Tail probabilities for infinite series of regularly varying random vectors

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Abstract

A random vector X with representation $X = \sum_{j\geq 0} A_j Z_j$ is considered. Here (Z_j) is a sequence of independent and identically distributed random vectors and (A_j) is a sequence of random matrices, "predictable" with respect to the sequence (Z_j). The distribution of Z_1 is assumed to be multivariate regular varying. Moment conditions on the matrices (A_j) are determined under which the distribution of X is regularly varying and, in fact, "inherits" its regular variation from that of (Z_j)'s. We compute the associated limiting measure. Examples include linear processes, random coefficient linear processes such as stochastic recurrence equations, random sums, and stochastic integrals.

Key words: infinite series, linear process, random sums, regular variation, stochastic recursion

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1 Introduction

A general and useful class of stochastic models is the class of random coefficient linear models. This is the class of d dimensional random vectors with the stochastic representation

$$X = \sum_{j=0}^{\infty} A_j Z_j. \tag{1.1}$$

Here $(Z_j)_{j\geq 0}$ is a sequence of independent and identically distributed (i.i.d.) random vectors in $\mathbb{R}^p$, which will be assumed to have a regularly varying law; a precise definition will be given below. A generic element of this sequence is denoted by Z. The sequence $(A_j)_{j\geq 0}$ consists of random matrices. We assume that there is a filtration $(\mathcal{F}_j,\, j \geq 0)$ such that

$$A_j \in \mathcal{F}_j, \quad Z_j \in \mathcal{F}_{j+1}, \quad \text{for } j \geq 0, \tag{1.2}$$

$$\mathcal{F}_j \ \text{ is independent of } \sigma(Z_j, Z_{j+1}, \ldots) \text{ for } j \geq 0. \tag{1.3}$$

In a sense, the sequence (A_j) is predictable with respect to the sequence (Z_j). An important particular case is that where the sequence (A_j) is independent of the sequence (Z_j); it reduces to the "predictable" framework by setting $\mathcal{F}_j = \sigma\big((A_k)_{k\geq 0},\, Z_0, \ldots, Z_{j-1}\big)$.

Naturally, the infinite series in (1.1) is assumed to converge. Sufficient conditions for such convergence will be provided in the sequel. We are interested in situations under which the random matrices (A_j) are "small" relatively to the noise vectors (Z_j), and the tail probabilities of the random vector X in (1.1) are "inherited" from the tail probabilities of the noise vectors. Many particular cases of the model (1.1) have been studied in literature; we discuss some of them below.

Probability distributions with regularly varying tails have become important building blocks in a wide variety of stochastic models. Evidence for power-tail distributions is well documented in a large number of applications including computer networks, telecommunications, finance, insurance, hydrology, atmospheric sciences, geology, ecology etc. Since the model described in (1.1) is multi-dimensional, we will use the notion of multivariate regular variation.

We say that a d-dimensional random vector Z has a regularly varying distribution if there exists a non-null Radon measure μ on $\overline{\mathbb{R}}^d \setminus \{0\}$ with $\mu(\overline{\mathbb{R}}^d \setminus \mathbb{R}^d) = 0$ such that

$$\frac{P(u^{-1}Z \in \cdot\,)}{P(|Z| > u)} \xrightarrow{v} \mu(\cdot) \tag{1.4}$$

on $\overline{\mathbb{R}}^d \setminus \{0\}$. Here $\xrightarrow{v}$ denotes vague convergence (see Kallenberg, 1983; Resnick, 1987, 2006). The limiting measure μ necessarily obeys a homogeneity property: there is an $\alpha > 0$ such that $\mu(uB) = u^{-\alpha}\mu(B)$ for all Borel sets $B \subset \overline{\mathbb{R}}^d \setminus \{0\}$. This follows from standard regular variation arguments (see e.g. Resnick, 2006;

Hult and Lindskog, 2006, Theorem 3.1). We will write $Z \in \mathrm{RV}(\mu, \alpha)$ for a random vector satisfying (1.4). For more on multivariate regular variation we refer to Basrak (2000) and Resnick (1987, 2006).

The class of stochastic models with representation (1.1) is very flexible and contains a wide range of models used in applications. Examples include:

Example 1.1 (Linear process)**.** Let (A_j) be a sequence of deterministic real-valued $d \times p$-matrices. Then, assuming convergence, $X_k = \sum_{j\in\mathbb{Z}} A_j Z_{k-j}$ is a linear process. It is, clearly, stationary. The (d-dimensional) marginal distribution of this process has the representation (1.1).

Example 1.2 (Random coefficient linear process)**.** This is a generalization of the linear process. Let $(\mathbf{A}_k, Z_k)_{k\in\mathbb{Z}}$ be a stationary sequence where each $\mathbf{A}_k = (A_{k,j})_{j\geq 0}$ is itself a sequence of random $d \times p$-matrices. Assuming, once again, convergence, the process $X_k = \sum_{j\geq 0} A_{k,j} Z_{k-j}$ is a random coefficient linear process. The stationarity of the sequence $(\mathbf{A}_k, Z_k)_{k\in\mathbb{Z}}$ implies that it is a stationary process, and its marginal distribution also has the representation (1.1).

Furthermore, let $S_n = X_1 + \cdots + X_n$ be the partial sum of the random coefficient linear process. Then

$$S_n = \sum_{k=1}^{n} \sum_{j\geq 0} A_{k,j} Z_{k-j} = \sum_{j\geq 0} \sum_{k=1}^{n} A_{k,k-j} Z_j =: \sum_{j\geq 0} B_{n,j} Z_j \, .$$

Of course, $(S_n)_{n\geq 1}$ is not, in general, a stationary process, but at each time n its marginal distribution has the representation (1.1). In both cases a certain "predictability" property of the sequence $(\mathbf{A}_k)$ with respect to the sequence (Z_k) is required for our result to be applicable.

Example 1.3 (SRE)**.** A very important particular case of the random coefficient linear process is the stationary solution of a stochastic recurrence equation (SRE).

Assume that $p = d$, and let $(Y_k, Z_k)_{k\in\mathbb{Z}}$ be a sequence of independent and identically distributed pairs of $d\times d$-matrices and d-dimensional random vectors. Put

$$\Pi_{n,m} = \begin{cases} Y_n \cdots Y_m, & n \leq m, \\ \mathrm{Id}, & n > m, \end{cases}$$

where Id is the $d \times d$ identity matrix. Under certain assumptions assuring existence of a stationary solution of a stochastic recurrence equation (SRE)

$$X_k = Y_k X_{k-1} + Z_k, \quad k \in \mathbb{Z}\, ,$$

this stationary solution can be represented by a random coefficient linear process with $A_{k,j} = \Pi_{k-j+1,k}$, $j \geq 0$ (see e.g Kesten, 1973). Then the marginal distribution of the stationary solution to the SRE is of the form (1.1). To satisfy the predictability assumption we need, additionally, to assume that the sequences (Y_k) and (Z_k) are independent.

Example 1.4 (Random sum)**.** Let $p = d$ and $A_j = \mathrm{Id}\, I\{1 \le j \le N\}$, where N is a positive integer-valued random variable which is a stopping time with respect a filtration $(\mathcal{G}_j,\, j \ge 0)$ to which the sequence (Z_j) is adapted, and such that $\mathcal{G}_j$ is independent of $\sigma(Z_{j+1}, Z_{j+2}, \ldots)$ for $j \ge 0$. Then $X = \sum_{j=1}^{N} Z_j$ is a random sum.

Example 1.5 (Stochastic integral)**.** Here is a modification of the random sum in the previous example. Let $(C_t)_{t\ge0}$ be a p-dimensional renewal reward process with renewal times $\tau_1, \tau_2, \ldots$. We take a version of (C_t) with right continuous paths with left limits. For some fixed time $T > 0$ let N_T be the number of renewals until time T. Suppose the jump distribution of (C_t) (that is the reward distribution) is regularly varying, i.e. $\Delta C_{\tau_j} = (C_{\tau_j} - C_{\tau_j -}) \stackrel{\mathrm{d}}{=} Z \in \mathrm{RV}(\mu)$. Let $\mathcal{G}_t = \sigma\big(\mathcal{A},\, C_s,\, 0 \le s \le t\big)$ for $t \ge 0$, where $\mathcal{A}$ is independent of $\sigma(C_s,\, s \ge 0)$. Let $(H_t)_{t\ge0}$ be a predictable with respect to this filtration $d \times p$-matrix valued process that is independent of (C_t). Then the vector valued stochastic integral

$$\int_0^T H_t dC_t$$

has representation (1.1) with $A_j = H_{\tau_j} I\{1 \le j \le N_T\}$ and $Z_j = \Delta C_{\tau_j}$.

In the next section we state the main theorem of this paper, giving sufficient conditions for convergence of the series X in (1.1) and for X acquiring the regular variation properties of the noise vectors (Z_j). We explore then the implications in the various special cases mentioned above. The assumptions in the main theorem turned out to be very tight, and improve the existing results in most special cases we are considering. The proof of the main theorem is given in Section 2.1.

2 Convergence and tail behavior

Consider the random vector X with stochastic representation (1.1). We will assume throughout most of this paper that

$$Z \in \mathrm{RV}(\mu, \alpha) \text{ and if } E|Z| < \infty, \text{ we assume additionally that } EZ = 0. \tag{2.1}$$

The assumption of the zero mean will allow us to work under relatively weak conditions. We will address separately what happens if the mean is not equal to zero.

For a matrix A we denote by $\|A\|$ the operator norm of A. For a vector $z \in \mathbb{R}^d$ we denote the Euclidean norm by $|z|$.

We start by considering a linear process. That is, suppose $(A_j)_{j\in\mathbb{Z}}$ is a deterministic sequence of matrices. Then the the following conditions are sufficient for a.s. convergence of the series (1.1):

$$\sum \|A_j\|^{\alpha-\varepsilon} < \infty, \text{ for some } \varepsilon > 0, \quad \text{if } 0 < \alpha \le 2, \tag{2.2}$$

$$\sum \|A_j\|^2 < \infty, \qquad \text{if } \alpha > 2 \tag{2.3}$$

(here, and throughout the paper, we omit the summation index whenever it is clear what it is). This follows from Mikosch and Samorodnitsky (2000) in the case $d = p = 1$; the extension to the vector case is immediate. By Fubini's theorem, in the case of random A_j's, *which are independent of the sequence* (Z_j), the following conditions are, therefore, sufficient for a.s. convergence of the series (1.1):

$$\sum \|A_j\|^{\alpha-\varepsilon} < \infty \text{ a.s. for some } \varepsilon > 0, \quad \text{if } 0 < \alpha \leq 2, \tag{2.4}$$

$$\sum \|A_j\|^2 < \infty \text{ a.s.} \qquad \text{if } \alpha > 2. \tag{2.5}$$

For linear processes the conditions (2.2) and (2.3) turn out to be sufficient for the sum X in (1.1) to acquire its regular variation from that of the (Z_j); in the case $d = p = 1$ this has been shown in Mikosch and Samorodnitsky (2000). It is clear that the conditions (2.4) and (2.5) will not suffice in the general case. If one looks at a general term in the sum (1.1), it is of the form $A_j Z_j$, and Z_j is regularly varying. The tail behavior of such a product is usually controlled by a moment condition on the matrix A_j, of the type:

$$E[\|A_j\|^{\alpha+\varepsilon}] < \infty, \quad \text{for some } \varepsilon > 0. \tag{2.6}$$

Then the term $A_j Z_j$ is regularly varying with limit measure $E[\mu \circ A_j^{-1}(\cdot)]$ (see Basrak et al., 2002, Proposition A.1). This is the measure such that

$$E[\mu \circ A_j^{-1}(B)] = E[\mu\{z : A_j z \in B\}].$$

This result is usually referred to as Breiman's theorem (Breiman, 1965). Clearly, requiring (2.6) for each j is too weak to control the tails of the infinite sum in (1.1). The above discussion shows, however, that we need to control both the small values of A_j's that persist for long stretches of time, and the large values of A_j's as well. This calls for a combination of different moment conditions. One such combination is presented in the following result.

Theorem 2.1. *Assume the "predictable" framework* (1.2) - (1.3). *Suppose that* (2.1) *holds and there is* $0 < \varepsilon < \alpha$ *such that*

$$\sum E\|A_j\|^{\alpha-\varepsilon} < \infty \quad \textit{and} \quad \sum E\|A_j\|^{\alpha+\varepsilon} < \infty, \quad \textit{if} \quad 0 < \alpha < 2, \tag{2.7}$$

$$E\left(\sum \|A_j\|^{\alpha-\varepsilon}\right)^{\frac{\alpha+\varepsilon}{\alpha-\varepsilon}} < \infty, \quad \textit{if} \quad \alpha = 2, \tag{2.8}$$

$$E\left(\sum \|A_j\|^2\right)^{\frac{\alpha+\varepsilon}{2}} < \infty, \quad \textit{if} \quad \alpha > 2. \tag{2.9}$$

Then the series (1.1) *converges a.s. and*

$$\frac{P(u^{-1}X \in \cdot\,)}{P(|Z| > u)} \xrightarrow{v} E\Big[\sum \mu \circ A_j^{-1}(\cdot)\Big], \tag{2.10}$$

on $\overline{\mathbb{R}}^d \setminus \{0\}$.

Remark 2.1. Note that in the context of Theorem 2.1 it is only notationally different to consider a two-sided sum

$$X = \sum_{j\in\mathbb{Z}} A_j Z_j \,,$$

as long as one modifies the "predictability" assumptions (1.2) - (1.3) appropriately. This can be done, for instance, by assuming that $\sigma(Z_j, Z_{j+1}, \ldots)$ is independent of $\sigma(\ldots, A_{-1}, A_0, \ldots, A_{j-1})$ for $j \geq 0$ and $\sigma(\ldots, Z_{-(j+1)}, Z_{-j})$ independent of $\sigma(A_{-(j-1)}, A_{-(j-2)}, \ldots)$ for $j \geq 1$.

Remark 2.2. If one removes the assumption of zero mean in the case where the mean is finite, it is clear that the conclusion of Theorem 2.1 will still hold under the following additional assumption on the sequence (A_j):

$$\text{the series } S_A = \sum A_j \text{ converges and } \lim_{u\to\infty} \frac{P(|S_A| > u)}{P(|Z| > u)} = 0\,. \tag{2.11}$$

Remark 2.3. In the univariate case $p = d = 1$, the limiting measure μ of Z has the representation

$$\mu(dx) = w\alpha x^{-\alpha-1}dx + (1-w)\alpha x^{-\alpha-1}dx \tag{2.12}$$

for some $w \in [0, 1]$. Then (2.10) becomes

$$\frac{P(X > u)}{P(|Z| > u)} \to \sum E\Big[|A_j|^\alpha (wI\{A_j > 0\} + (1-w)I\{A_j < 0\})\Big].$$

Next we consider the implications of Theorem 2.1 in some of the examples presented in Section 1.

Example 2.1 (Linear process)**.** Consider the linear process of the Example 1.1. Since (A_j) is a deterministic sequence, we immediately obtain the following statement.

Corollary 2.1. *Suppose that* (2.1) *holds and there is* $0 < \varepsilon < \alpha$ *such that*

$$\sum \|A_j\|^{\alpha-\varepsilon} < \infty, \quad 0 < \alpha \leq 2,$$
$$\sum \|A_j\|^2 < \infty, \quad \alpha > 2.$$

Then the series (1.1) *converges a.s. and*

$$\frac{P(u^{-1}X \in \cdot)}{P(|Z| > u)} \xrightarrow{v} \sum \mu \circ A_j^{-1}(\cdot)$$

on $\overline{\mathbb{R}}^d \setminus \{0\}$*.*

In the one-dimensional case $p = d = 1$ and $Z \in \mathrm{RV}(\alpha, \mu)$ with μ as in (2.12), we recover Lemma A.3 in Mikosch and Samorodnitsky (2000) under identical assumptions.

Example 2.2 (Random sum)**.** Consider the situation of Example 1.4: $d = p$ and $A_j = \mathrm{Id}\, I\{1 \le j \le N\}$, where N is a positive integer-valued random variable which is a stopping time with respect a filtration $(\mathcal{G}_j,\, j \ge 0)$ to which the sequence (Z_j) is adapted, and such that $\mathcal{G}_j$ is independent of $\sigma(Z_{j+1}, Z_{j+2}, \ldots)$ for $j \ge 0$. Then we have the following result, which can be thought of as a "tail Wald's identity".

Corollary 2.2. *Suppose that* (2.1) *holds and there is* $\tau > 0$ *such that*

$$EN < \infty \quad \text{if} \quad 0 < \alpha < 2 \tag{2.13}$$

$$EN^{1+\tau} < \infty \quad \text{if} \quad \alpha = 2 \tag{2.14}$$

$$EN^{\frac{\alpha}{2}+\tau} < \infty \quad \text{if} \quad \alpha > 2. \tag{2.15}$$

Then we have

$$\frac{P(u^{-1}X \in \cdot\,)}{P(|Z| > u)} \xrightarrow{v} EN\,\mu(\cdot)$$

on $\overline{\mathbb{R}}^d \setminus \{0\}$*. If* Z *has a finite mean, but* $EZ \neq 0$*, then the same conclusion is obtained if one replaces* (2.13) *(in the case of a finite mean),* (2.14)*, and* (2.15) *by the assumption*

$$\lim_{u\to\infty} \frac{P(N > u)}{P(|Z| > u)} = 0\,. \tag{2.16}$$

In the one-dimensional case $d = p = 1$, with the noise variables $(Z_k)_{k\in\mathbb{Z}}$ being nonnegative, and independent of N, we recover the results of Stam (1973) and Fay et al. (2006).

Example 2.3 (SRE)**.** Consider the stationary solution to a stochastic recurrence equation of Example 1.3. From Theorem 2.1 we obtain the following result.

Corollary 2.3. *Suppose that* $Z \in \mathrm{RV}(\mu, \alpha)$*, and*

$$E\|Y\|^\alpha < 1 \quad \text{and for some } \varepsilon > 0, \quad E\|Y\|^{\alpha+\varepsilon} < \infty\,.$$

Then the series (1.1) *converges a.s. and* (2.10) *holds.*

Note that we do not need to assume zero mean of the noise.

Specializing to the univariate case $p = d = 1$ and μ as in (2.12), we see that

$$
\begin{aligned}
\frac{P(X > x)}{P(|Z| > x)} &\to E\Big[\sum_{j\ge 0} \mu[z : \Pi_{1-j,0} z > 1]\Big] \\
&= E\Big[\sum_{j\ge 0} \mu[z > 0 : (Y_{1-j}\cdots Y_0)^+ z > 1] + \mu[z < 0 : (Y_{1-j}\cdots Y_0)^- z < -1]\Big] \\
&= \sum_{j\ge 0} \mu[z > 1] E[(Y_{1-j}\cdots Y_0)^+]^\alpha + \mu[z < -1] E[(Y_{1-j}\cdots Y_0)^-]^\alpha \\
&= \sum_{j\ge 0} w E[(Y_{1-j}\cdots Y_0)^+]^\alpha + (1-w) E[(Y_{1-j}\cdots Y_0)^-]^\alpha \\
&= w \sum_{j\ge 0} \sum_{k=0,1,\dots,j,\,\text{even}} \binom{j}{k} \big(E(Y^-)^\alpha\big)^k \big(E(Y^+)^\alpha\big)^{j-k} \\
&+ (1-w) \sum_{j\ge 0} \sum_{k=0,1,\dots,j,\,\text{odd}} \binom{j}{k} \big(E(Y^-)^\alpha\big)^k \big(E(Y^+)^\alpha\big)^{j-k} \\
&= w \sum_{k\ \text{even}} \frac{\big(E(Y^-)^\alpha\big)^k}{\big(1 - E(Y^+)^\alpha\big)^{k+1}} + (1-w) \sum_{k\ \text{odd}} \frac{\big(E(Y^-)^\alpha\big)^k}{\big(1 - E(Y^+)^\alpha\big)^{k+1}} \\
&= \frac{w\big(1 - E(Y^+)^\alpha\big) + (1-w) E(Y^-)^\alpha}{\big(1 - E(Y^+)^\alpha\big)^2 + \big(E(Y^-)^\alpha\big)^2}.
\end{aligned}
$$

In particular, if Y is nonnegative, then this reduces to

$$
\frac{P(X > x)}{P(|Z| > x)} \to w(1 - EY^\alpha)^{-1}.
$$

In this particular case we recover (under identical assumptions) the results of Grey (1994) or Konstantinides and Mikosch (2005, Proposition 2.2).

Remark 2.4. The most general results on the tail behavior of the series (1.1) so far have been the works of Resnick and Willekens (1991) and Wang and Tang (2006) who considered the one-dimensional case $p = d = 1$ and the sequence (A_j) being nonnegative and independent of the sequence (Z_j). Even in that particular case our conditions in the case $\alpha \ge 1$ are strictly weaker (the conditions are identical for $0 < \alpha < 1$).

3 Proof of Theorem 2.1

Let $(\hat{Z}_j)$ be a sequence with the same law as (Z_j), independent of the sequence (A_j) (we may need to enlarge the underlying probability space to construct such a sequence). The series $\sum_{j=0}^{\infty} A_j \hat{Z}_j$ converges a.s. by Fubini's theorem and (2.4)-(2.5). Furthermore, the sequences $(A_j \hat{Z}_j)$ and $(A_j Z_j)$ are tangent with respect to the filtration $\tilde{\mathcal{F}}_j = \sigma\big(\mathcal{F}_{j+1}, \hat{Z}_0, \dots, \hat{Z}_j\big)$ for $j \ge 0$. That is,

$\mathrm{Law}(A_jZ_j \mid \tilde{\mathcal{F}}_{j-1}) = \mathrm{Law}(A_j\hat{Z}_j \mid \tilde{\mathcal{F}}_{j-1})$. Note also that the sequence $(A_j\hat{Z}_j)$ is conditionally independent given the σ-field $\mathcal{G} = \sigma\big((A_k)_{k\ge 0}, (Z_k)_{k\ge 0}\big)$ (see Kwapień and Woyczyński, 1992, Section 4.3, for details). Therefore, Corollary 5.7.1 in Kwapień and Woyczyński (1992) guarantees a.s. convergence of the series in (1.1).

We will prove now (2.10), and we start with the univariate case $d = p = 1$, in which case the statement of the proposition reduces to

$$\lim_{x\to\infty} \frac{P(X > x)}{P(|Z| > x)} = \sum_{j=1}^{\infty} \Big(wE\big((A_j)_+\big)^\alpha + (1-w)E\big((A_j)_-\big)^\alpha \Big), \tag{3.1}$$

with $w = \lim P(Z > x)/P(|Z| > x)$ and where a_+ and a_- are, correspondingly, the positive part and the negative part of a real number a.

For finite $n \ge 1$ we have

$$\frac{P(\sum_{j=1}^n A_jZ_j > x)}{P(|Z| > x)} \to \sum_{j=1}^{n} \Big(wE\big((A_j)_+\big)^\alpha + (1-w)E\big((A_j)_-\big)^\alpha \Big)$$

by Lemma 3.4. Therefore it is sufficient to show that

$$\lim_{n\to\infty} \limsup_{x\to\infty} \frac{P(\sum_{j>n} A_jZ_j > x)}{P(|Z| > x)} = 0. \tag{3.2}$$

3.1 The case $0 < \alpha < 1$

We start with the case $0 < \alpha < 1$, in which case we will actually check that (3.2) holds with the sum of absolute values in the numerator. The first step is to show that for any $M > 0$ (3.2) holds with each A_j replaced by $\tilde{A}_j = A_j I\{|A_j| < M\}$ and, by scaling, it is enough to consider the case $M = 1$. We may decompose the probability as

$$\frac{P(\sum |\tilde{A}_jZ_j| > x)}{P(|Z| > x)} = \underbrace{\frac{P(\sum |\tilde{A}_jZ_j| > x, \bigvee |\tilde{A}_jZ_j| > x)}{P(|Z| > x)}}_{\text{I}} + \underbrace{\frac{P(\sum |\tilde{A}_jZ_j| > x, \bigvee |\tilde{A}_jZ_j| \le x)}{P(|Z| > x)}}_{\text{II}}, \tag{3.3}$$

with $\bigvee$ denoting maximum. For I an upper bound can be constructed as

$$\begin{aligned} \frac{P(\sum |\tilde{A}_jZ_j| > x, \bigvee |\tilde{A}_jZ_j| > x)}{P(|Z| > x)} &\le \frac{P(\bigvee |\tilde{A}_jZ_j| > x)}{P(|Z| > x)} \\ &\le \sum \frac{P(|\tilde{A}_jZ_j| > x)}{P(|Z| > x)} \\ &= \sum \frac{\int_0^1 P(y|Z_j| > x) P(|\tilde{A}_j| \in dy)}{P(|Z| > x)}. \end{aligned}$$

Using Potter's bounds (see e.g. Resnick, 1987) it follows that there exists $x_0 > 0$ such that $P(|Z| > x/y)/P(|Z| > x) \le c y^{\alpha-\varepsilon}$ for $x > x_0$ and $0 < y \le 1$. Hence, the last expression is bounded above by

$$c \sum_{j>n} \int_0^1 y^{\alpha-\varepsilon} P(|A_j| \in dy) \le c \sum_{j>n} E|A_j|^{\alpha-\varepsilon} \to 0 \quad \text{as } n \to \infty$$

by (2.7). For II, Markov's inequality implies

$$\begin{aligned} \mathrm{II} &\le \frac{P(\sum |\tilde{A}_j Z_j| I\{|\tilde{A}_j Z_j| \le x\} > x)}{P(|Z| > x)} \\ &\le \sum \frac{E[|\tilde{A}_j Z_j| I\{|\tilde{A}_j Z_j| \le x\}]}{xP(|Z| > x)} \\ &= \sum \int_0^1 y \frac{E[|Z_j| I\{|Z_j| \le x/y\}]}{xP(|Z| > x)} P(|A_j| \in dy). \end{aligned}$$

By Karamata's theorem (see Resnick, 1987)

$$E[|Z| I\{|Z| \le x\}] \sim \alpha(1-\alpha)^{-1} x P(|Z| > x)\,,$$

and there exists x_0 such that for $x > x_0$ the last expression is bounded from above by

$$c \sum_{j>n} \int_0^1 y\, y^{-1+\alpha-\varepsilon} P(|A_j| \in dy) \le c \sum_{j>n} E[|A_j|^{\alpha-\varepsilon}] \to 0 \quad \text{as } n \to \infty$$

by (2.7). Combining I and II proves (3.2) for $(\tilde{A}_j)$.

Next, for $M > 0$ and $v < x/M$ the remaining term can be bounded by

$$\begin{aligned} P\Big(\sum |A_j I\{|A_j| > M\} Z_j| > x\Big) &\le P\Big(\sum |A_j| I\{M < |A_j| < x/v\} |Z_j| > x/2\Big) \\ &\quad + P\Big(\sum |A_j| I\{|A_j| \ge x/v\} |Z_j| > x/2\Big). \end{aligned}$$

The result will follow once we show that, uniformly in n,

$$\lim_{M\to\infty} \limsup_{x\to\infty} \frac{P(\sum_{j>n} |A_j| I\{M \le |A_j| < x/v\} |Z_j| > x)}{P(|Z| > x)} = 0 \tag{3.4}$$

and

$$\lim_{M\to\infty} \limsup_{x\to\infty} \frac{P(\sum_{j>n} |A_j| I\{|A_j| \ge x/v\} |Z_j| > x)}{P(|Z| > x)} = 0. \tag{3.5}$$

Let us start with (3.4). Using the decomposition which lead to I and II above we see that it is sufficient to show

$$\lim_{M\to\infty} \limsup_{x\to\infty} \sum \frac{P(|A_j| I\{M \le |A_j| < x/v\} |Z_j| > x)}{P(|Z| > x)} = 0 \tag{3.6}$$

and

$$\lim_{M\to\infty}\limsup_{x\to\infty}\frac{P(\sum|A_j|I\{M\le|A_j|<x/v\}|Z_j|I\{|A_jZ_j|\le x\}>x)}{P(|Z|>x)}=0. \quad (3.7)$$

The term in (3.6) can be written as

$$\sum\frac{P(|A_j|I\{M\le|A_j|<x/v\}|Z_j|>x)}{P(|Z|>x)}=\sum\int_M^{x/v}\frac{P(|Z_j|>x/y)}{P(|Z|>x)}P(|A_j|\in dy).$$

For v and x sufficiently large Potter's bounds implies that this is bounded above by

$$c\sum\int_M^{x/v}y^{\alpha+\varepsilon}P(|A_j|\in dy)\le c\sum E|A_j|^{\alpha+\varepsilon}I\{A_j\ge M\}$$

Since $\sum E|A_j|^{\alpha+\varepsilon}<\infty$ by assumption (2.7) the sum converges to zero as $M\to\infty$.

Now we turn to (3.7). Markov's inequality gives

$$\begin{aligned}&\frac{P(\sum|A_j|I\{M\le|A_j|<x/v\}|Z_j|I\{|A_jZ_j|\le x\}>x)}{P(|Z|>x)}\\&\le\sum\frac{E[|A_j|I\{M\le|A_j|<x/v\}|Z_j|1\{|A_jZ_j|\le x\}]}{xP(|Z|>x)}\\&=\sum\int_M^{x/v}\frac{yE[|Z_j|I\{|Z_j|\le x/y\}]}{xP(|Z|>x)}P(|A_j|\in dy).\end{aligned}$$

Lemma 3.1 implies that for v and x sufficiently large we have the upper bound

$$c\sum\int_M^{x/v}y^{\alpha+\varepsilon}P(|A_j|\in dy)\le c\sum E|A_j|^{\alpha+\varepsilon}I\{|A_j|\ge M\},$$

which converges to zero.

Finally we want to show (3.5). By Potter's bounds, $P(|Z|>x)\ge x^{-\alpha-\varepsilon}$ for x sufficiently large. Hence,

$$\begin{aligned}\frac{P(\sum|A_j|I\{|A_j|\ge x/v\}|Z_j|>x)}{P(|Z|>x)}&\le\frac{P(\bigvee|A_j|>x/v)}{P(|Z|>x)}\\&\le\sum\frac{P(|A_j|>x/v)}{P(|Z|>x)}\\&\le\sum x^{\alpha+\varepsilon}P(|A_j|>x/v)\\&\le v^{\alpha+\varepsilon}\sum\int_{x/v}^{\infty}y^{\alpha+\varepsilon}P(|A_j|\in dy)\\&\le v^{\alpha+\varepsilon}\sum E|A_j|^{\alpha+\varepsilon}I\{|A_j|\ge x/v\},\end{aligned}$$

which converges to zero as $x\to\infty$. This completes the proof of (3.2) in the case $0<\alpha<1$.

3.2 The case $1 \le \alpha < 2$

Next, let $1 \le \alpha < 2$. Recall that we are assuming that $EZ_j = 0$ if $\alpha > 1$ or if $\alpha = 1$ and the mean is finite. Assume first that the law of Z is continuous and that the limit measure μ assigns positive weights to both $(-\infty, 0)$ and $(0, \infty)$.

We start as above in the case $0 < \alpha < 1$ (this time the absolute values stay outside of the sum) and establish (3.2) for $\tilde{A}_j$. We start with the same decomposition as in (3.3), except that we decompose not according to whether or not $\vee|\tilde{A}_j Z_j| > x$, but, rather, whether or not for some j, Z_j does not belong to the interval $[-h(x/\tilde{A}_j), x/\tilde{A}_j]$ for the function h in Lemma 3.2 in the case $\alpha > 1$ or $\alpha = 1$ and finite mean, or the function h in Lemma 3.3 in the case $\alpha = 1$ and infinite mean. Obviously, the argument for I in the decomposition (3.3) works for any $\alpha \ge 1$, and so it covers the case $1 \le \alpha < 2$. Let us consider the term II in that decomposition. Let $0 < \delta < 2 - \alpha$. By Markov's inequality,

$$\begin{aligned} &P\Big(\Big|\sum \tilde{A}_j Z_j I\{-h(x/\tilde{A}_j) \le Z_j \le x/\tilde{A}_j\}\Big| > x\Big) \\ &\quad \le \frac{1}{x^{\alpha+\delta}} E\Big|\sum \tilde{A}_j Z_j I\{-h(x/\tilde{A}_j) \le Z_j \le x/\tilde{A}_j\}\Big|^{\alpha+\delta}. \end{aligned} \tag{3.8}$$

By the predictability, zero mean of the Zs, and the property of the function h the above is a sum of martingale differences. Therefore, we can use the Burkholder-Davis-Gundy inequality (see e.g. Protter, 2004) to conclude that for large x the above is bounded by

$$\begin{aligned} &\frac{c}{x^{\alpha+\delta}} E\Big|\sum \Big(\tilde{A}_j Z_j I\{-h(x/\tilde{A}_j) \le Z_j \le x/\tilde{A}_j\}\Big)^2\Big|^{(\alpha+\delta)/2} \\ &\le \frac{c}{x^{\alpha+\delta}} \sum E\Big|\tilde{A}_j Z_j I\{-h(x/\tilde{A}_j) \le Z_j \le x/\tilde{A}_j\}\Big|^{\alpha+\delta}, \end{aligned}$$

where c is a finite positive constant that is allowed to change in the sequel. In the last step we used that the ℓ_q-norm is bounded by the ℓ_p-norm for $p < q$, i.e. $\|\cdot\|_{\ell_q} \le \|\cdot\|_{\ell_p}$. Conditioning on the A_j variables we see that,

$$\frac{II}{P(|Z| > x)} \le c \sum \int_0^1 y^{\alpha+\delta} \frac{E[|Z_j|^{\alpha+\delta} I\{-h(x/y) \le Z_j \le x/y\}]}{x^{\alpha+\delta} P(|Z| > x)} P(|A_j| \in dy).$$

Once again, by Karamata's theorem (Resnick, 1987)

$$E[|Z|^{\alpha+\delta} I\{|Z| \le x\}] \sim \alpha\delta^{-1} x^{\alpha+\delta} P(|Z| > x), \tag{3.9}$$

and there exists x_0 such that for $x > x_0$ the above expression is bounded by

$$c \sum_{j>n} \int_0^1 y^{\alpha+\delta} y^{-(\alpha+\delta)+\alpha-\varepsilon} P(|A_j| \in dy) \le c \sum_{j>n} E[|A_j|^{\alpha-\varepsilon}] \to 0 \quad \text{as } n \to \infty$$

by (2.7). Therefore, we have (3.2) for $(\tilde{A}_j)$.

Further, as in the case $0 < \alpha < 1$, we need to check (3.4) and (3.5). The argument for (3.5) works without changes in the present case, and the same is true about the first half of the argument for (3.4), presented in (3.6). Therefore, it remains only to consider the second half of (3.4), namely to prove that for v large enough,

$$\lim_{M\to\infty}\limsup_{x\to\infty}\frac{P\Big(\Big|\sum A_j I\big\{M\le |A_j|<\frac{x}{v}\big\}Z_j I\big\{-h\big(\frac{x}{A_j}\big)\le Z_j\le \frac{x}{A_j}\big\}\Big|>x\Big)}{P(|Z|>x)}=0.$$

The argument is similar to the one used above. Using Markov's inequality, predictability, zero mean and the definition of the function h allows us, once again, to use the Burkholder-Davis-Gundy inequality and see that for $0<\delta<2-\alpha$

$$\begin{aligned}
&P\Big(\Big|\sum A_j I\{M\le |A_j|<x/v\}Z_j I\{-h(x/A_j)\le Z_j\le x/A_j\}\Big|>x\Big) \qquad (3.10)\\
&\le \frac{c}{x^{\alpha+\delta}}\sum E\Big|A_jZ_jI\{M\le |A_j|<x/v\}I\{-h(x/A_j)\le Z_j\le x/A_j\}\Big|^{\alpha+\delta}\\
&:= III\,.
\end{aligned}$$

Notice that by (3.9) and v large enough, and using that $h(x)\le cx$,

$$\begin{aligned}
&E\Big|A_jZ_jI\{M\le |A_j|<x/v\}I\{-h(x/A_j)\le Z_j\le x/A_j\}\Big|^{\alpha+\delta}\\
&= E\Big(|A_j|^{\alpha+\delta}I\{M\le |A_j|<x/v\}E_Z|Z_j|^{\alpha+\delta}I\{-h(x/A_j)\le Z_j\le x/A_j\}\Big)\\
&\le cE\Big(|A_j|^{\alpha+\delta}I\{M\le |A_j|<x/v\}\Big(\frac{x}{|A_j|}\Big)^{\alpha+\delta}P_Z(|Z|>x/|A_j|)\Big)\\
&\le c\,x^{\alpha+\delta}P(|Z|>x)E\Big(|A_j|^{\alpha+\varepsilon}I\{|A_j|\ge M\}\Big)\,.
\end{aligned}$$

Here P_Z and E_Z indicate that the probability and expectation, respectively, are computed with respect to the Z variables (i.e. conditionally on A_j). In the last step we used independence and the fact that $P(|Z|>z)$ is regularly varying. Now we see that,

$$\frac{III}{P(|Z|>x)}\le c\sum E|A_j|^{\alpha+\varepsilon}I\{|A_j|\ge M\}\,,$$

which converges to zero by (2.7). Therefore, we have proved (3.2) for $1\le\alpha<2$, under the additional assumption that the law of Z is continuous and that the limit measure μ assigns positive weights to both $(-\infty,0)$ and $(0,\infty)$. In general, let $(\tilde Z_j)$ be an i.i.d. sequence independent of the sequences (A_j) and (Z_j) (we enlarge the probability space if necessary), such that each $\tilde Z_j$ is continuous, symmetric, and

$$\lim_{x\to\infty}\frac{P(|\tilde Z_j|>x)}{P(|Z|>x)}=1\,.$$

By symmetry and independence,

$$P\Big(\Big|\sum A_jZ_j\Big|>x\Big)\le 2P\Big(\Big|\sum A_j(Z_j+\tilde Z_j)\Big|>x\Big)\,.$$

However, the sequence $(Z_j+\tilde Z_j)$ satisfies the extra assumptions, and so we have established (3.2) for $1\le\alpha<2$ in full generality.

3.3 The case $\alpha \ge 2$

The proof of the relation (3.2) for $\alpha \ge 2$ proceeds similarly to the case $1 \le \alpha < 2$. Specifically, we need to estimate both the term II in (3.8) and the term III in (3.10). As in the case $1 \le \alpha < 2$ we may and will assume that the random variables (Z_j) satisfy the assumptions of Lemma 3.2. Furthermore, using part (iv) of Theorem 5.2.1 in Kwapień and Woyczyński (1992) we may assume that the sequence (Z_j) is independent of the sequence (A_j). Indeed, to achieve that we simply replace the sequence (Z_j) with the sequence $(\hat{Z}_j)$ defined in the beginning of the proof of the theorem and use the tangency.

We start with the case $\alpha = 2$. We estimate first II in (3.8). Starting with the Burkholder-Davis-Gundy inequality as before, we proceed as follows. Using Jensen's inequality

$$\sum\Big(\tilde{A}_j Z_j I\{-h(x/\tilde{A}_j) \le Z_j \le x/\tilde{A}_j\}\Big)^2$$
$$\le \Big(\sum \tilde{A}_j^2\Big)^{\delta/(2+\delta)} \Big(\sum \tilde{A}_j^2 |Z_j|^{2+\delta} I\{-h(x/\tilde{A}_j) \le Z_j \le x/\tilde{A}_j\}\Big)^{2/(2+\delta)} ,$$

and so, by conditioning,

$$II \le \frac{c}{x^{2+\delta}} E\left[\Big(\sum \tilde{A}_j^2\Big)^{\delta/2} \sum \tilde{A}_j^2 |Z_j|^{2+\delta} I\{-h(x/\tilde{A}_j) \le Z_j \le x/\tilde{A}_j\}\right]$$
$$= \frac{c}{x^{2+\delta}} E\left[\Big(\sum \tilde{A}_j^2\Big)^{\delta/2} \sum \tilde{A}_j^2 E_Z\big(|Z_j|^{2+\delta} I\{-h(x/\tilde{A}_j) \le Z_j \le x/\tilde{A}_j\}\big)\right] .$$

Using (3.9) we see that for large x this expression is bounded from above by

$$\frac{c}{x^{2+\delta}} E\left[\Big(\sum \tilde{A}_j^2\Big)^{\delta/2} \sum \tilde{A}_j^2 \left(\frac{x}{|\tilde{A}_j|}\right)^{2+\delta} P\big(|Z| > x/|\tilde{A}_j|\big)\right]$$
$$= cE\left[\Big(\sum \tilde{A}_j^2\Big)^{\delta/2} \sum |\tilde{A}_j|^{-\delta} P\big(|Z| > x/|\tilde{A}_j|\big)\right]$$
$$\le c\, P(|Z| > x) E\left[\Big(\sum \tilde{A}_j^2\Big)^{\delta/2} \sum |\tilde{A}_j|^{2-\varepsilon-\delta}\right]$$
$$\le c\, P(|Z| > x) E\left[\Big(\sum |\tilde{A}_j|^{2-\varepsilon-\delta}\Big)^{1+\delta/(2-\varepsilon-\delta)}\right]$$

(when writing sums as above and in the sequel we adopt the convention of not including the terms with $\tilde{A}_j = 0$). In the last two steps we used regular variation of $P(|Z| > z)$ and $\|\cdot\|_{\ell_q} \le \|\cdot\|_{\ell_p}$, for $p < q$, respectively. Choosing δ and ε small enough and using (2.8), we see that

$$\lim_{n\to\infty} \limsup_{x\to\infty} \frac{II}{P(|Z| > x)} = 0 .$$

The argument for III is similar; we present the main steps. Write

$$\begin{aligned}
III &\le \frac{c}{x^{\alpha+\delta}} E\Big[\Big(\sum A_j^2 I\{|A_j| \ge M\}\Big)^{\delta/2} \\
&\quad \times \sum A_j^2 I\{M \le |A_j| < x/v\} |Z_j|^{2+\delta} I\{-h(x/A_j) \le Z_j \le x/A_j\}\Big] \\
&\le c\, E\Big[\Big(\sum A_j^2 I\{|A_j| \ge M\}\Big)^{\delta/2} \\
&\quad \times \sum |A_j|^{-\delta} I\{M \le |A_j| < x/v\} P\big(|Z| > x/|A_j|\big)\Big] \\
&\le c\, P(|Z| > x) E\Big[\Big(\sum |A_j|^{2+\varepsilon-\delta} I\{|A_j| \ge M\}\Big)^{1+\delta/(2+\varepsilon-\delta)}\Big].
\end{aligned}$$

Choosing, for example, $\delta = \varepsilon$ small enough, we see by (2.8) that

$$\lim_{n\to\infty} \limsup_{x\to\infty} \frac{III}{P(|Z| > x)} = 0\,.$$

This establishes the statement (3.2) for $\alpha = 2$.

Next we look at the case $\alpha > 2$, and α not equal to an even integer. We start with the term II. Let $k = \lceil (\alpha+\delta)/2 \rceil$, i.e. the smallest integer greater than or equal to $(\alpha+\delta)/2$. Our assumption on α implies that for δ small enough,

$$2(k-1) < \alpha\,. \tag{3.11}$$

Proceeding from the Burkholder-Davis-Gundy bound on II, we have

$$\begin{aligned}
II &\le \frac{c}{x^{\alpha+\delta}} E\Big[\sum_{j_1} \cdots \sum_{j_k} \tilde{A}_{j_1}^2 \cdots \tilde{A}_{j_k}^2 Z_{j_1}^2 \cdots Z_{j_k}^2 \\
&\qquad \times I\{-h(x/\tilde{A}_{j_i}) \le Z_{j_i} \le x/\tilde{A}_{j_i} \quad \text{for } i = 1, \ldots, k\}\Big]^{(\alpha+\delta)/2k} \\
&\le \frac{c}{x^{\alpha+\delta}} E\Big[\sum |\tilde{A}_j|^{2k} |Z_j|^{2k} I\{-h(x/A_j) \le Z_j \le x/A_j\}\Big]^{(\alpha+\delta)/2k} \\
&\quad + \frac{c}{x^{\alpha+\delta}} E\Big[\sum_{(j_1,\ldots,j_k)\in D_k} \tilde{A}_{j_1}^2 \cdots \tilde{A}_{j_k}^2 Z_{j_1}^2 \cdots Z_{j_k}^2 \\
&\qquad \times I\{-h(x/\tilde{A}_{j_i}) \le Z_{j_i} \le x/\tilde{A}_{j_i} \quad \text{for } i = 1, \ldots, k\}\Big]^{(\alpha+\delta)/2k} \\
&:= II_a + II_b\,,
\end{aligned}$$

where $D_k = \{(j_1, \ldots, j_k)$ such that not all $j_1, \ldots, j_k$ are equal$\}$. Note that by the definition of k and again $\|\cdot\|_{\ell_q} \le \|\cdot\|_{\ell_p}$, for $p < q$,

$$\begin{aligned}
II_a &\le \frac{c}{x^{\alpha+\delta}} E \sum |\tilde{A}_j|^{\alpha+\delta} |Z_j|^{\alpha+\delta} I\{-h(x/A_j) \le Z_j \le x/A_j\} \\
&= \frac{c}{x^{\alpha+\delta}} E \sum |\tilde{A}_j|^{\alpha+\delta} E_Z\big(|Z_j|^{\alpha+\delta} I\{-h(x/A_j) \le Z_j \le x/A_j\}\big)\,,
\end{aligned}$$

and using once more (3.9) we can bound, for large x, this expression by

$$\frac{c}{x^{\alpha+\delta}} E \sum |\tilde{A}_j|^{\alpha+\delta} \left(\frac{x}{|\tilde{A}_j|}\right)^{\alpha+\delta} P\big(|Z| > x/|\tilde{A}_j|\big)$$
$$\leq c\, P(|Z| > x) E \sum |\tilde{A}_j|^{\alpha-\varepsilon} \leq c\, P(|Z| > x) E \left(\sum |\tilde{A}_j|^2\right)^{(\alpha-\varepsilon)/2} .$$

Therefore, by (2.9),

$$\lim_{n\to\infty} \limsup_{x\to\infty} \frac{II_a}{P(|Z| > x)} = 0 .$$

Next, it follows from independence and (3.11) that for some $0 < C < \infty$,

$$E\big(Z_{j_1}^2 \ldots Z_{j_k}^2\big) \leq C \tag{3.12}$$

for all $(j_1, \ldots, j_k) \in D_k$. Therefore,

$$II_b \leq \frac{c}{x^{\alpha+\delta}} E\Bigg[\sum_{(j_1,\ldots,j_k)\in D_k} \tilde{A}_{j_1}^2 \ldots \tilde{A}_{j_k}^2\Bigg]^{(\alpha+\delta)/2k} \leq \frac{c}{x^{\alpha+\delta}} E \left(\sum \tilde{A}_j^2\right)^{(\alpha+\delta)/2} ,$$

and so for δ small enough, by (2.9),

$$\lim_{x\to\infty} \frac{II_b}{P(|Z| > x)} = 0 .$$

This takes care of the term II, and the argument for III is, as we have seen a number of times before, is entirely similar.

Finally, let us consider the case $\alpha = 2m$ for some integer $m > 1$. This case is very similar to the case $\alpha > 2$ and not equal to an even integer, however (3.11) does not hold ($k = m + 1$ for small δ here). This does not make a difference as far as the term II_a above is concerned. For the term II_b we proceed as follows. Write $\hat{D}_k$ for the subset of D_k where exactly $k-1$ of the indices are equal. Note that for δ small enough, the bound (3.12) still holds for all $(j_1, \ldots, j_k) \in D_k \setminus \hat{D}_k$. Then

$$\begin{aligned}
II_b \leq{}& \frac{c}{x^{\alpha+\delta}} E\Bigg[\sum_{(j_1,\ldots,j_k)\in \hat{D}_k} \tilde{A}_{j_1}^2 \ldots \tilde{A}_{j_k}^2 Z_{j_1}^2 \ldots Z_{j_k}^2 \\
&\qquad \times I\{-h(x/\tilde{A}_{j_i}) \leq Z_{j_i} \leq x/\tilde{A}_{j_i} \quad \text{for } i = 1, \ldots, k\}\Bigg]^{(\alpha+\delta)/2k} \\
&+ \frac{c}{x^{\alpha+\delta}} E\Bigg[\sum_{(j_1,\ldots,j_k)\in D_k\setminus\hat{D}_k} \tilde{A}_{j_1}^2 \ldots \tilde{A}_{j_k}^2 Z_{j_1}^2 \ldots Z_{j_k}^2 \\
&\qquad \times I\{-h(x/\tilde{A}_{j_i}) \leq Z_{j_i} \leq x/\tilde{A}_{j_i} \quad \text{for } i = 1, \ldots, k\}\Bigg]^{(\alpha+\delta)/2k} \\
:={}& II_{b1} + II_{b2} .
\end{aligned}$$

The term II_{b2} is treated in the same way as the term II_b with $\alpha > 2$ not being an even integer. Furthermore,

$$II_{b1} \leq \frac{c}{x^{\alpha+\delta}} E\Big[\sum_{j_1}\sum_{j_2\neq j_1} \tilde{A}_{j_1}^2 |\tilde{A}_{j_2}|^\alpha Z_{j_1}^2 |Z_{j_2}|^\alpha I\Big\{-h\big(\frac{x}{\tilde{A}_{j_2}}\big) \leq Z_{j_2} \leq \frac{x}{\tilde{A}_{j_2}}\Big\}\Big]^{(\alpha+\delta)/2k}$$

$$\leq \frac{c}{x^{\alpha+\delta}} E\Big[\sum_{j_1}\sum_{j_2\neq j_1} \tilde{A}_{j_1}^2 |\tilde{A}_{j_2}|^\alpha E_Z\Big(Z_{j_1}^2 |Z_{j_2}|^\alpha I\Big\{-h\big(\frac{x}{\tilde{A}_{j_2}}\big) \leq Z_{j_2} \leq \frac{x}{\tilde{A}_{j_2}}\Big\}\Big)\Big]^{(\alpha+\delta)/2k}$$

$$= \frac{c}{x^{\alpha+\delta}} E\Big[\sum_{j_1}\sum_{j_2\neq j_1} \tilde{A}_{j_1}^2 |\tilde{A}_{j_2}|^\alpha E_Z\Big(|Z_{j_2}|^\alpha I\Big\{-h\big(\frac{x}{\tilde{A}_{j_2}}\big) \leq Z_{j_2} \leq \frac{x}{\tilde{A}_{j_2}}\Big\}\Big)\Big]^{(\alpha+\delta)/2k}.$$

By Karamata's theorem, $l(x) = E|Z|^\alpha I\{|Z| \leq x\}$ is a slowly varying at infinity function and, as such, it is bounded from above for large x by cx^ε. Therefore, for large x,

$$II_{b1} \leq \frac{c}{x^{\alpha+\delta}} E\Big[\sum_{j_1}\sum_{j_2\neq j_1} \tilde{A}_{j_1}^2 |\tilde{A}_{j_2}|^\alpha \left(\frac{x}{|\tilde{A}_{j_2}|}\right)^\varepsilon\Big]^{(\alpha+\delta)/2k}$$

$$= \frac{c}{x^{\alpha+\delta-\varepsilon(\alpha+\delta)/2k}} E\Big[\sum_{j_1} \tilde{A}_{j_1}^2 \sum_{j_2} |\tilde{A}_{j_2}|^{\alpha-\varepsilon}\Big]^{(\alpha+\delta)/2k}$$

$$\leq \frac{c}{x^{\alpha+\delta-\varepsilon(\alpha+\delta)/2k}} E\left(\sum \tilde{A}_j^2\right)^{(\alpha+2-\varepsilon)(\alpha+\delta)/(\alpha+2)},$$

and we see that for δ and ε small enough, and $\varepsilon < \delta(\alpha+1)/(\alpha+\delta)$, the power of x is larger than α, and we may use (2.9), to obtain

$$\lim_{x\to\infty} \frac{II_{b1}}{P(|Z| > x)} = 0$$

for all n. This completes the treatment of the term II in the case α an even integer greater than 2, and the term III is treated in the same way.

This proves the limit (3.2) in all cases and, hence, we have established the one-dimensional statement (3.1).

3.4 The general statement

Now we prove the general statement (2.10). For finite $n \geq 1$, Lemma 3.4 implies

$$\frac{P(u^{-1}\sum_{j\leq n} A_j Z_j \in \cdot)}{P(|Z| > u)} \xrightarrow{v} E\Big[\sum_{j\leq n} \mu \circ A_j^{-1}(\cdot)\Big].$$

As in the one-dimensional case considered above, (2.10) will follow once we check that

$$\lim_{n\to\infty} \limsup_{x\to\infty} \frac{P(|\sum_{j>n} A_j Z_j| > x)}{P(|Z| > x)} = 0.$$

Because of the finite-dimensionality, it is enough to prove the corresponding statement for each one of the $d \times p$ elements of the random matrices. We will check (in the obvious notation) that

$$\lim_{n\to\infty} \limsup_{x\to\infty} \frac{P(|\sum_{j>n} A_j^{(11)} Z_j^{(1)}| > x)}{P(|Z| > x)} = 0\,. \tag{3.13}$$

Note that $|A_j^{(11)}| \leq \|A_j\|$, and so the sequence $(A_j^{(11)})$ in (3.13) satisfies the 1-dimensional version of the assumptions (2.7) - (2.9). Only one thing prevents us from immediately applying the one-dimension statement (3.1), and it is the fact that the tail of $|Z^{(1)}|$ may happen to be strictly lighter than that of $\|Z\|$. To overcome this problem, let $(\tilde{Z}_j)$ be an independent copy of the sequence (Z_j), also independent of the sequence $(A_j^{(11)})$, and (ε_j) a sequence of i.i.d. Rademacher random variables independent of the rest of the random variables around. Then

$$\begin{aligned} P\Big(|\sum_{j>n} A_j^{(11)} Z_j^{(1)}| > x\Big) &\leq P\Big(|\sum_{j>n} A_j^{(11)} (Z_j^{(1)} + \varepsilon_j|\tilde{Z}_j|)| > x/2\Big) \\ &\quad + P\Big(|\sum_{j>n} A_j^{(11)} \varepsilon_j |\tilde{Z}_j|| > x/2\Big)\,, \end{aligned}$$

which allows us to apply (3.1) to each term above and prove (3.13). This completes the proof of the theorem. □

Remark 3.1. The argument leading to (3.1) and (3.2) can be also used to show various modifications of these two statements. For example, in (3.2) one can allow n and x to go to infinity at the same time to obtain

$$\lim_{x\to\infty} \frac{P(\sum_{j>n(x)} A_j Z_j > x)}{P(|Z| > x)} = 0 \tag{3.14}$$

for any function $n(x) \to \infty$ as $x \to \infty$. Further, only values of Z_j's comparable to the level x matter in (3.1), in the sense that

$$\lim_{\tau\to 0} \limsup_{x\to\infty} \frac{P(|\sum_{j=1}^{\infty} A_j Z_j 1(|Z_j| \leq \tau x)| > x)}{P(|Z| > x)} = 0\,. \tag{3.15}$$

Lemma 3.1. *Let $Z \in \mathrm{RV}(\alpha, \mu)$ be a random vector with $0 < \alpha < 1$. Then for each $y > 0$, $\varepsilon > 0$, and $c > 0$, there is x_0 such that*

$$\frac{|E[ZI\{|Z| \leq x/y\}]|}{xP(|Z| > x)} \leq c \max[y^{\alpha-1+\varepsilon}, y^{\alpha-1-\varepsilon}], \qquad x \geq x_0.$$

Proof. Karamata's theorem implies

$$E[|Z|I\{|Z| \leq x\}] \sim \alpha(1-\alpha)^{-1} xP(|Z| > x)$$

and the claim follows from Potter's bound. □

Lemma 3.2. *Let $Z \in \mathrm{RV}(\alpha, \mu)$ with $\alpha > 1$, or with $\alpha = 1$ and a finite mean, be a one-dimensional continuous random variable, such that μ assigns positive weights to both $(-\infty, 0)$ and $(0, \infty)$. Then there are numbers $K, C > 0$ and a function $h : [K, \infty) \to (0, \infty)$ satisfying $C^{-1} \leq h(x)/x \leq C$ and*

$$\int_x^\infty yP(Z \in dy) = \int_{-\infty}^{-h(x)} |y|P(Z \in dy)$$

for all $x \geq K$.

Proof. By the assumptions $B = \int_{-\infty}^0 |y|P(Z \in dy) \in (0, \infty)$. The function $G(x) = \int_x^\infty yP(Z \in dy)$ is continuous and decreases to zero. Let $K \geq 1$ be such that $G(x) \leq B/2$ for $x \geq K$, and define

$$h(x) = \inf\{t > 0 : \int_{-\infty}^{-t} |y|P(Z \in dy) = G(x)\} .$$

The existence of a number C in the statement of the lemma follows from the regular (or slow in the case $\alpha = 1$) variation and Karamata's theorem. □

The next lemma is a counterpart of the previous statement in the case of infinite means.

Lemma 3.3. *Let $Z \in \mathrm{RV}(\alpha, \mu)$ with $\alpha < 1$, or with $\alpha = 1$ and infinite absolute mean, be a one-dimensional continuous random variable, such that μ assigns positive weights to both $(-\infty, 0)$ and $(0, \infty)$. Then there are numbers $K, C > 0$ and a function $h : [K, \infty) \to (0, \infty)$ satisfying $C^{-1} \leq h(x)/x \leq C$ and*

$$\int_0^x yP(Z \in dy) = \int_{-h(x)}^0 |y|P(Z \in dy)$$

for all $x \geq K$.

Proof. The argument is similar to the one of the previous lemma. The function $G(x) = \int_0^x yP(Z \in dy)$ is continuous and increases to infinity. Let $K \geq 1$ be such that $G(x) \geq 1$ for $x \geq K$, and define

$$h(x) = \inf\{t > 0 : \int_{-t}^0 |y|P(Z \in dy) = G(x)\} .$$

Once again, the existence of a number C in the statement of the lemma follows from the regular (or slow in the case $\alpha = 1$) variation and Karamata's theorem. □

The following is a slight generalization of Proposition A.1 in (Basrak et al., 2002).

Lemma 3.4. *Let $Z_1, \ldots, Z_n$ be i.i.d. random vectors in* $\mathrm{RV}(\alpha, \mu)$*, $A_1, \ldots, A_n$ a sequence of random matrices such that for every $j = 1, \ldots, n$, Z_j is independent of $\sigma(A_1, \ldots, A_j)$. Assume, further, that for some $\varepsilon > 0$, $E\|A_j\|^{\alpha+\varepsilon} < \infty$ for $j = 1, \ldots, n$, Z_j. Then*

$$\frac{P(u^{-1}\sum_{j=1}^n A_j Z_j \in \cdot)}{P(|Z| > u)} \xrightarrow{v} E\Big[\sum_{j=1}^n \mu \circ A_j^{-1}(\cdot)\Big]. \tag{3.16}$$

Proof. Let $\hat{\mu}$ denote the measure in the right hand side of (3.16) (as usual, the origin is not a part of the space where $\hat{\mu}$ lives, so any mass at the origin is simply lost). Let $B \in \mathbb{R}^d$ be a bounded away from the origin Borel set, and assume that it is a $\hat{\mu}$-continuity set. We will use the notation $A^\varepsilon = \{x \in \mathbb{R}^d : d(x, A) < \varepsilon\}$ and $A_\varepsilon = \{x \in A : d(x, A^c) > \varepsilon\}$ for a set $A \in \mathbb{R}^d$ and $\varepsilon > 0$. Choose $\varepsilon > 0$ so small that B^ε is still bounded away from the origin. We have

$$\begin{aligned} P(u^{-1}\sum_{j=1}^n A_j Z_j \in B) &\le P\Big(\bigcup_{j=1}^n \{u^{-1} A_j Z_j \in \overline{B}^\varepsilon\}\Big) \\ &+ P\Big(\bigcup_{j_1=1}^n \bigcup_{j_2=j_1+1}^n \{u^{-1}|A_{j_1} Z_{j_1}| > \varepsilon/n,\, u^{-1}|A_{j_2} Z_{j_2}| > \varepsilon/n\}\Big). \end{aligned}$$

By the Portmanteau theorem,

$$\limsup_{u\to\infty} \frac{P\Big(\bigcup_{j=1}^n \{u^{-1} A_j Z_j \in \overline{B}_\varepsilon\}\Big)}{P(|Z| > u)} \le \hat{\mu}(\overline{B}^\varepsilon).$$

On the other hand, by Proposition A.1 in Basrak et al. (2002), for every $j_1 < j_2$ and $M > 0$,

$$\begin{aligned} &\limsup_{u\to\infty} \frac{P\Big(u^{-1}|A_{j_1} Z_{j_1}| > \varepsilon,\, u^{-1}|A_{j_2} Z_{j_2}| > \varepsilon\Big)}{P(|Z| > u)} \\ &\le \limsup_{u\to\infty} \frac{P\Big(|A_{j_1} Z_{j_1}| > M,\, |A_{j_2} Z_{j_2}| > u\varepsilon\Big)}{P(|Z| > u)} \\ &= \varepsilon^{-\alpha} E\left(|A_{j_2}|^\alpha I\{|A_{j_1} Z_{j_1}| > M\}\right), \end{aligned}$$

and letting $M \to \infty$ we obtain

$$\lim_{u\to\infty} \frac{P\Big(u^{-1}|A_{j_1} Z_{j_1}| > \varepsilon,\, u^{-1}|A_{j_2} Z_{j_2}| > \varepsilon\Big)}{P(|Z| > u)} = 0.$$

Using regular variation, we conclude that

$$\limsup_{u\to\infty} \frac{P(u^{-1}\sum_{j=1}^n A_j Z_j \in B)}{P(|Z| > u)} \le \hat{\mu}(\overline{B}^\varepsilon).$$

Letting $\varepsilon \to 0$ and using the fact that B is a $\hat{\mu}$-continuity set, we obtain

$$\limsup_{u\to\infty} \frac{P(u^{-1}\sum_{j=1}^n A_j Z_j \in B)}{P(|Z|>u)} \le \hat{\mu}(B)\,.$$

A matching lower bound follows in a similar way using the relation

$$P(u^{-1}\sum_{j=1}^n A_j Z_j \in B) \ge P\Big(\bigcup_{j=1}^n \{u^{-1}A_jZ_j \in B_\varepsilon\}\Big)$$
$$- P\Big(\bigcup_{j_1=1}^n \bigcup_{j_2=j_1+1}^n \{u^{-1}|A_{j_1}Z_{j_1}|>\theta,\, u^{-1}|A_{j_2}Z_{j_2}|>\theta\}\Big)\,,$$

where $\theta = \min\{\varepsilon/n, \inf_{x\in B}\|x\|\}$. □

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