AN ANALYTICAL APPROACH TO THE TWO-FLUID THEORY OF PRIGOGINE AND HERMAN FOR TOWN TRAFFIC

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ABSTRACT. We propose an analytical model for vehicular traffic at a fixed-cycle traffic light. It is found that in the long run the numbers of vehicles in the two opposing lanes are linearly dependent, confirming the empirical findings of Prigogine and Herman.

1. INTRODUCTION.

In a pioneering study Prigogine and Herman [5] developed a two-fluid model for town traffic, the two fluids consisting of moving cars and cars stopped as a result of congestion, traffic signals, stop signs or other traffic conditions. The main quantities of interest are the stop time per unit distance and the trip time per unit distance. Their theory was validated empirically in several cities, including the Detroit metropolitan area, London (U.K.), Melbourne (Australia) and Brussels. They found that these quantities are linearly dependent (see also [2])

In this paper we use an analytical approach to verify the findings of these authors. Specifically, we consider a fixed-cycle traffic light and cars moving in two opposing lanes (say, west to east and south to north). The phenomenon of congestion occurs in a sequence of cycles, each cycle consisting of a red phase of length r and a green phase of length g, where r and g are fixed numbers and $c = r + g$ is the cycle time. We denote by $Z_i(t)$ the number of cars in lane i ($i = 1$ is the east bound lane and $i = 2$ is the north bound lane). During a red phase $Z_1(t)$ is the number of cars waiting to cross the intersection (stopped cars), and $Z_2(t)$ is the number of cars in the process of crossing (moving cars). During a green phase the reverse statement holds. We propose a deterministic linear model in which the arrival of new cars at the traffic light and departures from the intersection occur at given constant rates in each lane. Our main objective is to study the nature of long run dependence between $Z_1(t)$ and $Z_2(t)$.

In the above description it is assumed that the red phase includes an amber phase and also the starting time of cars. Thus in practical terms the red phase is 'effectively red' phase. Also, we use the term 'cars' to denote vehicles of all types, although in some models it might be necessary to maintain the distinction between cars and trucks (for example). Finally, all vehicles are assumed to be of zero length, so that they appear as points on the time axis. This assumption is consistent with the fluid approach.

The first analytical model for queues at fixed-cycle traffic lights was proposed by Beckman, McGuire and Winsten [1]. Earlier authors had obtained empirical formulas using experimental data and computer simulation. For a survey of this earlier literature, see [4]. It should

be noted, however, that most such studies were formulated in discrete time, and were concerned with traffic moving in a given lane, the situation being symmetric for the opposing lane. By contrast our model is formulated in continuous time and is a two-dimensional one, emphasizing the interaction between the traffic in the two lanes.

In section 2 the model is formulated, in terms of the equations for $Z_1(t)$ and $Z_2(t)$. These are solved for finite t. As $t \to \infty$, randomness enters the system, for reasons explained there. We derive the joint distribution of the steady state variables Z_1, Z_2. In section 3 the linear relation between Z_1 and Z_2 is established. (Actually, there are two linear relations, that hold with probabilities r/c and g/c). This is in agreement with the findings of Prigogine and Herman [5], although strict comparison is not possible since our model is mathematical, while their model is based on the kinetic theory of multi lane traffic. We also derive the covariance between Z_1 and Z_2, which turns out to be negative.

2. THE TRAFFIC MODEL.

We denote by $Z_i(t)$ the number of vehicles at time t at the intersection crossing or waiting to cross in the the i-th lane ($i = 1, 2$). The model states that

$$Z_1(t) \;=\; \begin{cases} Z_1(n\,c) + \lambda_1(t - n\,c) & \text{if } n\,c < t \le n\,c + r\,, \\ [Z_1(n\,c) + \lambda_1\,r + (\lambda_1 - \mu_1)(t - n\,c - r)]^+ & \text{if } n\,c + r < t \le n\,c + c\,, \end{cases} \qquad (2.1)$$

$$Z_2(t) \;=\; \begin{cases} [Z_2(n\,c) + (\lambda_2 - \mu_2)(t - n\,c)]^+ & \text{if } n\,c < t \le n\,c + r\,, \\ [Z_2(n\,c) + (\lambda_2 - \mu_2)\,r]^+ + \lambda_2\,(t - n\,c - r) & \text{if } n\,c + r < t \le n\,c + c\,. \end{cases} \qquad (2.2)$$

where λ_i is the arrival rate of cars and μ_i the departure rate in lane i ($\lambda_i > 0$, $\mu_i > 0$). As first step in solving for $\{Z_1(t),\, Z_2(t)\}$ from the above equations we solve for the imbedded sequence $\{Z_1(n\,c),\, Z_2(n\,c),\; n = 0, 1, \ldots\}$. For convenience we assume $Z_1(0) = 0$ and $Z_2(0) = \lambda_2\,g$. We have the following.

Theorem 2.1. *Let* $\alpha_1 = \lambda_1\,c - \mu_1\,g$ *and* $\alpha_2 = \lambda_2\,c - \mu_2\,r$ *represent the net inputs during a cycle. Also assume that* $Z_1(0) = 0\,,\;\; Z_2(0) = \lambda_2\,g$.
(i) If $\alpha_i \le 0$*, then* $Z_i(n\,c) = Z_i(0)\,,\quad n = 0, 1, 2, \ldots\,,\quad i = 1, 2$.
(ii) If $\alpha_i > 0$*, then* $Z_i(n\,c) = Z_i(0) + n\,\alpha_i\,,\quad n = 0, 1, 2, \ldots\,,\quad i = 1, 2$.

Proof. From (2.1) we obtain

$$Z_1(n\,c + c) = [Z_1(n\,c) + \alpha_1]^+\,, \quad n = 0, 1, 2, \ldots\,, \qquad (2.3)$$

An induction argument yields the solution

$$Z_1(n\,c) = \max\{Z_1(0) + n\,\alpha_1\,, \quad m\,\alpha_1\,,\; (m = 0, 1, \ldots, n - 1)\}\,.$$

This leads to the desired results for $Z_1(n\,c)$. From (2.2) we obtain for $n = 0, 1, \ldots$

$$Z_2(n\,c + c) = [Z_2(n\,c) + (\lambda_2 - \mu_2)\,r]^+ + \lambda_2\,g\,. \qquad (2.4)$$

The substitution $\hat{Z}_2(n\,c) = Z_2(n\,c) - \lambda_2\,g\,,\;\; n = 0, 1, 2, \ldots$ reduces (2.4) to

$$\hat{Z}_2(n\,c + c) = [\hat{Z}_2(n\,c) + \alpha_2]^+\,, \quad n = 0, 1, 2, \ldots\,.$$

Proceeding as in (2.3) we find that for $n = 0, 1, 2, \ldots$

$$\hat{Z}_2(n\,c) = \begin{cases} \hat{Z}_2(0) & \text{if } \alpha_2 \leq 0\,, \\ \hat{Z}_2(0) + n\,\alpha_2 & \text{if } \alpha_2 > 0\,. \end{cases}$$

This is equivalent to the results for $Z_2(n\,c)$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

It follows from Theorem 2.1 that if $\alpha_i > 0$, then $Z_i(n\,c) \to \infty$ as $n \to \infty$ and equivalently $Z_i(t) \to \infty$ as $t \to \infty$, $i = 1, 2$. Accordingly we consider only the cases $\alpha_i \leq 0$ and proceed to obtain the limits for $Z_i(t)$ as $t \to \infty$. For the purpose of this paper we choose $Z_1(0) = 0$ and $Z_2(0) = \lambda_2\,g$.

The model states that during a red phase $Z_1(t)$ increases from $Z_1(0)$ to $Z_1(0) + \lambda_1\,r$ and then during the following green phase $Z_1(t)$ decreases to the value $Z_1(0)$. The reverse statement holds for $Z_2(t)$. This applies to the individual flow, that is the flow as observed by the driver of any given vehicle. The collective flow is characterized by the steady state behavior. Now as $t \to \infty$, t goes through a succession of red and green phases and the intuition tell us that a fraction r/c of time epochs is red and a fraction g/c is green. Thus in steady state randomness is introduced into the system. To be specific, let $N \equiv N(t) = [t/c]$, so that $N(t)$ is the number of completed cycles up to time t. Clearly $N(t) \to \infty$ as $t \to \infty$. In fact $N(t) \sim t/c$, $t \to \infty$. We are interested in the quantities $t - N(t)\,c$, $N(t)\,c < t \leq N(t)\,c + r$ and $t - N(t)\,c - r$, $N(t)\,c + r < t \leq N(t)\,c + c$. These are the distances between t and the last switching epoch (from green to red and from red to green respectively). From the theory of alternating renewal processes we find that

$$t - N(t)\,c \xrightarrow{d} U_r \quad \text{with probability } \frac{r}{c}\,, \tag{2.5}$$

$$t - N(t)\,c - r \xrightarrow{d} U_g \quad \text{with probability } \frac{g}{c}\,, \tag{2.6}$$

where for any $a > 0$, U_a represents a random variable with uniform density in $(0, a)$ (see [3]).

Theorem 2.2. *Choose $Z_1(0) = 0$, $Z_2(0) = \lambda_2\,g$ and assume that $\alpha_i \leq 0$, $i = 1, 2$. Then*

$$\{Z_1(t),\ Z_2(t)\} \xrightarrow{d} \{Z_1(\infty),\ Z_2(\infty)\}\,,$$

as $t \to \infty$, where

$$\{Z_1(\infty),\ Z_2(\infty)\} \stackrel{d}{=} \begin{cases} \{\lambda_1\,U_r,\ [\lambda_2\,g + (\lambda_2 - \mu_2)\,U_r]^+\} & \text{with probability } \dfrac{r}{c}\,, \\[4mm] \{[\lambda_1\,r + (\lambda_1 - \mu_1)\,U_g]^+,\ \lambda_2\,U_g\} & \text{with probability } \dfrac{g}{c}\,, \end{cases} \tag{2.7}$$

where in the right side we have a mixture of two distributions with weights r/c and g/c.

Proof. With the assumed choice of $Z_1(0)$ and $Z_2(0)$ the equations (2.1) and (2.2) reduce to

$$\{Z_1(t),\ Z_2(t)\} = \begin{cases} \{\lambda_1\,(t - n\,c),\ [\lambda_2\,g + (\lambda_2 - \mu_2)(t - n\,c)]^+\} & \text{if } n\,c < t \leq n\,c + r\,, \\[4mm] \{[\lambda_1\,r + (\lambda_1 - \mu_1)(t - n\,c - r)]^+,\ \lambda_2\,(t - n\,c - r)\} & \text{if } n\,c + r < t \leq n\,c + c\,. \end{cases}$$
$$\tag{2.8}$$

since $\lambda_2\, g + (\lambda_2 - \mu_2)\, r = \lambda_2\, c - \mu_2\, r = \alpha_2 \leq 0$. In view of (2.5) and (2.6) the steady state results now follow from (2.8). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We note that the components $Z_1(\infty)$, $Z_2(\infty)$ are dependent, since they are functions (in distribution) of the random variables U_r and U_g. We now explore the nature of this dependence. We first derive the distribution of $Z_1(\infty)$, $Z_2(\infty)$. To simplify the algebra we denote

$$Z_1 = \frac{Z_1(\infty)}{\lambda_1\, r}\,, \qquad Z_2 = \frac{Z_2(\infty)}{\lambda_2\, g}\,. \qquad (2.9)$$

Also, let

$$\rho_1 = \frac{\lambda_1}{\mu_1}\,, \quad \rho_2 = \frac{\lambda_2}{\mu_2}\,, \quad a_1 = \frac{g}{r}\frac{1 - \rho_1}{\rho_1}\,, \quad a_2 = \frac{r}{g}\frac{1 - \rho_2}{\rho_2}\,.$$

Since $\alpha_1 \leq 0$, $\alpha_2 \leq 0$, we have $\rho_1 \leq g/c$, $\rho_2 \leq r/c$, $a_1 \geq 1$, $a_2 \geq 1$. If $a_1 = a_2$ we have

$$\frac{\alpha_1}{\lambda_1\, r} = \frac{\alpha_2}{\lambda_2\, g}\,,$$

which means that the net inputs over a cycle are the same in both lanes, when they are scaled as in (2.9).

Finally, for two random variables X and Y we write $X \overset{d}{=} Y$ if they have the same distribution. In particular $U_a \overset{d}{=} a\, U_1$, $a > 0$.

With these notations the equations (2.7) can be written as

$$\{Z_1,\, Z_2\} \overset{d}{=} \begin{cases} \{U_1\,,\ (1 - a_2\, U_1)^+\} & \text{with probability } \dfrac{r}{c}\,, \\[2mm] \{(1 - a_1\, U_1)^+\,,\ U_1\} & \text{with probability } \dfrac{g}{c}\,. \end{cases} \qquad (2.10)$$

Theorem 2.3. *For $0 \leq z_i \leq 1$ and $\alpha_i \leq 0$, $i = 1, 2$, the joint distribution function of Z_1, Z_2 is given by*

$$F(z_1,\, z_2) = P[Z_1 \leq z_1\,,\ Z_2 \leq z_2] = \frac{r}{c\, a_2}(a_2\, z_1 + z_2 - 1)^+ + \frac{g}{c\, a_1}(a_1\, z_2 + z_1 - 1)^+. \quad (2.11)$$

Proof. From (2.10) we obtain

$$F(z_1,\, z_2) = \frac{r}{c}\, P[U_1 \leq z_1\,,\ (1 - a_2\, U_1)^+ \leq z_2] + \frac{g}{c}\, P[(1 - a_1\, U_1)^+ \leq z_1\,,\ U_1 \leq z_2]. \quad (2.12)$$

Now for any $a \geq 1$ we have

$$P[U_1 \leq z_1\,,\ (1 - a\, U_1)^+ \leq z_2] = P\left[\frac{1 - z_2}{a} \leq U_1 \leq z_1\right] = \left(z_1 - \frac{1 - z_2}{a}\right)^+.$$

Using this result in (2.12) we arrive at (2.11). $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 2.3 yields the marginal distributions of Z_1 and Z_2 and their moments. Of greater interest to us is the covariance of Z_1 and Z_2. In deriving an expression for this the result (2.11) is not useful since the density corresponding to F is given by

$$f(z_1,\, z_2) = \frac{\partial^2 F(z_1,\, z_2)}{\partial z_1\, \partial z_2} \equiv 0\,, \qquad 0 \leq z_1 \leq 1\,,\ 0 \leq z_2 \leq 1\,,$$

and thus the distribution is singular. We investigate the details of this situation in the next section.

3. The linear Dependence of Z_1 and Z_2.

We first prove the following.

Lemma 3.1. *Choose $Z_1(0) = 0$, $Z_2(0) = \lambda_2\, g$. Then*

$$\frac{Z_1(t)}{\lambda_1\, r} + \frac{Z_2(t)}{\lambda_2\, g} \leq 1\,, \tag{3.1}$$

where the equality holds if and only if $a_1 = a_2 = 1$ (or equivalently $\alpha_1 = \alpha_2 = 0$).

Proof. From (2.1) and (2.2) we have for $n\, c < t \leq n\, c + r$

$$\frac{Z_1(t)}{\lambda_1\, r} + \frac{Z_2(t)}{\lambda_2\, g} = \frac{t - n\, c}{r} + \left(1 - a_2 \frac{t - n\, c}{r}\right)^+ = \max\left\{\frac{t - n\, c}{r}\,,\ 1 - (a_2 - 1)\frac{t - n\, c}{r}\right\}\,.$$

For $t = n\, c + r$, (3.1) is trivially true. For $n\, c < t < n\, c + r$ we have

$$1 - (a_2 - 1)\frac{t - n\, c}{r} \leq 1\,,$$

where the equality holds if and only if $a_2 = 1$. Thus (3.1) holds for $n\, c < t < n\, c + r$. Similarly it holds for $n\, c + r < t \leq n\, c + c$ and the proof is complete. $\qquad\square$

In the limit as $t \to \infty$, Lemma 3.1 yields the result

$$Z_1 + Z_2 \leq 1\,,$$

which holds with probability one. Thus the distribution of $\{Z_1,\, Z_2\}$ is concetrated in the triagle $Z_1 \geq 0\,,\ Z_2 \geq 0\,,\ Z_1 + Z_2 \leq 1$ in the $(Z_1,\, Z_2)$ plane. A more precise statement is contained in the following, which essentially summarizes the conclusion from Lemma 3.1 and equation (2.10)

Theorem 3.2. *If $a_1 = a_2 = 1$, then with probability one*

$$Z_1 + Z_2 = 1\,. \tag{3.2}$$

More generally, we have

$$Z_1 + Z_2 \overset{d}{=} U_1 + (1 - a_2\, U_1)^+ \quad \text{with probability } \frac{r}{c}\,, \tag{3.3}$$

$$Z_1 + Z_2 \overset{d}{=} U_1 + (1 - a_1\, U_1)^+ \quad \text{with probability } \frac{g}{c}\,. \tag{3.4}$$

$\qquad\square$

Clearly, if $a_1 = a_2 = 1$ the equations (3.3) and (3.4) reduce to the single equation (3.2) This shows that the distribution of $\{Z_1,\, Z_2\}$ is concetrated on the diagonal of the unit square in the $(Z_1,\, Z_2)$ plane. In the general case the equations (3.3) and (3.4) represent two random straight lines, so the distribution of $\{Z_1,\, Z_2\}$ is concetrated on these two lines with probabilities r/c and g/c. We have thus confirmed our observation in section 2 that the distribution of $\{Z_1,\, Z_2\}$ is singular. Furthermore, the dependence between Z_1 and Z_2 is linear, in agreement with the statement of Prigogine and Herman [5].

Figure 1 illustrates the results of Theorem 3.2. The two straight lines correspond to two sample observations u', u'' of the random variable U_1. Their intercepts on each axis are respectively b_2 and b_1, where

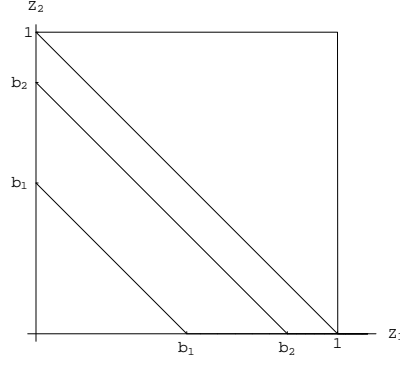$$b_1 = u' + (1 - a_1\, u')^+\,, \qquad b_2 = u'' + (1 - a_2\, u'')^+\,.$$

FIGURE 1. The linear dependence of Z_1 and Z_2 in the form of two random straight lines.

As indicated in the proof of Lemma 3.1, $b_1 \leq 1$, $b_2 \leq 1$. Also as a_i decreases to unity, $b_i \to 1$ and the corresponding line moves towards the diagonal.

Our final result is for the covariance of Z_1 and Z_2.

Theorem 3.3. *We have*

$$\mathrm{cov}(Z_1, Z_2) = -\frac{1}{12}\left[\frac{r}{c\,a_2^2}\,(3\,a_2 - 2) + \frac{g}{c\,a_1^2}\,(3\,a_1 - 2)\right] < 0\,. \tag{3.5}$$

Proof. From (2.10) we find that

$$\mathbf{E}(Z_1\,Z_2) - \mathbf{E}(Z_1)\,\mathbf{E}(Z_2) = \quad \frac{r}{c}\left\{\mathbf{E}[U_1\,(1 - a_2\,U_1)^+] - \mathbf{E}(U_1)\,\mathbf{E}(1 - a_2\,U_1)^+\right\}$$

$$+\frac{g}{c}\left\{\mathbf{E}[(1 - a_1\,U_1)^+\,U_1] - \mathbf{E}(1 - a_1\,U_1)^+\,\mathbf{E}(U_1)\right\}\,.$$

Now easy calculations show that for $a \geq 1$

$$\mathbf{E}(U_1) \quad = \frac{1}{2}\,, \tag{3.6}$$

$$\mathbf{E}[(1 - a\,U_1)^+] \quad = \frac{1}{2\,a}\,, \tag{3.7}$$

$$\mathbf{E}[U_1\,(1 - a\,U_1)^+] \quad = \frac{1}{6\,a^2}\,. \tag{3.8}$$

Using (3.6), (3.7) and (3.8) we arrive at the expression for $\mathrm{cov}(Z_1, Z_2)$. The inequality in (3.5) holds since $a_1 \geq 1$, $a_2 \geq 1$. $\qquad\square$

Remark 3.4. If $a_1 = a_2 = 1$. then (3.5) reduces to

$$\mathrm{cov}(Z_1, Z_2) = -\frac{1}{12}\,.$$

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