# Counterexamples related to a characterization of multivariate regular variation

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**Abstract**

In 2002 Basrak, Davis, and Mikosch showed that for noninteger-valued regular variation indices there is a characterization of (multivariate) regular variation for random vectors in terms of regular variation of its linear combinations (see Basrak et al., 2002a), similar in spirit to the Cramér-Wold characterization of convergence in distribution for random vectors. This characterization is of importance when studying stationary solutions to stochastic recurrence equations. In this paper we construct counterexamples showing that for integer-valued regular variation indices regular variation of all linear combinations does not imply multivariate regular variation. The construction is partly based on unpublished notes by Harry Kesten.

**Key words:** heavy-tailed distributions; linear combinations; multivariate regular variation

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# 1 Introduction

For $\mathbb{R}^d$-valued random vectors $\mathbf{X}_n$ and $\mathbf{X}$, the well-known Cramér-Wold Theorem says that a necessary and sufficient condition for $\mathbf{X}_n \xrightarrow{d} \mathbf{X}$ is that $\mathbf{x}'\mathbf{X}_n \xrightarrow{d} \mathbf{x}'\mathbf{X}$ for every $\mathbf{x} \in \mathbb{R}^d$. In (Basrak et al., 2002a) it was shown that, for noninteger-valued regular variation indices, there is a similar characterization of regular variation for a random vector in terms of regular variation of its linear combinations; meaning that for some $\alpha > 0$ and some function $L$ which is slowly varying at infinity,

$$\begin{cases} \text{for every } \mathbf{x} \neq \mathbf{0}, \quad \lim_{t\to\infty} t^\alpha L(t)\,\mathrm{P}(\mathbf{x}'\mathbf{X} > t) = w(\mathbf{x}) \text{ exists,} \\ \\ w(\mathbf{x}) > 0 \text{ for some } \mathbf{x} \neq \mathbf{0}. \end{cases} \tag{1}$$

If (1) holds, then nessecarily $w(u\mathbf{x}) = u^\alpha w(\mathbf{x})$ for all $\mathbf{x} \neq \mathbf{0}$ and $u > 0$. The interest in this condition originates from a classical result by Kesten (1973b) which (in short) says that, under mild conditions, the stationary solution $\mathbf{X}$ of a multivariate stochastic recurrence equation $\mathbf{X}_n = \mathbf{A}_n\mathbf{X}_{n-1} + \mathbf{B}_n$ satsifies (1), where $L(t) = 1$ and $\alpha$ is the unique solution to

$$\lim_{n\to\infty} \frac{1}{n}\,\mathrm{E}\ln \|\mathbf{A}_n \cdots \mathbf{A}_1\|^\alpha = 0.$$

A popular example is the stationary GARCH model which can be embedded in a stochastic recurrence equation (see Basrak et al., 2002b). Other examples where the condition (1) appears are the stochastic recurrence equations with heavy-tailed innovations studied in (Konstantinides and Mikosch, 2005) and the random coefficient AR($q$) models of Klüppelberg and Pergamentchtchikov (2004).

On the other hand, a random vector $\mathbf{X}$ is said to be regularly varying if there exist an $\alpha > 0$ and a probability measure $\sigma$ on $\mathcal{B}(\mathbb{S}^{d-1})$, the Borel $\sigma$-field of $\mathbb{S}^{d-1} = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = 1\}$, such that, for every $x > 0$, as $t \to \infty$,

$$\frac{\mathrm{P}(|\mathbf{X}| > tx, \mathbf{X}/|\mathbf{X}| \in \cdot\,)}{\mathrm{P}(|\mathbf{X}| > t)} \xrightarrow{v} x^{-\alpha}\sigma(\cdot) \quad \text{on } \mathcal{B}(\mathbb{S}^{d-1}). \tag{2}$$

Here $\xrightarrow{v}$ denotes vague convergence, $\alpha$ and $\sigma$ are called, respectively, the regular variation index and spectral measure of $\mathbf{X}$. For $\alpha \in (0, 2)$ this formulation of multivariate regular variation is a neccessary and sufficient condition for the convergence in distribution of normalized partial sums of iid random vectors to a stable random vector, see Rvačeva (1962). It is also used for the characterization of the maximum domain of attraction of extreme value distributions, Resnick (1987), and for weak convergence of point processes, see e.g. Davis and Hsing (1995); Davis and Mikosch (1998).

In (Basrak et al., 2002a, Theorem 1.1) it was proved that (2) implies (1) and the following statements hold:

(A) If $\mathbf{X}$ satisfies (1), where $\alpha$ is positive and noninteger, then (2) holds and the spectral measure $\sigma$ is uniquely determined.

(B) If $\mathbf{X}$ assumes values in $[0, \infty)^d$ and satisfies (1) for $\mathbf{x} \in [0, \infty)^d \setminus \{\mathbf{0}\}$, where $\alpha$ is positive and noninteger, then (2) holds and the spectral measure $\sigma$ is uniquely determined.

(C) If $\mathbf{X}$ assumes values in $[0,\infty)^d$ and satisfies (1), where $\alpha$ is an odd integer, then (2) holds and the spectral measure $\sigma$ is uniquely determined.

In Section 2 we construct a counterexample which shows that (A) cannot be extended to integer-valued regular variation indices without additional assumptions on the distribution of $\mathbf{X}$. In Section 3 we construct a counterexample which shows that (B) cannot be extended to integer-valued regular variation indices without additional assumptions on the distribution of $\mathbf{X}$. Whether (C) is true in the case of $\alpha$ belonging to the set of even integers is as far as the authors know still an open problem.

Let us point out that there are several equivalent formulations of (2); many of them are documented in (Basrak, 2000) and (Resnick, 2004). See also Basrak et al. (2002a), Hult (2003), Lindskog (2004) and Resnick (1987) for more on multivariate regular variation. For a detailed treatment of the concept of regularly varying functions, see the monograph Bingham et al. (1987).

In order to keep the computations as transparent as possible we will use a condition which is equivalent to (1), namely,

$$\begin{cases} \text{for every } \mathbf{x}\in\mathbb{S}^{d-1}, \quad \lim_{t\to\infty} t^\alpha L(t)\,\mathrm{P}(\mathbf{x}'\mathbf{X}>t)=w(\mathbf{x}) \text{ exists}, \\ w(\mathbf{x})>0 \text{ for some } \mathbf{x}\in\mathbb{S}^{d-1}. \end{cases} \tag{3}$$

# 2 Construction of the counterexamples

The constructions of the counterexamples corresponding to (A) and (B) for integer-valued regular variation indices are rather similar and consist of two steps. First we will find two bivariate regularly varying random vectors $\mathbf{X}_0$ and $\mathbf{X}_1$ with regular variation index $\alpha>0$ and spectral measures $\sigma_0$ and $\sigma_1$, with $\sigma_0\neq\sigma_1$, such that for every $\mathbf{x}\in\mathbb{S}$ (for (B) we restrict $\mathbf{x}$ to $\mathbb{S}\cap[0,\infty)^2$) and $t>1$,

$$\mathrm{P}(\mathbf{x}'\mathbf{X}_0>t)=\mathrm{P}(\mathbf{x}'\mathbf{X}_1>t). \tag{4}$$

Then we will construct the counterexamples by finding a random vector $\mathbf{X}$ (the vector $\mathbf{X}$ will have different distribution in (A) and (B)) such that, for every $\mathbf{x}\in\mathbb{S}$ ($\mathbf{x}\in\mathbb{S}\cap[0,\infty)^2$),

$$\lim_{t\to\infty} t^\alpha\,\mathrm{P}(\mathbf{x}'\mathbf{X}>t)=\lim_{t\to\infty} t^\alpha\,\mathrm{P}(\mathbf{x}'\mathbf{X}_0>t)=\lim_{t\to\infty} t^\alpha\,\mathrm{P}(\mathbf{x}'\mathbf{X}_1>t)=:w(\mathbf{x}) \tag{5}$$

and such that there are subsequences $(u_n)$, $(v_n)$, $u_n\uparrow\infty$, $v_n\uparrow\infty$, with the property that for every $S\in\mathcal{B}(\mathbb{S})$ with $\sigma_0(\partial S)=\sigma_1(\partial S)=0$,

$$\lim_{n\to\infty}\frac{\mathrm{P}(|\mathbf{X}|>u_n,\mathbf{X}/|\mathbf{X}|\in S)}{\mathrm{P}(|\mathbf{X}|>u_n)}=\sigma_0(S), \tag{6}$$

$$\lim_{n\to\infty}\frac{\mathrm{P}(|\mathbf{X}|>v_n,\mathbf{X}/|\mathbf{X}|\in S)}{\mathrm{P}(|\mathbf{X}|>v_n)}=\sigma_1(S). \tag{7}$$

The counterexamples are easily extended to $\mathbb{R}^d$-valued random vectors. Take $\mathbf{X}=(X^{(1)},X^{(2)})'$ as above and $\mathbf{Y}=(Y^{(1)},\dots,Y^{(d-2)})'$ independent of $\mathbf{X}$ with $\mathbf{Y}$ satisfying (1) with the same $\alpha$ as $\mathbf{X}$, $L(t)=1$, and limit function $w_{\mathbf{Y}}$.

Put $\mathbf{Z} = (X^{(1)}, X^{(2)}, Y^{(1)}, \ldots, Y^{(d-2)})'$. Then, by independence (c.f. Davis and Resnick, 1996, Lemma 2.1),

$$\begin{aligned}
&\lim_{t\to\infty} t^\alpha \operatorname{P}(\mathbf{z}'\mathbf{Z} > t) \\
&\quad = \lim_{t\to\infty} t^\alpha \operatorname{P}((z^{(1)}, z^{(2)})\mathbf{X} > t) + \lim_{t\to\infty} t^\alpha \operatorname{P}((z^{(3)}, \ldots, z^{(d)})\mathbf{Y} > t) \\
&\quad = w(z^{(1)}, z^{(2)}) + w_{\mathbf{Y}}(z^{(3)}, \ldots, z^{(d)}).
\end{aligned}$$

Hence, $\mathbf{Z}$ satisfies (1). However, $\mathbf{Z}$ does not satisfy (2). Indeed, assume on the contrary that $\mathbf{Z}$ satisfies (2) with spectral measure $\sigma_{\mathbf{Z}}$. Then since $\mathbf{X} = T(\mathbf{Z})$ with $T : \mathbb{R}^d \to \mathbb{R}^2$ is given by $T(\mathbf{z}) = (z^{(1)}, z^{(2)})'$ it follows that (e.g. Basrak et al., 2002b, Proposition A.1) $\mathbf{X}$ satisfies (2) for some spectral measure $\sigma$. This is a contradiction.

## 2.1 Construction of $\mathbf{X}_0$ and $\mathbf{X}_1$

We will now focus on the contruction of $\mathbf{X}_0$ and $\mathbf{X}_1$ in the counterexample corresponding to (A) when $\alpha$ is a positive integer.

Take $\alpha \in \{1, 2, \ldots\}$. We will construct two regularly varying random vectors $\mathbf{X}_0$ and $\mathbf{X}_1$ with regular variation index $\alpha$ and different spectral measures such that (4) is satisfied. A different construction in the case $\alpha = 1$ can be found in (Meerschaert and Scheffler, 2001, Example 6.1.35).

Let $\Theta_0$ be a $[0, 2\pi)$-valued random variable with density $f_0$ satisfying, for some $w > 0$, $f_0(\theta) > w$ for all $\theta \in [0, 2\pi)$. Take $v \in (0, w)$ and let $\Theta_1$ have density $f_1$ given by

$$f_1(\theta) = f_0(\theta) + v \sin((\alpha + 2)\theta), \quad \theta \in [0, 2\pi).$$

Let $R \sim \text{Pareto}(\alpha)$, i.e. $\operatorname{P}(R > x) = x^{-\alpha}$ for $x \geq 1$, be independent of $\Theta_i$, $i = 0, 1$, and put

$$\mathbf{X}_i \stackrel{d}{=} (R \cos \Theta_i, R \sin \Theta_i)'.$$

Obviously $\mathbf{X}_i$ is regularly varying with $\sigma_i(\cdot) = \operatorname{P}((\cos \Theta_i, \sin \Theta_i) \in \cdot)$. Take $\mathbf{x} \in \mathbb{S}$ and let $\beta \in [0, 2\pi)$ be given by $\mathbf{x} = (\cos \beta, \sin \beta)'$. Then, for $t > 1$,

$$\begin{aligned}
&\operatorname{P}(\mathbf{x}'\mathbf{X}_1 > t) - \operatorname{P}(\mathbf{x}'\mathbf{X}_0 > t) \\
&\quad = v \int_t^\infty \int_{\beta - \arccos(t/r)}^{\beta + \arccos(t/r)} \alpha r^{-\alpha - 1} \sin((\alpha + 2)\theta) \mathrm{d}\theta \mathrm{d}r \\
&\quad = -\frac{v\alpha}{\alpha + 2} \int_t^\infty r^{-\alpha - 1} \Big( \cos\{(\alpha + 2)(\beta + \arccos(t/r)\} \\
&\qquad\qquad - \cos\{(\alpha + 2)(\beta - \arccos(t/r))\} \Big) \mathrm{d}r \\
&\quad = \frac{2v\alpha}{\alpha + 2} \sin((\alpha + 2)\beta) \int_t^\infty r^{-\alpha - 1} \sin\{(\alpha + 2) \arccos(t/r)\} \mathrm{d}r
\end{aligned}$$

Using standard variable substitutions and trigonometric formulas the integral can be rewritten as follows:

$$\begin{aligned}
&\int_t^\infty r^{-\alpha-1}\sin\{(\alpha+2)\arccos(t/r)\}\mathrm{d}r \\
&\quad = t^{-\alpha}\int_0^1 r^{\alpha-1}\sin\{(\alpha+2)\arccos(r)\}\mathrm{d}r \\
&\quad = t^{-\alpha}\int_0^{\pi/2} \cos^{\alpha-1}(r)\sin((\alpha+2)r)\sin(r)\mathrm{d}r \\
&\quad = t^{-\alpha}\int_0^{\pi/2} \cos^{\alpha-1}(r)\cos((\alpha+1)r)\mathrm{d}r - t^{-\alpha}\int_0^{\pi/2}\cos^{\alpha}(r)\cos((\alpha+2)r)\mathrm{d}r.
\end{aligned}$$

The two last integrals equal zero for every $\alpha \in \{1, 2, \dots\}$, see Gradshteyn and Ryzhik (2000) p. 392. Hence, for $t > 1$, $\mathrm{P}(\mathbf{x}'\mathbf{X}_1 > t) = \mathrm{P}(\mathbf{x}'\mathbf{X}_0 > t)$, which proves (4).

The following construction of $\mathbf{X}$ satisfying (5)-(7) is based on unpublished notes by Harry Kesten (1973a) relating to Remark 4, p. 245, in (Kesten, 1973b). These notes were kindly provided to us by Laurens de Haan.

## 2.2 The counterexample

Consider the random vectors $\mathbf{X}_0$ and $\mathbf{X}_1$ above. Let $g_0$ and $g_1$ denote their densities. We construct a random vector $\mathbf{X}$ satisfying (3) which is not regularly varying; we will show that it satisfies (5), (6), and (7).

Take $\mathbf{y} \in \mathbb{R}^2$ with $|\mathbf{y}| > 1$. There exist unique integers $j, n \geq 1$ such that

$$|\mathbf{y}| \in (j!, (j+1)!] \quad \text{and} \quad j \in \Big\{\sum_{k=1}^{n-1} 2^k + 1, \dots, \sum_{k=1}^{n} 2^k\Big\}.$$

Let $\mathbf{X}$ a random vector on $\mathbb{R}^2$ with density $g$ given by,

$$\Big(1 - \Big(j - \sum_{k=1}^{n-1} 2^k\Big)2^{-n}\Big) g_{b(n)}(\mathbf{y}) + \Big(j - \sum_{k=1}^{n-1} 2^k\Big)2^{-n} g_{b(n+1)}(\mathbf{y}),$$

for $|\mathbf{y}| \in (j!, (j+1)!]$, where

$$b(n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

That is, the density $g$ is given by

$$g(\mathbf{y}) = \begin{cases}
g_0(\mathbf{y}) = 0, & |\mathbf{y}| \in (0, 1], \\
\frac{1}{2}g_0(\mathbf{y}) + \frac{1}{2}g_1(\mathbf{y}), & |\mathbf{y}| \in (1, 2], \\
g_1(\mathbf{y}), & |\mathbf{y}| \in (2, 3!], \\
\frac{1}{4}g_0(\mathbf{y}) + \frac{3}{4}g_1(\mathbf{y}), & |\mathbf{y}| \in (3!, 4!], \\
\frac{1}{2}g_0(\mathbf{y}) + \frac{1}{2}g_1(\mathbf{y}), & |\mathbf{y}| \in (4!, 5!], \\
\frac{3}{4}g_0(\mathbf{y}) + \frac{1}{4}g_1(\mathbf{y}), & |\mathbf{y}| \in (5!, 6!], \\
\text{etc.}
\end{cases}$$

Note that in each disc $|\mathbf{y}| \in (j!, (j+1)!]$ the density $g$ is a convex combination of the densities $g_0$ and $g_1$. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^2} g(\mathbf{y})\mathrm{d}\mathbf{y} &= \sum_{j=1}^{\infty} \int_{|\mathbf{y}|\in(j!,(j+1)!]} g(\mathbf{y})\mathrm{d}\mathbf{y} \\ &= \sum_{j=1}^{\infty} \int_{j!}^{(j+1)!} \alpha r^{-\alpha-1}\mathrm{d}r = \int_1^{\infty} \alpha r^{-\alpha-1}\mathrm{d}r = 1,\end{aligned}$$

so $g$ is indeed a probability density. Take $\mathbf{x} \in \mathbb{S}$ and $t \in ((j-1)!, j!]$. Then there are two possibilities:

$$\begin{aligned}&(i) \quad j-1 \in \Big\{ \sum_{k=1}^{n-1} 2^k + 1, \dots, \sum_{k=1}^{n} 2^k - 1 \Big\} \quad \text{or} \\ &(ii) \quad j-1 = \sum_{k=1}^{n-1} 2^k.\end{aligned}$$

Suppose $(i)$ holds. Then, with $\gamma = \Big(j - \sum_{k=1}^{n-1} 2^k\Big)2^{-n} \in [0,1]$, we have

$$\begin{aligned}\mathrm{P}(\mathbf{x}'\mathbf{X} > t) = &\big(1-\gamma+2^{-n}\big)\,\mathrm{P}(\mathbf{x}'\mathbf{X}_{b(n)} > t, |\mathbf{X}_{b(n)}| \in (t, j!]) \\ &+ \big(\gamma - 2^{-n}\big)\,\mathrm{P}(\mathbf{x}'\mathbf{X}_{b(n+1)} > t, |\mathbf{X}_{b(n+1)}| \in (t, j!]) \\ &+ (1-\gamma)\,\mathrm{P}(\mathbf{x}'\mathbf{X}_{b(n)} > t, |\mathbf{X}_{b(n)}| \in (j!, (j+1)!]) \\ &+ \gamma\,\mathrm{P}(\mathbf{x}'\mathbf{X}_{b(n+1)} > t, |\mathbf{X}_{b(n+1)}| \in (j!, (j+1)!]) \\ &+ \mathrm{P}(\mathbf{x}'\mathbf{X} > t, |\mathbf{X}| > (j+1)!).\end{aligned}$$

Hence,

$$\begin{aligned}\mathrm{P}(\mathbf{x}'\mathbf{X} > t) = 2^{-n}\Big\{ &\mathrm{P}(\mathbf{x}'\mathbf{X}_{b(n)} > t, |\mathbf{X}_{b(n)}| \in (t, j!]) \\ &- \mathrm{P}(\mathbf{x}'\mathbf{X}_{b(n+1)} > t, |\mathbf{X}_{b(n+1)}| \in (t, j!])\Big\} \\ &+ \underbrace{(1-\gamma)\,\mathrm{P}(\mathbf{x}'\mathbf{X}_{b(n)} > t) + \gamma\,\mathrm{P}(\mathbf{x}'\mathbf{X}_{b(n+1)} > t)}_{B_n} \\ &- (1-\gamma)\,\mathrm{P}(\mathbf{x}'\mathbf{X}_{b(n)} > t, |\mathbf{X}_{b(n)}| > (j+1)!) \\ &- \gamma\,\mathrm{P}(\mathbf{x}'\mathbf{X}_{b(n+1)} > t, |\mathbf{X}_{b(n+1)}| > (j+1)!) \\ &+ \mathrm{P}(\mathbf{x}'\mathbf{X} > t, |\mathbf{X}| > (j+1)!).\end{aligned}$$

We have,

$$\begin{aligned}&2^{-n}\Big\{ \mathrm{P}(\mathbf{x}'\mathbf{X}_{b(n)} > t, |\mathbf{X}_{b(n)}| \in (t, j!]) - \mathrm{P}(\mathbf{x}'\mathbf{X}_{b(n+1)} > t, |\mathbf{X}_{b(n+1)}| \in (t, j!])\Big\} \\ &\quad \le 2^{-n}\Big\{ \mathrm{P}(|\mathbf{X}_{b(n)}| > t) + \mathrm{P}(|\mathbf{X}_{b(n+1)}| > t)\Big\} = 2^{-n+1}t^{-\alpha}.\end{aligned}$$

Moreover, since $\mathrm{P}(\mathbf{x}'\mathbf{X}_1 > t) = \mathrm{P}(\mathbf{x}'\mathbf{X}_0 > t)$ we have $B_n = \mathrm{P}(\mathbf{x}'\mathbf{X}_0 > t)$. The absolute value of each of the remaining terms is less than or equal to $((j+1)!)^{-\alpha} \le (j\,j!)^{-\alpha} \le (j\,t)^{-\alpha}$ so we conclude that

$$t^{\alpha}|\,\mathrm{P}(\mathbf{x}'\mathbf{X} > t) - \mathrm{P}(\mathbf{x}'\mathbf{X}_0 > t)| \le 2^{-n+1} + 3j^{-\alpha}.$$

Since $j = j(t) \to \infty$ and $n = n(t) \to \infty$ as $t \to \infty$ we have,

$$\lim_{t\to\infty} t^{\alpha} \big| \mathrm{P}(\mathbf{x}'\mathbf{X} > t) - \mathrm{P}(\mathbf{x}'\mathbf{X}_0 > t) \big| = 0.$$

That is,

$$\lim_{t\to\infty} t^{\alpha} \mathrm{P}(\mathbf{x}'\mathbf{X} > t) = \lim_{t\to\infty} t^{\alpha} \mathrm{P}(\mathbf{x}'\mathbf{X}_0 > t) = \lim_{t\to\infty} t^{\alpha} \mathrm{P}(\mathbf{x}'\mathbf{X}_1 > t).$$

Suppose now that $(ii)$ holds. Then

$$\begin{aligned}
\mathrm{P}(\mathbf{x}'\mathbf{X} > t) &= \mathrm{P}(\mathbf{x}'\mathbf{X}_{b(n)} > t, |\mathbf{X}_{b(n)}| \in (t, j!]) \\
&\quad + (1-2^{-n}) \mathrm{P}(\mathbf{x}'\mathbf{X}_{b(n)} > t, |\mathbf{X}_{b(n)}| \in (j!, (j+1)!]) \\
&\quad + 2^{-n} \mathrm{P}(\mathbf{x}'\mathbf{X}_{b(n+1)} > t, |\mathbf{X}_{b(n+1)}| \in (j!, (j+1)!]) \\
&\quad + \mathrm{P}(\mathbf{x}'\mathbf{X} > t, |\mathbf{X}| > (j+1)!) \\
&= \mathrm{P}(\mathbf{x}'\mathbf{X}_{b(n)} > t) - \mathrm{P}(\mathbf{x}'\mathbf{X}_{b(n)} > t, |\mathbf{X}_{b(n)}| > (j+1)!) \\
&\quad + 2^{-n} \Big\{ \mathrm{P}(\mathbf{x}'\mathbf{X}_{b(n+1)} > t, |\mathbf{X}_{b(n+1)}| \in (j!, (j+1)!]) \\
&\qquad - \mathrm{P}(\mathbf{x}'\mathbf{X}_{b(n)} > t, |\mathbf{X}_{b(n)}| \in (j!, (j+1)!]) \Big\} \\
&\quad + \mathrm{P}(\mathbf{x}'\mathbf{X} > t, |\mathbf{X}| > (j+1)!).
\end{aligned}$$

By similar arguments as for case $(i)$ we get,

$$t^{\alpha} | \mathrm{P}(\mathbf{x}'\mathbf{X} > t) - \mathrm{P}(\mathbf{x}'\mathbf{X}_0 > t)| \le 2(2^{-n} + j^{-\alpha}).$$

It follows that,

$$\lim_{t\to\infty} t^{\alpha} \mathrm{P}(\mathbf{x}'\mathbf{X} > t) = \lim_{t\to\infty} t^{\alpha} \mathrm{P}(\mathbf{x}'\mathbf{X}_0 > t) = \lim_{t\to\infty} t^{\alpha} \mathrm{P}(\mathbf{x}'\mathbf{X}_1 > t),$$

which proves (5).

Finally, we find subsequences $(u_n)$ and $(v_n)$ satisfying (6) and (7). Take $S \in \mathcal{B}(\mathbb{S})$ with $\sigma_0(\partial S) = \sigma_1(\partial S) = 0$. Put

$$c_n = \sum_{k=1}^{2n} 2^k \quad \text{and} \quad d_n = \sum_{k=1}^{2n+1} 2^k.$$

Note that for $c_n! < |\mathbf{y}| \le (c_n+1)!$ we have $g(\mathbf{y}) = g_{b(2n+1)}(\mathbf{y}) = g_0(\mathbf{y})$, whereas for $d_n! < |\mathbf{y}| \le (d_n+1)!$ we have $g(\mathbf{y}) = g_{b(2n+2)}(\mathbf{y}) = g_1(\mathbf{y})$. It follows that, with $u_n = c_n!$,

$$\begin{aligned}
u_n^{\alpha} \mathrm{P}(|\mathbf{X}| > u_n, \mathbf{X}/|\mathbf{X}| \in S) &= u_n^{\alpha} \mathrm{P}\left(c_n! < |\mathbf{X}| \le (c_n+1)!, \mathbf{X}/|\mathbf{X}| \in S\right) \\
&\quad + u_n^{\alpha} \mathrm{P}\left(|\mathbf{X}| > (c_n+1)!, \mathbf{X}/|\mathbf{X}| \in S\right).
\end{aligned}$$

Since the second term is less than or equal to

$$u_n^{\alpha} \mathrm{P}\left(|\mathbf{X}| > (c_n+1)!\right) = (c_n!)^{\alpha} \left[(c_n+1)!\right]^{-\alpha} \to 0,$$

as $n \to \infty$, it follows that,

$$\begin{aligned}
\lim_{n\to\infty} u_n^{\alpha} \mathrm{P}(|\mathbf{X}| > u_n, \mathbf{X}/|\mathbf{X}| \in S) &= \lim_{n\to\infty} u_n^{\alpha} \left( u_n^{-\alpha} - [(c_n+1)!]^{-\alpha} \right) \sigma_0(S) \\
&= \sigma_0(S).
\end{aligned}$$

By a similar argument, with $v_n = d_n!$,

$$\lim_{n\to\infty} v_n^\alpha \,\mathrm{P}(|\mathbf{X}| > v_n, \mathbf{X}/|\mathbf{X}| \in S) = \sigma_1(S).$$

Thus, we have found sequences $(u_n)$ and $(v_n)$ satisfying (6) and (7) and the counterexample is complete.

# 3 Nonnegative components

In this section we construct a counterexample corresponding to (B) in the case of integer-valued regular variation indices $\alpha$.

The following construction of $\mathbf{X}_0$ and $\mathbf{X}_1$ was given in (Basrak et al., 2002a) for $\alpha = 2$ but can, as we will see, be extended to any positive integer $\alpha$. Take $\alpha \in \{1, 2, \dots\}$ and let $\Theta_0, \Theta_1$ be two $[0, \pi/2]$-valued random variables with unequal distributions satisfying,

$$\mathrm{E}(\cos^k \Theta_0 \sin^{\alpha-k} \Theta_0) = \mathrm{E}(\cos^k \Theta_1 \sin^{\alpha-k} \Theta_1), \quad k = 0, 1, \dots, \alpha. \tag{8}$$

Let $R$ be Pareto($\alpha$)-distributed and independent of $\Theta_i$, $i = 0, 1$, and put $\mathbf{X}_i \stackrel{d}{=} (R\cos\Theta_i, R\sin\Theta_i)'$. For $\mathbf{x} \in [0, \infty)^2 \backslash \{\mathbf{0}\}$ we have

$$\begin{aligned} t^\alpha \,\mathrm{P}(\mathbf{x}'\mathbf{X}_i > t) &= t^\alpha \,\mathrm{P}(x_1 R\cos\Theta_i + x_2 R\sin\Theta_i > t) \\ &= t^\alpha \int_1^\infty \mathrm{P}(x_1 \cos\Theta_i + x_2 \sin\Theta_i > t/r)\alpha r^{-\alpha-1} \mathrm{d}r \\ &= \int_0^{t^\alpha} \mathrm{P}((x_1 \cos\Theta_i + x_2 \sin\Theta_i)^\alpha > v)\mathrm{d}v \\ &= \sum_{k=1}^{\alpha} \binom{\alpha}{k} x_1^k x_2^{\alpha-k} \,\mathrm{E}(\cos^k \Theta_i \sin^{\alpha-k} \Theta_i) \end{aligned}$$

for $t$ sufficiently large. We can now apply the counterexample from Section 2.2, with $\mathbf{x} \in \mathbb{S}_+ = \mathbb{S} \cap [0, \infty)^2$ and new densities $g_0$ and $g_1$ of $\mathbf{X}_0$ and $\mathbf{X}_1$. It remains to show that we can find unequal distributions of the $[0, \pi/2]$-valued random variables $\Theta_0$ and $\Theta_1$ satisfying (8). Let $\Theta_0$ have density $f_0$ satisfying, for some $w > 0$, $f_0(\theta) > w$ for all $\theta \in [0, \pi/2]$. We will show that the density $f_1$ of $\Theta_1$ can be chosen as $f_1(\theta) = f_0(\theta) + vf(\theta)$, where $v \in (0, w)$ and $f$ is chosen such that $\sup_{\theta\in[0,\pi/2]} |f(\theta)| = 1$, $\int_0^{\pi/2} f(\theta)\mathrm{d}\theta = 0$, and (8) holds. Let

$$A := \mathrm{span}\{1, \sin^\alpha(\theta), \cos(\theta)\sin^{\alpha-1}(\theta), \dots, \cos^\alpha(\theta)\} \subset \mathbb{C}_2([0, \pi/2]),$$

where $\mathbb{C}_2([0, \pi/2])$ is the space of real-valued continuous functions on $[0, \pi/2]$ with the inner product $(h_1, h_2) = \int_0^{\pi/2} h_1(s)h_2(s)\mathrm{d}s$. For any nonzero $\widetilde{f} \notin A$ with $\widetilde{f} \in \mathbb{C}_2([0, \pi/2])$ we can choose

$$f := \frac{\widetilde{f} - \mathrm{Proj}_A(\widetilde{f})}{\sup_{\theta\in[0,\pi/2]} |\{\widetilde{f} - \mathrm{Proj}_A(\widetilde{f})\}(\theta)|}.$$

Then $f \perp A$, $f_1$ is a density function and (8) holds. Since $A$ is a finite-dimensional subspace of the infinite-dimensional space $\mathbb{C}_2([0, \pi/2])$ it is clear that its orthogonal complement is nonempty. This completes the counterexample.

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