SCHOOL OF OPERATIONS RESEARCH AND INDUSTRIAL ENGINEERING, CORNELL UNIVERSITY, ITHACA, NY, 14853

An Exact Optimal Solution to a Threshold Inventory Rationing Model for Multiple Priority Demand Classes

Technical Report No. 1450

September 2006

Oguzhan Vicil

Peter Jackson

An Exact Optimal Solution to a Threshold Inventory Rationing Model for Multiple Priority Demand Classes

Oğuzhan Vıcıl & Peter Jackson
School of OR&IE, Cornell University
Ithaca, NY 14853
ov24@cornell.edu • pj16@cornell.edu

September 10, 2006

Abstract

In this paper, we analyze the problem of stocking and dispensing inventory to satisfy customers who have contracted for different levels of parts availability. We consider a model consisting of multiple priority demand classes exhibiting mutually independent, stationary, Poisson demand processes, independent and identically distributed, non-zero, order lead times, an (S-1,S) ordering policy, and a threshold level-based allocation and backorder clearing policy. For the solution procedure, first we develop an efficient algorithm for the multiple class setting, called the Modified Bridge Algorithm-2, that significantly reduces the complexity of the solution procedure caused by high dimensionality. We then develop an efficient algorithm to determine steady state probabilities. After that, an optimization scheme is developed that minimizes total system stock for a given set of fill-rate constraints. Finally, we provide numerical results to suggest the cost savings that can be realized compared to conventional practices.

1 Model

Vicil and Jackson [5] analyze the same setting for two priority demand class setting. Their main main contribution is an exact analysis of the stationary probabilities and an efficient algorithm for finding the minimal stock required to satisfy demand class-specific fill-rate constraints. In this paper, we extend our analysis to a more general setting.

For the general *n-demand* classes, let us represent the customers as $1, 2, \ldots, n$ in terms of their priorities, where customer *type-1* and *type-n* have the lowest and the highest priorities respectively.

Each class has its own fill-rate service level requirement, where fill-rate is the fraction of demand satisfied directly from on-hand stock (physical inventory). For the general *n-demand* classes case, threshold levels are given by $(S^1, S^2, \ldots, S^n)$, where S^1 is the total system inventory.

As in the two priority demand classes setting, two allocation policies must be specified: one for when a demand occurs and one for when a unit of stock is delivered. We refer to the first as the rationing policy and the second as the backorder clearing mechanism. Both policies are governed by a threshold level. Hence, followed policy is:

Threshold Rationing Policy for n-demand class:

- On-hand stock $(OH) > S^2$: satisfy all customer demands on a FCFS basis;
- For $i = 2, 3, \ldots, n-1$:

  $S^{i+1} < OH \leqslant S^i$: satisfy *type-i,..., type-n* demands on a FCFS basis but backorder *type-1,..., type-(i-1)* demands;
- $0 < OH \leqslant S^n$: satisfy only *type-n* demands but backorder the others;
- $OH = 0$: backorder all types of demands.

Threshold Backorder Clearing Mechanism for n-demand class:

- an incoming unit from resupply system will satisfy an existing *type-n* backorder, if one exists;
- an incoming unit from resupply will satisfy an existing backorder of *type-i* customer, only if OH meets critical level S^{i+1} for that class.[1]

Hence, the optimization problem can be written as:

$$
\begin{aligned}
&\min\ S^1\\
&s.t.\\
&\beta^i(S^1, S^2, \ldots, S^n) \geqslant c^i \quad \text{for } i = 1, 2, \ldots, n\\
&\text{and} \qquad S^1 \geqslant S^2 \geqslant \ldots \geqslant S^n \geqslant 0.
\end{aligned}
$$

[1]In other words, if OH is at least S^{i+1} before the receipt of the unit.

Based on the ***PASTA*** principle, the fill-rate measures are related to the steady state distribution of an *on-hand* stock as:

$$\beta^i = P_\infty(OH > S^{i+1}) \quad \text{for } i = 1, 2, \ldots, n-1; \text{ and}$$
$$\beta^n = P_\infty(OH > 0).$$

For the *n-priority* demand class case, $n \geqslant 3$, let us denote the state random variable as $(OH, B^1, B^2, \ldots, B^n, R)$. The existence of steady state probabilities and general lead time solutions are given by the following theorems:

Theorem 1.1 *Assuming there are no orders in the pipeline at time zero, then for continuous or discrete (positively valued, and no probability mass at zero) lead time distributions,* $\lim_{t\to\infty} P_{(OH,B^1,B^2,\ldots,B^n,R)}(t) = \pi_{(OH,B^1,B^2,\ldots,B^n,R)}$ *exists and is well defined.*

Proof: Follows the same ideas as two priority class case, and, therefore the proof is omitted.

■

Theorem 1.2 *In steady state, for discrete or continuously distributed lead times with finite mean* T*, where there is no point mass at zero, the system of balance equations to be solved is the same as the balance equations of the problem with exponentially distributed lead time with rate* $\mu = 1/T$*. Hence the* ***solution*** *to steady state probabilities under CTMC setting with parameter* $\mu = 1/T$ *will be the* ***solution*** *to the main problem.*

Proof: Follows the same ideas as two priority class case, and, therefore the proof is omitted.

■

The main contribution of this paper is to provide an efficient algorithm, namely *Modified Bridge Algorithm-2* (MBA$^{(2)}$), to solve the optimization problem for n-priority demand classes. This algorithm is developed to solve for two customer class case, which in turn is used recursively to determine steady state OH probabilities corresponding to an n-priority demand class case, as well as within the search algorithm to determine optimal threshold levels.

Before presenting the analysis for the general n-priority demand class case, we study three priority demand classes and show how the dynamics of the system differ from the two demand classes case and provide an optimization scheme that uses the main framework of the two priority demand classes setting. Also, presenting the three customer case before the general priority class solution is necessary since the proof for the general optimization scheme requires an induction approach based on the three customer class solution.

The remainder of this thesis is organized as follows. We begin by describing the model and system dynamics for the three priority class case. In section 3, we describe our solution approach for the three class case in detail, developing a special technique to reduce the dimensionality of the problem. Then we derive exact steady state OH probabilities for given policy parameters and threshold levels, and from these we calculate the corresponding service levels associated with each customer class. Also in this section we describe an optimization algorithm that requires only two passes for the calculation of threshold levels that minimize total system inventory. In section 4, we describe the solution approach for the general multiple priority demand class, introducing the MBA$^{(2)}$ algorithm that significantly simplifies the solution procedure. Finally, we present a series of theoretical results that allow us to use MBA$^{(2)}$ recursively in the context of *Bridge Algorithm-n* and *Optimal Greedy Line Search-n* for n-priority class case.

Remarks:

1. The rest of the section is based on a CTMC framework due to Theorem 1.2. In other words, for a given discrete or continuously distributed lead time with finite mean T, where there is no point mass at zero, μ is set to be $1/T$. We solve the system for exponentially distributed lead times with this rate μ.

2. Most of the notational complexity of this section is required for the proofs. The solution procedure and optimization algorithms are quite simple to understand and implement.

2 The Model for Three Priority Demand Class

Previously, we had labeled the customers in increasing priority as *type-1*, *type-2*,..., and *type-n*. But for three priority class case, as we similarly did for two class case, let us represent the customers as *silver*, *gold* and *platinum*. The *platinum* customers contract for the highest fill-rate while the *silver* customers contract for the lowest. Hence, the followed policy is:

Threshold Rationing Policy:

- On-hand stock $(OH) > S^g$: satisfy all customer demands on a FCFS basis;
- $S^p < OH \leqslant S^g$: satisfy *gold* and *platinum* demands on a FCFS basis but backorder *silvers'*;
- $0 < OH \leqslant S^p$: satisfy *platinum* demands but backorder both *gold* and*silver* demands;
- $OH = 0$: backorder all types of demands.

Threshold Backorder Clearing Mechanism:

- an incoming unit from resupply system will satisfy an existing *platinum* backorder, if one exists;
- an incoming unit from resupply will satisfy an existing backorder of type ζ customer, where $\zeta \in \{silver, gold\}$ only if OH meets the critical level for that class. Otherwise, it is added to on-hand stock. Note that critical levels for *gold* and *silver* are S^p and S^g respectively, where incoming orders of those types are stopped when OH drops to those specified threshold levels.

Our objective is to determine threshold parameters (S^s, S^g, S^p) that minimize the total inventory investment S^s while satisfying fill-rate constraints for each customer type. The fill-rates are functions of the threshold parameters (S^s, S^g, S^p). The optimization problem can be written as:

$$
\begin{aligned}
&\min \ S^s \\
&s.t. \\
&\beta^s(S^s, S^g, S^p) \quad \geqslant \quad c^s \\
&\beta^g(S^s, S^g, S^p) \quad \geqslant \quad c^g \\
&\beta^p(S^s, S^g, S^p) \quad \geqslant \quad c^p
\end{aligned}
$$

$$
\text{and} \qquad S^s \geqslant S^g \quad \geqslant S^p \geqslant 0.
$$

At an arbitrary point in time, the system state information required to implement the policy can be characterized by (OH, B^s, B^g, B^p, R). From the ***PASTA*** principle:

$$
\begin{aligned}
\beta^s &= 1 - P_\infty(OH \leqslant S^g); \\
\beta^g &= P_\infty(S^p < OH \leqslant S^g); \\
\beta^p &= P_\infty(0 < OH).
\end{aligned}
$$

2.1 Transition Diagram

Transitions from one state to another are shown in Figure 1 for a sample of selected states. We use $(S^s, S^g, S^p) = (7, 4, 2)$ for our example. In Figure 1, to ease visualization, different symbols are used to represent states based on the value of OH.

As long as OH is at least $S^g + 1$, states are represented with circles. The system experiences demand at a rate of $\lambda = \lambda^s + \lambda^g + \lambda^p$, and all three demand types are satisfied immediately on a FCFS basis. When OH drops to the level S^g, then we stop serving silver demands and they are backordered. The transitions will take place towards the right on the B^s-axis whenever a *silver* demand is realized, increasing B^s by 1. On the other hand, there will be a downward movement at a rate $\lambda^g + \lambda^p$. When a *gold* or *platinum* arrival occurs, OH decreases by 1 unit until it drops to level S^p. The set of states corresponding to $S^p < OH \leqslant S^g$ are represented with squares.

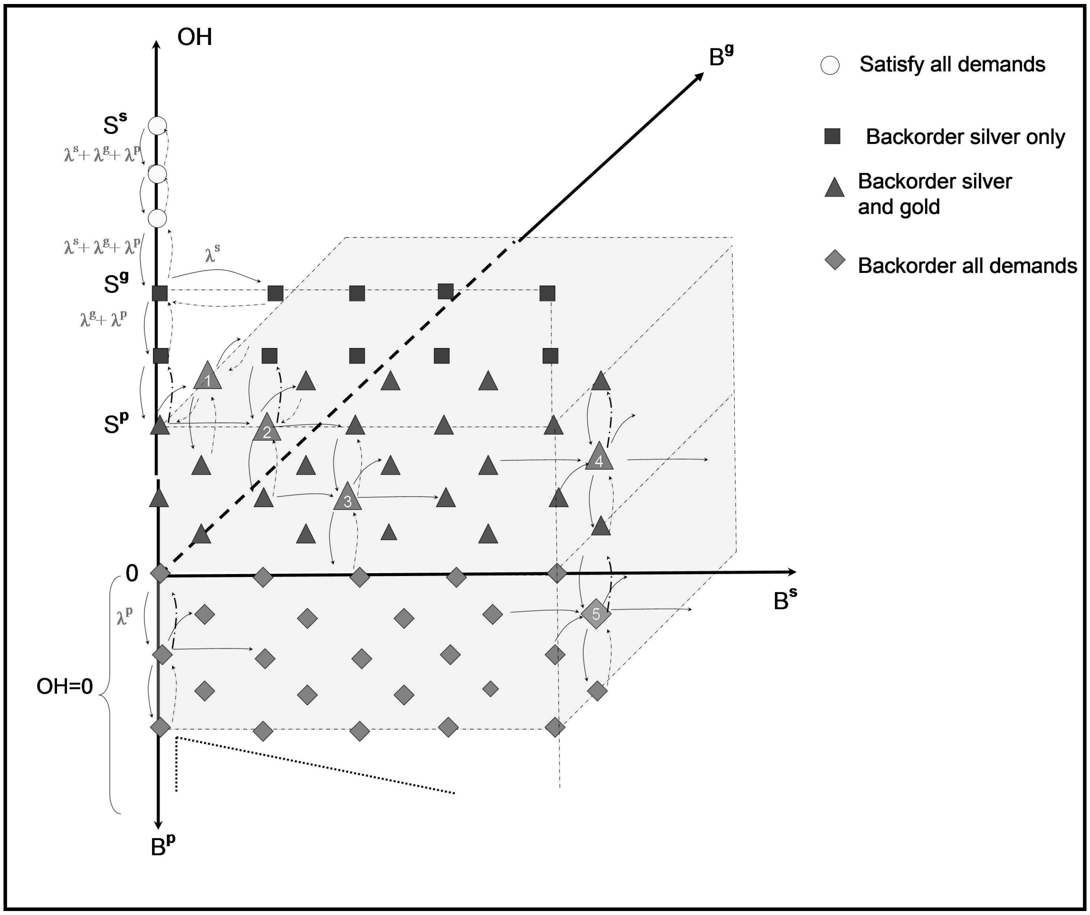

Figure 1: State transition diagram for 3 customer class case

When OH drops to S^p, then any incoming demand will cause an increase in the backorder level of its type except for *platinum* demands, which are served as long as OH is positive. Those states where only *platinum* customers are served, are represented by triangles. Note that when $OH \leqslant S^p$, the transitions take place in a three dimensional space. For example, whenever a *silver* (resp. *gold*) demand occurs, it will cause a movement on B^s-axis (resp. B^g-axis) causing an increase in its backorder level. Transitions in the B^s and B^g direction occur at rates of λ^s and λ^s respectively.

When OH is zero, then any incoming demand will cause an increase in the backorder level of its type. Those set of states are represented by diamonds. Note that triangles and diamonds correspond to states in a three dimensional space.

Consider states $(1, 2, 0, 0, 8), (1, 4, 1, 0, 11)$ and $(0, 4, 1, 1, 13)$ in Figure 1, represented by numbers $3, 4$ and 5 respectively. Those all correspond to states $OH < S^p$. When a unit is received from resupply, it will cause an upward transition which increases OH by 1. On the other hand, when a demand of type *silver,gold* or *platinum* occurs, then there will be a transition rightward, forward or downward respectively. Here, we define forward transition as the one that corresponds to an increase in B^g on B^g-axis.

Next, consider $(2,1,0,0,6)$, represented by number 2 in Figure 1. An incoming unit from resupply is added to on-hand stock causing an upward transition that makes *silver* backorder unsatisfied. On the other hand, for the state $(2,0,1,0,6)$, represented by number 1 in the figure, an incoming unit is used to satisfy an existing *gold* backorder, which causes a backward transition.

3 Solution Procedure for Three Priority Class Setting

Before going into the analysis for three priority class setting, let us first consider the values $S^s - S^g$ and $S^s - S^p$ that any feasible solution to the problem should satisfy.

3.1 Determination of the minimum values of $S^s - S^g$ and $S^s - S^p$ in a feasible solution

Proposition 3.1 *Consider two cases for a given resupply rate μ:*

a) three customer demand classes with threshold levels (S^s, S^g, S^p) and arrival rates $(\lambda^s, \lambda^g, \lambda^p)$;

b) two customer demand classes with threshold levels $(S^s - S^p, S^g - S^p)$ and arrival rates $(\lambda^s, \lambda^g + \lambda^p)$.

Denote on-hand stock for the former and the latter cases as OH^1 and OH^2 respectively. Then

$$P_\infty(OH^1 = S^p + k) = P_\infty(OH^2 = k) \quad \textit{for}\, k = 1, \ldots, S^s - S^p.$$

Proof: Consider three customer demand class case, and let $(OH^1, B^{s^1}, B^g, B^p, R^1)$ be the representation of a system state for a given (S^s, S^g, S^p). Let us group the states into one set according to following transformation:

$$(B^{s^1}, R^1) = \bigcup (OH^1, B^{s^1}, B^g, B^p, R^1) \tag{1}$$

For a given (B^{s^1}, R^1), the above operation is to unite all the states into one group that have the same (B^{s^1}, R^1) value. Notice that for $R^1 - B^{s^1} \leqslant S^s - S^p$, (B^{s^1}, R^1) corresponds to a unique state $(OH^1, B^{s^1}, B^g, B^p, R^1) = (S^s - R^1 + B^{s^1}, B^{s^1}, 0, 0, R^1)$ rather than a union of several states.

In addition, let us consider the two demand class case with threshold levels $(S^s - S^p, S^g - S^p)$, and arrival rates $(\lambda^s, \lambda^g + \lambda^p)$. Let us represent the states as $(OH^2, B^{s^2}, B^{g+p}, R^2)$.

Then there is a one-to-one matching between state space of two customer demand class setting $(OH^2, B^{s^2}, B^{g+p}, R^2)$ and (B^{s^1}, R^1) according to the following transformation:

$$\begin{aligned} (B^{s^1}, R^1) &\overset{1:1}{\longleftrightarrow} (OH^2, B^{s^2}, B^{g+p}, R^2) \\ &= \big((S^s - S^p - R^1 + B^{s^1})^+, B^{s^1}, (R^1 - B^{s^1} - S^s + S^p)^+, R^1\big) \end{aligned} \tag{2}$$

and the dynamics of the system are identical for both cases (i.e. jump probabilities and rates from one state to another). In other words, the two customer demand class case with threshold levels $(S^s - S^p, S^g - S^p)$, and arrival rates $(\lambda^s, \lambda^g + \lambda^p)$ is an exact characterization of the above system in Equation (1) resulting from a grouping operation.

By using Equations (1) and (2),

$$\begin{aligned} (OH^2, B^{s^2}, B^{g+p}, R^2) &= \big((S^s - S^p - R^1 + B^s)^+, B^{s^1}, (R^1 - B^s - S^s + S^p)^+, R^1\big) \\ &\equiv (B^{s^1}, R^1) \\ &= \bigcup (OH^1, B^{s^1}, B^g, B^p, R^1) \end{aligned}$$

Hence,

$$P_\infty(OH^2, B^{s^2}, B^{g+p}, R^2) \quad =$$

$$= \quad P_\infty\big((S^s - S^p - R + B^s)^+, B^{s^1}, (R - B^s - S^s + S^p)^+, R^1\big)$$

$$= \quad P_\infty(B^{s^1}, R^1)$$

$$= \quad \sum P_\infty(OH^1, B^{s^1}, B^g, B^p, R^1)$$

Recall that $R^1 - B^{s^1} < S^s - S^p$ for $OH^1 > S^p$, therefore there is a unique state $(OH^1, B^{s^1}, B^g, B^p, R^1)$ corresponding to state (B^{s^1}, R^1). In addition, it is also true that $OH^2 > 0$ for $R^2 - B^{s^2} < S^s - S^p$. As a result, we have the following relation for $OH^1 > S^p$,

$$P_\infty\Big\{(OH^2, B^{s^2}, B^{g+p}, R^2) = \big(S^s - S^p - R^1 + B^{s^1}, B^{s^1}, 0, R^1\big)\Big\}$$

$$= \quad P_\infty(B^{s^1}, R^1)$$

$$= \quad P_\infty\Big\{(OH^1, B^{s^1}, B^g, B^p, R^1) = (S^s - R^1 + B^{s^1}, B^{s^1}, 0, 0, R^1)\Big\}. \tag{3}$$

Hence the following relationship holds:

$$P_\infty(OH^1 = S^p + k) = P_\infty(OH^2 = k) \quad \text{for } k = 1, \ldots, S^s - S^p.$$

Furthermore, for $(OH^1, B^{s^1}) = (S^p, B^{s^1})$, $R^1 - B^{s^1} = S^s - S^p$. Hence, it is also true that there is a unique state

$$(OH^1, B^{s^1}, B^g, B^p, R^1) = (S^p, B^{s^1}, 0, 0, S^s - S^p + B^{s^1}),$$

corresponding to a state $(B^{s^1}, R^1) = (B^{s^1}, S^s - S^p + B^{s^1})$. In addition, it is also true that for $(B^{s^2}, R^2) = (B^{s^1}, S^s - S^p + B^{s^1})$,

$$(OH^2, B^{s^2}, B^{g+p}, R^2) = (0, B^{s^1}, 0, S^s - S^p + B^{s^1}).$$

Therefore, we also have the following:

$$P_\infty\Big\{(OH^2, B^{s^2}, B^{g+p}, R^2) = (0, B^{s^1}, 0, S^s - S^p + B^{s^1})\Big\}$$

$$= \quad P_\infty\Big\{(OH^1, B^{s^1}, B^g, B^p, R^1) = (S^p, B^{s^1}, 0, 0, S^s - S^p + B^{s^1})\Big\}. \tag{4}$$

■

Corollary 3.1 *For a given set of* (S^s, S^g, S^p) *and system parameters* $(\lambda^s, \lambda^g, \lambda^p, \mu)$*,*

$$\begin{aligned}(\beta^s)^1 &= P_\infty(OH^1 > S^g) = P_\infty(OH^2 > S^g - S^p);\\ (\beta^g)^1 &= P_\infty(S^p < OH^1 \leqslant S^g) = P_\infty(OH^2 > 0);\end{aligned}$$

where $P_\infty(OH^2 = k)$ *is the solution to two demand class case with threshold levels* $(S^s - S^p, S^g - S^p)$ *and system parameters* $(\lambda^s, \lambda^g + \lambda^p, \mu) \equiv (\lambda^s, \lambda^{g+p}, \mu)$*. Hence,*

$$\begin{aligned}(\beta^s)^1 &= (\beta^s)^2;\\ (\beta^g)^1 &= (\beta^{g+p})^2.\end{aligned}$$

Proof: Follows directly from Proposition 3.1.

■

Lemma 3.1 *Consider the three demand class setting with corresponding fill-rate requirements* c^s, c^g*, and* c^p*. Denote* $\tilde{S}^{s*}$ *and* $\tilde{S}^{(g+p)*}$ *as the threshold levels, found by applying Optimal Greedy Line Search algorithm to a two customer demand class setting with system parameters* $(\lambda^s, \lambda^g + \lambda^p, \mu)$ *and fill-rate constraints* c^s, c^g*. Then any feasible solution* (S^s, S^g, S^p) *to the three customer case* *must satisfy:*

$$S^s - S^p \geqslant \tilde{S}^{s*};$$

and

$$S^s - S^g \geqslant \tilde{S}^{s*} - \tilde{S}^{(g+p)*}.$$

Proof: For each feasible solution $(S^s, S^g, S^p) = [(S^s)^2 + S^p, (S^g)^2 + S^p, S^p]$ to the original problem, there corresponds a feasible solution $[(S^s)^2, (S^g)^2]$ to a two customer demand class setting with parameters $(\lambda^s, \lambda^g + \lambda^p)$ and fill-rate constraints c^s, c^g due to Corollary 3.1.

If $(S^s, S^g, S^p) = [(S^s)^2 + S^p, (S^g)^2 + S^p, S^p]$ is feasible, then $[(S^s)^2, (S^g)^2]$ is feasible. Then,

$$\begin{aligned} S^s - S^p &= [(S^s)^2 + S^p] - S^p \\ &= (S^s)^2, \end{aligned}$$

and

$$\begin{aligned} S^s - S^g &= [(S^s)^2 + S^p] - [(S^g)^2 + S^p] \\ &= (S^s)^2 - (S^g)^2. \end{aligned}$$

But it was shown in the previous discussions of Optimal Greedy Line Search algorithm that for a two customer class setting, any feasible solution $[(S^s)^2, (S^g)^2]$ should satisfy $(S^s)^2 - (S^g)^2 \geqslant S^{s*} - S^{(g+p)*}$, and $(S^s)^2 \geqslant S^{s*}$. Hence we conclude,

$$\begin{aligned} S^s - S^p &= (S^s)^2 \geqslant S^{s*}, \\ S^s - S^g &= (S^s)^2 - (S^g)^2 \geqslant S^{s*} - S^{(g+p)*}. \end{aligned}$$

■

Lemma 3.2 *For a given set of (S^s, S^g, S^p) and system parameters $(\lambda^s, \lambda^g, \lambda^p, \mu)$, the steady state probabilities of states corresponding to $OH \geqslant S^p$ and $(B^g, B^p) = (0, 0)$ can be found by applying the Bridge Algorithm to a two demand class setting with thresholds $(S^s - S^p, S^g - S^p)$ and system* *parameters* $(\lambda^s, \lambda^g + \lambda^p, \mu)$

Proof: For $OH > S^p$, the lemma is the immediate result of Proposition 3.1 and Equation (3). Next, it is left to show for the case $(OH, B^g, B^p) = (S^p, 0, 0)$. But it was also previously shown in Equation (4) that

$$P_\infty\Big\{(OH^2, B^{s^2}, B^{g+p}, R^2) = (0, B^{s^1}, 0, S^s - S^p + B^{s^1})\Big\}$$

$$= \quad P_\infty\Big\{(OH^1, B^{s^1}, B^g, B^p, R^1) = (S^p, B^{s^1}, 0, 0, S^s - S^p + B^{s^1})\Big\}.$$

■

From Lemma 3.2, it is clear that the residual difficulty of the three priority demand class case lies in computing $P_\infty(OH = k)$ for $k \leqslant S^p$.

3.2 Grouping states on a (B^s, B^g) plane

For $k = 1, \ldots, S^p$, observe that

$$P_\infty(OH = k) = \sum_{j=0}^{\infty} P_\infty\big(OH = k, B^s + B^g = j\big).$$

It is sufficient, therefore, to compute steady state probabilities of the system being in aggregate states of the form $\{OH = k, B^s + B^g = j\}$, for $k = 1, \ldots, S^p$ and $j = 0, 1, \ldots$.

In order to determine corresponding fill-rates for each customer demand class, we do not need to solve steady state probabilities of all states at once. What we need is the corresponding steady state OH probabilities to determine different class fill-rate measures. For a given (OH, B^p) we group the states (OH, B^s, B^g, B^p, R) into sets such that $B^s + B^g = k$, for $k = 0, 1, \ldots$. and $OH \leqslant S^p$.

The main goal in doing so is to decrease the dimensionality of the state space. Later we will show that this modified system correctly represents the original system. In order to clarify the ideas, let us refer to Figure 2. Two specific examples are shown, one for $OH = 1$ and one for $B^p = 1$. For $OH = 1$, the idea is first to categorize the system states (OH, B^s, B^g, B^p, R) such that

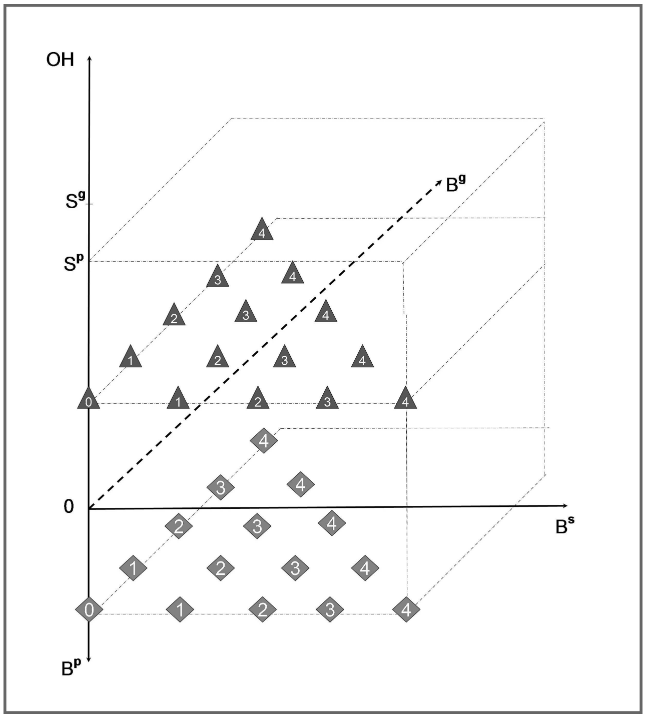

Figure 2: Labeling states

$B^s + B^g = k$,for $k = 0, 1, \ldots$, and label them k. The same operation is also applied for $B^p = 1$. Recall that Figure 2 represents a system for threshold levels $(S^s, S^g, S^p) = (7, 4, 2)$. For example, for a set $(OH, B^p) = (1, 0)$, let us consider state $(1, 0, 0, 0, 6)$ which is labeled as 0. If a *silver* customer demand occurs, then B^s will increase by 1, and new system state becomes $(1, 1, 0, 0, 7)$. On the other hand, if a *gold* customer demand occurs, then this order is also backordered, and new state becomes $(1, 0, 1, 0, 7)$. On the other hand, if a *platinum* customer demand occurs, then that order is immediately satisfied from on-hand stock, decreasing OH by one, resulting in a system state $(0, 0, 0, 0, 7)$. However, if a unit is received from resupply, then this unit is added to on-hand stock. There will be a upward transition into state $(2, 0, 0, 0, 6)$. Notice that states $(1, 1, 0, 0, 7)$ and $(1, 0, 1, 0, 7)$ are both indexed by 1 in the figure. For $OH < S^p$, any type of *silver* or *gold* demand will increase the state index by 1, while a unit receipt from resupply or a *platinum* demand arrival has no effect on this index. This only changes OH or B^p levels.

Next, for $OH \leqslant S^p$ we group the states on a (B^s, B^g) plane that have the same labels and represent the system states as $(OH, B^{s \oplus g}, B^p, R)$. So it becomes

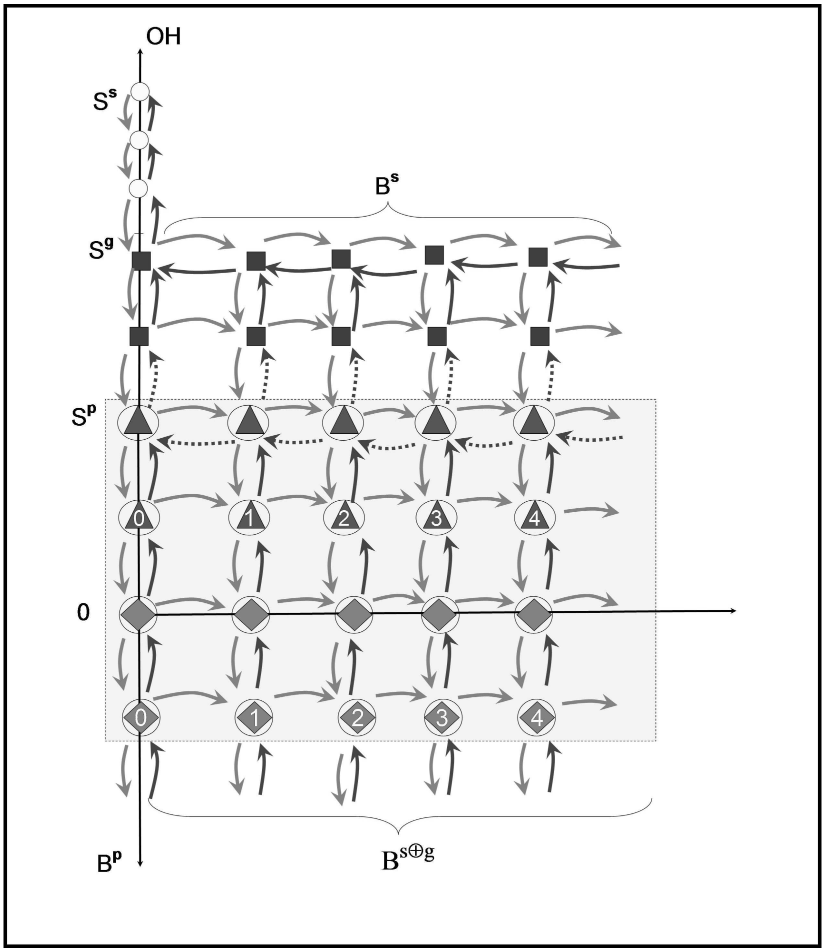

Figure 3: State transition diagram after grouping

$$(OH, B^{s\oplus g}, B^p, R) \equiv \bigcup_{B^s+B^g=B^{s\oplus g}} (OH, B^s, B^g, B^p, R). \tag{5}$$

The resulting transition diagram can be seen in Figure 3. In the figure, those states are represented as circled shapes, i.e. circled triangles and circled diamonds. To make the $s \oplus g$ operation clearer, the set of indexed states in Figure 3 corresponds to the same ones as shown in Figure 2.

3.3 Simplification of State Representations

Working in four and five dimensional settings can be confusing and the notational complexity increases the difficulty of analysis. Therefore, for a given (S^s, S^g, S^p) we present a transformation

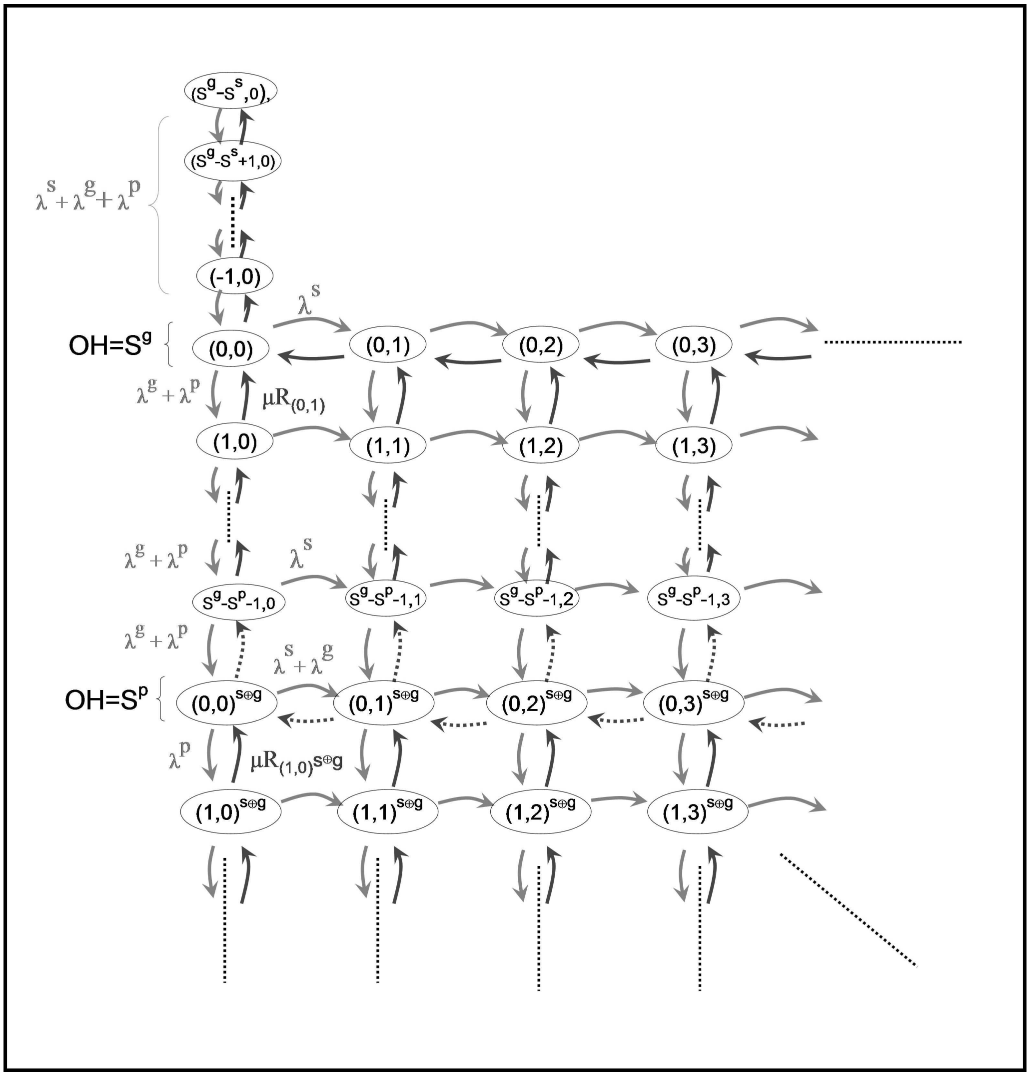

Figure 4: Simplified state transition diagram after grouping

performed on the five dimensional (OH, B^s, B^g, B^p, R) representation for $OH > S^p$ and another performed on the four dimensional $(OH, B^{s\oplus g}, B^p, R)$ representation for $OH \leqslant S^p$.

For $OH > S^p$, the state random variable for the simplified transition diagram can be represented as (X, Y) , where

$$
\begin{aligned}
X &= S^g - OH; \\
Y &= B^s.
\end{aligned}
$$

As a result, for the relation $(OH, B^s, B^g, B^p, R) \equiv (X, Y)$, we have the following one-to-one transformation between the grouped (Fig. 3) and the simplified transition diagram (Fig. 4):

$$\begin{aligned} \big(X, Y\big) &= \big(S^g - OH, B^s\big) \quad \text{for } OH > S^p; \\ \big(OH, B^s, B^g, B^p, R\big) &= \big(S^g - X, Y, 0, 0, X + Y + S^s - S^g\big) \quad \text{for} X < S^g - S^p. \end{aligned}$$

On the other hand, for $OH \leqslant S^p$, for the relation $(OH, B^{s\oplus g}, B^p, R) \equiv (X, Y)^{s\oplus g}$, we have the following one-to-one transformation between the grouped and the simplified transition diagram:

$$\begin{aligned} \big(X, Y\big)^{s\oplus g} &= \big(S^p - OH + B^p, B^{s\oplus g}\big) \quad \text{for } OH \leqslant S^p; \\ \big(OH, B^{s\oplus g}, B^p, R\big) &= \big((S^p - X)^+, Y, (X - S^p)^+, X + Y + S^s - S^p\big). \end{aligned}$$

Let us conclude the above discussions by writing the final result relationship between the simplified and the original state variables (OH, B^s, B^g, B^p, R). Using Equation (5), for a given (S^s, S^g, S^p):

$$\begin{aligned} \big(X, Y\big) &\equiv \big(S^g - X, Y, 0, 0, X + Y + S^s - S^g\big) \quad \text{for } X < S^g - S^p; \\ \big(X, Y\big)^{s\oplus g} &\equiv \bigcup_{B^s + B^g = Y} \big((S^p - X)^+, B^s, B^g, (X - S^p)^+, X + Y + S^s - S^p\big). \end{aligned} \tag{6}$$

The resulting transition diagram is shown in Figure 4. For the (X, Y) representation, column index Y represents the number of existing *silver* backorders, while it represents the total *silver* and *gold* backorders in the $(X, Y)^{s\oplus g}$ representation. Notice that bi-directional transitions between columns occur only when OH is equal to a threshold level.

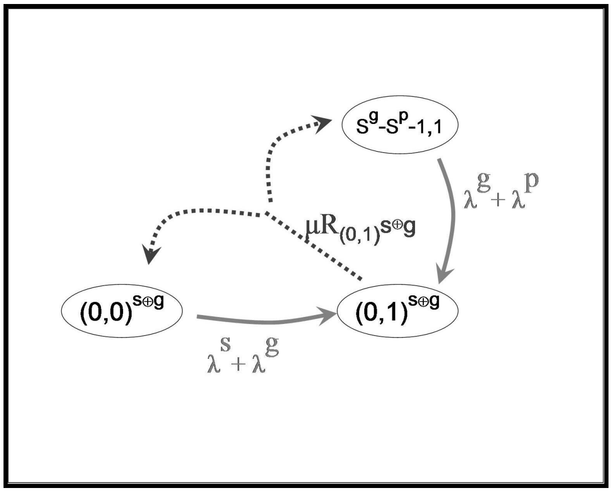

Figure 5: Portion of flows on state $(0,1)^{s\oplus g}$

Remark: It is not possible to write down the steady state balance equations involving states $(0,j)^{s\oplus g}$, $j \geqslant 1$ from studying Figure 4. If the current state is $(0,j)^{s\oplus g} \equiv (S^p, j, 0, 0, S^s - S^p + j)$, then when a unit is received from resupply, it will be added to OH stock and the new system state becomes

$$(S^p + 1, j, 0, 0, S^s - S^p + j - 1) \equiv (S^g - S^p - 1, j).^2$$

For all other cases, the unit will be used to fulfill an existing *gold* backorder since $B^g > 0$ for $B^s < j$, resulting in a system state $(0, j-1)^{s\oplus g}$. Related transitions shown in Figure 4 can be seen more explicitly in Figure 5. Therefore, in steady state, total flow from state $(0,1)^{s\oplus g}$ into states $(0,0)^{s\oplus g}$ and $(S^g - S^p - 1, j)$ is equal to $\pi_{(0,1)^{s\oplus g}} \mu \mathrm{R}_{(0,1)^{s\oplus g}}$. However, the split of this flow among states $(0,0)^{s\oplus g}$ and $(S^g - S^p - 1, j)$ cannot be determined from studying the simplified state transition diagram alone. We will explain the procedure of writing the balance equations for states $(0,j)^{s\oplus g}$, $j \geqslant 1$ in the next section.

3.4 Calculation of Steady State Probabilities under $(OH, B^{s\oplus g}, B^p, R)$ Setting

Next, we determine the steady state probabilities corresponding to $OH \leqslant S^p$. Steady state probabilities $\pi_{(S^p,k,0,0,S^s-S^p+k)}$ for $k \geqslant 1$, are known due to Lemma 3.2. Note that $(0,0)^{s\oplus g}$ corresponds

[2] Recall that (i,j) corresponds to a simplified state representation for $OH > S^p$.

to a unique state $(S^p, 0, 0, 0, S^s - S^p)$ in the original representation (other ones generally correspond to a union of more than one states). Hence, steady state probability $\pi_{0,0)^{s\oplus g}} = \pi_{(S^p,0,0,0,S^s-S^p)}$ is also known due to Lemma 3.2.

The following theorems are analogous to the one that was stated for two priority class setting:

Theorem 3.1 *In steady state, the following equation holds:*

$$\pi_{(1,0)^{s\oplus g}} \cdot \mu \cdot R_{(1,0)^{s\oplus g}} \quad = \quad \pi_{(0,0)^{s\oplus g}} \cdot \lambda^p \cdot [X_1^{(0)}]^{s\oplus g}$$

$$where \quad [X_1^{(0)}]^{s\oplus g} = P_{(1,0)^{s\oplus g}}\big[\tau_{(0,0)^{s\oplus g}}(1) < min\{\tau_{(0,1)^{s\oplus g}}(1), \tau_{(0,1)}(1)\}\big].$$

Proof: See Appendix 1.

■

The above theorem allows us to calculate $\pi_{(i,0)^{s\oplus g}}$, $i \geqslant 2$, recursively from the following balance equations by having the knowledge of $\pi_{(1,0)^{s\oplus g}}$:

$$\pi_{(i,0)^{s\oplus g}} \cdot \big[\lambda \,+\, \mu \cdot R_{(i,0)^{s\oplus g}}\big] \;=\; \pi_{(i-1,0)^{s\oplus g}} \cdot \lambda^p \,+\, \pi_{(i+1,0)^{s\oplus g}} \cdot \mu \cdot R_{(i+1,0)^{s\oplus g}}$$

The following theorem establishes the relation of $\pi_{(1,j)^{s\oplus g}}$ with $\pi_{(0,j)^{s\oplus g}}$ and $\pi_{(i,j-1)^{s\oplus g}}$ for $i, j \geqslant 1$. Eventually, this allows us to obtain $\pi_{(i,j)^{s\oplus g}}$, $i \geqslant 2$, values recursively from the balance equations on a given *column* j.

Theorem 3.2 *In steady state, the following equation holds for $j \geqslant 1$:*

$$\pi_{(1,j)^{s\oplus g}} \mu R_{(1,j)^{s\oplus g}} \quad = \quad \pi_{(0,j)^{s\oplus g}} \lambda^p X_1^{(j)} + \sum_{i=1}^{\infty} \pi_{(i,j-1)^{s\oplus g}} \, \lambda^{s+g} \, [X_i^{(j)}]^{s\oplus g}$$

$$where \quad [X_i^{(j)}]^{s\oplus g} = P_{(i,j)^{s\oplus g}}\big[\tau_{(0,j)^{s\oplus g}}(1) < min\{\tau_{(0,j+1)^{s\oplus g}}(1), \tau_{(0,j+1)}(1)\}\big].$$

Proof: See Appendix 2.

∎

The next section establishes the procedure of calculating $\pi_{(0,j)^{s\oplus g}}$, $j \geqslant 1$ from balance equations.

3.5 Calculating $\pi_{(0,j)^{s\oplus g}}$, $j \geqslant 1$ from Balance Equations

Proposition 3.2 *For a generic state $\pi_{(0,j)^{s\oplus g}}$, $j \geqslant 1$, the following relation holds:*

$$
\begin{aligned}
\pi_{(0,j)^{s\oplus g}}\left(\lambda^s+\lambda^g+\lambda^p+\mu R_{(0,j)^{s\oplus g}}\right) \quad &= \\
&= \pi_{(0,j-1)^{s\oplus g}}\left(\lambda^s+\lambda^g\right) \\
&\quad + \pi_{(1,j)^{s\oplus g}}\,\mu\,R_{(1,j)^{s\oplus g}} \\
&\quad + \pi_{(S^g-S^p-1,j)}\,(\lambda^g+\lambda^p) \\
&\quad + \left(\pi_{(0,j+1)^{s\oplus g}}-\pi_{(S^p,j+1,0,0,S^s-S^p+j+1)}\right)\mu\,R_{(0,j+1)^{s\oplus g}}.
\end{aligned}
\tag{7}
$$

Proof: For a given set of (S^s, S^g, S^p) and system parameters $(\lambda^s, \lambda^g, \lambda^p, \mu)$, $\pi_{(0,0)^{s\oplus g}} \equiv \pi_{(S^p,0,0,0,S^s-S^p)}$ can be calculated by using Lemma 3.2. Now, let us consider a generic state $(0,j)^{s\oplus g}$, $j \geqslant 1$. Recall that

$$
(0,j)^{s\oplus g} \quad \equiv \bigcup_{B^s+B^g=j} (S^p, B^s, B^g, 0, S^s-S^p+j).
$$

We decompose the RHS of above equation as:

$$
\bigcup_{B^s+B^g=j} (S^p, B^s, B^g, 0, S^s-S^p+j) =
$$

$$= \left\{(S^p, j, 0, 0, S^s - S^p + j)\right\} \bigcup \left\{ \bigcup_{\substack{B^s+B^g=j \\ B^s<j}} (S^p, B^s, B^g, 0, S^s - S^p + j)\right\}.$$

If the current state is $(0, j)^{s\oplus g} \equiv (S^p, j, 0, 0, S^s - S^p + j)$, then when a unit is received from resupply, it will be added to OH stock and the new system state becomes $(S^p + 1, j, 0, 0, S^s - S^p + j - 1) \equiv (S^g - S^p - 1, j)$.[3] For all other cases, the unit will be used to fulfill an existing *gold* backorder since $B^g > 0$ for $B^s < j$, resulting in a system state $(0, j-1)^{s\oplus g}$. Using those relations, we get

$$\pi_{(0,j)^{s\oplus g}} = \pi_{(S^p,j,0,0,S^s-S^p+j)} + \sum_{\substack{B^s+B^g=j \\ B^s<j}} \pi_{(S^p,B^s,B^g,0,S^s-S^p+j)}. \tag{8}$$

Now $\pi_{(1,0)^{s\oplus g}}$ can be determined by using Theorem 3.1. Also, based on the balance equations on the first column, we can obtain the $\pi_{(i,0)^{s\oplus g}}$, $i > 1$, values recursively.

Now, refer to Figure 4 and using Theorem 3.1, write the balance equation for a generic state $\pi_{(0,j)^{s\oplus g}}$:

$$\begin{aligned} \pi_{(0,j)^{s\oplus g}}\left(\lambda^s + \lambda^g + \lambda^p + \mu R_{(0,j)^{s\oplus g}}\right) &= \\ &= \pi_{(0,j-1)^{s\oplus g}}\left(\lambda^s + \lambda^g\right) \\ &\quad + \pi_{(1,j)^{s\oplus g}}\, \mu\, R_{(1,j)^{s\oplus g}} \\ &\quad + \pi_{(S^g-S^p-1,j)}\, (\lambda^g + \lambda^p) \\ &\quad + \left(\pi_{(0,j+1)^{s\oplus g}} - \pi_{(S^p,j+1,0,0,S^s-S^p+j+1)}\right) \mu\, R_{(0,j+1)^{s\oplus g}}. \end{aligned}$$

The last term follows from the relation in Equation (8). Equation (7) allows $\pi_{(0,j+1)^{s\oplus g}}$ to be computed knowing all other terms in the equation.

[3] Recall that (i, j) corresponds to a simplified state representation for $OH > S^p$.

$\blacksquare$

The only thing left is to calculate $[X_i^{(j)}]^{s\oplus g}$ values, which are used in Theorems 3.1 and 3.2. After that we are ready to determine steady state OH probabilities.

3.6 Solving for $[X_i^{(j)}]^{s\oplus g}$

Let us define the corresponding transition probabilities in the DTMC for $i \geqslant 1$ as follows:

$$[\alpha_i^{(j)}]^{s\oplus g} = P\big[\xi_1 = (i-1,j)^{s\oplus g} \mid \xi_0 = (i,j)^{s\oplus g}\big];$$

$$[\beta_i^{(j)}]^{s\oplus g} = P\big[\xi_1 = (i+1,j)^{s\oplus g} \mid \xi_0 = (i,j)^{s\oplus g}\big];$$

$$[\gamma_i^{(j)}]^{s\oplus g} = P\big[\xi_1 = (i,j+1)^{s\oplus g} \mid \xi_0 = (i,j)^{s\oplus g}\big].$$

Recall that,

$$[X_i^{(j)}]^{s\oplus g} = P_{(i,j)^{s\oplus g}}\big[\tau_{(0,j)^{s\oplus g}}(1) < min\{\tau_{(0,j+1)^{s\oplus g}}(1), \tau_{(0,j+1)}(1)\}\big].$$

Now, refering to Figure 4 we write the following recursion for $[X_i^{(j)}]^{s\oplus g}$:

$$[X_1^{(j)}]^{s\oplus g} = [\alpha_1^{(j)}]^{s\oplus g} + [\beta_1^{(j)}]^{s\oplus g}\,[X_2^{(j)}]^{s\oplus g};$$

and for $k \geqslant 1$,

$$[X_k^{(j)}]^{s\oplus g} = [\alpha_k^{(j)}]^{s\oplus g}\,[X_{k-1}^{(j)}]^{s\oplus g} + [\beta_k^{(j)}]^{s\oplus g}\,[X_{k+1}^{(j)}]^{s\oplus g}.$$

The above recursions have the same form as for the two-demand class setting, for which we have proven that a closed form solution exists. Therefore, let us define a set function f as:

$$f: \quad (\lambda', \lambda'', \mu, R^0) \longmapsto (\mathbf{X}, \mathbf{R})$$

where for all $i \geqslant 1,\ j \geqslant 0$:

$$R_{(i,j)} \ = \ R^0 \ + \ i \ + \ j;$$

$$\alpha_i^{(j)} \ = \ \frac{\mu R_{(i,j)}}{\lambda' + \lambda'' + \mu R_{(i,j)}};$$

$$\beta_i^{(j)} \ = \ \frac{\lambda'}{\lambda' + \lambda'' + \mu R_{(i,j)}};$$

$$\gamma_i^{(j)} \ = \ \frac{\lambda''}{\lambda' + \lambda'' + \mu R_{(i,j)}};$$

and for all $j = 1, 2, \ldots,$

$$\mathbf{X}_1^{(j)} \ = \ \alpha_1^{(j)} \ + \sum_{u=1}^{\infty} \frac{\alpha_u^{(j)} \left(\prod_{v=1}^{u-1} \beta_v^{(j)} \alpha_v^{(j)} \right)}{b_{u-1}^{(j)} b_u^{(j)}};$$

$$\mathbf{X}_k^{(j)} \ = \ \frac{\prod_{v=1}^{k} \alpha_v^{(j)}}{b_k^{(j)}} \ + \ \frac{\beta_k^{(j)} b_{k-1}^{(j)}}{b_k^{(j)}} \, \mathbf{X}_{k+1}^{(j)}, \qquad \text{for } k \geqslant 1$$

where

$$\begin{aligned} b_0^{(j)} &= 1, \\ b_1^{(j)} &= 1, \\ b_k^{(j)} &= b_{k-1}^{(j)} - \beta_{k-1}^{(j)} \alpha_k^{(j)} b_{k-2}^{(j)} \qquad \text{for } k \geqslant 2. \end{aligned}$$

According to the above set function f, we have

$$\left(\mathbf{X}^{s \oplus g}, \mathbf{R}\right) \ = \ f(\lambda^p, \lambda^s + \lambda^g, \mu, S^s - S^p). \tag{9}$$

For the simplified state representation $\pi_{(i,j)^{s\oplus g}}$. $i, j \geqslant 0$, necessary balance equations are given by:

1. For $\pi_{(i,0)^{s\oplus g}},\ i = 2, \cdots, \mathrm{M}_{\max}$:[4]

$$\pi_{(i,0)^{s\oplus g}} \cdot \left[\lambda \,+\, \mu \cdot R_{(i,0)^{s\oplus g}}\right] \,=\, \pi_{(i-1,0)^{s\oplus g}} \cdot \lambda^p \,+\, \pi_{(i+1,0)^{s\oplus g}} \cdot \mu \cdot R_{(i+1,0)^{s\oplus g}};$$

2. For $\pi_{(0,j)^{s\oplus g}},\ j \geqslant 1$:

$$\begin{aligned}
\pi_{(0,j)^{s\oplus g}}\left(\lambda^s + \lambda^g + \lambda^p + \mu R_{(0,j)^{s\oplus g}}\right) &= \\
&= \pi_{(0,j-1)^{s\oplus g}}\left(\lambda^s + \lambda^g\right) \\
&\quad + \pi_{(1,j)^{s\oplus g}}\, \mu\, R_{(1,j)^{s\oplus g}} \\
&\quad + \pi_{(S^g - S^p - 1,j)}\, \left(\lambda^g + \lambda^p\right) \\
&\quad + \left(\pi_{(0,j+1)^{s\oplus g}} - \pi_{(S^p,j+1,0,0,S^s - S^p + j+1)}\right) \mu\, R_{(0,j+1)^{s\oplus g}};
\end{aligned}$$

3. For $\pi_{(k,j)^{s\oplus g}},\ k \geqslant 2, j \geqslant 1$:

$$\pi_{(k-1,j)^{s\oplus g}} \cdot \left[\lambda \,+\, \mu \cdot R_{(k-1,j)^{s\oplus g}}\right] \,=$$

$$=\, \pi_{(k-2,j)^{s\oplus g}} \cdot \lambda^p \,+\, \pi_{(k,j)^{s\oplus g}} \cdot \mu \cdot R_{(k,j)^{s\oplus g}} \,+\, \pi_{(k-1,j-1)^{s\oplus g}} \cdot \left(\lambda^s + \lambda^g\right).$$

[4] $\mathrm{M}_{\max}$ will be defined later.

3.7 Determining steady state OH probabilities

Let us summarize the steps to determine steady state OH probabilities in Figure 4 for a given value of threshold parameters (S^s, S^g, S^p) and system parameters $(\lambda^s, \lambda^g, \lambda^p, \mu)$. Although this is an infinite state system, we do not need to solve all the steady state probabilities for all practical problems. For example, in order to capture at least 99.9% of the probabilities, we can use the following quantity as a bound in our algorithm: $\mathrm{M}_{\max} = Poissinv\big(0.999, (\lambda^s + \lambda^g + \lambda^p)/\mu\big)$.

Define *Bridge Algorithm-n* as the one that corresponds to solving OH probabilities for n-demand class setting.

Remark: From now on, for both three and multiple priority demand classes cases, while determining steady state OH probabilities for a given threshold levels, we will assume that $S^s > S^p > S^p$ and $S^1 > S^2 > \ldots > S^n$. This is a valid assumption since if $S^i = S^{i+1}$ for some $1 \leqslant i \leqslant n-1$, then this system is equivalent to a system with thresholds $\big(S^1, \ldots, S^i, S^{i+2}, \ldots, S^n\big)$ and system parameters $(\lambda^1, \ldots, \lambda^i + \lambda^{i+1}, \lambda^{i+2}, \ldots, \lambda^n, \mu)$. Hence the original problem can be transformed into a system such that $S^1 > S^2 > \ldots > S^n$. This is mainly done to simplify the proofs.

Bridge Algorithm-3:

1. Apply *Bridge Algorithm-2* to a two class setting with threshold levels $(S^s - S^p, S^g - S^p)$ with parameters $(\lambda^s, \lambda^g + \lambda^p, \mu)$, and represent the obtained (simplified) steady state probabilities as $\pi_{(i,j)'}$. Then:

   a) $\pi_{(k,0)} = \pi_{(k,0)'}$ for $k = -1, -2, \cdots, -(S^s - S^g)$,

   b) $\pi_{(i,j)} = \pi_{(i,j)'}$ for $0 \leqslant i \leqslant S^g - S^p - 1$, and $0 \leqslant j \leqslant \mathrm{M}_{\max}$,

   c) $\pi_{(S^p, j, 0, 0, S^s - S^p + j)} = \pi_{(S^g - S^p, j)'}$ for $j \geqslant 0$,

   d) $\pi_{(0,0)^{s \oplus g}} = \pi_{(S^g - S^p, 0)'}$;

2. Use Theorem 3.1 to calculate $\pi_{(1,0)^{s \oplus g}}$;

3. Set

   a) $\lambda = \lambda^s + \lambda^g + \lambda^p$;

b) $\left(\mathbf{X}^{s\oplus g}, \mathbf{R}^{\mathbf{s\oplus g}}\right) = f(\lambda^p, \lambda^s + \lambda^g, \mu, S^s - S^p)$;

4. Calculate $\pi_{(i,0)^{s\oplus g}}$ recursively for $i = 2, \cdots, \mathrm{M_{max}}$ from balance equations;

$$\pi_{(i,0)^{s\oplus g}} \cdot \left[\lambda \, + \, \mu \cdot R_{(i,0)^{s\oplus g}}\right] \, = \, \pi_{(i-1,0)^{s\oplus g}} \cdot \lambda^p \, + \, \pi_{(i+1,0)^{s\oplus g}} \cdot \mu \cdot R_{(i+1,0)^{s\oplus g}}$$

5. Calculate $\pi_{(0,1)^{s\oplus g}}$ from the balance equation in (7);

6. Set $j \, = \, 1$;

   While $1 \leqslant j \leqslant \mathrm{M_{max}}$:

   - Use Theorem 3.2 to calculate $\pi_{(1,j)^{s\oplus g}}$;
   - Calculate $\pi_{(k,j)^{s\oplus g}}$ recursively from balance equations for $2 \leqslant k \leqslant \mathrm{M_{max}} - (S^s - S^p)$;

$$\pi_{(k-1,j)^{s\oplus g}} \cdot \left[\lambda \, + \, \mu \cdot R_{(k-1,j)^{s\oplus g}}\right] \, =$$

$$= \, \pi_{(k-2,j)^{s\oplus g}} \cdot \lambda^p \, + \, \pi_{(k,j)^{s\oplus g}} \cdot \mu \cdot R_{(k,j)^{s\oplus g}} \, + \, \pi_{(k-1,j-1)^{s\oplus g}} \cdot \left(\lambda^s + \lambda^g\right)$$

   - Calculate $\pi_{(0,j+1)^{s\oplus g}}$ from balance equation in (7);
   - Set $j \, = \, j + 1$.

7. $\pi_{OH=S^g-k} = \pi_{(k,0)}$ for $k \, = \, -1, -2, \cdots, -(S^s - S^g)$;

8. $\pi_{OH=i} = \sum_{j=0}^{\mathrm{M_{max}}} \pi_{(S^g-i,j)}$ for $S^p < i \leqslant S^g$;

9. $\pi_{OH=i} = \sum_{j=0}^{\mathrm{M_{max}}} \pi_{(S^p-i,j)^{s\oplus g}}$ for $0 < i \leqslant S^p$;

10. $\pi_{OH=0} = 1 - \sum_{k=1}^{S^s} \pi_{OH=k}$.

3.8 Determining the Optimal Threshold Levels (S^{s*}, S^{g*}, S^{p*})

Our main goal is to find the minimum inventory investment S^{s*} such that (S^{s*}, S^g, S^p) satisfies fill-rate constraints for all customer classes. Due to Lemma 3.1, any feasible solution (S^s, S^g, S^p)

to a three customer case should satisfy:

$$\begin{aligned} S^s - S^p &\geqslant \tilde{S}^{s*}; \\ S^s - S^g &\geqslant \tilde{S}^{s*} - \tilde{S}^{(g+p)*}, \end{aligned}$$

where $(\tilde{S}^{s*}, \tilde{S}^{(g+p)*})$ are the threshold levels, found by applying Optimal Greedy Line Search algorithm to a two customer demand class setting with parameters $(\lambda^s, \lambda^g + \lambda^p, \mu)$ and fill-rate constraints c^s, c^g.

Let,

$$\begin{aligned} \Delta'^* &= \tilde{S}^{s*} - \tilde{S}^{(g+p)*}; \\ \Delta''^* &= \tilde{S}^{(g+p)*}. \end{aligned}$$

Next, among the set of feasible solutions (S^s, S^g, S^p), let us fix $\Delta'^* = S^s - S^g$, $\Delta''^* = S^g - S^p$, and then determine the optimal value of S^p. In other words, for a given set of (Δ'^*, Δ''^*), we seek to find the smallest value of S^s such that $S^p = S^s - \Delta'^* - \Delta''^*$, and (S^s, S^g, S^p) satisfies the *platinum* constraint. The following analog of the proposition for a two customer class case will be used for this purpose. This also allows us to calculate the steady state probabilities of the transition diagram *only once*.

Proposition 3.3 *For a given set of system parameters $(\lambda^s, \lambda^g, \lambda^p, \mu)$, if $\Delta' = S^s - S^g$ and $\Delta'' = S^g - S^p$ are kept constant for two cases, say $\left[(S^s)^0, (S^g)^0, (S^p)^0\right]$ and $\left[(S^s)^1, (S^g)^1, (S^p)^1\right]$, then*

$$\begin{aligned} \pi^0_{(i,j)} &= \pi^1_{(i,j)} \quad \forall i, j; \\ \pi^0_{(i,j)^{s \oplus g}} &= \pi^1_{(i,j)^{s \oplus g}} \quad \forall i, j. \end{aligned}$$

Note that what those states refer to, in terms of real system states (OH, B^s, B^g, B^p, R), depend on the actual values of (S^s, S^g, S^p) in the two cases.

Proof: Proof will be given later for general n-demand class setting.

∎

An upper bound for the total inventory investment to satisfy all constraints is given by

$$S_{UB}^{s} = Poissinv(c^{p}, \lambda/\mu), \text{ where } \lambda = \lambda^{s} + \lambda^{g} + \lambda^{p}.$$

Hence, if threshold levels are set to $(S_{UB}^{s}, 0, 0)$ then all constraints are satisfied while all customer types are receiving the same amount of *platinum* service. That is $\beta^{s} = \beta^{g} = \beta^{p} \geqslant c^{p}$.

On the other hand, if we set $S^{s} = S_{UB}^{s}$, $S^{g} = S_{UB}^{s} - \Delta'^{*}$, and $S^{p} = S_{UB}^{s} - \Delta'^{*} - \Delta''^{*}$ then the following lemma proves that all constraints are satisfied. In this case, *silver* and *gold* customers receive their appropriate levels of services while *platinum* customer receive higher service than required. We are going to base our optimal search algorithm on this.

Lemma 3.3 *For $S_{UB}^{s} = Poissinv(c^{p}, \lambda/\mu)$, where $\lambda = \lambda^{s} + \lambda^{g} + \lambda^{p}$, if we set $S^{s} = S_{UB}^{s}$, $S^{g} = S_{UB}^{s} - \Delta'^{*}$, and $S^{p} = S_{UB}^{s} - \Delta'^{*} - \Delta''^{*}$, then all constraints are satisfied.*

Proof: We now demonstrate starting point that $(S_{UB}^{s}, S_{UB}^{s} - \Delta'^{*}, S_{UB}^{s} - \Delta'^{*} - \Delta''^{*})$ is feasible. It is true that $(\Delta'^{*} + \Delta''^{*}, \Delta''^{*}, 0)$ satisfies c^{s} and c^{g} constraints by construction due to Proposition 3.1. Intuitive explanation is as follows: for $0 < OH \leqslant \Delta''^{*}$, incoming *gold* and *platinum* customers are satisfied on a FCFS basis while for $OH = 0$, an incoming unit from resupply is added to OH stock only after all existing *gold* and *platinum* backorders are satisfied. Hence, this is equivalent to a system of two customer demand classes with arrival rates $(\lambda^{s}, \lambda^{g} + \lambda^{p})$ and fill rate constraints (c^{s}, c^{g}).

Hence, provided fill rates have the following relationships $c^{g} \geqslant \beta^{s} \geqslant c^{s}$ and $c^{p} \geqslant \beta^{g} = \beta^{p} \geqslant c^{g}$ for the threshold values $(\Delta'^{*} + \Delta''^{*}, \Delta''^{*}, 0)$. On the other hand, $S_{UB}^{s} \geqslant \Delta'^{*} + \Delta''^{*}$ since $(S_{UB}^{s}, 0, 0)$ provides $\beta^{s} = \beta^{g} = \beta^{p} \geqslant c^{p}$.

Let $w = S_{UB}^{s} - \Delta'^{*} - \Delta''^{*}$, then it is also true that

$$(\Delta'^{*} + \Delta''^{*} + w, \Delta''^{*} + w, w) = (S_{UB}^{s}, S_{UB}^{s} - \Delta'^{*}, S_{UB}^{s} - \Delta'^{*} - \Delta''^{*})$$

also satisfies c^{s} and c^{g} constraints since we are providing more stock to the system. On the other hand, let us consider $(S_{UB}^{s}, 0, 0)$ that already satisfies all fill rate constraints. By providing more inventory to be used by *gold* and *platinum* customers, $(S^s_{UB}, S^s_{UB} - \Delta'^*, S^s_{UB} - \Delta'^* - \Delta''^*)$ is guaranteed to satisfy c^g and c^p constraints. As a result, from the above discussion we can conclude that $(S^s_{UB}, S^s_{UB} - \Delta'^*, S^s_{UB} - \Delta'^* - \Delta''^*)$ satisfies all fill rate constraints.

■

The above lemma shows that $(S^s_{UB}, S^s_{UB} - \Delta'^*, S^s_{UB} - \Delta'^* - \Delta''^*)$ is feasible by construction. Recall that among the set of feasible solutions (S^s, S^g, S^p), we fix $S^s - S^g = \Delta'^*$, $S^g - S^p = \Delta''^*$, and then determine the minimum value of S^p. In other words, for a given set of (Δ'^*, Δ''^*), we seek to find the smallest value of S^s such that $S^p = S^s - \Delta'^* - \Delta''^*$, and (S^s, S^g, S^p) satisfies the *platinum* constraint.

The relationship between the *platinum* fill-rate provided and OH probability as

$$\beta^p = 1 - P\big(OH\ =\ 0\big)\ =\ P\big(OH\ >\ 0\big),$$

and *platinum* constraint to be satisfied is $\beta^p \geqslant c^p$.

Let us apply *Bridge Algorithm-3* to the threshold levels $(S^s_{UB}, S^s_{UB} - \Delta'^*, S^s_{UB} - \Delta'^* - \Delta''^*)$. So, the sum of steady state probabilities on each row of the simplified diagram corresponds to the steady state OH probability that the row refers to. For example, if we consider *row* $1^{s\oplus g}$, then that row corresponds to $OH = S^p - 1$. Let $\pi_{row\ i^{s\oplus g}} = \sum_j \pi_{(i,j)}$. Hence, $\pi_{OH=S^g-1} = \pi_{row\ 1^{s\oplus g}} = \sum_j \pi_{(1,j)^{s\oplus g}}$. In a similar way that we derived optimal algorithm for two demand case, from this discussion, it is clear that we need the index of total number of rows necessary to satisfy the *platinum* constraint. In other words, starting from initial value of $\beta^p_{temp}\ =\ \beta^g\ =\ P_\infty(OH > S^s_{UB} - \Delta'^*)$, we continue to sum the row probabilities until $\beta^p_{temp} \geqslant c^p$ is satisfied for the first time. Then in the final situation, the corresponding number of rows that are used will give us the minimum value of S^p to satisfy the gold constraint. Therefore it will be the minimum S^{p*} for a given set of (Δ'^*, Δ''^*) due to Proposition 3.3, which makes it possible to express S^{p*} independent of the total inventory investment S^s for a given set of (Δ'^*, Δ''^*).

The following algorithm summarizes the necessary steps to find the optimal inventory investment using only a single pass:

Optimal Greedy Line Search-3:

Step 1: Set $S^s_{UB} = Poissinv(c^p, \lambda/\mu)$, where $\lambda = \lambda^s + \lambda^g + \lambda^p$;

Step 2: Apply *Optimal Greedy Line Search-2* algorithm to a two customer demand class setting with parameters $(\lambda^s, \lambda^g + \lambda^p)$ and fill-rate constraints c^s, c^g to determine $(\tilde{S}^{s*}, \tilde{S}^{(g+p)*})$.

- Set $\Delta'^* = \tilde{S}^{s*} - \tilde{S}^{(g+p)*}$;
- Set $\Delta''^* = \tilde{S}^{(g+p)*}$.

Step 3: Apply *Bridge Algorithm-3* to the threshold levels $(S^s_{UB}, S^s_{UB} - \Delta'^*, S^s_{UB} - \Delta'^* - \Delta''^*)$;

Step 4:

- Set $\beta^p_{temp} = \beta^g = P_\infty(OH > S^s_{UB} - \Delta'^* - \Delta''^*)$;
- set $i = 0$;
- While $\beta^p_{temp} < c^p$;
  - $\beta^p_{temp} = \beta^p_{temp} + \pi_{row\ i^{s \oplus g}}$;
  - $i = i + 1$;

Step 5:

- Set $S^{p*} = i$;
- Set $S^{g*} = S^{p*} + \Delta''^*$;
- Set $S^{s*} = S^{p*} + \Delta''^* + \Delta'^*$.

Step 6: Actual service levels provided to all customer types are given by:

- $\beta^s = P_\infty(OH > S^s_{UB} - \Delta'^*)$;
- $\beta^g = P_\infty(OH > S^s_{UB} - \Delta'^* - \Delta''^*)$;
- $\beta^p = \beta^p_{temp}$.

Lemma 3.4 *For a given set of (S^s, S^g, S^p) threshold values, if the system does not satisfy platinum fill rate constraint c^p, then decreasing S^g or S^p while keeping S^s constant will still not satisfy the c^p constraint.*

Proof: Will be given later for general n demand class setting.

$\blacksquare$

Theorem 3.3 *Denote the threshold levels found by applying Optimal Greedy Line Search-3 as S^{s*}, S^{g*} and S^{p*}. There is no (S^s, S^g, S^p) such that $S^s < S^{s*}$ and all fill-rate constraints are satisfied. Hence, the Optimal Greedy Line Search-3 provides an optimal inventory investment S^{s*} that satisfies all fill-rate constraints.*

Proof: Let us denote the set of Δ values obtained by applying *Optimal Greedy Line Search-3* algorithm as Δ'^*, Δ''^* and Δ'''^*, where $\Delta'''^* = S^{p*}$. By construction, Δ'''^* is the minimum value of S^{p*} for a given set of (Δ'^*, Δ''^*). It is clear from the algorithm that while keeping Δ'^*, Δ''^* fixed, any selection of $\Delta''' < \Delta'''^*$ will violate c^p constraint. Hence, for $\Delta''' = \Delta'''^* - 1$, set of threshold values $(S^{s*} - 1, S^{g*} - 1, S^{p*} - 1)$ is infeasible due to the violation of c^p constraint.

Now, let us assume that there exists (S^s, S^g, S^p) such that $S^s < S^{s*}$ and satisfies all fill-rate constraints. Then, $(S^{s*} - 1, S^g, S^p)$ is also feasible since $S^s \leqslant S^{s*} - 1$ and we are allocating more inventory to be used by the system.

Let $u = S^{s*} - S^s$ where $u \geqslant 1$. In order (S^s, S^g, S^p) to be feasible, the following relationship should hold due to Lemma 3.1:

$$\begin{aligned} S^s - S^p &\geqslant \Delta'^* + \Delta''^*; \\ S^s - S^g &\geqslant \Delta'^*. \end{aligned}$$

Therefore,

$$\begin{aligned} S^{s*} - S^p &\geqslant \Delta'^* + \Delta''^* + u \;=\; S^{s*} - S^{p*} + u; \\ S^{s*} - S^g &\geqslant \Delta'^* + u \;=\; S^{s*} - S^{g*} + u. \end{aligned}$$

From the above relationships, we get

$$\begin{aligned} S^{p*} - S^{p} &\geqslant u; \\ S^{g*} - S^{g} &\geqslant u. \end{aligned}$$

Now, if $(S^{s*}-1, S^{g*}-1, S^{p*}-1)$ is infeasible due to the violation of c^p constraint, then $(S^{s*}-1, S^{g*}-u, S^{p*}-u)$ and, hence, $(S^{s*}-1, S^{g}, S^{p})$ are also infeasible due to Lemma 3.4. (That is because, for a given system inventory $S^{s*}-1$ we are allocating more units to be used by *silver* and *gold* customers which in turn leads to a lower service rate for *platinum* customers). But this contradicts the previous conclusion that $(S^{s*}-1, S^{g}, S^{p})$ is feasible. Therefore, there cannot be any (S^{s}, S^{g}, S^{p}) such that $S^{s} < S^{s*}$ and satisfies all fill-rate constraints.

■

4 General Multiple Priority Demand Class Solution

For the general *n-demand* classes, we will use the original representation for customer types. Hence, the customers are represented as $1, 2, \ldots, n$ in terms of their priorities, where customer *type-1* and *type-n* have the lowest and the highest priorities, respectively. Recall that the optimization problem can be written as:

$$\begin{aligned} &\min\ S^1 \\ &s.t. \\ &\beta^i(S^1, S^2, \ldots, S^n) \geqslant c^i \quad \text{for } i = 1, 2, \ldots, n \\ &\text{and} \quad S^1 \geqslant S^2 \geqslant \ldots \geqslant S^n \geqslant 0. \end{aligned}$$

From the ***PASTA*** principle, we have:

$$\begin{aligned} \beta^i &= P_\infty(OH > S^{i+1}) \quad \text{for } i = 1, 2, \ldots, n-1; \text{ and} \\ \beta^n &= P_\infty(OH > 0). \end{aligned}$$

Next, in order to simplify the solution method and use the main ideas developed for the *Bridge Algorithm-2* for the n demand classes case, we first present a *Modified Bridge Algorithm-2* and then use this within the above *Bridge Algorithm-3*.

Modified Bridge Algorithm-2:

INPUT: $\tilde{\pi}'$, $\tilde{\pi}''$, λ', λ'', $\tilde{\lambda}$, μ, R^0, $\mathrm{M}_{\max}$

where, $\tilde{\pi}'$, $\tilde{\pi}''$: $1 \times (\mathrm{M}_{\max} + 1)$ vector,

1. Set $\big(\mathbf{X}, \mathbf{R}\big) \;=\; f(\lambda', \lambda'', \mu, R^0)$ by using Equation (9);

2. Set $\pi_{(0,0)} = \tilde{\pi}'_0$;

3. Calculate $\pi_{(1,0)}$ from the following equation:[5]

$$\pi_{(0,0)}\, \lambda'\, \mathbf{X}_1^{(0)} \;=\; \pi_{(1,0)}\, \mu\, R_{(1,0)};$$

4. For $i = 2 : \mathrm{M}_{\max}$, calculate $\pi_{(i,0)}$ recursively from the following (balance) equations:

$$\pi_{(i-1,0)}\, (\lambda' + \lambda'' + \mu\, R_{(i-1,0)}) \;=\; \pi_{(i-2,0)}\, \lambda' \;+ \pi_{(i,0)}\, \mu\, R_{(i,0)};$$

5. Calculate $\pi_{(0,1)}$ from the following (balance) equation:

$$\pi_{(0,0)}\, (\lambda' + \lambda'' + \mu\, R_{(0,0)}) \;=\; \tilde{\pi}''_0\, \tilde{\lambda} + \pi_{(1,0)}\, \mu\, R_{(1,0)} \;+ (\pi_{(0,1)} \;-\; \tilde{\pi}'_1)\, \mu\, R_{(0,1)};$$

6. Calculate $\pi_{(1,1)}$ from the following equation:[6]

$$\pi_{(1,1)}\, \mu\, R_{(1,1)} \;=\; \pi_{(0,1)}\, \lambda'\, \mathbf{X}_1^{(1)} \;+\; \sum_{i=1}^{\mathrm{M}_{\max}} \pi_{(i,0)}\, \lambda''\, \mathbf{X}_i^{(1)};$$

7. For $i = 2 : \mathrm{M}_{\max}$, calculate $\pi_{(i,1)}$ recursively from the following (balance) equations:

$$\pi_{(i-1,1)}\, (\lambda' + \lambda'' + \mu\, R_{(i-1,1)}) \;=\; \pi_{(i-2,1)}\, \lambda' \;+ \pi_{(i-1,0)}\, \lambda'' \;+\; \pi_{(i,1)}\, \mu\, R_{(i,1)};$$

[5] This follows from Theorem 3.1.
[6] This follows from Theorem 3.2.

8. For $j = 2 : \mathrm{M}_{\max}$,

   a) Calculate $\pi_{(0,j)}$ from the following (balance) equations:

$$\begin{aligned}\pi_{(0,j-1)}\,(\lambda' + \lambda'' + \mu\, R_{(0,j-1)}) \;&=\; \pi_{(0,j-2)}\,\lambda'' \;+\; \tilde{\pi}''_{j-1}\,\tilde{\lambda} \\ &\quad + \pi_{(1,j-1)}\,\mu\, R_{(1,j-1)} \\ &\quad + (\pi_{(0,j)} \;-\; \tilde{\pi}'_j)\,\mu\, R_{(0,j)};\end{aligned}$$

   b) For $i = 2 : \mathrm{M}_{\max}$, calculate $\pi_{(i,1)}$ recursively from the following (balance) equations:

$$\pi_{(i-1,j)}\,(\lambda' + \lambda'' + \mu\, R_{(i-1,j)}) \;=\; \pi_{(i-2,j)}\,\lambda' \;+ \pi_{(i-1,j-1)}\,\lambda'' \;+\; \pi_{(i,j)}\,\mu\, R_{(i,j)};$$

   c) Calculate $\pi_{(1,j)}$ from the following equation:

$$\pi_{(1,j)}\,\mu\, R_{(1,j)} \;=\; \pi_{(0,j)}\,\lambda'\,\mathbf{X}_1^{(j)} \;+\; \sum_{i=1}^{\mathrm{M}_{\max}} \pi_{(i,j-1)}\,\lambda''\,\mathbf{X}_i^{(j)}.$$

Let us define the above algorithm as a set function:

$$\Pi = \mathrm{MBA}^{(2)}\,(\tilde{\pi}',\; \tilde{\pi}'',\; \lambda',\; \lambda'',\; \tilde{\lambda},\; \mu,\; R^0,\; \mathrm{M}_{\max}),$$

where $\mathrm{MBA}^{(2)}$ stands for *Modified Bridge Algorithm-2* and Π is the resulting steady state probability vector. Now, let us rewrite the *Bridge Algorithm-3* using *Modified Bridge Algorithm-2*:

Bridge Algorithm-3:

INPUT: $S^s,\ S^g,\ S^p,\ \lambda^s,\ \lambda^g,\ \lambda^p,\ \mu$

1. Use Palm's Theorem to calculate $\pi_{OH=S^g+k}$ for $k = 1, 2, \ldots, S^s - S^g$ according to the following formula:

$$\pi_{OH=S^g+k} = \mathit{Poisspdf}\big(S^s - S^g - k, (\lambda^s + \lambda^g + \lambda^p)/\mu\big);$$

2. Set

$$\begin{aligned}
\tilde{\pi}'_0 &= \mathit{Poisspdf}\big(S^s - S^g, (\lambda^s + \lambda^g + \lambda^p)/\mu\big);\\
\tilde{\pi}''_0 &= \mathit{Poisspdf}\big(S^s - S^g - 1, (\lambda^s + \lambda^g + \lambda^p)/\mu\big);\\
\tilde{\pi}'_j = \tilde{\pi}''_j &= 0 \quad \text{for} j = 1, 2, \ldots, \mathrm{M_{max}};\\
\lambda' &= \lambda^g + \lambda^p;\\
\lambda'' &= \lambda^s;\\
\tilde{\lambda} &= \lambda^s + \lambda^g + \lambda^p;\\
R^0 &= S^s - S^g;\\
\mathrm{M_{max}} &= 100 * \mathit{Poissinv}\big(0.999, (\lambda^s + \lambda^g + \lambda^p)/\mu\big);
\end{aligned}$$

3. Set $\bar{\Pi} = \mathrm{MBA}^{(2)}\ (\tilde{\pi}',\ \tilde{\pi}'',\ \lambda',\ \lambda'',\ \tilde{\lambda},\ \mu,\ R^0,\ \mathrm{M_{max}})$;

   Hence:

   a) $\pi_{(i,j)} = \bar{\pi}_{(i,j)}$ for $0 \leqslant i \leqslant S^g - S^p - 1$, and $0 \leqslant j \leqslant \mathrm{M_{max}}$;

   b) $\tilde{\pi}'_j = \bar{\pi}_{(S^g - S^p, j)}$ for $0 \leqslant j \leqslant \mathrm{M_{max}}$;[7]

   c) $\tilde{\pi}''_j = \bar{\pi}_{(S^g - S^p - 1, j)}$ for $0 \leqslant j \leqslant \mathrm{M_{max}}$;

   d) $\pi_{OH=S^g-k} = \sum_{j=0}^{\mathrm{M_{max}}} \pi_{(k,j)}$ for $0 \leqslant k \leqslant S^g - S^p - 1$;

[7] This relation will be proved later during the discussion of *Bridge Algorithm-n*.

Then set:

$$\begin{aligned}
\lambda' &= \lambda^p; \\
\lambda'' &= \lambda^s + \lambda^g; \\
\tilde{\lambda} &= \lambda^g + \lambda^p; \\
R^0 &= S^s - S^p; \\
\mathrm{M}_{\max} &= \mathrm{M}_{\max} - (S^g - S^p);
\end{aligned}$$

4. Set $\bar{\Pi} = \mathrm{MBA}^{(2)}\ (\tilde{\pi}',\ \tilde{\pi}'',\ \lambda',\ \lambda'',\ \tilde{\lambda},\ \mu,\ R^0,\ \mathrm{M}_{\max});$

   Hence:

   a) $\pi_{(i,j)^{s\oplus g}} = \bar{\pi}_{(i,j)}$ for $0 \leqslant i \leqslant S^p - 1$, and $0 \leqslant j \leqslant \mathrm{M}_{\max}$;

   b) $\pi_{OH=S^p-k} = \sum_{j=0}^{\mathrm{M}_{\max}} \pi_{(k,j)^{s\oplus g}}$ for $0 \leqslant k \leqslant S^p - 1$;

5. Set $\pi_{OH=0} = 1 - \sum_{i=1}^{S^s} \pi_{OH=i}$.

The following proposition allows us to use *Modified Bridge Algorithm-2* recursively in the context of *Bridge Algorithm-n* to determine steady state probabilities corresponding to n customer demand classes.

Proposition 4.1 *For a given resupply rate μ, consider two cases:*

a) n customer demand classes with threshold levels $(S^1, S^2, \ldots, S^n)$ and arrival rates $(\lambda^1, \lambda^2, \ldots, \lambda^n)$;

b) k, $k \geqslant 2$ customer demand classes with threshold levels $(S^1 - S^{k+1}, S^2 - S^{k+1}, \ldots, S^k - S^{k+1})$ and arrival rates $(\lambda^1, \lambda^2, \ldots, \lambda^{k-1}, \sum_{u=k}^{n} \lambda^u)$.

Denote state vectors for the former and the latter cases as

$$(OH^a, B^{1^a}, B^{2^a}, \ldots, B^{n^a}, R^a)$$

and

$$(OH^b, B^{1^b}, B^{2^b}, \ldots, B^{(k-1)^b}, B^{(k)+(k+1)+\cdots+(n)^b}, R^b),$$

respectively.

Then, for $j = 1, \ldots, S^1 - S^{k+1}$*:*

$$P_\infty(OH^a = S^{k+1} + j, B^{1^a} = v_1, B^{2^a} = v_2, \ldots, B^{(k-1)^a} = v_{k-1}, 0, 0, \ldots, 0, R^a = v_R)$$

$$= P_\infty(OH^b = j, B^{1^b} = v_1, B^{2^b} = v_2, \ldots, B^{(k-1)^b} = v_{k-1}, 0, R^b = v_R).$$

Hence,

$$P_\infty(OH^a = S^{k+1} + j) = P_\infty(OH^b = j) \quad for\, j = 1, \ldots, S^1 - S^{k+1}.$$

Proof: See Appendix 3.

■

Corollary 4.1 *Consider two cases:*

(a) n *customer demand classes with threshold levels* $(S^{1^a}, S^{2^a}, \ldots, S^{n^a})$ *and system parameters* $(\lambda^1, \lambda^2, \ldots, \lambda^n, \mu)$*;*

(b) n *customer demand classes with threshold levels* $(S^{1^b}, S^{2^b}, \ldots, S^{n^b})$ *and system parameters* $(\lambda^1, \lambda^2, \ldots, \lambda^n, \mu)$*;*

If $S^{i^a} - S^{i+1^a} = S^{i^b} - S^{i+1^b}$ *for all* $1 \leqslant i \leqslant k, \quad k \leqslant n-1,$ *then*

$$P_\infty(OH^a = S^{k+1^a} + j) = P_\infty(OH^b = S^{k+1^b} + j) \quad for\, j = 1, \ldots, S^{1^a} - S^{k+1^a}.$$

Proof: Let us consider two systems (for a fixed resupply rate μ):

1) $(S^{1^a} - S^{k+1^a}, S^{2^a} - S^{k+1^a}, \ldots, S^{k^a} - S^{k+1^a})$ with arrival rates $(\lambda^1, \lambda^2, \ldots, \lambda^{k-1}, \sum_{u=k}^{n} \lambda^u)$, and denote on-hand stock as OH'.

2) $(S^{1^b} - S^{k+1^b}, S^{2^b} - S^{k+1^b}, \ldots, S^{k^b} - S^{k+1^b})$ with arrival rates $(\lambda^1, \lambda^2, \ldots, \lambda^{k-1}, \sum_{u=k}^{n} \lambda^u)$, and denote on-hand stock as OH''

The above systems are identical, therefore $P_\infty(OH' = j) = P_\infty(OH'' = j)$ for all j. By applying Proposition 4.1 to the parts (a) and (b) of Corollary 4.1, we get

$$\begin{aligned} P_\infty(OH^a = S^{k+1^a} + j) &= P_\infty(OH' = j), \\ P_\infty(OH^b = S^{k+1^b} + j) &= P_\infty(OH'' = j) \text{ for } j = 1, \ldots, S^{1^a} - S^{k+1^a}. \end{aligned}$$

As a result, we have the following

$$P_\infty(OH^a = S^{k+1^a} + j) = P_\infty(OH^b = S^{k+1^b} + j) \quad \text{for } j = 1, \ldots, S^{1^a} - S^{k+1^a}.$$

■

Proposition 4.2 *For a given set of system parameters* $(\lambda^1, \lambda^2, \ldots, \lambda^n, \mu)$, *if* $\Delta^1, \Delta^2, \ldots, \Delta^{n-1}$, *are kept constant for two cases, say* $\left(S^{1^a}, S^{2^a}, \ldots, S^{n^a}\right)$ *and* $\left(S^{1^b}, S^{2^b}, \ldots, S^{n^b}\right)$, $S^{1^a} \leqslant S^{1^b}$ *and* $\Delta^{i^a} = S^{i^a} - S^{(i+1)^a} = S^{i^b} - S^{(i+1)^b} = \Delta^{i^b}$ *for* $1 \leqslant i \leqslant n-1$, *then for each state of* $\left(S^{1^a}, S^{2^a}, \ldots, S^{n^a}\right)$ *there corresponds a unique state in* $\left(S^{1^b}, S^{2^b}, \ldots, S^{n^b}\right)$ *for which the following relation holds:*

$$\pi_{\mathrm{A}} = \pi_{\mathrm{B}}$$

where

$$\begin{aligned} z &= S^{1^b} - S^{1^a}; \\ \mathrm{A} &= \left(OH^a = j, B^{1^a} = v_1, B^{2^a} = v_2, \ldots, B^{(n-1)^a} = v_{n-1}, B^{(n)^a} = v_n, R^a = v_R\right); \\ \mathrm{B} &= \left(OH^b = j + (z - v_n)^+, B^{1^b} = v_1, B^{2^b} = v_2, \ldots, B^{(n-1)^b} = v_{n-1},\right. \\ &\qquad \left. B^{(n)^b} = (v_n - z)^+, R^b = v_R\right). \end{aligned}$$

Proof: If $\Delta^1, \Delta^2, \ldots, \Delta^{n-1}$ values are kept constant, then for two cases it is clear that the relationship between A and B is one-to-one.

Now, consider a generic state

$$\mathrm{A} = \big(OH^a = j, B^{1^a} = v_1, B^{2^a} = v_2, \ldots, B^{(n-1)^a} = v_{n-1}, B^{(n)^a} = v_n, R^a = v_R\big),$$

and suppose that some event occurs which causes a transition to a state

$$\mathrm{A}' = \big(OH^a = j', B^{1^a} = v'_1, B^{2^a} = v'_2, \ldots, B^{(n-1)^a} = v'_{n-1}, B^{(n)^a} = v'_n, R^a = v'_R\big).$$

Since $\Delta^1, \Delta^2, \ldots, \Delta^{n-1}$ are kept constant, considering state

$$\begin{aligned} \mathrm{B} \quad = \quad & \big(OH^b = j + (z - v_n)^+, B^{1^b} = v_1, B^{2^b} = v_2, \ldots, B^{(n-1)^b} = v_{n-1}, \\ & B^{(n)^b} = (v_n - z)^+, R^b = v_R\big), \end{aligned}$$

the same event will cause a transition to a state B$'$ for which the same relationship between A and B is still preserved among A$'$ and B$'$ as follows:

$$\begin{aligned} \mathrm{B}' \quad = \quad & \big(OH^b = j + (z - v'_n)^+, B^{1^b} = v'_1, B^{2^b} = v'_2, \ldots, B^{(n-1)^b} = v'_{n-1}, \\ & B^{(n)^b} = (v'_n - z)^+, R^b = v'_R\big). \end{aligned} \tag{10}$$

In order to make this clearer, let us suppose $v_n = 0$ and $S^{(k+1)^a} \leqslant j \leqslant S^{k^a}$. Hence,

$$\mathrm{A} = \big(OH^a = j, B^{1^a} = v_1, B^{2^a} = v_2, \ldots, B^{(n-1)^a} = v_{n-1}, B^{(n)^a} = 0, R^a = v_R\big),$$

and

$$\begin{aligned} \mathrm{B} \quad = \quad & \big(OH^b = S^{1^b} - S^{1^a} + j, B^{1^b} = v_1, B^{2^b} = v_2, \ldots, B^{(n-1)^b} = v_{n-1}, \\ & B^{(n)^b} = 0, R^b = v_R\big). \end{aligned}$$

Let us also suppose that a *type-i* customer demand occurs for which $i < k$. Then according to the proposed policy, since $S^{(k+1)^a} \leqslant j \leqslant S^{k^a} < S^{i^a}$, that demand is backordered and new state becomes,

$$\begin{aligned} \mathrm{A}' &= \big(OH^a = j, B^{1^a} = v_1, B^{2^a} = v_2, \ldots, B^{i^a} = v_i + 1, \ldots, B^{k^a} = v_k, \\ &\quad \ldots, B^{(n-1)^a} = v_{n-1}, B^{(n)^a} = 0, R^a = v_R + 1\big). \end{aligned}$$

Considering the state B, since $OH^b = S^{1^b} - S^{1^a} + j$ and $S^{(k+1)^a} \leqslant j \leqslant S^{k^a} < S^{i^a}$,

$$\begin{aligned} S^{(k+1)^a} + S^{1^b} - S^{1^a} &\leqslant OH^b = S^{1^b} - S^{1^a} + j \\ &\leqslant S^{k^a} + S^{1^b} - S^{1^a} \\ &< S^{i^a} + S^{1^b} - S^{1^a}. \end{aligned}$$

Since $\Delta^{i^a} = S^{i^a} - S^{(i+1)^a} = S^{i^b} - S^{(i+1)^b} = \Delta^{i^b}$ for $1 \leqslant i \leqslant n-1$, we have

$$S^{1^a} - S^{i^a} = \sum_{m=1}^{i-1} \Delta^{i^a} = \sum_{m=1}^{i-1} \Delta^{i^b}.$$

Therefore,

$$\begin{aligned} S^{i^a} + S^{1^b} - S^{1^a} &= S^{1^b} - \sum_{m=1}^{i-1} \Delta^{i^a} \\ &= S^{1^b} - \sum_{m=1}^{i-1} \Delta^{i^b} \\ &= S^{i^b}, \end{aligned}$$

providing us the following relation

$$OH^b = S^{1^b} - S^{1^a} + j < S^{i^b}.$$

Hence, when a *type-i* customer demand occurs, that demand is backordered and new state becomes,

$$\begin{aligned} \mathrm{B}' &= \big(OH^b = S^{1^b} - S^{1^a} + j, B^{1^b} = v_1, B^{2^b} = v_2, \ldots, B^{i^b} = v_i + 1, \ldots, B^{k^b} = v_k, \\ &\quad \ldots, B^{(n-1)^b} = v_{n-1}, B^{(n)^b} = 0, R^a = v_R + 1\big). \end{aligned}$$

As a result, the same relationship between A and B is still preserved among A′ and B′ as in

Equation (10). Other kind of events and the resulting transitions follow similar ideas, therefore omitted.

Furthermore, the rates of flow from A to A′ and B to B′ are identical. Therefore, the balance equations to be solved is identical for those two cases. As a result, the steady state probabilities π_{A} and π_{B} are identical.

■

Corollary 4.2 *Consider two cases:*

a) $k, k \geqslant 2$ *customer demand classes with threshold levels* $(S^1, S^2, \ldots, S^k)$ *and system parameters* $(\lambda^1, \lambda^2, \ldots, \lambda^k, \mu)$,

b) k, $k \geqslant 2$ *customer demand classes with threshold levels* $(S^1+z, S^2+z, \ldots, S^k+z)$ *and system parameters* $(\lambda^1, \lambda^2, \ldots, \lambda^k, \mu)$, *for* $z \geqslant 1$.

Denote the on-hand stock for the former and the latter cases by OH^a *and* OH^b *respectively. Then,*

$$P_\infty(OH^a = j) = P_\infty(OH^b = z + j) \quad \textit{for}\, j = 1, \ldots, S^1.$$

Proof: For $j > 0$, it is true for both cases that $B^{k^a} = B^{k^b} = 0$. Then by using Proposition 4.2 for which the one-to-one transformation between two cases are

$$\begin{aligned}
\mathrm{A} &= \big(OH^a = j, B^{1^a} = v_1, B^{2^a} = v_2, \ldots, B^{(k-1)^a} = v_{k-1}, B^{(k)^a} = 0, R^a = v_R\big); \\
\mathrm{B} &= (OH^b = j + z, B^{1^b} = v_1, B^{2^b} = v_2, \ldots, B^{(k-1)^b} = v_{k-1}, B^{(k)^b} = (0, R^b = v_R\big),
\end{aligned}$$

we get

$$P_\infty(OH^a = j) = P_\infty(OH^b = z + j) \quad \text{for}\, j = 1, \ldots, S^1.$$

■

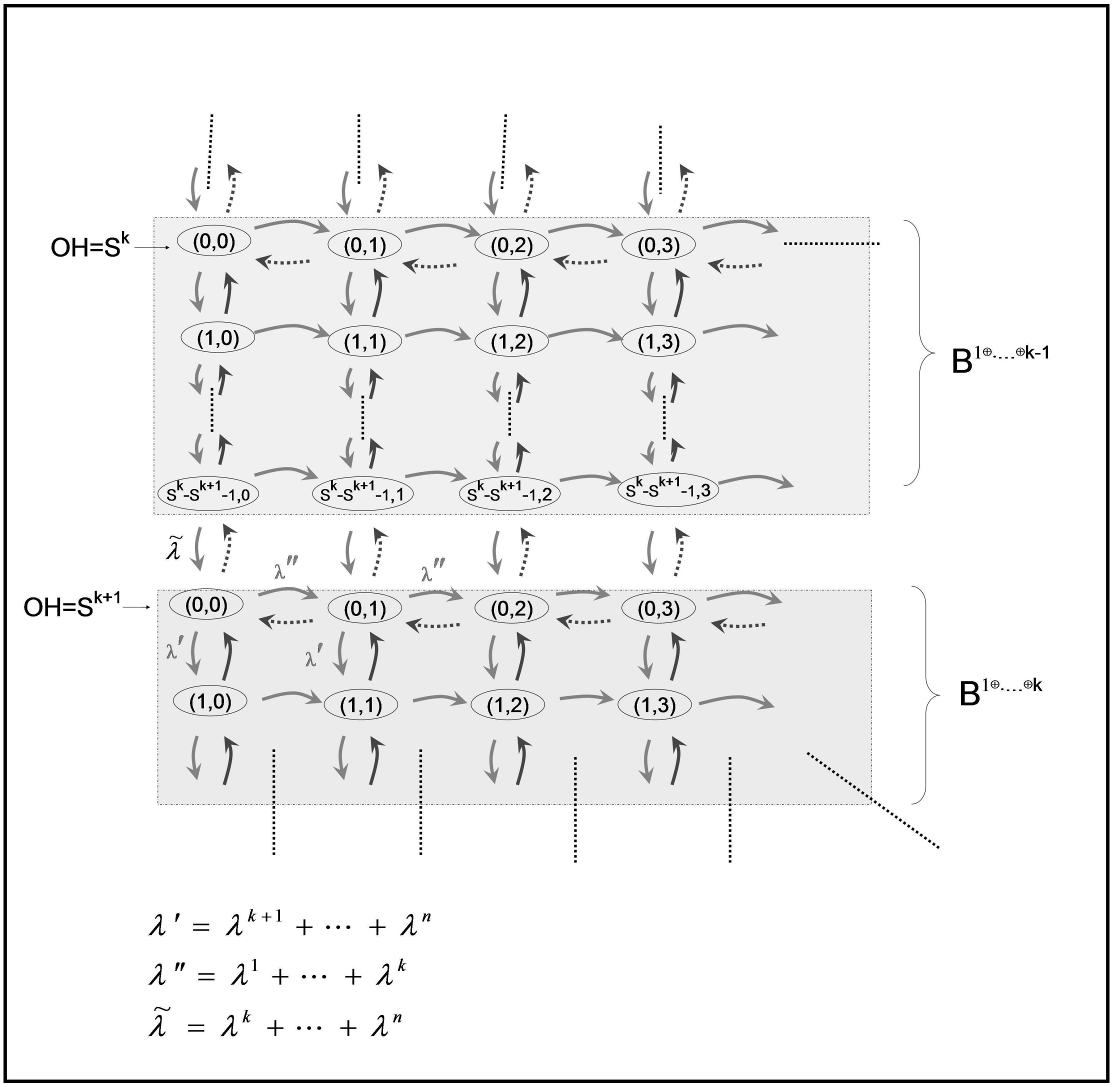

Figure 6: Simplified state transition diagram after grouping for general priority demand classes

4.1 Updating Threshold Level Steady State Probabilities from Balance Equations

Let us refer to Figure 6, and consider a generic state $(0,j)^{1\oplus\cdots\oplus k}$, $j \geqslant 1$. Recall that

$$(0,j)^{1\oplus\cdots\oplus k} \quad \equiv \bigcup_{B^1+\cdots+B^k=j} (S^{k+1}, B^1, \ldots, B^k, 0, \ldots, 0, S^1 - S^{k+1} + j).$$

We decompose the RHS of above equation as:

$$\bigcup_{B^1+\cdots+B^k=j} (S^{k+1}, B^1, \ldots, B^k, 0, \ldots, 0, S^1-S^{k+1}+j) =$$

$$\begin{aligned} = & \left\{ \bigcup_{\substack{B^1+\cdots+B^k=j \\ 0<B^k\leqslant j}} (S^{k+1}, B^1, \ldots, B^k, 0, \ldots, 0, S^1 - S^{k+1} + j) \right\} \\ & \bigcup \left\{ \bigcup_{\substack{B^1+\cdots+B^{k-1}=j \\ B^k=0}} (S^{k+1}, B^1, \ldots, B^k, 0, \ldots, 0, S^1 - S^{k+1} + j) \right\}. \end{aligned}$$

If the current state is

$$\begin{aligned} (0,j)^{1\oplus\cdots\oplus k} \equiv & \Big\{ (S^{k+1}, B^1, \ldots, B^k, 0, \ldots, 0, S^1 - S^{k+1} + j) \quad | \\ & B^1 + \cdots + B^{k-1} = j,\ B^k = 0 \Big\}, \end{aligned}$$

then when a unit is received from resupply, it will be added to OH stock and the new system state becomes

$$\Big\{ (S^{k+1} + 1, B^1, \ldots, B^k, 0, \ldots, 0, S^1 - S^{k+1} + j) \mid B^1 + \cdots + B^{k-1} = j,\ B^k = 0 \Big\},$$

which is $\in (S^k - S^{k+1} - 1, j)^{1\oplus\cdots\oplus k-1}$. For all other cases, the unit will be used to fulfill an existing *type-k* backorder since $B^1+\cdots+B^k = j$ and $0 < B^k \leqslant j$, resulting in a system state $(0, j-1)^{1\oplus\cdots\oplus k}$.

Using those relations, we get

$$\begin{aligned} \pi_{(0,j)^{1\oplus\cdots\oplus k}} = & \sum_{\substack{B^1+\cdots+B^k=j \\ 0<B^k\leqslant j}} \pi_{(S^{k+1},B^1,\ldots,B^k,0,\ldots,0,S^1-S^{k+1}+j)} \\ & + \sum_{\substack{B^1+\cdots+B^{k-1}=j \\ B^k=0}} \pi_{(S^{k+1},B^1,\ldots,B^k,0,\ldots,0,S^1-S^{k+1}+j)}. \end{aligned} \tag{11}$$

For the sake of notational simplicity, let us use the following terms:

$$\begin{aligned}
\tilde{\pi}'_j &= \sum_{\substack{B^1+\cdots+B^{k-1}=j \\ B^k=0}} \pi_{(S^{k+1},B^1,\ldots,B^k,0,\ldots,0,S^1-S^{k+1}+j)}, \\
\tilde{\pi}''_j &= (S^k - S^{k+1} - 1, j)^{1\oplus\cdots\oplus k-1}.
\end{aligned} \tag{12}$$

Hence the first term on the RHS of Equation (11) can be expressed as:

$$\sum_{\substack{B^1+\cdots+B^k=j \\ 0<B^k\leqslant j}} \pi_{(S^{k+1},B^1,\ldots,B^k,0,\ldots,0,S^1-S^{k+1}+j)} = \pi_{(0,j)^{1\oplus\cdots\oplus k}} - \tilde{\pi}'_j.$$

Now, referring to Figure 6 and Equation (12), write the balance equation for a generic state $\pi_{(0,j)^{1\oplus\cdots\oplus k}}$:

$$\begin{aligned}
\pi_{(0,j)^{1\oplus\cdots\oplus k}}\left(\lambda' + \lambda'' + \mu R_{(0,j)^{1\oplus\cdots\oplus k}}\right) &= \pi_{(0,j-1)^{s\oplus g}}\lambda'' \\
&+ \pi_{(1,j)^{1\oplus\cdots\oplus k}}\, \mu\, R_{(1,j)^{1\oplus\cdots\oplus k}} \\
&+ \tilde{\pi}''_j\, \tilde{\lambda} \\
&+ \left(\pi_{(0,j+1)^{1\oplus\cdots\oplus k}} - \tilde{\pi}'_j\right) \mu\, R_{(0,j+1)^{1\oplus\cdots\oplus k}}.
\end{aligned} \tag{13}$$

From the above equation, we can confirm that Step 8.a of MBA$^{(2)}$ provides a correct calculation of threshold level steady state probabilities for a general priority demand class scenario.

Remark: According to Figure 6, it is shown that conditioned on the current state $(0,j)^{1\oplus\cdots\oplus k}$, when there is a demand of any type that belongs to *type-m*, $m \in \{k+1,\ldots,n\}$, then there will be a transition into state $(1,j)^{1\oplus\cdots\oplus k}$, based on the implicit assumption that $S^{k+1} > S^{k+2}+1$.

However, for the case $S^{k+1} = S^{k+2}+1$, the balance equation stated in Equation (13) is still correct and it does not create any problem in the implementation of MBA$^{(2)}$ in *Bridge Algorithm-n* and *Optimal Greedy Line Search-n* algorithms. This is because in both of the *Bridge Algorithm-n* and *Optimal Greedy Line Search-n* algorithms, MBA$^{(2)}$ is used recursively to determine $P_\infty(OH = z)$, $S^{k+1} < z \leqslant S^k$ at each k^{th} iteration. And, as was proved earlier, we have the following relation between each iteration:

$$(S^k - S^{k+1}, j)^{1\oplus\cdots\oplus k} \quad = \quad \left\{ \bigcup_{\substack{B^1+\cdots+B^k=j \\ B^{k+1}=0}} (S^{k+2}, B^1, \ldots, B^k, B^{k+1}, 0, \ldots, 0, S^1 - S^{k+2} + j) \right\};$$

Hence,

$$\pi_{(S^k - S^{k+1}, j)^{1\oplus\cdots\oplus k}} \quad = \quad \sum_{\substack{B^1+\cdots+B^k=j \\ B^{k+1}=0}} \pi_{(S^{k+2}, B^1, \ldots, B^k, B^{k+1}, 0, \ldots, 0, S^1 - S^{k+2} + j)}.$$

If the current state belongs to the set

$$\left\{ \bigcup_{\substack{B^1+\cdots+B^k=j \\ B^{k+1}=0}} (S^{k+2}, B^1, \ldots, B^k, B^{k+1}, 0, \ldots, 0, S^1 - S^{k+2} + j) \right\},$$

then when a unit is received from resupply it will be added to on-hand stock which causes an upward transition in Figure 6.

Therefore, if $S^{k+1} = S^{k+2}+1$ then the second term in the balance equations of (13) will be written in terms of

$$\sum_{\substack{B^1+\cdots+B^k=j \\ B^{k+1}=0}} (S^{k+2}, B^1, \ldots, B^k, B^{k+1}, 0, \ldots, 0, S^1 - S^{k+2} + j),$$

which is equivalent to $\pi_{(S^k-S^{k+1},j)^{1\oplus\cdots\oplus k}}$. Hence, the expression in (13) is still correct.

4.2 Calculating $\tilde{\pi}'_j$ at the $k-1^{th}$ grouping operation

In the previous section, at the k^{th} grouping operation, the balance equations contain the term $\tilde{\pi}'_j$, which is given as:

$$\tilde{\pi}'_j = \sum_{\substack{B^1+\cdots+B^{k-1}=j \\ B^k=0}} \pi_{(S^{k+1},B^1,\ldots,B^k,0,\ldots,0,S^1-S^{k+1}+j)}.$$

Now, let us assume that grouping operations have been performed only for $m = 1,\ldots,k-1$. Hence we have representations for $(i,j)^{1\oplus\cdots\oplus m}$ for $m = 1,\ldots,k-1$. Let us also suppose that steady state probabilities are already determined for those states. Then, the state $(S^k - S^{k+1}, j)^{1\oplus\cdots\oplus k-1}$ corresponds to:

$$(S^k - S^{k+1}, j)^{1\oplus\cdots\oplus k-1} \equiv \bigcup_{B^1+\cdots+B^{k-1}=j} (S^{k+1}, B^1,\ldots,B^{k-1},0,\ldots,0,S^1-S^{k+1}+j).$$

Hence,

$$\pi_{(S^k-S^{k+1},j)^{1\oplus\cdots\oplus k-1}} = \sum_{B^1+\cdots+B^{k-1}=j} \pi_{(S^{k+1},B^1,\ldots,B^{k-1},0,\ldots,0,S^1-S^{k+1}+j)}.$$

But this is equivalent to $\tilde{\pi}'_j$ that is given above. Therefore, when we consider the balance equations written for the k^{th} grouping operation in Equation (13), we can conclude that

$$\pi_{(S^k-S^{k+1},j)^{1\oplus\cdots\oplus k-1}} = \tilde{\pi}'_j. \tag{14}$$

4.3 Calculating $\mathrm{X}^{1\oplus\cdots\oplus\mathrm{k}}$ at the k^{th} grouping operation

Let us refer to Figure 6 and consider the states $OH \leqslant S^{k+1}$. According to the figure, grouping operations have been performed only for $m = 1, \ldots, k$. Hence, the downward and rightward rates (for states $OH \leqslant S^{k+1}$) are denoted as λ' and λ'' respectively.

In a similar way as we have defined for two and three priority class, let us define the following one step transition probabilities:

$$[\alpha_i^{(j)}]^{1\oplus\cdots\oplus k} = P\big[\xi_1 = (i-1, j)^{1\oplus\cdots\oplus k} \mid \xi_0 = (i, j)^{1\oplus\cdots\oplus k}\big];$$

$$[\beta_i^{(j)}]^{1\oplus\cdots\oplus k} = P\big[\xi_1 = (i+1, j)^{1\oplus\cdots\oplus k} \mid \xi_0 = (i, j)^{1\oplus\cdots\oplus k}\big];$$

where for all $i \geqslant 1,\ j \geqslant 0$:

$$[\alpha_i^{(j)}]^{1\oplus\cdots\oplus k} = \frac{\mu R_{(i,j)^{1\oplus\cdots\oplus k}}}{\lambda' + \lambda'' + \mu R_{(i,j)^{1\oplus\cdots\oplus k}}};$$

$$(\beta_i^{(j)})^{1\oplus\cdots\oplus k} = \frac{\lambda'}{\lambda' + \lambda'' + \mu R_{(i,j)^{1\oplus\cdots\oplus k}}}.$$

And define,

$$[X_i^{(j)}]^{1\oplus\cdots\oplus k} = P_{(i,j)^{1\oplus\cdots\oplus k}}\big[\tau_{(0,j)^{1\oplus\cdots\oplus k}}(1) < \min\{\tau_{(0,j+1)^{1\oplus\cdots\oplus m}}(1)\ :\ m = 1, \ldots, k\}\big].$$

Now, write the recursions for $[X_i^{(j)}]^{1\oplus\cdots\oplus k}$:

$$[X_1^{(j)}]^{1\oplus\cdots\oplus k} = [\alpha_1^{(j)}]^{1\oplus\cdots\oplus k} + [\beta_1^{(j)}]^{1\oplus\cdots\oplus k}\,[X_2^{(j)}]^{1\oplus\cdots\oplus k};$$

and for $k \geqslant 1$,

$$[X_k^{(j)}]^{1\oplus\cdots\oplus k} = [\alpha_k^{(j)}]^{1\oplus\cdots\oplus k}\,[X_{k-1}^{(j)}]^{1\oplus\cdots\oplus k} + [\beta_k^{(j)}]^{1\oplus\cdots\oplus k}\,[X_{k+1}^{(j)}]^{1\oplus\cdots\oplus k}.$$

In addition, for $i \geqslant 1$, $j \geqslant 0$, $R_{(i,j)^{1\oplus\cdots\oplus k}} = S^1 - S^{k+1} + i + j$. Therefore, the above recursions are identical to the ones studied previously. Hence, the set function f defined in Equation (9) provides the correct values for a given set of inputs as follows:

$$\left(\mathbf{X}^{1\oplus\cdots\oplus k}, \mathbf{R}^{\mathbf{1}\oplus\cdots\oplus\mathbf{k}}\right) \quad = \quad f(\lambda', \lambda'', \mu, S^1 - S^{k+1}).$$

Due to the above relations, it is also legitimate to use Theorems 3.1 and 3.2 in the context of MBA$^{(2)}$ algorithm. Hence, we can conclude that MBA$^{(2)}$ is a general algorithm that can be used recursively for a general priority class solution.

4.4 Determining Steady State OH probabilities for $(S^1, S^2, \ldots, S^n)$

Let us consider n-priority demand class with threshold levels $(S^1, S^2, \ldots, S^n)$ and system parameters $(\lambda^1, \lambda^2, \ldots, \lambda^n, \mu)$. Next, we provide the *Bridge Algorithm-n* for general priority class scenario by using MBA$^{(2)}$ recursively:

Bridge Algorithm-n:

INPUT: S^1, $S^2, \ldots, S^n$, λ^1, $\lambda^2, \ldots, \lambda^n$, μ

Step 1: Set $\lambda = \sum_{v=1}^{n} \lambda^v$ and use Palm's Theorem to calculate $\pi_{OH=S^2+k}$ for $k = 1, 2, \ldots, S^1 - S^2$ according to the following formula:

$$\pi_{OH=S^2+k} = Poisspdf\left(S^1 - S^2 - k, \lambda/\mu\right);$$

Step 2: Set

$$\begin{aligned}
\tilde{\pi}'_0 &= Poisspdf\big(S^1 - S^2, \lambda/\mu\big);\\
\tilde{\pi}''_0 &= Poisspdf\big(S^1 - S^2 - 1, \lambda/\mu\big);\\
\tilde{\pi}'_j = \tilde{\pi}''_j &= 0 \quad \text{for } j = 1, 2, \ldots, \mathrm{M}_{\max};\\
\lambda' &= \lambda - \lambda^1;\\
\lambda'' &= \lambda^1;\\
\tilde{\lambda} &= \lambda;\\
R^0 &= S^1 - S^2;\\
\mathrm{M}_{\max} &= 10.Poissinv\big(0.999, \lambda/\mu\big);\\
S^{n+1} &= 0;\\
k &= 2;
\end{aligned}$$

Step 3: While $k \leqslant n$;

Set:

a) $\bar{\Pi} = \mathrm{MBA}^{(2)}\ (\tilde{\pi}',\ \tilde{\pi}'',\ \lambda',\ \lambda'',\ \tilde{\lambda},\ \mu,\ R^0,\ \mathrm{M}_{\max})$;

b) $\pi_{(i,j)} = \bar{\pi}_{(i,j)}$ for $0 \leqslant i \leqslant S^k - S^{k+1} - 1$, and $0 \leqslant j \leqslant \mathrm{M}_{\max}$;

c) $\tilde{\pi}'_j = \bar{\pi}_{(S^k - S^{k+1}, j)}$ for $0 \leqslant j \leqslant \mathrm{M}_{\max}$;[8]

d) $\tilde{\pi}''_j = \bar{\pi}_{(S^k - S^{k+1} - 1, j)}$ for $0 \leqslant j \leqslant \mathrm{M}_{\max}$;

e) $\pi_{OH = S^k - v} = \sum_{j=0}^{M_{max}} \pi_{(v,j)}$ for $0 \leqslant v \leqslant S^k - S^{k+1} - 1$;

[8]Follows from Equation (14).

Then set:

$$\begin{aligned}
\pi_{OH=S^k-v} &= \sum_{j=0}^{M_{max}} \pi_{(v,j)} \quad \text{for } 0 \leqslant v \leqslant S^k - S^{k+1} - 1;\\
\tilde{\lambda} &= \lambda - \sum_{v=1}^{k-1} \lambda^v;\\
\lambda' &= \lambda - \sum_{v=1}^{k} \lambda^v;\\
\lambda'' &= \sum_{v=1}^{k} \lambda^v;\\
R^0 &= S^1 - S^{k+1};\\
\mathrm{M}_{\max} &= \mathrm{M}_{\max} - (S^k - S^{k+1});\\
\\
k &= k+1;
\end{aligned}$$

Step 4: Set $\pi_{OH=0} = 1 - \sum_{v=1}^{n} \pi_{OH=v}$.

In the next section, we provide a computationally efficient algorithm to determine threshold levels $(S^{1*}, S^{2*}, \ldots, S^{n*})$ that provides the minimum system stock S^1 that satisfies all fill-rate constraints. Note that *Bridge Algorithm-n* is used implicitly in the *Optimal Greedy Line Search-n*.

4.5 Determining the Optimal Threshold Levels $(S^{1*}, S^{2*}, \ldots, S^{n*})$

Optimal Greedy Line Search-n:

INPUT: $c^1,\ c^2, \ldots, c^n,\ \lambda^1,\ \lambda^2, \ldots, \lambda^n,\ \mu$

Step 1: Set $S^1_{UB} = Poissinv(c^n, \lambda/\mu)$, where $\lambda = \sum_{k=1}^{n} \lambda^k$;

Step 2: Set

a) $\Delta^{1*} = 1 + Poissinv(c^1, \lambda/\mu)$;

b) $\beta^1 = Poisscdf\left(\Delta^{1*} - 1, \lambda/\mu\right)$;

Step 3: Set

a)

$$
\begin{aligned}
\tilde{\pi}'_0 &= Poisspdf\left(\Delta^{1*}, \lambda/\mu\right); \\
\tilde{\pi}''_0 &= Poisspdf\left(\Delta^{1*} - 1, \lambda/\mu\right); \\
\tilde{\pi}'_j = \tilde{\pi}''_j &= 0 \quad \text{for} j = 1, 2, \ldots, \mathrm{M_{max}}; \\
\lambda' &= \lambda - \lambda^1; \\
\lambda'' &= \lambda^1; \\
\tilde{\lambda} &= \lambda; \\
S^2 &= S^1_{UB} - \Delta^{1*}; \\
R^0 &= \Delta^{1*}; \\
\mathrm{M_{max}} &= \mathrm{M_{max}} - \Delta^{1*};
\end{aligned}
$$

b) $\Pi = \mathrm{MBA}^{(2)}\ (\tilde{\pi}',\ \tilde{\pi}'',\ \lambda',\ \lambda'',\ \tilde{\lambda},\ \mu,\ R^0,\ \mathrm{M_{max}})$;

c) $i = 0$;

$\beta^2_{temp} = \beta^1$;

While $\beta^2_{temp} < c^2$;

* $\beta^2_{temp} = \beta^2_{temp} + \pi_{rowi}$;
* $i = i + 1$;

d) $\Delta^{2*} = i$;

e) $\beta^2 = \beta^2_{temp}$;

f) $z = 2$;

Step 4: While $z \leqslant n - 1$;

a) Set

$$\begin{aligned}
\tilde{\pi}'_j &= \pi_{(\Delta^{z*},j)} \quad \text{for } 0 \leqslant j \leqslant \mathrm{M}_{\max}; \\
\tilde{\pi}''_j &= \pi_{(\Delta^{z}-1,j)} \quad \text{for } 0 \leqslant j \leqslant \mathrm{M}_{\max}; \\
\tilde{\lambda} &= \lambda - \sum_{k=1}^{z-1} \lambda^k; \\
\lambda' &= \lambda - \sum_{k=1}^{z} \lambda^k; \\
\lambda'' &= \sum_{k=1}^{z} \lambda^k; \\
S^{z+1} &= S^1_{UB} - \sum_{k=1}^{z} \Delta^{k*}; \\
R^0 &= \sum_{k=1}^{z} \Delta^{k*}; \\
\mathrm{M}_{\max} &= \mathrm{M}_{\max} - \Delta^{z*};
\end{aligned}$$

b) Set $\Pi = \mathrm{MBA}^{(2)}\ (\tilde{\pi}',\ \tilde{\pi}'',\ \lambda',\ \lambda'',\ \tilde{\lambda},\ \mu,\ R^0,\ \mathrm{M}_{\max})$;

c) $i = 0$;

$\beta^{z+1}_{temp} = \beta^z$;

While $\beta^{z+1}_{temp} < c^{z+1}$;

* $\beta^{z+1}_{temp} = \beta^{z+1}_{temp} + \pi_{rowi}$;
* $i = i + 1$;

d) $\Delta^{z+1*} = i$;

e) $\beta^{z+1} = \beta^{z+1*}_{temp}$;

f) $z = z + 1$;

Step 5: Apply Corollary 4.2 to normalize and then set $S^{k*} = \sum_{v=k}^{n} \Delta^{v*}$ for $1 \leqslant k \leqslant n$.

■

Now, we are ready to prove that MBA$^{(2)}$ provides a feasible solution for each priority class (in an orderly fashion) at each iteration in the above *The Optimal Greedy Line Search-n* algorithm. In other words, when it is solved for $z = m - 1$, then the temporary solution $(S^1_{UB}, S^2, \ldots, S^m)$ satisfies all fill rate constraints with $\beta^m = \ldots = \beta^n > c^n$, while $\beta^1, \ldots, \beta^{m-1}$ are just enough to meet fill rate constraints $c^1, c^2, \ldots, c^{m-1}$.

Theorem 4.1 MBA$^{(2)}$ *provides a feasible solution for each priority class at each iteration in the above The Optimal Greedy Line Search-n algorithm.*

Proof: Consider three cases:

(a) n customer demand classes with threshold levels $(S^1, S^2, \ldots, S^n)$ and system parameters $(\lambda^1, \lambda^2, \ldots, \lambda^n, \mu)$;

(b) k, $k \geqslant 2$ customer demand classes with threshold levels $(S^1 - S^{k+1}, S^2 - S^{k+1}, \ldots, S^k - S^{k+1})$ and system parameters $(\lambda^1, \lambda^2, \ldots, \lambda^{k-1}, \sum_{u=k}^{n} \lambda^u, \mu)$;

(c) k, $k \geqslant 2$ customer demand classes with threshold levels $(S^1, S^2, \ldots, S^k)$ and system parameters $(\lambda^1, \lambda^2, \ldots, \lambda^{k-1}, \sum_{u=k}^{n} \lambda^u, \mu)$.

Denote the on-hand stock for those cases by OH^a, OH^b and OH^c respectively. Applying Proposition 4.1 to (a) and (b), we get

$$P_\infty(OH^a = S^{k+1} + j) = P_\infty(OH^b = j) \quad \text{for } j = 1, \ldots, S^1 - S^{k+1}.$$

Then using Corollary 4.2 for (b) and (c), we get

$$P_\infty(OH^c = S^{k+1} + j) = P_\infty(OH^b = j) \quad \text{for } j = 1, \ldots, S^1 - S^{k+1}.$$

From the above relationships, we get

$$P_\infty(OH^a = S^{k+1} + j) = P_\infty(OH^c = S^{k+1} + j) \quad \text{for } j = 1, \ldots, S^1 - S^{k+1}.$$

Hence, this shows that at each iteration MBA$^{(2)}$ provides a feasible solution for a particular class. For example, for *type-k* priority class, $\beta^k = P_\infty(OH^a > S^{k+1} + j)$ and MBA$^{(2)}$ provides a feasible solution by solving case (c).

■

Theorem 4.2 *Let us denote the threshold levels found by applying the Optimal Greedy Line Search-n algorithm as* $(S^{1*}, S^{2*}, \ldots, S^{n*})$. *There is no* $(S^1, S^2, \ldots, S^n)$ *such that* $S^1 < S^{1*}$ *and satisfies all fill-rate constraints. Hence, the Optimal Greedy Line Search-n provides an optimal inventory investment* S^{1*} *that satisfies all fill-rate constraints.*

Proof: The proof is by induction. As we have already proved, the induction hypothesis holds for $m = 3$.

Assume it holds for $m = 3, \ldots, n-1$ customer classes. In other words, *Optimal Greedy Line Search-m* gives the optimal inventory investment for $m = 4, \ldots, n-1$ customer classes.

Let us denote the set of Δ^{i*} values obtained by applying *Optimal Greedy Line Search-n* algorithm as $\Delta^{1*}, \ldots, \Delta^{n*}$ where $\Delta^{i*} = S^i - S^{i+1}$.

Conjecture: Any feasible solution $(S^1, S^2, \ldots, S^n)$ to a *n demand classes* case will satisfy

$$S^1 - S^{m+1} \geqslant \sum_{i=1}^{m} \Delta^{i*} \qquad \text{for } \ 1 \leqslant m \leqslant n-1. \tag{15}$$

For $1 \leqslant m \leqslant n-1$, let us consider the system of m *customer demand classes* with threshold levels $(S^1 - S^{m+1}, S^2 - S^{m+1}, \ldots, S^m - S^{m+1})$ and arrival rates $(\lambda^1, \lambda^2, \ldots, \lambda^{m-1}, \sum_{u=m}^{n} \lambda^u)$. Corresponding fill rate constraints are $c^1, \ldots, c^m$.

By the induction hypothesis, the *Optimal Greedy Line Search-m* gives the optimal inventory investment for $m = 4, \ldots, n-1$ customer classes. Hence, the minimum inventory investment for the m *customer demand classes* is given by $(S^1 - S^{m+1})^* = \sum_{i=1}^{m} \tilde{\Delta}^{i*}$, where $\tilde{\Delta}^{i*}$ values are obtained by the algorithm. Therefore, any feasible solution $(S^1 - S^{m+1})$ will satisfy $(S^1 - S^{m+1}) \geqslant \sum_{i=1}^{m} \tilde{\Delta}^{i*}$.

But those $\tilde{\Delta}^{i*}$ values are the same Δ^{i*} values obtained by the *Optimal Greedy Line Search-n* algorithm applied to an n *customer demand classes* case. Therefore, the conjecture should hold for any feasible solution to an n *customer demand classes*.

It is clear from the *Optimal Greedy Line Search-n* algorithm that while keeping $\Delta^{1*}, \ldots, \Delta^{n-1*}$ fixed, any selection of $\Delta^n < \Delta^{n*}$ will violate the c^n constraint. Therefore, for $\Delta^n = \Delta^{n*} - 1$, the system will be infeasible for the corresponding threshold values $(S^{1*} - 1, S^{2*} - 1, \ldots, S^{n*} - 1)$.

Now, let us assume that $\exists\, (S^1, S^2, \ldots, S^n)$ such that $S^1 < S^{1*}$ and satisfies all fill-rate constraints. Let $u = S^1 - S^{1*}$ where $u \geqslant 1$.

Then $(S^{1*} - 1, S^2, \ldots, S^n)$ is also feasible since $S^1 \leqslant S^{1*} - 1$ and we are providing more inventory to be used by the system.

From Equation (15),

$$S^1 - S^{m+1} \geqslant \sum_{i=1}^{m} \Delta^{i*} \qquad \text{for} \;\; 1 \leqslant m \leqslant n-1.$$

Therefore,

$$S^{1*} - S^{m+1} \geqslant \sum_{i=1}^{m} \Delta^{i*} + u \;=\; S^{1*} - S^{m+1*} + u \qquad \text{for} \;\; 1 \leqslant m \leqslant n-1.$$

$$\Longrightarrow \quad S^{m+1*} - S^{m+1} \geqslant u, \quad u \geqslant 1.$$

Consider $(S^{1*} - 1, S^{2*} - 1, \ldots, S^{n*} - 1)$ which is infeasible due to the violation of c^n constraint. Then $(S^{1*} - 1, S^2, \ldots, S^n)$ is also infeasible since for a given system stock, we are allocating more units to be used by *type-1*, *type-2*,..., *type-n-1* customers which leads to a lower service rate for *type-n* customer that was already receiving lower service than c^n.

But this contradicts the previous conclusion that $(S^{1*}-1, S^2, \ldots, S^n)$ is feasible. Therefore, there cannot be any $(S^1, S^2, \ldots, S^n)$ such that $S^1 < S^{1*}$ and satisfies all fill-rate constraints.

$\blacksquare$

5 Numerical Analysis

In this section, as we did for the two priority class case, we compare the performance of the proposed inventory rationing policy with the current industry practices, namely *round-up* and *separate stock* strategies. In Table 1, we analyze the situation for various service contracts. We want to observe the trade off pattern between different strategies. Service contracts are based on fill-rate constraints for each customer class. Corresponding total inventory levels are presented for each policy. In Table 2, for the same scenarios presented in Table 1, we present the corresponding threshold levels for each priority class as well as the actual fill-rates provided to them.

Table 1: Optimal target stock levels under different stocking policies

				Fill-rate Constraints			Target Stock Level		
λ^s	λ^g	λ^p	T	c^s	c^g	c^p	Threshold rat.	Sep. stock	Round-up
3.0	1.5	1.0	10	60%	80%	95%	**60**	67	68
3.0	1.5	1.0	4	60%	80%	95%	**26**	32	31
3.0	1.5	1.0	2	60%	80%	95%	**15**	18	18
5.0	3.0	3.5	4	60%	85%	90%	**51**	59	56
2.8	0.56	0.28	10	60%	80%	90%	**40**	45	45

Table 2: Actual provided fill-rates under Threshold Rationing Policy

				Threshold levels			Provided Fill-rates		
λ^s	λ^g	λ^p	T	S^s	S^g	S^p	β^s	β^g	β^p
3.0	1.5	1.0	10	60	2	1	63.9 %	86.0 %	97.9 %
3.0	1.5	1.0	4	26	2	1	63.7 %	87.2 %	98.3 %
3.0	1.5	1.0	2	15	2	1	68.9 %	90.4 %	98.9 %
5.0	3.0	3.5	4	41	2	0	65.2 %	92.3 %	92.3 %
2.8	0.56	0.28	10	40	1	0	64.5 %	93.3 %	93.3 %

In Table 1, for the first three scenarios different order lead times are considered in order to see their effect on the target system stock. The first scenario has the longest lead time. In this case, the separate stock policy is superior to the round-up policy. This implies that the benefit from the pooling effect is diminished by providing 95% fill-rate to all three customer types for such a long lead time. On the other hand, for T=4 we can see that round-up policy becomes advantageous over the separate stock strategy while both policies give the same target stock level for T=2. However, as can easily be seen, the threshold rationing policy is superior to both of the other strategies in all the scenarios.

In Table 2, we provide the corresponding actual fill-rates that each priority class customer is experiencing along with the threshold levels obtained by the *Optimal Line Search-3* algorithm for threshold rationing policy. One can see that in all cases, the actual fill-rates are greater than what those customers have contracted for as service agreements. The fourth and fifth scenarios have very interesting results. Both of these cases have a S^p threshold level of zero, which means we do not need to allocate additional inventory for *platinum* customers over the *gold* ones because *gold* customers are already receiving fill-rates greater than the *platinum* fill-rate constraints. Additionally, when all the cases are considered, we can see that allocation of even a few units for higher priority customers according to the proposed policy has significant impacts on the fill-rates provided. For example, for the first scenario, demands from all three customer classes are satisfied on a first come-first serve basis as long as on-hand stock is greater than 2, where the target system stock is 60. The *silver* fill-rate is 63.9% for this setting. Those 2 units which are allocated for the use of *gold* and *platinum* customers serve as a great protection for the incoming demands of those higher priority class. This protection, in combination with the backorder clearing rules, increases the fill-rates from 63.9% to 86.0%. Furthermore, an additional protection of 1 unit for *platinum* customers increases its fill-rate all the way up to 97.9%.

5.1 Comparison with the Existing Heuristics

The only published heuristic related to our model, known to us, is that presented by Dekker et al. [1] and Deshpande et al. [2]. Both consider only fixed lead time for replenishment orders under the two priority demand class setting. Although Deshpande et al. [2] use a different reasoning for the heuristic approach, they obtain essentially the same result. Therefore, in this part of the analysis, we will focus on the performance of heuristics with respect to our exact analysis for *two priority* demand class setting.

The idea is quite simple. In order to determine the service level for both customer classes, Dekker et al [1] use the following reasoning. At an arbitrary point in time t, inventory position is equal to S^s. By time $t+T$, where T is the lead time duration, all the outstanding orders will have arrived. Therefore, stockout for *silver* customers will occur if and only if the total demand during $[t, t+T)$ is greater than or equal to $S^s - S^g$. Total demand during a lead time is Poisson distributed with parameter $(\lambda^s + \lambda^g)T$; therefore the fill-rate for *silver* customers is given by

$$\beta^s = \sum_{i=0}^{S^s - S^g - 1} \frac{e^{-(\lambda^s+\lambda^g)T}[(\lambda^s+\lambda^g)T]^i}{i!} \tag{16}$$

On the other hand, in order to calculate the service level for *gold* customers, the *hitting time* concept of Nahmias and Demmy [3] is used. H is denoted as the non-negative random variable representing the time until the first $S^s - S^g$ demands have arrived. Then, at time $t+T$, a stockout for *gold* will occur if $H < T$ and the *gold* demand during $[t+H, t+T)$ is greater than or equal to S^g. Since total demand process is a Poisson Process with parameter $\lambda^s + \lambda^g$, H is Erlang-$(S^s - S^g)$ distributed with parameter $\lambda^s + \lambda^g$.

Conditioning on H, fill-rate for *gold* customers can be obtained by

$$\beta^g = 1 - \int_0^T (\lambda^s + \lambda^g)^{S^s - S^g} \frac{y^{S^s - S^g - 1}}{(S^s - S^g - 1)!} \; e^{-(\lambda^s + \lambda^g)y}$$

$$\cdot \left\{ 1 - \sum_{i=0}^{S^g - 1} \frac{e^{-\lambda^g (T-y)} [\lambda^g (T-y)]^i}{i!} \right\} \mathrm{d}y \tag{17}$$

On the other hand, in order to get a tractable solution, Deshpande et al. [2] analyze the same system with an alternative backorder clearing mechanism and then use the threshold values obtained from this setting in the original system. Interestingly, their alternative clearing mechanism turns out to be same system as given by Dekker et al. [1].

Deshpande et al. [2] also provides a simplified version of the expression in Equation (17) by the following reasoning. Given that a demand arrival occurs, the probability of it being a *gold* demand is $p^g = \frac{\lambda^g}{\lambda^s + \lambda^g}$. In order to observe a stockout for *gold* customers at time $t + T$, first, total demand during $[t, t+T)$ should be at least S^s and then *total* gold arrivals during $[t+H, t+T)$ should be greater than or equal to S^g. In addition, provided that total demand during some interval is n, the probability of having exactly n_i *gold* demands is a simple binomial expression,

$$\text{Binom}\,(p^g; n; n_i) \; = \; \frac{n!}{n!(n-n_i)!}(p^g)^{n_i}(1-p^g)^{n-n_i}.$$

Then, Equation (17) is equivalent to

$$\beta^g \;\; = 1 - \sum_{x=S^s}^{\infty} \sum_{z=S^g}^{x-S^s+S^g} \text{Binom}\,(p^g; x - S^s + S^g; z) \cdot \text{Poisspdf}[x, (\lambda^s + \lambda^g)T].$$

However, there are problems with this reasoning, hence it is only an approximation. The first reason is that it does not take into account the proposed threshold rationing policy; it assumes that the first $S^s - S^g$ demands are satisfied immediately regardless of their type. Secondly, it ignores the fact that the sequence of events affects system state at a particular point in time. These can be explained by the following examples:

Example 1: Let us consider the system with $S^s = 10, S^g = 5$, and at time t, $OH = 4$, and there are no backorders of any type. This also implies that there are 6 units in resupply. Also let us assume that there is a sequence of 5 *silver* demands and the $[S^s - S^g = 5]^{\text{th}}$ demand occurs at time $t + H$. This is followed by a sequence of 6 *gold* demands, and 6 units receipt from resupply, in that order. Note that those 6 units are the ones that are in resupply at time t. According to this information and the proposed rationing policy, at time $t+T$, OH inventory will be equal to 4, which implies that there is no stockout situation for *gold* customers. System state is $(OH, B^s, B^g, R) = (4, 5, 0, 11)$.

On the other hand, according to the proposed heuristic above, since total demand during $[t, t+T)$ is 11, which is more than S^s, and since there are more than $S^g = 5$ *gold* demands during $[t+H, t+T)$, there should be a stockout situation for *gold* customers at time $t+T$. However, this contradicts the fact that OH stock at time $t+T$ is 4.

Example 2: Let us consider the same initial state at time t as the previous example $(OH, B^s, B^g, R) = (4, 0, 0, 6)$. Now consider the following events in their respective order: sequences of 5 *silver* demands, receipts of 3 units from resupply, 6 *gold demands*, and receipts of 3 units from resupply. As a result, system state at time $t+T$ is $(OH, B^s, B^g, R) = (2, 3, 0, 11)$, which is different than the previously obtained state $(OH, B^s, B^g, R) = (4, 5, 0, 11)$. However, during $[t, t+T)$, in both of the examples, there are a total of 5 *silver* demands, 6 *gold* demands, and receipts of 6 units from resupply. As can be seen here, the sequence of events affects the system state, a fact ignored by the heuristic approach.

Now, let us refer to the heuristic results provided by Dekker et al. [1]. The information in Table 3 is taken from their paper.

Table 3: Numerical examples due by Dekker et al. (1998)

S^s	S^g	λ^s	λ^g	T	β^g exact	β^g approx.
2	1	2.727	0.273	0.5	96.5 %	93.7 %
3	2	5.455	0.545	0.5	99.7 %	98.3 %
4	1	1.5	1.5	0.5	96.8 %	96.2 %
10	2	6.0	6.0	0.5	98.1 %	96.3 %

For all the cases shown in Table 3, the lead time T is fixed at a half-period, and, arguably the heuristic approach provides a reasonably good approximation of the *gold* fill-rate. In order to investigate the accuracy of the heuristic approach, we implement a controlled experiment for which $\lambda^s = \lambda^g = 1.5$ for all scenarios.

Table 4: Comparison of the heuristic with an exact solution

S^s	S^g	λ^s	λ^g	T	β^g exact	β^g approx.
4	2	1.5	1.5	0.5	98.6 %	97.8 %
5	2	1.5	1.5	1	96.1 %	92.5 %
9	2	1.5	1.5	2	96.6 %	92.9 %
11	1	1.5	1.5	3.5	77.0 %	62.2 %
18	2	1.5	1.5	5	94.2 %	85.2 %
26	3	1.5	1.5	8	96.0 %	79.7 %
32	2	1.5	1.5	10	91.0 %	73.1 %

We can see from the results presented in Table 4, that as long as lead time is sufficiently short, the heuristic provides a reasonable approximation. However, as soon as the lead time exceeds 3.5 periods, we observe a significant deviation from the true fill-rate figures.

Furthermore, Deshpande et al. [2] also perform a set of experiments for different values of λ^s/λ^g ratio for a given lead time duration, and obtain what appear to be nice results for their optimization problem. However, as we can observe from the above results that the heuristic approach does not provide a good approximation in general. This is mainly due to the fact that, not only demand rates ratio λ^s/λ^g and lead times are important but also $\lambda^s T$ and $\lambda^g T$ values are important.

References

[1] Dekker, R., M. J. Kleijn, P. J. de Rooij. 1998. A spare parts stocking policy based on equipment criticality. *International Journal of Production Economics* 56-57: 69-77

[2] Deshpande, V., M. A. Cohen, K. Donohue. 2003. A threshold rationing policy for service differentiated demand classes. *Management Science* 49(6): 683-703

[3] Nahmias, S., W. S. Demmy. 1981. Operating characteristics of an inventory system with rationing. *Management Science* 27(11): 1236-1245

[4] Tijms, H.C. 1986. Stochastic Modelling and Analysis: A Computational Approach. John Wiley & Sons, Great Britain, 1986.

[5] Vicil, O., P. Jackson. 2006. An Exact Optimal Solution to a Threshold Inventory Rationing Model for Two Priority Demand Classes. Technical Report No. 1439, School of ORIE, Cornell University, Ithaca, NY.

6 APPENDICES

Appendix 1

Theorem 3.1: *In steady state, the following equation holds:*

$$\pi_{(1,0)^{s\oplus g}} \cdot \mu \cdot R_{(1,0)^{s\oplus g}} \quad = \quad \pi_{(0,0)^{s\oplus g}} \cdot \lambda^p \cdot [X_1^{(0)}]^{s\oplus g} \tag{18}$$

where $\quad [X_1^{(0)}]^{s\oplus g} = P_{(1,0)^{s\oplus g}}\big[\tau_{(0,0)^{s\oplus g}}(1) < min\{\tau_{(0,1)^{s\oplus g}}(1), \tau_{(0,1)}(1)\}\big].$

Proof: We are going to use similar arguments to those used in two priority class setting.

Recall that we assumed the limiting probabilities of the levels, computed using Palm's Theorem, that contain states $(0,0)$, and $(0,1)$ are non-negligible. This guarantees that steady state probabilities for those states are non-negligible. Hence, state $(0,1)$ is recurrent.

Let us define a cycle as the sequence of transitions starting from state (p,r) until re-entering it for the first time. Recall that the expected number of direct transitions from state (i,j) to state (s,t) for a defined cycle will be given by:

$$\begin{aligned}
E_{(p,r)}[Z_{(i,j),(s,t)}] &= \sum_{m=1}^{\infty} P_{(p,r)}[\tau_{(p,r)}(1) > m+1, \xi_m = (i,j), \xi_{m+1} = (s,t)] \\
&= \sum_{m=1}^{\infty} \Big\{ P_{(p,r)}[\xi_{m+1} = (s,t) \quad | \quad \xi_m = (i,j), \tau_{(p,r)}(1) > m+1] \\
&\qquad \cdot P_{(p,r)}[\xi_m = (i,j), \tau_{(p,r)}(1) > m+1] \Big\} \\
&= \sum_{m=1}^{\infty} \Big\{ P[\xi_{m+1} = (s,t) \quad | \quad \xi_m = (i,j), \tau_{(p,r)}(1) > m+1] \\
&\qquad \cdot P_{(p,r)}[\xi_m = (i,j), \tau_{(p,r)}(1) > m+1] \Big\}
\end{aligned}$$

$$= \quad P_{(i,j),(s,t)} \sum_{m=1}^{\infty} P_{(p,r)}[\xi_m = (i,j), \tau_{(p,r)}(1) > m+1]$$

$$= \quad P_{(i,j),(s,t)} \cdot \nu_{(i,j)} \tag{19}$$

Thus the expected number of transitions from state $(1,0)^{s\oplus g}$ to state $(0,0)^{s\oplus g}$ in a cycle, defined as the sequence of transitions starting from state $(0,1)$ until re-entering it for the first time, will be:

$$\begin{aligned}
E_{(0,1)}[Z_{(1,0)^{s\oplus g},(0,0)^{s\oplus g}}] &= \nu_{(1,0)^{s\oplus g}} \cdot P_{(1,0)^{s\oplus g},(0,0)^{s\oplus g}} \qquad (20)\\
&= \sum_{m=1}^{\infty} P_{(0,1)}[\tau_{(0,1)}(1) > m+1, \xi_m = (1,0)^{s\oplus g}, \xi_{m+1} = (0,0)^{s\oplus g}]\\
&= \sum_{m=1}^{\infty} \Big\{ P_{(0,1)}[\tau_{(0,1)}(1) > m+1, \xi_1 = (0,0), \xi_m = (1,0)^{s\oplus g}, \xi_{m+1} = (0,0)^{s\oplus g}]\\
&\quad + P_{(0,1)}[\tau_{(0,1)}(1) > m+1, \xi_1 \neq (0,0), \xi_m = (1,0)^{s\oplus g}, \xi_{m+1} = (0,0)^{s\oplus g}] \Big\}\\
&= \sum_{m=1}^{\infty} P_{(0,1)}[\tau_{(0,1)}(1) > m+1, \xi_1 = (0,0), \xi_m = (1,0)^{s\oplus g}, \xi_{m+1} = (0,0)^{s\oplus g}],
\end{aligned}$$

since $P_{(0,1)}[\tau_{(0,1)}(1) > m+1, \xi_1 \neq (0,0), \xi_m = (1,0)^{s\oplus g}, \xi_{m+1} = (0,0)^{s\oplus g}] = 0$.

In addition, during a cycle at some point in time, the process should have passed through states $(0,0)^{s\oplus g}$ and $(1,0)^{s\oplus g}$ sequentially for any direct visit from state $(1,0)^{s\oplus g}$ to $(0,0)^{s\oplus g}$. In addition, in order to prevent multiple counts for the same event, we condition the event after passing through states $(0,0)^{s\oplus g}$ and $(1,0)^{s\oplus g}$ sequentially as follows:

$$E_{(0,1)}[Z_{(1,0)^{s\oplus g},(0,0)^{s\oplus g}}] = \nu_{(1,0)^{s\oplus g}} \cdot P_{(1,0)^{s\oplus g},(0,0)^{s\oplus g}}$$

$$= \sum_{m=1}^{\infty}\sum_{k=0}^{m-1} P_{(0,1)}\Big[\tau_{(0,1)}(1) > m+1, \xi_1 = (0,0), \xi_k = (0,0)^{s\oplus g}, \xi_{k+1} = (1,0)^{s\oplus g},$$

$$\big(\tau_{(0,1)^{s\oplus g}}(1) > m+1 \mid \xi_{k+1} = (1,0)^{s\oplus g}\big), \xi_m = (1,0)^{s\oplus g}, \xi_{m+1} = (0,0)^{s\oplus g}\Big]$$

$$= \sum_{m=1}^{\infty}\sum_{k=0}^{m-1} \Big\{P_{(0,1)}\big[\tau_{(0,1)}(1) > m+1,\ \big(\tau_{(0,1)^{s\oplus g}}(1) > m+1 \mid \xi_{k+1} = (1,0)^{s\oplus g}\big),$$

$$\xi_m = (1,0)^{s\oplus g}, \xi_{m+1} = (0,0)^{s\oplus g} \mid \tau_{(0,1)}(1) > k+1,$$

$$\xi_1 = (0,0),\ \xi_k = (0,0)^{s\oplus g},\ \xi_{k+1} = (1,0)^{s\oplus g}\big]$$

$$\cdot P_{(0,1)}\big[\tau_{(0,1)}(1) > k+1, \xi_1 = (0,0), \xi_k = (0,0)^{s\oplus g}, \xi_{k+1} = (1,0)^{s\oplus g}\big]\Big\}.$$

By the Markov property applied to the conditional probability above, we have:

$$E_{(0,1)}[Z_{(1,0)^{s\oplus g},(0,0)^{s\oplus g}}] \quad =$$

$$= \sum_{m=1}^{\infty}\sum_{k=0}^{m-1} \Big\{P\big[\tau_{(0,1)}(1) > m+1,\ \big(\tau_{(0,1)^{s\oplus g}}(1) > m+1 \mid \xi_{k+1} = (1,0)^{s\oplus g}\big),$$

$$\xi_m = (1,0)^{s\oplus g}, \xi_{m+1} = (0,0)^{s\oplus g} \mid \tau_{(0,1)}(1) > k+1, \xi_{k+1} = (1,0)^{s\oplus g}\big]$$

$$\cdot P_{(0,1)}\big[\tau_{(0,1)}(1) > k+1, \xi_1 = (0,0), \xi_k = (0,0)^{s\oplus g}, \xi_{k+1} = (1,0)^{s\oplus g}\big]\Big\}$$

$$= \sum_{m=1}^{\infty}\sum_{k=0}^{m-1} \Big\{P_{(1,0)^{s\oplus g}}\big[\tau_{(0,1)}(1) > m-k,\ \tau_{(0,1)^{s\oplus g}}(1) > m-k,\ \xi_{m-k-1} = (1,0)^{s\oplus g},$$

$$\xi_{m-k} = (0,0)^{s\oplus g}\big]$$

$$\cdot P_{(0,1)}\big[\tau_{(0,1)}(1) > k+1, \xi_1 = (0,0), \xi_k = (0,0)^{s\oplus g}, \xi_{k+1} = (1,0)^{s\oplus g}\big]\Big\}.$$

Because state $(0,1)$ is recurrent, the expression is finite and so by Fubini's Theorem we may reverse the order of summation,

$$E_{(0,1)}[Z_{(1,0)^{s\oplus g},(0,0)^{s\oplus g}}] \quad =$$

$$= \sum_{k=0}^{\infty} \sum_{m=k+1}^{\infty} \Big\{ P_{(1,0)^{s\oplus g}}\big[\tau_{(0,1)}(1) > m-k,\ \tau_{(0,1)^{s\oplus g}}(1) > m-k,$$

$$\xi_{m-k-1} = (1,0)^{s\oplus g}, \xi_{m-k} = (0,0)^{s\oplus g}\big]$$

$$\cdot P_{(0,1)}\big[\tau_{(0,1)}(1) > k+1, \xi_1 = (0,0), \xi_k = (0,0)^{s\oplus g}, \xi_{k+1} = (1,0)^{s\oplus g}\big]\Big\}$$

$$= \sum_{k=0}^{\infty} \Bigg\{ P_{(0,1)}\big[\tau_{(0,1)}(1) > k+1, \xi_1 = (0,0), \xi_k = (0,0)^{s\oplus g}, \xi_{k+1} = (1,0)^{s\oplus g}\big]$$

$$\cdot \sum_{m=k+1}^{\infty} P_{(1,0)^{s\oplus g}}\big[\tau_{(0,1)}(1) > m-k,\ \tau_{(0,1)^{s\oplus g}}(1) > m-k,$$

$$\xi_{m-k-1} = (1,0)^{s\oplus g}, \xi_{m-k} = (0,0)^{s\oplus g}\big]\Bigg\}.$$

A change in variable, $h = m - k$, results in:

$$E_{(0,1)}[Z_{(1,0)^{s\oplus g},(0,0)^{s\oplus g}}] =$$

$$= \sum_{k=0}^{\infty} \Bigg\{ P_{(0,1)}\big[\tau_{(0,1)}(1) > k+1, \xi_1 = (0,0), \xi_k = (0,0)^{s\oplus g}, \xi_{k+1} = (1,0)^{s\oplus g}\big]$$

$$\cdot \sum_{h=1}^{\infty} P_{(1,0)^{s\oplus g}}\big[\tau_{(0,1)}(1) > h,\ \tau_{(0,1)^{s\oplus g}}(1) > h,\ \xi_{h-1} = (1,0)^{s\oplus g}, \xi_h = (0,0)^{s\oplus g}\big]\Bigg\}.$$

The inner summation is equal to something with which we are familiar:

$$\sum_{h=1}^{\infty} P_{(1,0)^{s\oplus g}}\big[\tau_{(0,1)}(1) > h,\ \tau_{(0,1)^{s\oplus g}}(1) > h,\ \xi_{h-1} = (1,0)^{s\oplus g}, \xi_h = (0,0)^{s\oplus g}\big]$$

$$= P_{(1,0)^{s\oplus g}}\big[\tau_{(0,0)^{s\oplus g}}(1) < \min\{\tau_{(0,1)}(1),\ \tau_{(0,1)^{s\oplus g}}(1)\}\big]. \tag{21}$$

In addition, equation (21) does not depend on k. Consequently, we can write:

$$E_{(0,1)}[Z_{(1,0)^{s\oplus g},(0,0)^{s\oplus g}}] =$$

$$
\begin{aligned}
&= P_{(1,0)^{s\oplus g}}\big[\tau_{(0,0)^{s\oplus g}}(1) < \min\{\tau_{(0,1)}(1),\ \tau_{(0,1)^{s\oplus g}}(1)\}\big] \\
&\quad \cdot\left\{\sum_{k=0}^{\infty} P_{(0,1)}\big[\tau_{(0,1)}(1) > k+1, \xi_1 = (0,0), \xi_k = (0,0)^{s\oplus g}, \xi_{k+1} = (1,0)^{s\oplus g}\big]\right\} \\
&= P_{(1,0)^{s\oplus g}}\big[\tau_{(0,0)^{s\oplus g}}(1) < \min\{\tau_{(0,1)}(1),\ \tau_{(0,1)^{s\oplus g}}(1)\}\big] \cdot E_{(0,1)}[Z_{(0,0)^{s\oplus g},(1,0)^{s\oplus g}}] \\
&= P_{(1,0)^{s\oplus g}}\big[\tau_{(0,0)^{s\oplus g}}(1) < \min\{\tau_{(0,1)}(1),\ \tau_{(0,1)^{s\oplus g}}(1)\}\big] \cdot \nu_{(0,0)^{s\oplus g}} \cdot P_{(0,0)^{s\oplus g},(1,0)^{s\oplus g}}.
\end{aligned}
\tag{22}
$$

If we combine equations (20) and (22), we get:

$$
\begin{aligned}
\nu_{(1,0)^{s\oplus g}} \cdot P_{(1,0)^{s\oplus g},(0,0)^{s\oplus g}} &= \\
&= \nu_{(0,0)^{s\oplus g}} \cdot P_{(0,0)^{s\oplus g},(1,0)^{s\oplus g}} \cdot P_{(1,0)^{s\oplus g}}\big[\tau_{(0,0)^{s\oplus g}}(1) < \min\{\tau_{(0,1)}(1),\ \tau_{(0,1)^{s\oplus g}}(1)\}\big].
\end{aligned}
\tag{23}
$$

Dividing both sides of equation (23) by the expected cycle length $E_{(0,1)}\tau_{(0,1)}(1)$, we get:

$$
\begin{aligned}
\tilde{\pi}_{(1,0)^{s\oplus g}} \cdot P_{(1,0)^{s\oplus g},(0,0)^{s\oplus g}} &= \\
&= \tilde{\pi}_{(0,0)^{s\oplus g}} \cdot P_{(0,0)^{s\oplus g},(1,0)^{s\oplus g}} \cdot P_{(1,0)^{s\oplus g}}\big[\tau_{(0,0)^{s\oplus g}}(1) < \min\{\tau_{(0,1)}(1),\ \tau_{(0,1)^{s\oplus g}}(1)\}\big].
\end{aligned}
\tag{24}
$$

In addition, the relationship between the DTMC steady state probability $\tilde{\pi}_i$(based on jump probability) in the imbedded M.C. and CTMC steady state probability π_i is given by:

$$
\pi_i = \frac{\tilde{\pi}_i \cdot \omega_i}{\sum_j \tilde{\pi}_j \cdot \omega_j}
$$

where $\omega(i)$ is the mean time spent in state i.

Using the above relation in Equation (24) results in:

$$
\begin{aligned}
\pi_{(1,0)^{s\oplus g}} \cdot \mu \cdot R_{(1,0)^{s\oplus g}} &= \\
&= \pi_{(0,0)^{s\oplus g}} \cdot \lambda^p \cdot P_{(1,0)^{s\oplus g}}\big[\tau_{(0,0)^{s\oplus g}}(1) < \min\{\tau_{(0,1)}(1),\ \tau_{(0,1)^{s\oplus g}}(1)\}\big].
\end{aligned}
$$

■

Appendix 2

Theorem 3.2: *In steady state, the following equation holds for $j \geqslant 1$:*

$$
\pi_{(1,j)^{s\oplus g}} \mu R_{(1,j)^{s\oplus g}} = \pi_{(0,j)^{s\oplus g}} \lambda^p X_1^{(j)} + \sum_{i=1}^{\infty} \pi_{(i,j-1)^{s\oplus g}}\ \lambda^{s+g}\ [X_i^{(j)}]^{s\oplus g} \tag{25}
$$

$$
\text{where} \quad [X_i^{(j)}]^{s\oplus g} = P_{(i,j)^{s\oplus g}}\big[\tau_{(0,j)^{s\oplus g}}(1) < min\{\tau_{(0,j+1)^{s\oplus g}}(1), \tau_{(0,j+1)}(1)\}\big].
$$

Proof: We provide a brief sketch of the proof since most of the arguments follow similar analysis to the two priority demand class setting. Therefore, proofs for same arguments will be omitted, and the results will be written directly. Let us define the cycle as the sequence of transitions starting from state $(0, j+1)$ until re-entering it for the first time (provided that $(0, j+1)$ is recurrent).

Due to the structure of the transition diagram, before each one-step transition from $(1, j)^{s\oplus g}$ to $(0, j)^{s\oplus g}$, there must have been a matching one step transition from one of the set $\{(0, j)^{s\oplus g}\} \cup \{(i, j-1)^{s\oplus g} :\ i \geqslant 1\}$ into the set $\{(i, j)^{s\oplus g} :\ i \geqslant 1\}$. In other words, for each one step transition from $(1, j)^{s\oplus g}$ to $(0, j)^{s\oplus g}$, previously there must had been exactly one transition of the form: $(0, j)^{s\oplus g} \longrightarrow (1, j)^{s\oplus g}$ or $(i, j-1)^{s\oplus g} \longrightarrow (i, j)^{s\oplus g}$ for some $i \geqslant 1$, ***which is not followed by a visit to state*** $(0, j+1)^{s\oplus g}$ ***in a cycle.*** This is because, before each one-step transition from $(1, j)^{s\oplus g}$

to $(0,j)^{s\oplus g}$, there may be several transitions from one of the set $\{(0,j)^{s\oplus g}\}\cup\{(i,j-1)^{s\oplus g} : i \geqslant 1\}$ into the set $\{(i,j)^{s\oplus g} : i \geqslant 1\}$ due to the usage of the bridge between $(0,j)^{s\oplus g}$ and $(0,j+1)^{s\oplus g}$.

As a result,we have the following form:

$$\begin{aligned} E_{(0,j+1)}[Z_{(1,j)^{s\oplus g},(0,j)^{s\oplus g}}] &= \nu_{(1,j)^{s\oplus g}}\, P_{(1,j)^{s\oplus g},(0,j)^{s\oplus g}} \\ &= \sum_{m=1}^{\infty} P_{(0,j+1)}\left[\tau_{(0,j+1)}(1) > m+1, \xi_1 = (0,j), \xi_m = (1,j)^{s\oplus g}, \xi_{m+1} = (0,j)^{s\oplus g}\right] \\ &= \mathrm{A}_1 + \mathrm{B}_1 \end{aligned} \tag{26}$$

where

$$\begin{aligned} \mathrm{A}_1 = \sum_{m=1}^{\infty}\sum_{k=0}^{m-1}\Bigg\{P_{(0,j+1)}\Big[&\tau_{(0,j+1)}(1) > m+1, \xi_1 = (0,j), \xi_k = (0,j)^{s\oplus g}, \\ &\xi_{k+1} = (1,j)^{s\oplus g}, \big(\tau_{(0,j+1)^{s\oplus g}}(1) > m+1 \mid \xi_{k+1} = (1,j)^{s\oplus g}\big), \\ &\xi_m = (1,j)^{s\oplus g}, \xi_{m+1} = (0,j)^{s\oplus g}\Big]\Bigg\}, \end{aligned}$$

and

$$\begin{aligned} \mathrm{B}_1 = \sum_{i=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{m-1}\Bigg\{P_{(0,j+1)}\Big[&\tau_{(0,j+1)}(1) > m+1, \xi_1 = (0,j), \xi_k = (i,j-1)^{s\oplus g}, \\ &\xi_{k+1} = (i,j)^{s\oplus g}, \big(\tau_{(0,j+1)^{s\oplus g}}(1) > m+1 \mid \xi_{k+1} = (i,j)^{s\oplus g}\big), \\ &\xi_m = (1,j)^{s\oplus g}, \xi_{m+1} = (0,j)^{s\oplus g}\Big]\Bigg\}. \end{aligned}$$

The above expressions for A_1 and B_1 have the same forms as in the proof of Theorem 3.1. Using similar analysis, we can obtain

$$\begin{aligned} \mathrm{A}_1 = \ & \nu_{(0,j)^{s\oplus g}} \cdot P_{(0,j)^{s\oplus g},(1,j)^{s\oplus g}} \\ & \cdot P_{(1,j)^{s\oplus g}}\left[\tau_{(0,j)^{s\oplus g}}(1) < \min\{\tau_{(0,j+1)}(1),\ \tau_{(0,j+1)^{s\oplus g}}(1)\}\right], \end{aligned} \tag{27}$$

and

$$
\begin{aligned}
\mathrm{B}_1 &= \sum_{i=1}^{\infty} \nu_{(i,j-1)^{s\oplus g}} \cdot P_{(i,j-1)^{s\oplus g},(i,j)^{s\oplus g}} \\
&\quad \cdot P_{(i,j)^{s\oplus g}}\big[\tau_{(0,j)^{s\oplus g}}(1) < \min\{\tau_{(0,j+1)}(1),\ \tau_{(0,j+1)^{s\oplus g}}(1)\}\big].
\end{aligned}
\tag{28}
$$

Combining Equations (26), (27) and (28),

$$
\begin{aligned}
\nu_{(1,j)^{s\oplus g}}\, P_{(1,j)^{s\oplus g},(0,j)^{s\oplus g}} &= \\
&= \nu_{(0,j)^{s\oplus g}} \cdot P_{(0,j)^{s\oplus g},(1,j)^{s\oplus g}} \\
&\quad \cdot P_{(1,j)^{s\oplus g}}\big[\tau_{(0,j)^{s\oplus g}}(1) < \min\{\tau_{(0,j+1)}(1),\ \tau_{(0,j+1)^{s\oplus g}}(1)\}\big] \\
&\quad + \sum_{i=1}^{\infty} \nu_{(i,j-1)^{s\oplus g}} \cdot P_{(i,j-1)^{s\oplus g},(i,j)^{s\oplus g}} \\
&\quad \cdot P_{(i,j)^{s\oplus g}}\big[\tau_{(0,j)^{s\oplus g}}(1) < \min\{\tau_{(0,j+1)}(1),\ \tau_{(0,j+1)^{s\oplus g}}(1)\}\big].
\end{aligned}
$$

Dividing both sides of the above equation by the expected cycle length $E_{(0,j+1)}\tau_{(0,j+1)}(1)$, and then using the relationship between the DTMC steady state probability $\tilde{\pi}_i$(based on jump probability) in the imbedded M.C. and CTMC steady state probability π_i, we get

$$
\pi_{(1,j)^{s\oplus g}} \mu R_{(1,j)^{s\oplus g}} = \pi_{(0,j)^{s\oplus g}} \lambda^p X_1^{(j)} + \sum_{i=1}^{\infty} \pi_{(i,j-1)^{s\oplus g}}\ \lambda^{s+g}\ [X_i^{(j)}]^{s\oplus g} \tag{29}
$$

where $[X_i^{(j)}]^{s\oplus g} = P_{(i,j)^{s\oplus g}}\big[\tau_{(0,j)^{s\oplus g}}(1) < min\{\tau_{(0,j+1)^{s\oplus g}}(1), \tau_{(0,j+1)}(1)\}\big].$

■

Appendix 3

Proposition 4.1: *For a given resupply rate μ, consider two cases:*

a) *n-customer demand classes with threshold levels $(S^1, S^2, \ldots, S^n)$ and arrival rates $(\lambda^1, \lambda^2, \ldots, \lambda^n)$;*

b) *k, $k \geqslant 2$ customer demand classes with threshold levels $(S^1 - S^{k+1}, S^2 - S^{k+1}, \ldots, S^k - S^{k+1})$ and arrival rates $(\lambda^1, \lambda^2, \ldots, \lambda^{k-1}, \sum_{u=k}^{n} \lambda^u)$.*

Denote state vectors for the former and the latter cases as

$$(OH^a, B^{1^a}, B^{2^a}, \ldots, B^{n^a}, R^a)$$

and

$$(OH^b, B^{1^b}, B^{2^b}, \ldots, B^{(k-1)^b}, B^{(k)+(k+1)+\cdots+(n)^b}, R^b)$$

respectively.

Then for $j = 1, \ldots, S^1 - S^{k+1}$:

$$P_\infty(OH^a = S^{k+1} + j, B^{1^a} = v_1, B^{2^a} = v_2, \ldots, B^{(k-1)^a} = v_{k-1}, 0, 0, \ldots, 0, R^a = v_R)$$

$$= P_\infty(OH^b = j, B^{1^b} = v_1, B^{2^b} = v_2, \ldots, B^{(k-1)^b} = v_{k-1}, 0, R^b = v_R).$$

Hence,

$$P_\infty(OH^a = S^{k+1} + j) = P_\infty(OH^b = j) \quad \text{for } j = 1, \ldots, S^1 - S^{k+1}.$$

Proof: For the n customer demand classes case, let us group the states into one set according to following transformation:

$$(B^{1^a}, B^{2^a}, \ldots, B^{(k-1)^a}, R^a) = \bigcup (OH^a, B^{1^a}, B^{2^a}, \ldots, B^{n^a}, R^a) \tag{30}$$

For $R^a - \sum_{i=1}^{k-1} B^{i^1} \leqslant S^1 - S^{k+1}$, $(B^{1^a}, B^{2^a}, \ldots, B^{(k-1)^a}, R^a)$ corresponds to a unique state

$$(OH^a, B^{1^a}, B^{2^a}, \dots, B^{n^a}, R^a) = (S^1 - R^a + \sum_{i=1}^{k-1} B^{i^1}, B^{1^a}, B^{2^a}, \dots, B^{(k-1)^a}, 0, 0, \dots, 0, R^a),$$

rather than a union of several states.

In addition, let us consider k, $k \geqslant 2$ customer demand classes with threshold levels $(S^1 - S^{k+1}, S^2 - S^{k+1}, \dots, S^k - S^{k+1})$ and arrival rates $(\lambda^1, \lambda^2, \dots, \lambda^{k-1}, \sum_{u=k}^{n} \lambda^u)$. Denote the state vector as

$$\big(OH^b, B^{1^b}, B^{2^b}, \dots, B^{(k-1)^b}, B^{(k)+(k+1)+\dots+(n)^b}, R^b\big)$$

Then there is a one-to-one matching between state space of k, $k \geqslant 2$ customer demand class setting $\big(OH^b, B^{1^b}, B^{2^b}, \dots, B^{(k-1)^b}, B^{(k)+\dots+(n)^b}, R^b\big)$ and
$(B^{1^a}, B^{2^a}, \dots, B^{(k-1)^a}, R^a)$ according to the following transformation:

$$(B^{1^a}, B^{2^a}, \dots, B^{(k-1)^a}, R^a) \quad \xleftrightarrow{1:1} \quad \big(OH^b, B^{1^b}, B^{2^b}, \dots, B^{(k-1)^b}, B^{(k)+\dots+(n)^b}, R^b\big)$$

$$= \Big((S^1 - S^{k+1} - R^a + \sum_{i=1}^{k-1} B^{i^a})^+, B^{1^a}, B^{2^a}, \dots, B^{(k-1)^a}, (R^a - \sum_{i=1}^{k-1} B^{i^a} - S^1 + S^{k+1})^+, R^a\Big). \tag{31}$$

and the dynamics of the system are identical for both cases (i.e. jump probabilities and rates from one state to another). In other words, k, $k \geqslant 2$ customer demand classes with threshold levels $(S^1 - S^{k+1}, S^2 - S^{k+1}, \dots, S^k - S^{k+1})$ and arrival rates $(\lambda^1, \lambda^2, \dots, \lambda^{k-1}, \sum_{u=k}^{n} \lambda^u)$ is the exact characterization of the above system resulted from a grouping operation.

By using Equations (30) and (31),

$$\big(OH^b, B^{1^b}, B^{2^b}, \dots, B^{(k-1)^b}, B^{(k)+(k+1)+\dots+(n)^b}, R^b\big) =$$

$$
\begin{aligned}
&= \Big((S^1 - S^{k+1} - R^a + \sum_{i=1}^{k-1} B^{i^a})^+, B^{1^a}, B^{2^a}, \ldots, B^{(k-1)^a}, \\
&\qquad (R^a - \sum_{i=1}^{k-1} B^{i^a} - S^1 + S^{k+1})^+, R^a\Big) \\
&\equiv (B^{1^a}, B^{2^a}, \ldots, B^{(k-1)^a}, R^a) \\
&= \bigcup (OH^a, B^{1^a}, B^{2^a}, \ldots, B^{n^a}, R^a)
\end{aligned}
$$

Hence,

$$P_\infty\big(OH^b, B^{1^b}, B^{2^b}, \ldots, B^{(k-1)^b}, B^{(k)+(k+1)+\cdots+(n)^b}, R^b\big) =$$

$$
\begin{aligned}
&= P_\infty\Big((S^1 - S^{k+1} - R^a + \sum_{i=1}^{k-1} B^{i^a})^+, B^{1^a}, B^{2^a}, \ldots, B^{(k-1)^a}, \\
&\qquad (R^a - \sum_{i=1}^{k-1} B^{i^a} - S^1 + S^{k+1})^+, R^a\Big) \\
&\equiv P_\infty(B^{1^a}, B^{2^a}, \ldots, B^{(k-1)^a}, R^a) \\
&= \bigcup P_\infty(OH^a, B^{1^a}, B^{2^a}, \ldots, B^{n^a}, R^a).
\end{aligned}
$$

It is true that $R^a - \sum_{i=1}^{k-1} B^{i^a} \leqslant S^1 - S^{k+1}$ for $OH^a > S^{k+1}$, therefore there is a unique state $(OH^a, B^{1^a}, B^{2^a}, \ldots, B^{n^a}, R^a)$ corresponding to state $(B^{1^a}, B^{2^a}, \ldots, B^{(k-1)^a}, R^a)$. It is also true that $OH^b > 0$ for $R^b - \sum_{i=1}^{k-1} B^{i^b} \leqslant S^1 - S^{k+1}$. As a result, we have the following relation for $OH^a > S^{k+1}$,

$$P_\infty\Big\{\big(OH^b, B^{1^b}, B^{2^b}, \ldots, B^{(k-1)^b}, B^{(k)+\cdots+(n)^b}, R^b\big) =$$

$$= \quad (S^1 - S^{k+1} - R^a + \sum_{i=1}^{k-1} B^{i^a}, B^{1^a}, \ldots, B^{(k-1)^a}, 0, R^a)\}$$

$$= \quad P_\infty(B^{1^a}, B^{2^a}, \ldots, B^{(k-1)^a}, R^a)$$

$$= \quad P_\infty\Big\{(OH^a, B^{1^a}, B^{2^a}, \ldots, B^{n^a}, R^a) =$$

$$= (S^1 - R^a + \sum_{i=1}^{k-1} B^{i^a}, B^{1^a}, B^{2^a}, \ldots, B^{(k-1)^a}, 0, 0, \ldots, 0, R^a)\Big\}. \tag{32}$$

Hence, the following holds: Then for $j = 1, \ldots, S^1 - S^{k+1}$:

$$P_\infty(OH^a = S^{k+1} + j, B^{1^a} = v_1, B^{2^a} = v_2, \ldots, B^{(k-1)^a} = v_{k-1}, 0, 0, \ldots, 0, R^a = v_R)$$

$$= P_\infty(OH^b = j, B^{1^b} = v_1, B^{2^b} = v_2, \ldots, B^{(k-1)^b} = v_{k-1}, 0, R^b = v_R).$$

■

Furthermore, for $(OH^a, B^{1^a}, B^{2^a}, \ldots, B^{(k-1)^a} = (S^{k+1}, B^{1^a}, B^{2^a}, \ldots, B^{(k-1)^a}),$

$$R^a - \sum_{i=1}^{k-1} B^{i^a} = S^1 - S^{k+1}.$$

Hence, it is also true that there is a unique state

$$(OH^a, B^{1^a}, B^{2^a}, \ldots, B^{n^a}, R^a) =$$

$$= (S^{k+1}, B^{1^a}, B^{2^a}, \ldots, B^{(k-1)^a}, 0, 0, \ldots, 0, S^1 - S^{k+1} + \sum_{i=1}^{k-1} B^{i^a}),$$

corresponding to the state

$$(B^{1^a}, B^{2^a}, \ldots, B^{(k-1)^a}, R^a) = (B^{1^a}, B^{2^a}, \ldots, B^{(k-1)^a}, S^1 - S^{k+1} + \sum_{i=1}^{k-1} B^{i^a}).$$

In addition, it is also true that for

$$(B^{1^b}, B^{2^b}, \ldots, B^{(k-1)^b}, R^b) = (B^{1^a}, B^{2^a}, \ldots, B^{(k-1)^a}, S^1 - S^{k+1} + \sum_{i=1}^{k-1} B^{i^a}),$$

$$\big(OH^b, B^{1^b}, B^{2^b}, \ldots, B^{(k-1)^b}, B^{(k)+\cdots+(n)^b}, R^b\big) =$$

$$= (0, B^{1^a}, B^{2^a}, \ldots, B^{(k-1)^a}, 0, S^1 - S^{k+1} + \sum_{i=1}^{k-1} B^{i^a}).$$

Therefore, we also have the following:

$$P_\infty\Big\{\big(OH^b, B^{1^b}, B^{2^b}, \ldots, B^{(k-1)^b}, B^{(k)+\cdots+(n)^b}, R^b\big) =$$

$$= (0, B^{1^a}, B^{2^a}, \ldots, B^{(k-1)^a}, 0, S^1 - S^{k+1} + \sum_{i=1}^{k-1} B^{i^a})\Big\}$$

$$\begin{aligned} = \quad & P_\infty\Big\{(OH^a, B^{1^a}, B^{2^a}, \ldots, B^{n^a}, R^a) = \\ & = \quad S^{k+1}, B^{1^a}, B^{2^a}, \ldots, B^{(k-1)^a}, 0, 0, \ldots, 0, S^1 - S^{k+1} + \sum_{i=1}^{k-1} B^{i^a})\Big\}. \end{aligned} \tag{33}$$