(X)$ and $Y_t + Y_t^{-1}$ relate, in the sense that the former operator is associated to the α_j^i terms and the latter associated to the $d_{j,k}$ terms. We know of no conceptual reason to expect (5.19) to hold; yet it can be demonstrated via direct (if sometimes long) q-computations. Let us show it.

Let us define $s_j^i = e_{i+1,j} - \alpha_j^{i+1} d_{i+1,i+1}$. We will show (5.19) by showing, for a fixed i, both α_j^i and s_j^i / s_i^i are defined by the same recursion:

$$s_i^i / s_i^i = \alpha_i^i = 1$$

$$\frac{s_k^i / s_i^i}{s_{k+1}^i / s_i^i} \left(= \frac{s_k^i}{s_{k+1}^i} \right) = \frac{\alpha_k^i}{\alpha_{k+1}^i} = -\frac{[2(i+k)]}{[2(i-k)]} \qquad k < i$$

Proof. (Recursion for α_k^i)

The base case is immediate. To establish the recursion, assume we have

$$\frac{\alpha_k^i}{\alpha_{k+1}^i} = -\frac{[2(i+k)]}{[2(i-k)]}$$

for some i and all $k = 1, ..., i-1$. Then since we have

$$(q^{2i}X - q^{-2i}X^{-1})(q^{-2i}X - q^{2i}X^{-1}) \sum_{k=1}^{i} \alpha_k^i (X^{2k-1} - X^{-2k+1}) = \sum_{k=1}^{i+1} \alpha_k^{i+1} (X^{2k-1} - X^{-2k+1})$$

we have the recursion $\alpha_k^{i+1} = \alpha_{k-1}^i - \langle 4i \rangle \alpha_k^i + \alpha_{k+1}^i$, and thus

$$\frac{\alpha_k^{i+1}}{\alpha_{k+1}^{i+1}} = \frac{\alpha_{k-1}^i / \alpha_k^i - \langle 4i \rangle + \alpha_{k+1}^i / \alpha_k^i}{1 - \langle 4i \rangle \alpha_{k+1}^i / \alpha_k^i + \alpha_{k+2}^i / \alpha_k^i}$$

$$= \frac{-[2(i-k+1)]/[2(i+k-1)] - \langle 4i \rangle - [2(i-k)]/[2(i+k)]}{1 + \langle 4i \rangle [2(i-k)]/[2(i+k)] + [2(i-k)][2(i-k-1)]/[2(i+k)][2(i+k+1)]}$$

which can be shown through a long computation to simplify to

$$-\frac{[2(i+1+k)]}{[2(i+1-k)]}$$

as desired. ∎

Before we move on to the recursion defining the s^i_j, let us pause here to note that the above recursion gives us the following recognization of α^i_k as a certain q−binomial coefficient:

$$\alpha^i_k = (-1)^{i-k}\frac{[2(i+k)][2(i+k+1)]\cdots[2(2i-1)]}{[2(i-k)][2(i-k-1)]\cdots[2]} = \binom{2i-1}{i-k}_{q^2} \tag{5.20}$$

Proof. (Recursion for s^i_j)

First note that s^i_j depends ostensibly on t_1 and λ. Expanding the definition of $e_{i+1,j}$ shows that, in fact, there is no λ dependence:

$$s^i_j = \sum_{k=1}^{i+1} \alpha^{i+1}_k d_{k,j} - \alpha^{i+1}_j d_{i+1,i+1}$$

$$= \sum_{k=j+1}^{i+1} \alpha^{i+1}_k d_{k,j} + \alpha^{i+1}_j (d_{j,j} - d_{i+1,i+1})$$

$$= -\sum_{k=j+1}^{i+1} \alpha^{i-1}_k \langle 2k-1\rangle [2j-1]\overline{t_1} + \alpha^{i+1}_j (q^{2(2j-1)}t_1^{-1} + q^{-2(2j-1)}t_1 - q^{2(2i+1)}t_1^{-1} - q^{-2(2i+1)}t_1)$$

Further, it is convenient to introduce $t^+_1 := t_1 + t_1^{-1}$. If we do this, decomposing s^i_j into t^+_1 and $\overline{t_1}$ components, we have

$$\begin{aligned} s^i_j = {} & \frac{1}{2}\alpha^{i+1}_j\Big(\langle 2(2j-1)\rangle - \langle 2(2i+1)\rangle\Big)t^+_1 \\ & + \left(\frac{1}{2}\alpha^{i+1}_j\Big([2(2i+1)] - [2(2j-1)]\Big) - [2j-1]\sum_{k=j+1}^{i+1}\alpha^{i+1}_k\langle 2k-1\rangle\right)\overline{t_1} \end{aligned} \tag{5.21}$$

In particular, we can compute a closed form for s^i_i:

$$s^i_i = -\frac{1}{2}[2(2i+1)][4i]t^+_1 + \left(-\frac{1}{2}[2(2i+1)]\langle 4i\rangle + [2i-1]\langle 2i+1\rangle\right)\overline{t_1}$$

with which we can turn the desired statement $s^i_j = \alpha^i_j s^i_i$ into a pair of q-identities that we must establish, namely

$$\alpha^{i+1}_j\Big(\langle 2(2j-1)\rangle - \langle 2(2i+1)\rangle\Big) = -\alpha^i_j[2(2i+1)][4i] \tag{5.22}$$

$$\frac{1}{2}\alpha_j^{i+1}\Big([2(2i+1)]-[2(2j-1)]\Big)-[2j-1]\sum_{k=j+1}^{i+1}\alpha_k^{i+1}\langle 2k-1\rangle$$
$$=\alpha_j^i\left(-\frac{1}{2}[2(2i+1)]\langle 4i\rangle+[2i-1]\langle 2i+1\rangle\right) \quad (5.23)$$

Crucially, the ratios $\alpha_k^{i+1}/\alpha_k^i$ admit a simple form - this follows directly from the closed form produced by the recursion they satisfy - explicitly, they are

$$\frac{\alpha_k^i}{\alpha_k^{i-1}}=-\frac{[4i][2(2i+1)]}{[2(i+k)][2(i+1-k)]}$$

with these, one sees that (5.22) is reduced to an explicit calculation and (5.23) becomes a straightforward induction in j, the details of which we omit.

□

Given (5.19), we observe that since $\det(E_{i+1}) = (-1)^{i+1}\det(E_{i+1}^{(3)}) = -s_i^i\det(E_i) = -[4i][2(2i+1)]A_{i+1}\det(E_i)$, we have the following for all $i \geq 1$:

$$\det(E_{i+1}) = -[2(2i)][2(2i+1)]A_{i+1}$$

and since in the base case we have

$$\det(E_1)=\begin{vmatrix} 0 & 1 \\ [2] & d_{1,1}\end{vmatrix}=-[2]$$

we arrive at the promised closed form for $\det(E_i)$, and thus $F^i(-q^2,\lambda)$:

$$F^i(-q^2,\lambda)=\frac{(-1)^i\left(\prod_{j=1}^{2i-1}[2j]\right)\left(\prod_{k=2}^{i}A_k\right)}{\prod_{k=1}^{i}(q^{2(2k-1)}t_1^{-1}+q^{-2(2k-1)}t_1+\lambda+\lambda^{-1})} \quad (5.24)$$

which, combining with (5.9) as in the general case, finishes the proof.

The proof of (5.18) from Theorem 21 is, comparatively, much quicker:

Proof. We first show that (5.24) implies the following shift equation in λ for our generating function:

$$\frac{F^i(-q^2, q^2\lambda)}{F^i(-q^2, q^{-2}\lambda)} = \frac{\lambda + t_1}{1+\lambda t_1}\frac{q^{2i} + q^{-2i}\lambda t_1}{q^{2i}\lambda + q^{-2i}t_1} \tag{5.25}$$

As the denominator of (5.24) does not depend on λ, we are led to the equation

$$\begin{aligned}\frac{F^i(-q^2, q^2\lambda)}{F^i(-q^2, q^{-2}\lambda)} &= \prod_{j=1}^{i} \frac{(q^{2(2k-1)}t^{-1} + q^{-2(2k-1)}t + q^{-2}\lambda + q^2\lambda^{-1})}{(q^{2(2k-1)}t^{-1} + q^{-2(2k-1)}t + q^{2}\lambda + q^{-2}\lambda^{-1})} \\ &= \prod_{j=1}^{i} \frac{(q^{2(k-1)}\lambda + q^{-2(k-1)}t)(q^{2k} + q^{-2k}\lambda t)}{(q^{2(k-1)} + q^{-2(k-1)}\lambda t)(q^{2k}\lambda + q^{-2k}t)} \\ &= \frac{\lambda + t}{1+\lambda t}\frac{q^{2i} + q^{-2i}\lambda t}{q^{2i}\lambda + q^{-2i}t}\end{aligned}$$

Rearranging this equation and taking the coefficient of λ^n, we are led to the following three-term recurrence in the $\{\widetilde{c_{n,i-1}}\}$ for a fixed i:

$$(q^{2i}t^{-1} + q^{-2i}t)[2n]\widetilde{c_{n,i-1}} = [2(n+1-i)]\widetilde{c_{n+1,i-1}} + [2(n-1+i)]\widetilde{c_{n-1,i-1}}$$

and dividing through by $c_{n,i-1}$ gives us

$$(q^{2i}t^{-1} + q^{-2i}t)\frac{\widetilde{c_{n,i-1}}}{c_{n,i-1}} = \frac{[2(n+i)]}{[2n]}\frac{\widetilde{c_{n+1,i-1}}}{c_{n+1,i-1}} + \frac{[2(n-i)]}{[2n]}\frac{\widetilde{c_{n-1,i-1}}}{c_{n-1,i-1}}$$

This three-term recurrence relation can readily be shown through a calculation to coincide with the Pieri relations (4.2) for the sequence $\{\pi_{n-i}(q^{2i}t^{-1}; q^{4i}|q^4)\}$, or equivalently the sequence $\{\frac{p_{n-i}(q^{2i}t^{-1};q^{4i}|q^4)}{p_{n-i}(q^{2i};q^{4i}|q^4)}\}$. Thus the proof will be completed provided we establish the two base cases for each i

$$\widetilde{c_{i,i-1}} = \left(\prod_{j=2}^{i} A_j\right) c_{i,i-1}$$

$$\widetilde{c_{i+1,i-1}} = \left(\prod_{j=2}^{i} A_j\right) \frac{q^{2i}t_1^{-1} + q^{-2i}t_1}{q^{2i} + q^{-2i}} c_{i+1,i-1}$$

This becomes straightforward once one establishes the following formulas

$$a_{i,i} = \prod_{j=2}^{i} A_j \qquad a_{i+1,i} = (A_1 - A_{i+1}) \prod_{j=2}^{i} A_j$$

which are quick consequences of the induction defining the $a_{n,k}$, and then uses the equations

$$\widetilde{c_{i,i-1}} = a_{i,i} c_{i,i-1} \qquad \widetilde{c_{i+1,i-1}} = a_{i+1,i+1} c_{i+1,i-1} - a_{i+1,i} c_{i,i-1}$$

which give an explicit verification of the base cases. □

5.4.5 Alternate proof: $t_2 = 1$ case

We can alternately derive (5.18) from (5.17) by recognizing the function of λ in the latter as being a simple variant of a standard generating function for the *continuous* q-*ultraspherical polynomials* $C_n(X; \beta|q)$.

The polynomials $C_n(X; \beta|q)$ are defined in terms of the Macdonald polynomials $p_n(X; \beta|q)$ by the formula

$$C_n(X; \beta|q) = \frac{(1-\beta)(1-\beta q)\cdots(1-\beta q^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)} p_n(X; \beta|q). \tag{5.26}$$

The $C_n(X; \beta|q)$ obey the following generating function identity, which is well known (see, e.g., [28])

$$\sum_{n=0}^{\infty} C_n(X; \beta|q)\lambda^n = \frac{(\lambda\beta X; q)_\infty (\lambda\beta X^{-1}; q)_\infty}{(\lambda X; q)_\infty (\lambda X^{-1}; q)_\infty} \tag{5.27}$$

which, when $q = q^4$, $\beta = q^{4i}$ becomes

$$\sum_{n=0}^{\infty} C_n(X; q^{4i}|q^4)\lambda^n = \frac{1}{\prod_{k=0}^{i-1}(1 - q^{4k}\lambda X)(1 - q^{4k}\lambda X^{-1})}.$$

Before we start, let us note that we will work with the following equivalent version of (5.17), where λ has been replaced by $-\lambda$ to ease the amount of signs, and we have shifted the index of i by 1:

$$\widetilde{c_{n,i-1}} = [\lambda^n]\left(\frac{c_{i,i-1}\prod_{k=2}^{i} A_k}{\prod_{k=1}^{i}(\lambda+\lambda^{-1}-(q^{2(2k-1)}t^{-1}+q^{-2(2k-1)}t))}\right) \tag{5.28}$$

Consider the product appearing in the denominator of (5.28):

$$\prod_{k=1}^{i}(\lambda+\lambda^{-1}-(q^{2(2k-1)}t^{-1}+q^{-2(2k-1)}t))$$

Introduce the variables $\mu = q^{-2(i-1)}\lambda$ and $X = q^{2i}t^{-1}$, and this becomes

$$= q^{-2i(i-1)}\mu^{-i}\prod_{k=1}^{i}(1-q^{4(k-1)}\mu X)(1-q^{4(i-k)}\mu X^{-1})$$

$$= q^{-2i(i-1)}\mu^{-i}\prod_{k=0}^{i}(1-q^{4k}\mu X)(1-q^{4k}\mu X^{-1})$$

Thus by employing (5.27), we can rewrite (5.28) (after returning to λ, t variables) as

$$\widetilde{c_{n,i-1}} = c_{i,i-1}\left(\prod_{k=2}^{i} A_k\right)[\lambda^n]\left(\sum_{n=0}^{\infty} q^{-2n(i-1)}C_n(q^{2i}t^{-1};q^{4i}|q^4)\lambda^{n+i}\right)$$

or, more simply,

$$\widetilde{c_{n,i-1}} = c_{i,i-1}\left(\prod_{k=2}^{i} A_k\right)q^{-2(n-i)(i-1)}C_{n-i}(q^{2i}t^{-1};q^{4i}|q^4) \tag{5.29}$$

It is a consequence of (5.26) as well as the evaluation formula for $p_n(\beta^{1/2};\beta|q)$ (Theorem 17) that

$$q^{-2(n-i)(i-1)}c_{i,i-1}C_{n-i}(q^{2i}t^{-1};q^{4i}|q^4) = c_{n,i-1}\frac{p_{n-i}(q^{2i}t^{-1};q^{4i}|q^4)}{p_{n-i}(q^{2i};q^{4i}|q^4)}$$

which allows us to conclude

$$\widetilde{c_{n,i-1}} = \left(\prod_{k=2}^{i} A_k\right)\frac{p_{n-i}(q^{2i}t^{-1};q^{4i}|q^4)}{p_{n-i}(q^{2i};q^{4i}|q^4)}c_{n,i-1}$$

which is exactly (5.18).

CHAPTER 6

BS CONJECTURE (QUASI-CLASSICAL)

This chapter will be concerned with the BS conjecture of before, at the special value $q = -1$. Our goal will be to reduce this conjecture to a statement which can be checked computationally for a fixed knot K which admits a knot group on 2 or 3 generators.

In [34], the authors relate the BS conjecture at $q = -1$ to a conjecture made previously by Brumfiel and Hilden about the peripheral map

$$H\alpha_K : H\mathbb{Z}^2 \to H\pi_1(S^3 \setminus K).$$

Theorem 22. *(Berest, Samuelson, 2018) Let* $X, Y \in H\mathbb{Z}^2$ *be the class of the meridan and longitude, respectively. If we assume the ring map*

$$(X - X^{-1}) : H^+\pi_1(S^3 \setminus K) \to H^+\pi_1(S^3 \setminus K)[X^{\pm 1}]$$

$$a \mapsto (X - X^{-1})a$$

is injective, then the BS conjecture at $q = -1$ *holds provided*

$$Y \in H^+\pi_1(S^3 \setminus K)[X^{\pm 1}] \tag{6.1}$$

where we have identified X *and* Y *with their images under* α_K*.*

In the paper establishing this theorem, the authors show that all torus knots, 2-bridge knots and some invertible torus knots satisfy (6.1) - a class strictly larger than for which the general conjecture (at all q) is known. Notably, this class lacked an example of a non-invertible knot.

6.1 Case of pretzel knots

In this section we describe the approach to produce code which will verify (6.1) for the $(3,5,7)$ and $(3,5,9)$-pretzel knots, known to be non-invertible by work of Trotter [32]. The result is the following

Theorem 23. *For both the* $(3,5,7)$ *and* $(3,5,9)$*-pretzel knots, (6.1) holds.*

Proof. We can reduce the decision problem defining (6.1) to that of checking whether or not two explicit modules over polynomial rings intersect or not, which can be calculated efficiently using the software Macaulay2. Efficiency will be important, as the generators of our modules over polynomial rings will quickly become of very large degree.

Step 1. Recall the presentation of $\pi := \pi_1(S^3 \setminus K_{p,q,r})$, where $K_{(p,q,r)}$ is the (p,q,r)-pretzel knot and $p = 2k+1, q = 2l+1, r = 2m+1$:

$$\pi = \langle a, b, c | r_1, r_2, r_3 \rangle$$

$$r_1 : (ab^{-1})^m a (ab^{-1})^{-m} = (bc^{-1})^{k+1} c (bc^{-1})^{-k-1}$$

$$r_2 : (bc^{-1})^k b (bc^{-1})^{-k} = (ca^{-1})^{l+1} a (ca^{-1})^{-l-1}$$

$$r_3 : (ca^{-1})^l c (ca^{-1})^{-l} = (ab^{-1})^{m+1} b (ab^{-1})^{-m-1}.$$

A meridian-longitude pair is given by $(m, l) = (a, w)$, where

$$w = (ab^{-1})^{-m}(bc^{-1})^{k+1}(ca^{-1})^{-l}(ab^{-1})^{m+1}(bc^{-1})^{-k}(ca^{-1})^{l+1}.$$

Step 2. Employ the structure theorems for HF_3 (Theorems 6, 7) to produce it explicitly in Macaulay2. This amounts to producing a free rank-8 module H

over a polynomial algebra in 6 generators, along with a matrix $M : H \otimes H \to H$ describing the multiplication in this algebra.

Computational note: The multiplication table is large (a 64×8 matrix), but thankfully due to the rules defining the multiplication, it is quite sparse. One may naturally worry about the possibility of introducing an error in such a large table; a good consistency check (which we perform) is to verify the associativity of multiplication defined via M for a few triples of elements of H, chosen at random. Additionally, in the particular case of our pretzel knot computation, we can exploit the fact that all of our generators are conjugate. This has the nice effect that three of the generators of the underlying polynomial algebra coincide in the quotient and thus we may write everything with two fewer variables.

Step 3. Since the natural surjection $p : F_3 \to \pi$ induces a surjection $Hp : HF_3 \to H\pi$, we have $Y \in H^+\pi[X^{\pm 1}]$ iff $\tilde{Y} \in R + \ker Hp$ for some R which maps onto $H^+\pi[X^{\pm 1}]$ and some $\tilde{Y}$ which maps onto Y. Using the result (10) on the structure of such ideals, we can generate given r_1, r_2, r_3 a particular $R + \ker Hp$. To do this quickly, we write a method which can take a word in the generators a, b, c and produce its image in HF_3. Finally, we generate a $\tilde{Y}$ and check the inclusion.

Note: Here is where most of the computational difficulty is encountered. First of all, since the presentation is Wirtinger, one may remove any one of the relations. The length of the relations does not make computing any particular generator of $\ker p$ difficult, but since there are enough generators, it is lengthy to check if any given element is in $\ker p$. Add to this the fact that the longitude is (already for the smallest k, l and m) of quite high degree and the computation becomes demanding. Most of the effort seems to lie in computing Grobner bases for all of these modules, which is not something at the moment seems to have an easy solution.

One may hope that an alternate presentation may ameliorate some of these difficulties, but after exhausting many natural attempts it appears as if the difficulty is innate. Introducing, e.g. $x = a, y = ab^{-1}, z = bc^{-1}$, the gains one makes in simplicity of presentation are weighed against the fact that now one has lost the virtue of having all 3 generators conjugate, and so the polynomial algebra underlying everything again expands.

The full Macaulay 2 code can be found in the appendix. □

6.2 Case of virtual knots

One way to produce examples of knots satisfying (6.1) is to use coverings of knot groups. Suppose that $\pi_1, (m_1, l_1)$ is one knot group with peripheral system for which (6.1) is known and $\pi_2, (m_2, l_2)$ for which it is unknown. If there exists an epimorphism

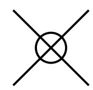
$$p : \pi_1 \to \pi_2$$

$$m_1 \mapsto m_1, l_1 \mapsto m_2^a l_2$$

then since $p(H^+\pi_1) = H^+\pi_2$ and $p(X_1) = X_2$, we have necessarily that $Y_2 \in H^+\pi_2[X_2]$.

Coverings of knots, both preserving and not preserving peripheral structure in this way, were studied by Silver and Whitten in [30]. Either notion of covering naturally induces a partial order on knots; the authors were then compelled to examine how this partial order extended to the more general class of virtual knots.

Briefly, virtual knots are equivalence classes of knot diagrams which have in addition to the usual over/under-crossings a "virtual crossing"



Associated to these virtual knots are fundamental groups with peripheral data, defined in a fashion naturally extending the typical Wirtinger presentation of the usual knot group; see [16] for a full discussion.

In his work, Silver shows that for the figure-8 knot, a knot which does not cover any other knot (in any sense considered), there exists a sequence of virtual knots which it does cover. We discover that the method of proof explicated in this work can be extended to more generally cover any 2-bridge knot, and thus arrive at the

Theorem 24. *Given any 2-bridge knot* k*, there exists an infinite family of virtual knots* k_q*,* $q \in \mathbb{N}$ *such that the knot group of* k *maps onto the knot group of* k_q*. If the peripheral systems of the two groups are* (m, l) *and* (m_q, l_q)*, then this covering takes* $m \to m_q, l \to l_q$.

Proof. The fundamental group of a 2-bridge knot k is given by

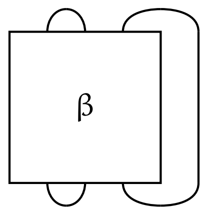
$$\pi_k = \langle a, b, \mid aw = wb \rangle$$

where w is a word in a, b. We will produce a particular virtual knot k_q with fundamental group

$$\pi_{k_q} = \langle a, b, \mid aw = wb, a^q b a^{-q} = b \rangle$$

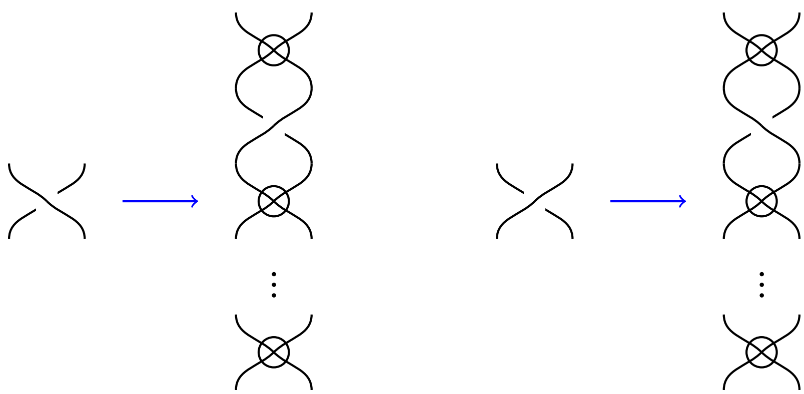
in such a way that it will be clear the image of the peripheral system of π_k under the natural projection to π_{k_q} maps onto the peripheral system of the latter.

Let us assume we have a 4-plat presentation for k, that is, a presentation as k as the closure of a 3-braid β in the following way:



Without loss of generality, we may assume $\beta = \alpha\sigma_{2,3}^{\pm 1}$

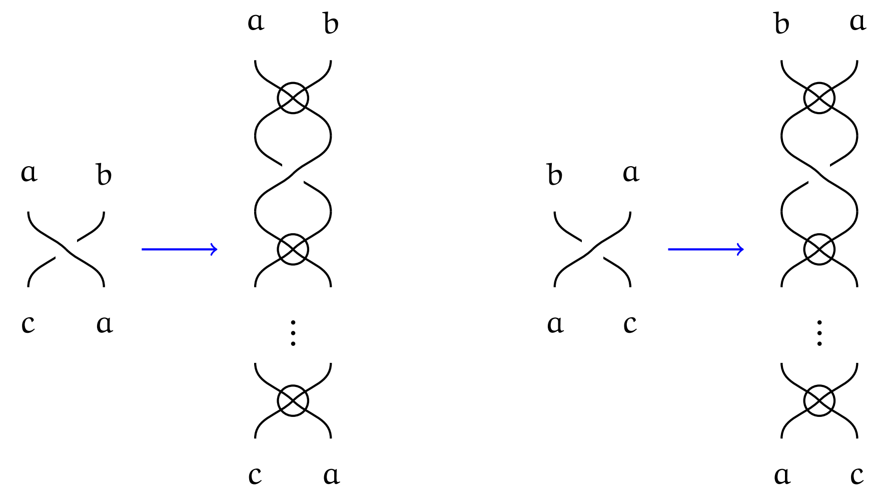
We now aim to "virtualize" this crossing in a controlled way; that is, make one of the following two replacements:



and thus our k_q will be just k virtualized with one chain of length q. Let us label the curves appearing in both diagrams:

We now collect the following lemmas:

Lemma 10. *If the classical region contributes the Wirtinger relation* $c = aba^{-1}$ *or* $c = a^{-1}ba$, *then the virtualized region contributes* $c = a^{-q+1}ba^{q-1}$ *or* $c = a^{q-1}ba^{-q+1}$, *respectively. All of the new arcs introduced in the virtual region may be written in terms of* a, b *and* c.



Lemma 11. *When computing a longitude* l *for the fixed meridian* a*, if in the classical case one gets*

$$l = w_1 a^{\pm 1} w_2 a^{-s}$$

with the isolated a *referring to passing through the crossing underneath the* a *strand, in the virtual case we have*

$$l = w_1 a^{\pm(-q+1)} w_2 a^{-s+q}$$

Lemma 12. *One may always swap labels of* b *and* c *in such a way that* a *and* b *are generators of* π_k*, and the Wirtinger presentation for* π_k *omitting the relation arising from this crossing is of the standard form*

$$\pi_k = \langle a, b \mid aw = wb \rangle$$

Note: Lemmas 10 and 11 hold for any crossing in any knot diagram, but Lemma 12 is highly specialized to the current case. The proofs of the lemmas are straightforward. From the lemmas, the proof of the result follows quickly:

Combining Lemmas 10 and 12, we see that π_{k_q} has the presentation:

$$\pi_{k_q} = \langle a, b \mid aw = wb, c = a^{q-1}ba^{-q+1} \rangle$$

(Here we are assumming the original crossing gave the relation $c = a^{-1}ba$, but the other case is identical.) As $aw = wb$ must implicitly imply $c = a^{-1}ba$, we may rewrite this last relation as

$$a^{-1}ba = a^{q-1}ba^{-q+1} \Rightarrow b = a^{q}ba^{-q}$$

and so we have the presentation

$$\pi_{k_q} = \langle a, b \mid aw = wb, b = a^{q}ba^{-q} \rangle$$

.

Combining Lemmas 11 and 12, if we assume the original longitude is of the form $w_1 a w_2 a^{-s}$, then the longitude of the virtualization is $w_1 a a^{-q} w_2 a^{-s+q}$. But as w_2 is a word in a, b and a^{-q} commutes with both a and b in π_{k_q}, we have

$$w_1 a a^{-q} w_2 a^{-s+q} = w_1 a w_2 a^{-s} \in \pi_{k_q}$$

and so the virtual longitude is exactly the reduction of the classical one. $\square$

APPENDIX A

CODE FOR PRETZEL KNOTS

```
{*

Here we create HF_3 and the multiplication table.
This is "Step 2."

*}

kk = QQ
R = kk[t_0,t_1,t_2,t_3,t_4,t_5]

Hpi = R^8

m1 = Hpi_{0}
m2 = Hpi_{1}
m3 = Hpi_{2}
m4 = Hpi_{3}
m5 = Hpi_{4}
m6 = Hpi_{5}
m7 = Hpi_{6}
m8 = Hpi_{7}

m9 = Hpi_{1}
m10 = matrix((t_0^2-1)*Hpi_0)
m11 = matrix(t_3*Hpi_0 + Hpi_4)
```

```
m12 = matrix(t_4*Hpi_0 + Hpi_5)
m13 = matrix((t_0^2-1)*Hpi_2 - t_3*Hpi_1)
m14 = matrix((t_0^2-1)*Hpi_3 - t_4*Hpi_1)
m15 = matrix(Hpi_7 + t_3*Hpi_3 - t_4*Hpi_2)
m16 = matrix((t_0^2-1)*Hpi_6 - t_3*Hpi_5 + t_4*Hpi_4)

m17 = Hpi_{2}
m18 = matrix(t_3*Hpi_0 - Hpi_4)
m19 = matrix((t_1^2-1)*Hpi_0)
m20 = matrix(t_5*Hpi_0 + Hpi_6)
m21 = matrix(t_3*Hpi_2 - (t_1^2-1)*Hpi_1)
m22 = matrix(t_3*Hpi_3 - t_5*Hpi_1 - Hpi_7)
m23 = matrix((t_1^2-1)*Hpi_3 - t_5*Hpi_2)
m24 = matrix(t_3*Hpi_6 - (t_1^2-1)*Hpi_5 + t_5*Hpi_4)

m25 = Hpi_{3}
m26 = matrix(t_4*Hpi_0 - Hpi_5)
m27 = matrix(t_5*Hpi_0 - Hpi_6)
m28 = matrix((t_2^2-1)*Hpi_0)
m29 = matrix(Hpi_7 + t_4*Hpi_2 - t_5*Hpi_1)
m30 = matrix(t_4*Hpi_3 - (t_2^2-1)*Hpi_1)
m31 = matrix(t_5*Hpi_3 - (t_2^2-1)*Hpi_2)
m32 = matrix(t_4*Hpi_6 - t_5*Hpi_5 + (t_2^2-1)*Hpi_4)

m33 = Hpi_{4}
m34 = matrix(t_3*Hpi_1 - (t_0^2-1)*Hpi_2)
```

```
m35 = matrix((t_1^2-1)*Hpi_1 - t_3* Hpi_2)
m36 = matrix(t_5*Hpi_1 - t_4*Hpi_2 + Hpi_7)
m37 = matrix((t_3^2-(t_0^2-1)*(t_1^2-1))*Hpi_0)
m38 = matrix((t_4*t_3-(t_0^2-1)*t_5)*Hpi_0
- t_4*Hpi_4 + t_3*Hpi_5 - (t_0^2-1)*Hpi_6)
m39 = matrix((t_4*(t_1^2-1)-t_3*t_5)*Hpi_0
+ (t_1^2-1)*Hpi_5 - t_3*Hpi_6 - t_5*Hpi_4)
m40 = matrix(((t_1^2-1)*t_4-t_3*t_5)*Hpi_1
- (t_3*t_4-(t_0^2-1)*t_5)*Hpi_2
+ (t_3^2-(t_0^2-1)*(t_1^2-1))*Hpi_3)

m41 = Hpi_{5}
m42 = matrix(t_4*Hpi_1 - (t_0^2-1)*Hpi_3)
m43 = matrix(t_5*Hpi_1 - t_3*Hpi_3 - Hpi_7)
m44 = matrix((t_2^2-1)*Hpi_1 - t_4*Hpi_3)
m45 = matrix((t_3*t_4-(t_0^2-1)*t_5)*Hpi_0
- t_3*Hpi_5 + t_4*Hpi_4 + (t_0^2-1)*Hpi_6)
m46 = matrix((t_4^2-(t_0^2-1)*(t_2^2-1))*Hpi_0)
m47 = matrix((t_4*t_5 - t_3*(t_2^2-1))*Hpi_0
- t_4*Hpi_6 + t_5*Hpi_5 - (t_2^2-1)*Hpi_4)
m48 = matrix((t_5*t_4-t_3*(t_2^2-1))*Hpi_1
- (t_4^2-(t_0^2-1)*(t_2^2-1))*Hpi_2
+ (t_3*t_4-t_5*(t_0^2-1))*Hpi_3)

m49 = Hpi_{6}
m50 = matrix(Hpi_7 + t_4*Hpi_2 - t_3*Hpi_3)
```

```
m51 = matrix(t_5*Hpi_2 - (t_1^2-1)*Hpi_3)
m52 = matrix((t_2^2-1)*Hpi_2 - t_5*Hpi_3)
m53 = matrix(((t_1^2-1)*t_4-t_3*t_5)*Hpi_0
- (t_1^2-1)*Hpi_5 + t_3*Hpi_6 + t_5*Hpi_4)
m54 = matrix((t_5*t_4 - t_3*(t_2^2-1))*Hpi_0
- t_5*Hpi_5 + t_4*Hpi_6 + (t_2^2-1)*Hpi_4)
m55 = matrix((t_5^2-(t_1^2-1)*(t_2^2-1))*Hpi_0)
m56 = matrix((t_5^2-(t_1^2-1)*(t_2^2-1))*Hpi_1
- (t_4*t_5-t_3*(t_2^2-1))*Hpi_2
+ (t_4*(t_1^2-1)-t_3*t_5)*Hpi_3)

m57 = Hpi_{7}
m58 = matrix((t_0^2-1)*Hpi_6 - t_3*Hpi_5 + t_4*Hpi_4)
m59 = matrix(t_3*Hpi_6 - (t_1^2-1)*Hpi_5 + t_5*Hpi_4)
m60 = matrix(t_4*Hpi_6 - t_5*Hpi_5 + (t_2^2-1)*Hpi_4)
m61 = matrix(((t_1^2-1)*t_4-t_3*t_5)*Hpi_1
- (t_3*t_4-(t_0^2-1)*t_5)*Hpi_2
+ (t_3^2-(t_0^2-1)*(t_1^2-1))*Hpi_3)
m62 = matrix((t_5*t_4-t_3*(t_2^2-1))*Hpi_1
- (t_4^2-(t_0^2-1)*(t_2^2-1))*Hpi_2
+ (t_3*t_4-t_5*(t_0^2-1))*Hpi_3)
m63 = matrix((t_5^2-(t_1^2-1)*(t_2^2-1))*Hpi_1
- (t_4*t_5-t_3*(t_2^2-1))*Hpi_2
+ (t_4*(t_1^2-1)-t_3*t_5)*Hpi_3)
m64 = matrix((t_4^2*(t_1^2-1)
+t_3^2*(t_2^2-1)+t_5^2*(t_0^2-1)
```

```
-(t_0^2-1)*(t_1^2-1)*(t_2^2-1)
-t_3*t_5*t_4-t_4*t_3*t_5)*Hpi_0)

MT1 = matrix( m1 | m2 | m3 | m4 | m5 | m6 | m7 | m8 )
MT2 = matrix( m9 | m10 | m11 | m12 | m13 | m14 | m15 | m16 )
MT3 = matrix( m17 | m18 | m19 | m20 | m21 | m22 | m23 | m24 )
MT4 = matrix( m25 | m26 | m27 | m28 | m29 | m30 | m31 | m32 )
MT5 = matrix( m33 | m34 | m35 | m36 | m37 | m38 | m39 | m40 )
MT6 = matrix( m41 | m42 | m43 | m44 | m45 | m46 | m47 | m48 )
MT7 = matrix( m49 | m50 | m51 | m52 | m53 | m54 | m55 | m56 )
MT8 = matrix( m57 | m58 | m59 | m60 | m61 | m62 | m63 | m64 )

MT = matrix( MT1 | MT2 | MT3 | MT4 | MT5 | MT6 | MT7 | MT8 )

{*

A skippable assocativity check:
This is "Step 2"

*}

MT(Hpi_2**MT(Hpi_3**Hpi_4)) - MT(MT(Hpi_2**Hpi_3)**Hpi_4)
MT(Hpi_3**MT(Hpi_4**Hpi_5)) - MT(MT(Hpi_3**Hpi_4)**Hpi_5)
MT(Hpi_1**MT(Hpi_4**Hpi_5)) - MT(MT(Hpi_1**Hpi_4)**Hpi_5)

{*
```

```
Recording words in three letters as lists of integer pairs.

The first integer may be one of 0, 1 or 2, corresponding
to the 3 generators,
while the second integer, the exponent, is unrestricted.

(Currently storing words relevant to pretzel knots)
This is "Step 3"

*}

xmy = {(0,1),(1,-1)}
ymx = {(1,1),(0,-1)}
ymz = {(1,1),(2,-1)}
zmy = {(2,1),(1,-1)}
zmx = {(2,1),(0,-1)}
xmz = {(0,1),(2,-1)}

k = 1
l = 2
m = 3

p = 2*k+1
q = 2*l+1
r = 2*m+1
```

```
{*

All methods in this block are subordinate to the last method,
 asRingElt,
which creates the corresponding element of Hpi from a list
 formatted as above.
This is "Step 3"

*}

f = (i,j) -> if (j == 0) then Hpi_0 else (
    if (j == -1 or j == 1) then (t_i*Hpi_0 + j*Hpi_(i+1)) else (
     if j < 0 then MT( (t_i*Hpi_0 - Hpi_(i+1))**f(i, j+1) ) else (
MT( (t_i*Hpi_0 + Hpi_(i+1))**f(i, j-1) ) )))

List _ Sequence := (x,y) -> apply(toList y, i -> x#i)

isFormatted = method()
isFormatted List := L -> if length L == 0 then true else (
    if instance(L#0, Sequence) then (
if length(L#0) == 2 then (
    if (L#0#0 == 0 or L#0#0 == 1 or L#0#0 == 2) then isFormatted L_(1
    else false )
else false )
```

```
    else false )

asRingElt = method()
asRingElt List := L -> if isFormatted L then (
    if length(L) == 0 then Hpi_0 else (
MT(asRingElt(L_(0 .. length(L)-2))**f(L#(length(L)-1)))))

{*

Here we use the methods to convert our strings into ring elements.
We then use them to create the relations/longitude elements.

Specifically, we are creating the relations/longitude for Trotter's
presentation of the (p,q,r) pretzel knot from the paper
 "Invertible Knots Exist".
This is "Step 3"

*}

xmy = asRingElt xmy
ymx = asRingElt ymx
ymz = asRingElt ymz
zmy = asRingElt zmy
zmx = asRingElt zmx
xmz = asRingElt xmz
```

```
raise = (r, n) -> if n == 1 then r else MT(r**raise(r, n-1))

r1 = MT(raise(xmy, m)**MT(f(0,1)**raise(ymx, m)))
- MT(raise(ymz, k+1)**MT(f(2,1)**raise(zmy, k+1)))
r2 = MT(raise(ymz, k)**MT(f(1,1)**raise(zmy, k)))
- MT(raise(zmx, l+1)**MT(f(0,1)**raise(xmz, l+1)))
r3 = MT(raise(zmx, l)**MT(f(2,1)**raise(xmz, l)))
- MT(raise(xmy, m+1)**MT(f(1,1)**raise(ymx, m+1)))
l = MT(
MT(
MT(
MT(
MT(raise(ymx, m)**raise(ymz, k+1))
**raise(xmz, l))
**raise(xmy, m+1))
**raise(zmy, k))
**raise(zmx, l+1))

rels = {r1, r2, r3}

{*

The output of this code creates the two-sided ideal generated by
the relations r_1, r_2 and r_3 within H\pi, the ideal
itself given by the image of the relation matrix rMatrix.
```

```
This is "Step 3"

*}

rMatrix = L -> if length(L) == 0 then 0*Hpi_{0} else (
    M = 0*Hpi_{0};
    for i to 7 do (for j to 7 do M = M | matrix(
    MT(Hpi_i**MT(L#0**Hpi_j))));

    M | rMatrix(L_(1 .. length(L)-1)))


{*

Here we pose the important question: Does the longitude element
in Hpi lie in the sum of the spaces Hplus
and the relation ideal Irel?

This is "Step 3".

*}

Irel = image rMatrix(rels)
Hplus = image matrix ( Hpi_{0} | Hpi_{7} | Hpi_{1} |
 matrix((t_0^2-1)*Hpi_6 - t_3*Hpi_5 + t_4*Hpi_4) )
```

```
L = image matrix l

isSubset(L, Hplus + Irel)
```

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