Optimal bidding in Multi-unit Procurement Auctions

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Abstract

In a multi-unit reverse auction, the single buyer wants to buy multiple units of a single object from one or more potential suppliers. This paper introduces a new auction mechanism called the quasi-uniform-price auction, in which the per-unit price of a bidder’s payment is equal to the highest rejected bid submitted by other bidders. We find the optimal-response bidding strategies for this auction as well as Discriminatory auctions, uniform-price first-rejected auctions, uniform-price last-accepted auctions and Vickrey auctions. We present an asymmetric bidder model and obtain analytical expressions for the Nash equilibrium bidding strategy. Using this model we study the impact of the auction mechanism on profit allocation and the incentives for acquisitions and expansions.
1 Introduction and Literature Review

Background

For ages, auctions have been used for the sale of a unique object with uncertain valuation such as antiques, coins and paintings. Government and other utility companies have also used auctions for the privatization of publicly-owned resources and the allocation of rights. As a result, the study of auctions has been focused on the sale of a single object, for which a mathematically elegant theory of auctions has developed.

The industrial procurement process has typically involved negotiating with a limited number of suppliers to determine the final terms of agreement. Since this process is typically expensive and requires patience, it is not feasible to negotiate with more than a few suppliers. However, with the emergence of the Internet, it has become possible to involve many suppliers from all over the world when a well-defined formal mechanism is employed (Jap (2003)). A large pool of suppliers bidding in an auction increases the power of the buyer, and reduces her payment. Thus, the procurement auction provides the buyer with an attractive alternative to the classical negotiation process.

A procurement auction involves a single buyer and multiple sellers. It is also referred to as a reverse auction since classical auction theory assumes a single seller and multiple buyers. A procurement auction for buying a single item, by symmetry, inherits most of the theoretical properties of the traditional forward auction for selling a single item. Industrial procurement typically involves multiple units of the same item, but there has been rather limited attention in the auction theory literature to the multi-unit auction, whether in the reverse or forward setting.

Multi-Unit Auctions

In this paper, we consider a procurement auction in which one buyer wants to purchase a fixed number $Q$ of indivisible and identical units from one or more risk-neutral sellers. The number $N$ of sellers is known, and sellers have private values for each unit. Each seller
submits multiple bids, each one for a single unit, and the buyer accordingly makes allocation and payment. In this paper, we consider standard auctions in which the $Q$ lowest bids are accepted for allocation.

The standard multi-unit auctions differ in their payment schemes. In a Discriminatory auction, a seller is paid the sum of his accepted bids. It is also known in the literature as a “pay-your-bid” auction. In a uniform-price auction, all accepted bids will receive the same price per unit. This price is the lowest rejected bid in the uniform-price - first rejected (UPFR) auction, and the highest accepted bid in the uniform-price - last accepted (UPLA) auction. The UPFR auction is simply referred to as the uniform-price auction in some literature. In the multi-unit Vickrey auction, a bidder winning $q$ units receives the sum of the lowest $q$ rejected bids submitted by bidders other than himself. In addition, we propose a new auction rule called the quasi-uniform-price (QUP) auction, in which a bidder receives, for each of his accepted units, the lowest rejected bid submitted by another bidder. Thus, the per-unit price for a bidder is never equal to any of his own bids.

This paper addresses a variety of multi-unit standard procurement auctions. We study optimal bidding for a risk-neutral bidder. We provide a computational approach for optimal-response bidding, and first-order necessary conditions that apply to both optimal-response bidding and equilibrium strategies. We then present an asymmetric bidder model, in which one bidder is bigger than all the other bidders, and investigate the impact of auction formats on capacity expansion and acquisition decisions.

**Literature Review**

Vickrey (1961), through his seminal paper, initiates a fascinating web of research on auction theory. The early auction theory papers focus on the efficiency of allocation and the optimality of seller’s revenue in the sale of a single unit. Riley and Samuelson (1981) provide the conditions for revenue equivalence, and Myerson (1981) characterizes the optimal auction. Milgrom and Weber (1982) show that the revenue to the auctioneer is higher in the first
price auction than the second price auction when the bidders’ values are not private but instead are positively correlated.

We present a brief review of multi-unit auctions focusing on bidding strategies, optimality for the buyer and efficiency. Due to a vast number of research papers on multi-unit auctions, our review excludes many related and relevant bodies of literature including sequential auctions, combinatorial auctions, open auctions, double auctions, common values, experimental economics and statistical methods. Excellent reviews on auction theory include McAfee and McMillan (1987), Rothkopf and Harstad (1994) and Klemperer (2000).

In a multi-unit auction where each bidder is interested in only one unit, most results for the single-unit auction can be extended including revenue equivalence (Weber (1983)). The optimal auction mechanism design of Myerson (1981) is extended by Branco (1996) to multi-unit auctions with a unit-demand symmetric-bidder model. When each bidder bids for only one unit, it can be shown that the outcomes of standard symmetric-bidder auctions are efficient and thus the allocation is the same. As a result, a majority of studies on multi-unit auctions are based on the one-bid-per-bidder restriction. For example, Vulcano et al. (2002) study a variation of the traditional revenue management problem using a multi-period auction, in which a fixed number of units are auctioned over time, and each bidder wants only one unit. Keeping the same auction environment, Van Ryzin and Vulcano (2004) analyze a retailer’s joint inventory-pricing problem.

However, if each bidder demands more than one unit, the ex-post allocation depends on the choice of auction mechanism, resulting in the non-equivalence of revenue. Auctions become difficult to analyze when bidders demand more than one unit. (A notable exception is the multi-unit Vickrey auction, due to Vickrey (1961) himself, which is easy to analyze because of its incentive compatibility (truth-telling) and efficiency properties.)

Given the difficulty in analyzing multi-unit auctions, it is not surprising that a large number of papers consider the cases in which each bidder demands only two units. Under certain assumptions, the first-order necessary conditions for optimal bidding strategies are given for Discriminatory auctions (Engelbrecht-Wiggans and Kahn (1998)), UPFR auctions
Engelbrecht-Wiggans and Kahn (1998) show that a bidder might submit two identical bids that correspond to different values in a Discriminatory auction. Such “bid-pooling” does not occur in UPFR auctions since the first bid of each bidder is truth-telling (similar to the second-price single-unit auction) and the second bid is shaded. Katzman (2003b) characterizes bidding strategy equilibria in Vickrey, Discriminatory and UPFR auctions using first order conditions. Maskin and Riley (1989) provide characterizations of optimal selling procedure and incentive compatibility. Krishna and Perry (1998) show that the Vickrey auction maximizes the auctioneer’s revenue among all efficient mechanisms. Ausubel and Crampton (2002) show that efficiency does not hold in an UPFR auction, and in a Discriminatory auction, holds under very restrictive assumptions. As the number of bidders increases to infinity, asymptotic efficiency is obtained in a Discriminatory auction (Katzman (2003a), Swinkels (1999), and Swinkels (2001)). Nautz (1995) and Nautz and Wolfstetter (1997) analyze Discriminatory and uniform-price auctions under a restrictive assumption that a bidder’s bidding behavior does not affect market-clearing prices.

In recent years, auctions have received much attention in the operations management community. Auctions have been used to address operational issues in companies such as Home Depot (Elmaghraby and Keskinocak (2003)), Mars (Hohner et al. (2003)) and Sears Logistics Services (Ledyard et al. (2002)). Auction mechanisms have been applied to the traditional revenue management problem (Vulcano et al. (2002)), retailer’s joint inventory-pricing problem (Van Ryzin and Vulcano (2004)), procurement of multiple items with constraints on suppliers’ capacity (Gallien and Wein (2001)), procurement with multiple attributes (Beil and Wein (2003)), supplier profit maximization in a decentralized supply chain (Deshpande and Schwarz (2002)), and online auctions (Bapna et al. (2003) and Pinker et al. (2003)). Chen (2001) and Seshadri and Zemel (2003) study auctioning a supply contract as opposed to a specified number of items, which extends earlier works of Dasgupta and Spulber.
We mention existing operations-related studies related to sequential or simultaneous auctions such as Jin and Wu (2002), Elmaghraby (2003) and Feng and Chatterjee (2002).

**Summary and Contributions**

This paper makes the following contributions.

First, it introduces a new multi-unit auction mechanism called a quasi-uniform-price (QUP) auction, which can be considered a hybrid between the multi-unit Vickrey auction, and the UPFR. It combines the simplicity of uniform price auctions with a property of the Vickrey auction that a bidder’s profit depends on his bids only though the quantity of his winning bids.

Second, in five standard auctions, we provide first-order necessary conditions for bidders’ optimal bidding strategies, and show that a bidder’s profit is separable in his bids. It generalizes the result of Draaisma and Noussair (1997) to an arbitrary number of units demanded by a bidder, and extends to auctions other than the UPLA auction. We formulate the bidder’s problem as a constrained optimization problem and find expressions for the derivatives, enabling computation. (Draaisma and Noussair (1997) ignore the constraint in their derivation of the first-order conditions.)

Third, we identify sufficient conditions to ensure that the constraints of the bidder’s optimization problem are not binding. In this case, the first-order conditions are presented. It generalizes earlier results most of which are restricted to the case where a bidder demands at most 2 units.

Fourth, we consider the asymmetric model where there are many small bidders but only one big bidder. For this model we study the outcomes of the Vickrey, QUP and UPFR auctions, which are the three multi-unit generalizations of the classical single-unit second-price auction. We obtain analytical expressions for the optimal equilibrium bidding strategy for all three auctions. This result is the first analytical solution to multi-unit auctions in which bidders submit bids for more than 2 items.
Finally, we use this model to analyze a market with one large seller and multiple small sellers. The Vickrey auction is preferable for both the buyer and the big bidder, while the QUP and UPFR auctions are preferable for the small bidders, measured by the seller profit per unit and the buyer’s purchase price. The Vickrey auction favors the large seller at the expense of the small sellers. However the Vickrey auction creates a strong incentive for sellers to grow, leading to a market dominated by a small number of large suppliers. When a seller decides to grow, the buyer would like to create an incentive for capacity expansion rather than acquisition. However the selection of auction mechanism has only a minimal impact on that decision. The decision to expand capacity or acquire competitors is driven by the relative costs of the two alternatives, and by the supplier’s budget for investment. A supplier with limited resources is more likely to add capacity, whereas one with deeper pockets is more likely to acquire competitors.

Organization

In Section 2, we outline standard modeling assumptions for multi-unit standard auctions, such as Discriminatory auctions, uniform-price (UPFR and UPLA) auctions and Vickrey auctions. We formally define the quasi-uniform-price (QUP) auction. We show that the optimal bidding strategy of a bidder is separable and monotone with respect to his values. Section 3 presents the optimal response bidding strategies when bid-ordering constraints are not binding. Section 4 presents an asymmetric bidder model where all bidders have unit capacity with the exception of one bidder with a large capacity, and study the large bidder’s decision to either acquire a small bidder or increase capacity. We conclude in Section 5.

2 The Model

Description

In this paper, we consider multi-unit sealed-bid private-value reverse auctions. One buyer wishes to purchase $Q \geq 2$ units of an object from one or more of $N$ potential sellers (bidders),
indexed by $n = 1, 2, \ldots, N$. Each bidder submits exactly $Q$ bids, some of which may be infinite. The marginal production cost of the $j$'th unit of seller $n$ is denoted by $v^n_j$, and we refer to $v^n = (v^n_1, v^n_2, \ldots, v^n_Q)$ as seller $n$’s value vector. Only bidder $n$ knows his own value vector $v^n$. However, all the other bidders have the same information concerning seller $n$’s value vector (e.g., a common distribution function). We assume $v^n$ is stochastically independent of the value vectors $v^n'$ of any other bidder $n' \neq n$. Bidder $n$ submits the bid vector $b^n = (b^n_1, b^n_2, \ldots, b^n_Q)$. We assume without loss of generality that $b^n_1 \leq b^n_2 \leq \ldots \leq b^n_Q$. We also assume that $v^n_1 \leq v^n_2 \leq \ldots \leq v^n_Q$. Both of these are standard assumptions in auction theory. When the $j$'th bid $b^n_j$ is added to the bids that bidder $n$ wins, the marginal production cost incurred is $v^n_j$. A vector $b^n$ satisfying this chain constraint is said to be proper.

**Standard Auctions**

In standard auctions, the $Q$ lowest bids are deemed winning and awarded sales. We let $R^n_j$ be the $j$'th smallest competing bid (bid submitted by bidders other than $n$), and let $R^n = (R^n_1)$. Then, $b^n_j$ competes with $R^n_{Q-j+1}$, i.e., exactly one of $b^n_j$ and $R^n_{Q-j+1}$ is deemed winning.

Standard auctions are differentiated by the payment scheme (as a function of bid vectors). Let $q_n$ represent the number of bidder $n$’s winning bids. Assuming the buyer procures all $Q$ units, $\sum_n q_n = Q$. In a Discriminatory auction, each seller $n$ receives the sum $\sum_{j=1}^{q_n} b^n_j$ of his winning bids. In the UPFR auction, each winning bid receives the per-unit price equal to the lowest losing bid, $\min\{b^n_{q_n+j}, R^n_{Q-q_n+j+1}\}$. By contrast, in the UPLA auction, the per-unit price is equal to the highest winning bid, $\max\{b^n_{q_n}, R^n_{Q-q_n}\}$. In a Vickrey auction, the total payment to seller $n$ is the sum of the $q_n$ lowest losing bids that are submitted by sellers other than $n$. Thus, bidder $n$ receives a sum $\sum_{j=1}^{q_n} R^n_{Q-q_n+j}$ for $q_n$ units.

**Quasi-Uniform-Price Auction**

We propose a standard auction called the quasi-uniform-price (QUP) auction, which is a modification of the UPFR auction. Bidder $n$ receives a per-unit price for $q_n$ units; this price
is equal to the lowest rejected bid $R_{Q-q_n+1}^n$ submitted by other bidders $\{1, 2, \ldots, N\}\{n\}$. Thus, the total payment he receives is $q_nR_{Q-q_n+1}^n$. We note that the payment depends on bidder $n$’s bid vector $b^n$ only through the quantity $q_n$. This property is shared with the Vickrey auction, but not with the Discriminatory or either of the uniform-price auctions. The QUP auction is not a uniform price auction because the price for the seller submitting the lowest rejected bid is different from the price for every other seller. In order for this mechanism to be well defined, we assume that at least two bidders have a finite bid rejected. This condition is satisfied if every bidder submits $Q$ finite bids.

We assume that all bidders are risk-neutral. Bidder $n$’s optimal bidding strategy maps a value vector to a bid vector, and is denoted by $\beta^n(v^n) = (\beta^n_1(v^n), \ldots, \beta^n_Q(v^n))$. Let $\pi^n(b^n|v^n)$ denote the expected profit of bidder $n$ when his value vector is $v^n$ and his bid vector is $b^n$. The expectation is taken over the distribution $R^n$ of competing bids facing bidder $n$. We assume that the distribution $R^n_j$ of the $j$’th competing bid has contiguous support, a probability density function $f_{R^n_j}$ and a cumulative density function $F_{R^n_j}$. We study bidder $n$’s optimal response $\beta^n$, such that $b^n = \beta^n(v^n)$ maximizes $\pi^n(b^n|v^n)$ for fixed $v^n$ subject to $b^n_1 \leq \cdots \leq b^n_Q$. (In general, the proper bid vector $b^n$ maximizing $\pi^n(b^n|v^n)$ may not be unique. We let $\beta^n(v^n)$ be any maximizer.)

**Profit Functions**

In a standard auction, the probability that bidder $n$’s $j$’th unit will be sold is the probability that $b^n_j$ is smaller than $R^n_{Q-j+1}$. (We ignore the event $b^n_j = R^n_{Q-j+1}$, which has probability 0 since we assume $R^n_{Q-j+1}$ has a probability density.) The expectation of the total value of units sold amounts to $\sum_j v^n_j \cdot F_{R^n_{Q-j+1}}(b^n_j)$, where $F_{R^n_j}(b) = 1 - F_{R^n_j}(b)$ is the complementary cumulative distribution function of $R^n_j$. Let $P^n(b^n)$ be the expected payment to bidder $n$ for a given bid vector $b^n$. It is determined by bidder $n$’s bid vector as well as the distribution of other bidders’ bid vectors, and is independent of the value vector $v^n$. The exact expression
for \( P^n \) depends on the auction format. Conditioned on the number of units awarded to bidder \( n \), \( P^n(b^n) \) is assumed to be nondecreasing in \( b^n \). The expected profit of seller \( n \) is

\[
\pi^n(b^n|v^n) = P^n(b^n) - \sum_j v^n_j \cdot F_{R_{Q-j+1}}(b^n_j).
\] (2.1)

**Bidders’ Payments and Separability**

The expected payment function \( P^n(b^n) \) is defined for all proper bid vectors of \( n \). For the five types of standard auctions we consider, we now present expressions for \( P^n(b^n) \). In a Discriminatory auction, it is easy to verify

\[
P^n(b^n) = \sum_{j=1}^Q b^n_j \cdot F_{R_{Q-j+1}}(b^n_j).
\] (2.2)

In any standard auction, bidder \( n \) wins exactly \( j \) units provided both \( b^n_j < R^n_{Q-j+1} \) and \( R^n_{Q-j} < b^n_{j+1} \) hold, ignoring ties. Suppose bidder \( n \) wins \( j \) bids. In an UPFR auction, the uniform price is \( b^n_j \) if \( b^n_j \in (R^n_{Q-j}, R^n_{Q-j+1}) \), and \( R^n_{Q-j+1} \) if \( R^n_{Q-j+1} \in (b^n_j, b^n_{j+1}) \). It follows that his expected payment is

\[
P^n(b^n) = \sum_{j=1}^{Q-1} j b^n_{j+1} \cdot \{F_{R^n_{Q-j}}(b^n_{j+1}) - F_{R^n_{Q-j+1}}(b^n_{j+1})\} + \sum_{j=1}^Q j \int_{b^n_j}^{b^n_{j+1}} ydF_{R^n_{Q-j+1}}(y),
\] (2.3)

where we let \( b^n_{Q+1} = \infty \) for notational convenience. Similarly, in an UPLA auction,

\[
P^n(b^n) = \sum_{j=1}^Q j b^n_j \cdot \{F_{R^n_{Q-j}}(b^n_j) - F_{R^n_{Q-j+1}}(b^n_j)\} + \sum_{j=1}^Q j \int_{b^n_j}^{b^n_{j+1}} ydF_{R^n_{Q-j}}(y).
\] (2.4)

In a Vickrey auction, the payment associated with bidder \( n \) who wins \( j \) units is the sum of \( R^n_{Q-j+1} \), \( R^n_{Q-j+2} \), …, \( R^n_Q \). The payment includes the \( R^n_{Q-j+1} \) term if and only if \( b^n_j < R^n_{Q-j+1} \). It follows that

\[
P^n(b^n) = \sum_{j=1}^Q \int_{b^n_j}^{\infty} ydF_{R^n_{Q-j+1}}(y).
\]

In a QUP auction, if \( j \) is the number of bidder \( n \)'s accepted bids, then he receives \( R^n_{Q-j+1} \) per unit. Thus,

\[
P^n(b^n) = \sum_{j=1}^Q j E[R^n_{Q-j+1}|R^n_{Q-j+1} \geq b^n_j, R^n_{Q-j} \leq b^n_{j+1}] \cdot P[R^n_{Q-j+1} \geq b^n_j, R^n_{Q-j} \leq b^n_{j+1}].
\] (2.5)
The following proposition shows that all of the above expressions of $P^n(b^n)$ are separable. We recall that a function $g : \mathbb{R}^N \rightarrow \mathbb{R}$ is separable if there exist a set of functions $g_1, g_2, \ldots, g_N$ such that $g(x_1, \ldots, x_N) = g_1(x_1) + \cdots + g_N(x_N)$ for any $(x_1, x_2, \ldots, x_N)$ in the domain of $g$.

**Proposition 2.1.** In a Discriminatory auction, UPFR auction, UPLA auction, Vickrey auction and QUP auction, the expected payment function $P^n(b^n)$ of bidder $n$ for a fixed value vector is separable with respect to a proper bid vector $b^n$. Furthermore, if $\frac{\partial}{\partial b^n_1} \pi^n(b^n|v^n)$ exists, it depends on the value vector $v^n$ only through $v^n_j$.

We note that the separability of $P^n(b^n)$ in the UPLA auction was noted by Draaisma and Noussair (1997) for the case of $Q = 2$.

**Proof.** We prove here the case for the QUP auction. All the other cases are easily verifiable. Let $f_{R^n_{Q-j},R^n_{Q-j+1}}$ be the joint density probability distribution function of $R^n_{Q-j}$ and $R^n_{Q-j+1}$. We have

\[
E[R^n_{Q-j+1} | R^n_{Q-j} \geq b^n_j, R^n_{Q-j} \leq b^n_{j+1}] \cdot P[R^n_{Q-j+1} \geq b^n_j, R^n_{Q-j} \leq b^n_{j+1}]
\]

\[
= \int_{b^n_j}^{b^n_{j+1}} \int_0^{y} ydF_{R^n_{Q-j},R^n_{Q-j+1}}(w,y)
\]

\[
= \int_{b^n_j}^{b^n_{j+1}} \int_0^{y} ydF_{R^n_{Q-j},R^n_{Q-j+1}}(w,y) - \int_0^{b^n_j} ydF_{R^n_{Q-j+1}}(y)
\]

where the last equality follows since $R^n_{Q-j} \leq R^n_{Q-j+1}$ and $b^n_j \leq b^n_{j+1}$. By substitution, the expected payment function (2.5) in a QUP auction becomes

\[
P^n(b^n) = \sum_{j=1}^{Q} j \cdot \left( \int_{b^n_j}^{b^n_{j+1}} \int_0^{y} ydF_{R^n_{Q-j},R^n_{Q-j+1}}(w,y) - \int_0^{b^n_j} ydF_{R^n_{Q-j+1}}(y) \right). \quad (2.6)
\]

This expression is separable. The last statement follows from (2.1). \qed

Therefore, given bidder $n$’s value vector, the determination of his optimal-response bid vector $b^n$ reduces to maximization of a separable function subject to a chain constraint $b^n_1 \leq \cdots \leq b^n_Q$.

It is shown in Section 3.2 that in some standard auctions, the expected payment function may not be separable.
Monotonicity of the Optimal Response

From (2.2)-(2.6), it is easy to find sufficient conditions for the partial differentiability of the payment function. The partial derivative of $P^m$ with respect $b^m_j$ exists if the CDF of $R^Q_{Q-j+1}$ is continuously differentiable at $b^m_j$ in the Discriminatory, Vickrey and QUP auctions. In the UPFR and UPLA auctions, we additionally require the continuous differentiability of $F_{R^Q_{Q-j+2}}$ and $F_{R^Q_{Q-j}}$, respectively.

We define the following two assumptions that will enable us to do derivative-based analysis for the rest of the paper.

**Assumption 2.2.** For any $n$ and $j$, $R^Q_{Q-j+1}$ has a contiguous support.

**Assumption 2.3.** For any $n$ and $j$, the optimal bid $\beta^m_j(v^m)$ is in the interior of the support of its competing bid $R^m_{Q-j+q}$.

The following proposition shows that under a weak condition, the optimal bidding strategy $\beta^m$ is a monotone function of the value vector $v^m$. Generally, in the auction literature, the monotonicity of bidding functions is an assumption as opposed to a derived result.

**Theorem 2.4.** Suppose that the distribution of bidder $n$’s vector of competing bids $R^m$ has a known density in a multi-unit standard auction. Then, any continuous optimal response function $\beta^m$ satisfies $\beta^m(v') \leq \beta^m(v'')$ whenever $v' \leq v''$. If in addition Assumptions 2.2 and 2.3 hold and $v'_j < v''_j$ for some $j$, then $\beta^m_j(v') < \beta^m_j(v'')$.

**Proof.** See Appendix A.1.

In auctions that have a reserve price $R$, bids that are greater than or equal to $R$ cannot win. Payments are computed assuming that an ample number of artificial bids equal to $R$ were submitted by a third party. The first assertion in Theorem 2.4 and its proof hold in the presence of a reserve price. Note that we allow the support of the density of competing bids to include values that are greater than $R$. 
Value Vector and Optimal Profit

Let \( U_n(v^n) = \pi_n(\beta_n(v^n)|v^n) \) be the optimal profit function for a given value vector \( v^n \). It is reasonable to expect that when production costs are high, the expected profit is low. Furthermore, the expected profit is convex in the values (i.e., the production costs).

**Proposition 2.5.** The optimal profit function \( U_n(v^n) \) is nonincreasing and convex in \( v^n \).

**Proof.** For distinct value vectors \( v' \) and \( v'' \) of bidder \( n \), let \( \hat{b}' = \beta_n(v') \) and \( \hat{b}'' = \beta_n(v'') \). For any \( \lambda \in [0,1] \), define \( \hat{v} = \lambda v' + (1-\lambda)v'' \) and \( \hat{b} = \beta_n(\hat{v}) \). Then,

\[
U_n(\hat{v}) = P_n(\hat{b}) - \sum_j \hat{v}_j \cdot F_{R_{Q_j-1}}(\hat{b}_j)
\]

\[
= P_n(\hat{b}) - \sum_j (\lambda v'_j + (1-\lambda)v''_j) \cdot F_{R_{Q_j-1}}(\hat{b}_j)
\]

\[
= \lambda [P_n(\hat{b}) - \sum_j v'_j \cdot F_{R_{Q_j-1}}(\hat{b}_j)] + (1-\lambda) [P_n(\hat{b}) - \sum_j v''_j \cdot F_{R_{Q_j-1}}(\hat{b}_j)]
\]

\[
= \lambda \pi_n(\hat{b}|v') + (1-\lambda) \pi_n(\hat{b}|v'') \leq \lambda U_n(v') + (1-\lambda)U_n(v'').
\]

Thus, \( U_n \) is convex. By the choice of \( b' \), we have \( U_n(v') = \pi_n(b'|v') \geq \pi_n(b''|v') \). The definition of the profit function \( \pi_n \) implies \( \pi_n(b''|v') \geq \pi_n(b''|v'') \) for \( v' \leq v'' \). Thus, we conclude \( U_n(v') \geq \pi_n(b''|v'') = U_n(v'') \).

\[
\square
\]

### 3 Optimal Bidding Strategies

Section 3.1 computes the partial derivative of the profit function with respect to bids in five standard auctions. It shows that with the exception of the Vickrey auction, truth-telling is not an optimal strategy, and thus, supply reduction (equivalent to demand reduction in forward auctions) occurs. Section 3.2 presents first-order conditions for “optimal response” bidding in standard auctions, and discusses computation of the optimal response. It includes an example with the Poisson bid distribution. In Section 3.3 we find sufficient conditions for the bid-ordering constraint to be non-binding in a QUP auction. We assume Assumption 2.2 holds throughout this section.

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3.1 Partial Derivatives in Multi-Unit Auctions

This section presents the partial derivatives $\frac{\partial}{\partial b_n^j} \pi_n(b^n|v^n)$ of bidder $n$’s profit function in five standard multi-unit auctions. We give heuristic arguments based on first principles. Appendix A.2 gives rigorous derivations using derivatives.

**Discriminatory Auction**

If $b_n^j + \epsilon \leq R_{Q-j+1}$, and we increase $b_n^j$ by $\epsilon > 0$, then the payment bidder $n$ receives increases by $\epsilon$. However, bidder $n$ loses his $j$’th bid if $b_n^j \leq R_{Q-j+1} < b_n^j + \epsilon$. Thus the change in profit is

$$\epsilon \cdot 1[b_n^j \leq R_{Q-j+1} - \epsilon] - (b_n^j - v_n^j) \cdot 1[b_n^j \leq R_{Q-j+1} < b_n^j + \epsilon] + o(\epsilon).$$

Dividing the expression by $\epsilon$ and taking a limit as $\epsilon \to 0^+$, we get

$$\frac{\partial \pi_n(b^n|v^n)}{\partial b_n^j} = (v_n^j - b_n^j) \cdot f_{R_{Q-j+1}}(b_n^j) + F_{R_{Q-j+1}}(b_n^j). \quad (3.7)$$

**UPFR Auction**

Decreasing the $j$’th bid $b_n^j$ by $\epsilon > 0$ affects the price when $b_n^j$ is the first rejected bid. If $b_n^j - \epsilon$ is still the first rejected bid, then bidder $n$’s profit decreases by $\epsilon$ for each of his $j - 1$ winning units. However, if $b_n^j - \epsilon$ becomes accepted, then the change in bidder $n$’s price is at most $\epsilon$, but he is awarded an extra unit. Thus, the change in the profit is

$$-(j - 1) \cdot \epsilon \cdot 1[R_{Q-j+1}^n \leq v_n^j \leq R_{Q-j+2}^n] + (R_{Q-j+1}^n - v_n^j) \cdot 1[R_{Q-j+1}^n - \epsilon \leq R_{Q-j+1}^n \leq R_{Q-j+2}^n] + o(\epsilon).$$

Taking the limit of this expression divided by $-\epsilon$ as $\epsilon \to 0^+$ gives

$$\frac{\partial \pi_n(b^n|v^n)}{\partial b_n^j} = (v_n^j - b_n^j) \cdot f_{R_{Q-j+1}}(b_n^j) + (j - 1) \cdot \{F_{R_{Q-j+1}}(b_n^j) - F_{R_{Q-j+2}}(b_n^j)\}. \quad (3.8)$$

**UPLA Auction**

Increasing the $j$’th bid $b_n^j$ by $\epsilon > 0$ affects the profit only if $b_n^j$ is the last accepted bid. The change in the profit is

$$j \cdot \epsilon \cdot 1[R_{Q-j}^n \leq b_n^j \leq R_{Q-j+1}^n - \epsilon] - (b_j - v_j) \cdot 1[b_j \leq R_{Q-j+1} < b_j + \epsilon] + o(\epsilon)$$
and the partial derivative of \( \pi^n(b^n|v^n) \) becomes
\[
\frac{\partial \pi^n(b^n|v^n)}{\partial b^n_j} = (v^n_j - b^n_j) \cdot f_{R^n_{Q-j+1}}(b^n_j) + j \cdot \{ F_{R^n_{Q-j}}(b^n_j) - F_{R^n_{Q-j+1}}(b^n_j) \}.
\] (3.9)

**Vickrey Auction**

Increasing \( j \)’th bid \( b^n_j \) by \( \epsilon > 0 \) affects bidder \( n \)’s profit only if it changes the number of units awarded to him, which happens when \( b^n_j \leq R^n_{Q-j+1} < b^n_j + \epsilon \). Thus the change in his profit is
\[
(v^n_j - R^n_{Q-j+1}) \cdot 1[b^n_j \leq R^n_{Q-j+1} < b^n_j + \epsilon],
\]
which implies
\[
\frac{\partial \pi^n(b^n|v^n)}{\partial b^n_j} = (v^n_j - b^n_j) \cdot f_{R^n_{Q-j+1}}(b^n_j).
\] (3.10)

Note that the Vickrey derivative implies that the bidder has an incentive to bid true values. Also note that each derivative is a function only of \( v^n_j \) and \( b^n_j \), as Proposition 2.1 implies.

**QUP Auction**

Perturbing the \( j \)’th bid \( b^n_j \) upward by \( \epsilon > 0 \) changes the profit of bidder \( n \) if and only if the number of his winning bids decreases from \( j \). The loss of the \( j \)’th bid occurs when \( b^n_j \leq R^n_{Q-j+1} < b^n_j + \epsilon \). His payment changes from \( j \cdot R^n_{Q-j+1} \) to \( (j - 1)R^n_{Q-j+2} \). Thus, the change in the profit is
\[
[(j - 1)R^n_{Q-j+2} - j \cdot R^n_{Q-j+1} + v^n_j] \cdot 1[b^n_j \leq R^n_{Q-j+1} < b^n_j + \epsilon].
\]
Taking the limit of this expression divided by \( \epsilon \) as \( \epsilon \to 0^+ \), we obtain
\[
\frac{\partial \pi^n(b^n|v^n)}{\partial b^n_j} = \alpha \cdot \{ (j - 1)\rho(j, b^n_j) - (b^n_j - v^n_j) \},
\] (3.11)
where \( \alpha = f_{R^n_{Q-j+1}}(b^n_j) \) and
\[
\rho(j, b) = E_n[R^n_{Q-j+2}|R^n_{Q-j+1} = b] - b.
\] (3.12)
The expression $\rho(j, b)$ is the expected gap between the first and second rejected bids submitted by other bidders, conditioned on the first rejected bid being equal to seller $n$’s $j$’th bid.

**Yet Another Uniform-Price Auction**

We can define another variant of the uniform-price auction, in which the per-unit price is the second lowest rejected price. We call this auction a *Next-To-Last-Accepted* Uniform Price Auction. This auction is a standard auction, and is analogous to the third-price auction in the single-unit case. (See Wolfstetter (1995) for an analysis of the third-price auction.) The partial derivative of $\pi^n(b^n|v^n)$, if exists, is given by

$$
\frac{\partial \pi^n(b^n|v^n)}{\partial b^n_j} = (v^n_j - E[\min\{R^n_{Q-j+2}, b^n_{j+1}\}]) \cdot \frac{f_{R^n_{Q-j+1}}(b^n_j)}{F_{R^n_{Q-j+2}}(b^n_j) - F_{R^n_{Q-j+3}}(b^n_j)}
$$

for $j \geq 2$. Thus, $\pi^n(b^n|v^n)$ is not separable in $b^n$, and the conclusion of Proposition 2.1 does not hold in some standard auctions.

### 3.2 First-Order Conditions in Standard Auctions

Using the expressions of the partial derivatives developed in Section 3.1, we present bidding strategies based on the first-order conditions. An example of Poisson bid distribution is also given in this section.

**Optimal Bidding Problem and Local Optimality**

Given a value vector $v^n$, bidder $n$’s optimal bidding problem is

$$
\max_{b^n} \quad \pi^n(b^n|b^n) = P^n(b^n) - \sum_j v^n_j \cdot F_{R^n_{Q-j+1}}(b^n_j)
$$

s. t. \quad 0 \leq b^n_1 \leq b^n_2 \leq \cdots \leq b^n_Q.

By Proposition 2.1, if bidder $n$’s optimal bid $b^n = \beta^n(v^n)$ satisfies the bid-ordering constraints $b^n_1 \leq \cdots \leq b^n_Q$ with strict inequalities, then bidder $n$’s optimal bid $b^n_j = \beta^n_j(v^n)$ can be
expressed as a function of $v^n_j$ only, i.e., bidder $n$’s bid for each unit depends on his marginal value for that unit, and not on his value for other units. A sufficient condition for redundant bid-ordering constraints in the QUP auction is given Section 3.3. However, in general, the bid-ordering chain constraints are not redundant, and the problem does not decompose into a set of single dimensional problems. (In analyzing the UPLA auction, Draaisma and Noussair (1997) assume the independence of $b^n_j$’s, and ignore the bid-ordering constraints. See Huh et al. (2004) for details.)

As noted in Section 2, the problem of finding the optimal bid vector becomes a maximization of a separable function with a chain constraint. Results in Section 3.1 show that the partial derivatives of the profit function in five standard auctions are easily obtainable. Using these derivatives, Hochbaum and Queyranne (2003) present an efficient algorithm of finding the minimizer in the case of convex objective functions. Even if the objective function is not convex, a locally-optimal solution can be found very efficiently from an arbitrary starting point (Huh et al. (2003)). Using an ample set of randomly-selected starting points, we have a viable computational approach to computing, with high probability, an “optimal response” bid for bidder $n$, given distributional information on the bids of competing bidders.

For any bid vector $b^n$ and $t \geq 0$, let $C_t(b^n) = \{ j : b^n_j = t \}$ be a cluster of indices such that their corresponding entries in $b^n$ are equal to $t$. A cluster is either empty or of the form \( \{ j_1, j_1 + 1, \ldots, j_2 \} \) where $j_1 \leq j_2$. We have the following characterization of the local optimality of the optimal bidding problem by applying Kuhn-Tucker conditions.

**Proposition 3.1.** Bidder $n$’s bid vector $b^n$ is a locally optimal solution to the bidding problem if and only if each nonempty cluster $C_t(b^n) = \{ j_1, j_1 + 1, \ldots, j_2 \}$, $t \geq 0$, satisfies the following condition:

$$
\sum_{j=j_1}^{j_2} \sum_{j_1}^{j_2'} \partial_{b^n_j} \pi^n(b^n|v^n) \geq 0 \quad \text{for } j' = j_1, j_1 + 1, \ldots, j_2 \quad \text{and}
$$

$$
\sum_{j=j'}^{j_2} \sum_{j_1}^{j_2} \partial_{b^n_j} \pi^n(b^n|v^n) \leq 0 \quad \text{for } t > 0 \text{ and } j' = j_1, j_1 + 1, \ldots, j_2.
$$

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In all standard auctions presented in this paper (except the Next-To-Last-Accepted Uniform Price Auction), \( \pi^n(b^n|v^n) \) is separable in \( b^n \). Thus, we can write \( \pi^n(b^n|v^n) = \sum_{j=1}^{Q} \pi^n_j(b^n_j|v^n) \). If the \( b^n_j \)'s maximize \( \pi^n_j(b^n_j|v^n) \) for each \( j \), and form a proper bid vector, then this bid vector maximizes \( \pi^n \).

**Corollary 3.2.** Suppose bid vector \( b^n \) satisfies \( \frac{\partial}{\partial b^n_j} \pi^n(b^n|v^n) = 0 \) for \( j = 1, 2, \cdots, Q \), and \( 0 < b^n_1 < b^n_2 < \cdots < b^n_Q \). Then, \( b^n \) satisfies the first-order condition of the optimal bidding problem. Furthermore, the condition \( \frac{\partial}{\partial b^n_j} \pi^n(b^n|v^n) = 0 \) is equivalent to the following:

\[
\tilde{b}_j^n = \begin{cases} 
  v^n_j + (j-1)\rho(j, b^n_j) & \text{for QUP auction;} \\
  v^n_j + \frac{F^n_{Q,j+1}(b^n_j)}{F^n_{Q,j+1}(b^n_j)} & \text{for Discriminatory auction;} \\
  v^n_j + (j-1) \cdot \frac{F^n_{Q,j+1}(b^n_j) - F^n_{Q,j+2}(b^n_j)}{F^n_{Q,j+1}(b^n_j)} & \text{for UPFR auction;} \\
  v^n_j + j \cdot \frac{F^n_{Q,j+1}(b^n_j) - F^n_{Q,j+2}(b^n_j)}{F^n_{Q,j+1}(b^n_j)} & \text{for UPLA auction;} \\
  v^n_j & \text{for Vickrey auction.}
\end{cases}
\]

**Proof.** The first part follows from Proposition 3.1. The second part is obtained by setting \( \frac{\partial}{\partial b^n_j} \pi^n(b^n|v^n) = 0 \) in equations (3.7), (3.8), (3.9), (3.10) and (3.11).

We note that determining a bid vector satisfying the first-order condition of optimal bidding is different from finding a Nash equilibrium bidding strategy. Typically the distribution of the vector \( R^n \) of competing bids facing bidder \( n \) will depend on the type of auction.

There are some indications that QUP auctions might be more stable than Discriminatory, UPLA and UPFR auctions. If the density of \( R^n_j \) has a continuous \( p \)'th derivative for all \( j \), then the function mapping bids to values has continuous \( p \)'th derivative for Discriminatory, UPLA and UPFR auctions, but it has continuous \((p+1)\)'th derivative for a QUP auction.

The above proposition shows that in the case of non-binding constraints, a bidder sometimes makes a bid higher than his value in a QUP auction. This supply reduction is equivalent to demand reduction in multi-unit forward auctions. In general, supply reduction is known to exist in Discriminatory and uniform-price auctions (Ausubel and Crampton (2002)). The following proposition shows supply reduction in QUP auctions.
Proposition 3.3. Suppose bidder $n$’s value vector $v^n$ satisfies $v^n_1 \leq \cdots \leq v^n_Q$. The optimal bid vector $b^n$ for all five auctions satisfies $b^n_j \geq v^n_k$ for all $k \leq j$. Furthermore, $b^n_1 = v^n_1$ in the Vickrey, UPFR and QUP auctions.

Proof. See Appendix A.3 for proof. \qed

We note that the truth-telling property for the first unit is shared between the UPFR and QUP auction.

**Poisson Bid Distribution Example**

**Proposition 3.4.** Suppose that the competing bids $R^n_1, R^n_2, \ldots, R^n_Q$ of bidder $n$ are distributed as a Poisson distribution with rate parameter $\beta^{-1}$, i.e., $R^n_j$ has a gamma distribution with parameters $j$ and $\beta$. The expressions for $b^n_j$ in Corollary 3.2 become:

$$
    b^n_j = \begin{cases} 
        v^n_j + (j - 1)\beta & \text{for QUP auction;} \\
        v^n_j + \frac{\beta}{(b^n_j)^{j-1}} \sum_{k=0}^{Q-j} \frac{(Q-j)!}{k!} (b^n_j)^k \beta^{Q-j-k} & \text{for Discriminatory auction;} \\
        \frac{Q-j+1}{Q-2j+2} v^n_j & \text{for UPFR auction;} \\
        v^n_j + j\beta & \text{for UPLA auction;} \\
        v^n_j & \text{for Vickrey auction.} 
    \end{cases} 
$$

(3.14)

The above bidding strategies are given in simple and closed forms except for the Discriminatory auction. In particular, the amount of bid inflation $(b^n_j - v^n_j)$ depends only on the bid index $j$ and a Poisson parameter $\beta$. Proving the above proposition involves algebraic rearrangement, the details of which are provided in Appendix A.4.

### 3.3 Sufficient Conditions for Optimal Bidding in a QUP Auction

We identify sufficient conditions for the QUP auction bidding strategy given in Corollary 3.2 to produce an optimal bid vector. Since seller $n$’s expected profit $\pi^n(b^n|v^n)$ given his value vector $v^n$ is separable in $b^n_j$’s, we can write $\pi^n(b^n|v^n) = \sum_{j=1}^Q \pi^n_j(b^n_j|v^n)$ for some $\pi^n_j$’s. From

$$
    \frac{\partial}{\partial b^n_j} \pi^n(b^n|v^n) = f_{R^n_{Q-j+1}}(x) \{ (x - v^n_j) - (j - 1)\rho(j, x) \},
$$

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it follows

\[ \pi^n_j(b^n_j | v^n) = \int_{b^n_j}^{\infty} f_{R^n_{2-j+1}}(x) \{ (j-1)\rho(j, x) - (x - v^n_j) \} dx. \]  \hspace{1cm} (3.15)

It is desirable to have the concavity of \( \pi^n \) since the strict concavity of \( \pi^n \) implies the uniqueness of the optimal bidding strategy. However, even without the concavity of \( \pi^n \), we can still compute the optimal bidding strategy under a certain condition on \( \rho \).

**Proposition 3.5.** For a QUP auction, suppose for \( j = 1, 2, \ldots, Q \),

\[ \frac{\partial}{\partial b^n_j} \rho(j, b^n_j) \leq \frac{1}{j-1}, \]  \hspace{1cm} (3.16)

and \( \rho(j, b^n_j) \) is nondecreasing in \( j \) for a fixed \( b^n_j \). Then, a bidding strategy obtained by (3.13) is optimal, i.e., \( b^n_j = v^n_j + (j-1)\rho(j, b^n_j) \) is optimal.

**Proof.** We first claim that \( \pi^n_j(b^n_j | v^n) \) is quasi-concave in \( b^n_j \), i.e. there exists \( b_o \) depending on \( j \) such that \( \pi^n_j(b^n_j | v^n) \) is weakly increasing for \( b^n_j \leq b_o \) and weakly decreasing for \( b^n_j \geq b_o \). To show this, consider (3.11). Since \( \frac{\partial}{\partial b^n_j} \rho(j, b^n_j) \leq \frac{1}{j-1} \), we get \( (j-1)\frac{\partial}{\partial b^n_j} \rho(j, b^n_j) - 1 \leq 0 \), implying that \( (j-1)\rho(j, b^n_j) - (b^n_j - v^n_j) \) is weakly decreasing in \( b^n_j \). Since \( \alpha \geq 0 \), we obtain the quasi-concavity of \( \pi^n_j(\cdot | v^n) \).

It follows from (3.16) that \( \pi^n_j(\cdot | v^n) \) is maximized when \( b^n_j = v^n_j + (j-1)\rho(j, b^n_j) \). If \( \rho(j-1, b^n_j) \leq \rho(j, b^n_j) \) for \( j = 1, \ldots, Q \), then the monotonicity of values \( v^n_{j-1} \leq v^n_j \) implies \( b^n_{j-1} \leq b^n_j \). Thus, the bid-ordering constraints \( b^n_1 \leq b^n_2 \leq \cdots \leq b^n_Q \) are non-binding. Thus, an optimal bidding strategy is given by \( b^n_j = v^n_j + (j-1)\rho(j, b^n_j) \) for each \( j \).

Appendix A.5 illustrates that both of the conditions in Proposition 3.5 is necessary. In the Poisson bid distribution example, \( \rho(j, b^n_j) \) is independent of both parameters. It follows that the optimal bidding strategy is \( b^n_j = v^n_j + (j-1)\beta \) as in (3.14).

The sufficient condition presented in this section enables us to find a Nash equilibrium in the model presented in the subsequent section.
4 Asymmetric Bidder Model

Comparison among multi-unit auctions is difficult given that equilibrium bidding strategies are not easy to compute in general. In this section, we present a model which allows us to compare three types of multi-unit auctions, the QUP, UPFR and Vickrey auctions. These auctions are generalizations of single-unit second-price auctions. We present some computational results addressing acquisitions and expansion of bidders.

The buyer wishes to purchase $Q$ units, and there are $N$ sellers. We assume $Q < N$. One of the sellers, corresponding to $n = 1$, is big, and submits $K$ finite bids. The remaining sellers $n = 2, 3, \cdots, N$ are small, and submit only 1 finite bid. A small seller $n$, where $n \geq 2$, has a value $v_n^1$ which is independently and identically drawn from the Uniform(0,1) distribution. The big seller’s values $\{v_1^1, v_1^2, \cdots, v_K^1\}$ are generated by first independently drawing $K$ numbers from the Uniform(0,1) distribution and then sorting them in ascending order. Thus, the expected average value of each bidder (whether small or big) is 0.5. We refer to this model as the asymmetric bidder model. We will investigate the bidding strategies and expected per-unit profits of small and big sellers.

If the quantity $Q$ is 1, QUP, Vickrey and UPFR auctions all become the single-unit second-price auction. We remark that our model is a generalization of multi-unit auctions with unit demands.

4.1 Bidding Strategies

In this section, we derive equilibrium bidding strategies in these three standard auctions. In all three auctions, it is every seller’s weakly dominant strategy to bid his value truthfully for the first unit, i.e., $b_1^n = v_1^n$, for all $n = 1, \cdots, N$. For small sellers, the first unit is the only unit. The proof resembles the incentive compatibility of the single-unit second-price sealed-bid auction. The following theorem presents the optimal bidding strategy of the big seller and small sellers in each auction.
Theorem 4.1. In the asymmetric model, a small seller $n$'s weakly dominant strategy is truth-telling, i.e., $b_n^1 = v_n^1$. The big seller’s unique optimal response to small sellers’ truth-telling strategy is given by

$$b_j^1 = \beta_j^1(v^1) = \begin{cases} 
    v_j^1, & \text{in Vickery;} \\
    \frac{N-Q+j-1}{N-Q+2j-2} v_j^1 + \frac{j-1}{N-Q+2j-2}, & \text{in QUP;} \\
    \min\{\frac{Q-j+1}{Q-2j+2} v_j^1, 1\}, & \text{if } j < \frac{Q+2}{2} \text{ in UPFR;} \\
    1, & \text{otherwise in UPFR.}
\end{cases} \quad (4.17)$$

The bidding strategies in the above theorem form a Nash equilibrium.

Proof. It is easy to verify that truth-telling is the small sellers’ optimal response. We find the optimal response of the big seller.

In a Vickrey auction, it is a weakly dominant strategy of any seller to bid his values. Since the distribution of order statistics of bids submitted by small and big sellers is positive and continuous with support $(0, 1)$, the standard dominance proof establishes that truth-telling is the unique optimal response.

In a QUP auction, we first verify the preconditions of Proposition 3.5. Given $R_{Q-j+1}^1 = b_j^1$, $R_{Q-j+2}^1$ is distributed as the first order statistics of $(N-1) - (Q-j+1)$ Uniform($b_j^1, 1$) distributions. It follows

$$\rho(j, b_j^1) = E_{R}[R_{Q-j+2}^1| R_{Q-j+1}^1 = b_j^1] - b_j^1 = \frac{1 - b_j^1}{N-Q+j-2}.$$ 

It is easy to see that $\rho(j, b_j^1)$ is strictly decreasing in $b_j^1$ and increasing in $j$. By Proposition 3.5 and (3.13), we get

$$b_j^1 = v_j^1 + (j-1)\rho(j, b_j^1) = v_j^1 + (j-1)\frac{1 - b_j^1}{N-Q+j-2},$$

which is equivalent to (4.17). We note that $b_j^1$ is a weighted average of $v_j^1$ and 1.

Now consider UPFR. We recall from (3.8)

$$\frac{\partial \pi_1(b_1^1|v^1)}{\partial b_j^1} = (v_j^1 - b_j^1) \cdot f_{R_{Q-j+1}^1}(b_j^1) + (j-1) \cdot P[R_{Q-j+2}^1 \leq b \leq R_{Q-j+1}^1]$$
The formulas for the distribution of order statistics of uniform random variables imply

\[
f_{R_{Q-j+1}}(b) = \frac{(N - 1)!}{(Q - j)! (N - Q + j - 2)!} b^{Q-j}(1 - b)^{N-Q+j-2}
\]

\[
P[R_{Q-j+1} \leq b \leq R_{Q-j+2}] = \binom{N - 1}{Q - j + 1} b^{Q-j+1}(1 - b)^{N-Q+j-2}.
\]

Thus,

\[
\frac{\partial \pi^1(b^j_1 | v^j_1)}{\partial b^j_1} = (v^j_1 - b^j_1) \cdot \frac{(N - 1)!}{(Q - j)! (N - Q + j - 2)!} (b^j_1)^{Q-j}(1 - b^j_1)^{N-Q+j-2}
\]

\[
+ (j - 1) \binom{N - 1}{Q - j + 1} (b^j_1)^{Q-j+1}(1 - b^j_1)^{N-Q+j-2}
\]

\[
= \binom{N - 1}{Q - j + 1} (b^j_1)^{Q-j}(1 - b^j_1)^{N-Q+j-2} ((Q - j + 1)v^j_1 + (Q - 2j + 2)b^j_1).
\]

The first two factors are strictly positive. If \( j < \frac{Q+2}{2} \), the third factor is strictly positive for \( b^j_1 < \frac{Q-j+1}{Q-2j+2} v^j_1 \), and strictly negative for \( b^j_1 > \frac{Q-j+1}{Q-2j+2} v^j_1 \). It follows that the optimal \( b^j_1 \) is \( \min \{ \frac{Q-j+1}{Q-2j+2} v^j_1, 1 \} \).

If \( j \geq \frac{Q+2}{2} \), then \( Q - 2j + 2 \leq 0 \). Thus, for any \( v^j_1, b^j_1 \in [0, 1] \), the third factor \( (Q - j + 1)(v^j_1 - b^j_1) + (j - 1)b^j_1 \geq -(Q - 2j + 2)b^j_1 \) is nonnegative, implying \( \pi^1(b^j_1 | v^j_1) \) is nondecreasing in \( b^j_1 \). Thus, we set \( b^j_1 = 1 \).

We observe that the bidding functions (4.17) in both the Vickrey auction and the UPFR auction are independent of the number \( N \) of bidders. As \( N \) increases, the QUP auction bid \( b^j_1 \) converges to the Vickrey truth-telling bid \( v^j_1 \), whereas the UPFR auction bid does not.

**Corollary 4.2.** In the asymmetric model, the Vickrey auction has a weakly dominant and efficient strategy equilibrium; however, QUP and UPFR auctions are inefficient in general.

**Proof.** In QUP and UPFR auctions, small bidders bid their values, but the second bid of the big bidder is strictly greater than the corresponding value. Thus, an inefficient allocation occurs with a positive probability. \( \square \)
A consequence of the above corollary is the difference in allocation among these auctions. As a result, unless \( K = 1 \) the celebrated Revenue Equivalence Theorem does not apply. The total cost of goods produced, as well as the expected payments by the buyer, may not be the same for the different auctions.

### 4.2 Computation

Using bidding strategies of the QUP, Vickrey and UPFR auctions presented in the previous section, we can efficiently compute outcomes of auctions, and use monte-carlo methods to estimate the expected outcomes of auctions. In this section, we use computational results to compare these three auctions with respect to the payment by the buyer and the profits of sellers, and study how the choice of auction mechanism impacts incentives for acquisition and expansion.

Computational results are obtained by randomly generating value vectors of bidders, and applying Theorem 4.1 to find corresponding bid vectors. The bid vectors are used to simulate auction outcomes to determine allocation and payment, from which bidders’ profits are computed. The expected payments and profits are based on a sample size of 200,000 observations.

#### 4.2.1 Acquisition

In the first experiment, we fix the aggregated capacity of all sellers, and vary the size of the big seller. The buyer wants to purchase \( Q = 50 \) units, with the overall capacity being 100 units.\(^1\) We let \( K \) be the size of the big seller. Thus there are \( 100 - K \) small seller. Increasing \( K \) corresponds to the big seller increasing his/her size by acquiring small sellers in the market.

The buyer’s total payment in the Vickrey auction is typically lower than it is in the UPFR and QUP auctions. (See Figures 1a, 1b and 1c.) Thus, the buyer has an incentive to

\(^1\)While we report results for one problem size, we have done more extensive experiments and the conclusions that we draw hold consistently.
Figure 1a: Acquisition Case (Vickrey): $Q = 50$ and $N + K = 101$.

Figure 1b: Acquisition Case (UPFR): $Q = 50$ and $N + K = 101$.

Figure 1c: Acquisition Case (QUP auction): $Q = 50$ and $N + K = 101$.
use Vickrey auctions on a short-term cost basis. This observation may be surprising when you consider the payment rule of the Vickrey auction – if the big seller has $q_1$ winning bids, then he is paid the sum of $q_1$ lowest rejected bids submitted by small sellers. However, bids in Vickrey auctions are lower than those in the other auctions because of the truth-telling bidding strategy.

In the long term, acquisitions increase the total payment by the buyer. This increase occurs in all three auctions, and is attributed to the decreased level of competition due to acquisitions.

The big seller has a profit incentive to acquire in all of the auctions, indicated by his increasing per-unit profits as $K$ grows. But that incentive is much stronger with the Vickrey auction than it is with the QUP or UPFR auctions. In the Vickrey auction, acquisition does not exert any externality to small bidders. The increase in the buyer’s payment is entirely transferred to the big seller. This contrasts strongly with the UPLA and QUP auctions. We define the competitiveness of a seller to be his profit per unit of capacity, divided by the profit per unit of capacity of all sellers combined. In the QUP and UPFR auctions, acquisitions reduce the big seller’s competitiveness and increase the competitiveness of the small sellers. The opposite occurs in the Vickrey auction. If competitiveness is important then acquisition activity, which is costly to the buyer, is less likely to occur in the UPFR or QUP auction than in the Vickrey auction.

To see why acquisition does not exert any externality to small bidders in the Vickrey auction, recall that in a Vickrey auction bidders will bid their values, and the $Q$ smallest values are the winners. A small bidder will win if his value is among the winning values. If he wins his payment will be the smallest value that is not a winner. Acquisitions effect neither the probability of winning, nor the payment if he wins. (This logic applies to broader contexts than the one we are discussing.)

To summarize, short-term profits make the Vickrey auction a tempting alternative for the buyer. However if acquisitions (or, equivalently form our point of view, mergers) among suppliers are a possibility, the Vickrey auction is dangerous for the buyer. It dramatically
increases the rewards associated with mergers and acquisitions, and undermines her long-term profitability. The UPLA and QUP auctions enhance the competitiveness of small suppliers.

4.2.2 Expansion

Suppose now that the big seller expands by adding capacity rather than acquiring other sellers. We fix the number of small sellers at 50. Thus, the total number of sellers is $N = 51$. We vary the size $K$ of the big seller. The buyer wants to purchase $Q = 50$ units.

As the big seller expands, his total profit increases in all auctions, but the profit per unit of capacity of all sellers decreases, strongly benefitting the buyer (see Figures 2a, 2b and 2c). The big seller’s expansion strongly enhances his competitiveness in the Vickrey auction. In UPFR and QUP auctions, the impact on competitiveness is lower, and it benefits the small sellers.

To summarize, if the buyer is convinced that sellers will prefer capacity expansion to acquisitions or mergers, and she believes that small suppliers will not leave the market, she will prefer the Vickrey auction. It is more profitable in the short run, and it encourages expansion more strongly, creating long-term benefits as well. However if the buyer is concerned that sellers will grow through acquisitions rather than expansions, or she believes that capacity expansion will drive small suppliers out of the market, she has a conflict between short-term and long-term objectives. The Vickrey auction is more attractive in terms of short term profit, but it strengthens the incentives that the sellers have to grow, contributing to the emergence of a small number of powerful sellers and driving the buyer’s costs up in the long run.

4.2.3 Choosing between Acquisition and Expansion

The buyer has a clear incentive to predict whether sellers will prefer capacity expansion to acquisitions (or, equivalently, mergers). We now suppose that both acquisitions and capacity
Figure 2a: Expansion Case (Vickrey): $Q = 50$ and $N = 51$.

Figure 2b: Expansion Case (UPFR): $Q = 50$ and $N = 51$.

Figure 2c: Expansion Case (QUP auction): $Q = 50$ and $N = 51$. 

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expansions are available to the big seller, and study the total profit of the big seller after a combination of mergers and expansions has been made.

The quantity the buyer wants to purchase is fixed at $Q = 50$. Initially, there are 99 small sellers and one big seller. Thus, $N = 100$. The big bidder’s initial capacity is $K = 1$. If the big seller makes $s$ acquisitions and $t$ expansions, the total capacity of all sellers becomes $N = t + 100$, and the size of the big seller becomes $K = s + t + 1$. The number of bidders participating becomes $100 - s$. We assume that the capacity $K = s + t + 1$ of the big bidder is at most half of the total capacity of all sellers. Thus, $s$ and $t$ should satisfy $2s + t \leq 98$.

Figure 3 illustrates the impact of the big seller’s action in the Vickrey auction. The horizontal and vertical axes of Subfigure (a) corresponds to the number of expansions and the number of acquisitions. The contours show the total expected profit of the big seller. Clearly the big seller’s profit increases with his size. Subfigure (b) is the contour graph of the buyer’s total payment, which increases with acquisitions but decreases with expansions.

The preference of the big seller depends on the relative costs of acquisitions and expansions. When the cost of an acquisition is low, the big seller prefers acquisitions to expansions, and vice versa. Subfigure (c) illustrates the big seller’s optimal combination of acquisitions and expansions when each expansion takes 1 budget unit and each acquisition takes 1.5 budget units, and the seller is budget-constrained. (This analysis is based on the profit of the big seller only, and ignores the pre-acquisition profits of small sellers.) The big seller wants to expand as opposed to acquire, when the budget is small (less than 20 units). However when the budget is big enough, the big seller wants to acquire exclusively, taking as much market share as possible. Subfigure (d) shows the total (not per-unit) profits of the big seller and the buyer. The big seller’s profit increases steadily with his budget. The cost of the buyer decreases when the big seller has a small budget and expands, but it increases when the budget is big.

Figures 4 and 5 show similar qualitative results for the UPFR and QUP auctions. A closer observation shows that the Vickrey auction has lower payment by the buyer and higher profit for the big seller, consistent with earlier observations. In spite of the fact that the choice
of auction mechanism has a major impact on the incentive that the big seller has to grow, the choice of auction mechanism has a negligible impact on the big seller’s choice between acquisitions and expansions. This decision driven by the relative costs of acquisitions and expansions and the big seller’s investment budget.

5 Conclusion

In this paper we consider a multi-item reverse auction, in which a single buyer wants to acquire a total of $Q$ units from a collection of suppliers. We propose a new multi-unit auction called the Quasi-Uniform-Price auction, which inherits properties of both the Vickrey and UPFR auctions. For the QUP, Vickrey, UPFL, UPLA and Discriminatory auctions we formulate the problem of finding an optimal response to bidding strategies of competitors as an optimization problem with simple inequality chain constraints. We show that the objective function is separable and we find its derivative, enabling numerical solution. In the likely case that the constraints are loose we find first-order necessary conditions. For the QUP auction we find a sufficient condition for the constraints to be loose. We demonstrate the usefulness of these results by obtaining analytical expressions for Nash Equilibrium bidding strategies for an interesting multi-unit auction, under the QUP, Vickrey and UPFR protocols.

The solution to the multi-unit auction has interesting managerial implications. In terms of short-term acquisition costs the buyer has a clear incentive to use the Vickrey auction. However the Vickrey auction strongly rewards bidders for expanding, both through acquisitions and through capacity expansions, relative to the QUP and UPLA auctions. If economics strongly favor capacity expansion rather than acquisitions, and capacity expansion will not drive small suppliers out of the market, the Vickrey auction will increases the rate of expansion, benefitting the buyer in the long run as well. However if there is a significant danger of acquisitions among suppliers, or that capacity expansion will eliminate small suppliers, Vickrey auctions may give rise to a market with a small number of very large suppliers, which
Figure 3: The Impact of the Acquisition and Expansions of the Big Seller in the Vickrey Auction. Initially, $N = 100$, $Q = 50$, and $K = 1$. 
Figure 4: The Impact of the Acquisitions and Expansions of the Big Seller in the First-Rejected Uniform Price Auction. Initially, $N = 100$, $Q = 50$, and $K = 1$. 

(a) Expected Profit of Big Seller  
(b) Expected Total Payment of the Buyer  
(c) Optimal Strategy of Big Seller with Investment Budget Constraint  
(d) The Impact of Budget Constraint of the Buyer
Figure 5: The Impact of the Acquisitions and Expansions of the Big Seller in the QUP auction. Initially, $N = 100$, $Q = 50$, and $K = 1$. 
would be very expensive for the buyer in the long run. The buyer cannot influence the seller’s acquire-versus-expand decision by selecting the auction mechanism. That decision is driven by the relative costs of acquisitions and expansion, and by the seller’s investment budget. A seller with a modest budget will use capacity expansion exclusively, but a deep-pocketed seller will devote the entire budget to acquisitions.

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References


A Appendix

A.1 Proof of Theorem 2.4

We first prove a slightly weaker result.

Lemma A.1. Given ordered value vectors \( v' \leq v'' \) where \( v'_j < v''_j \) for some \( j \), we have \( \beta^n_j(v') \leq \beta^n_j(v'') \).

Proof. Define \( b' = \beta^n(v') \) and \( b'' = \beta^n(v'') \). Let \( \bar{b} = b' \vee b'' \) and \( \underline{b} = b' \wedge b'' \) be the component-wise maximum and minimum of \( b' \) and \( b'' \). The separability of \( P^n(\cdot) \) implies \( P^n(b') + P^n(b'') = P^n(\bar{b}) + P^n(\underline{b}) \). It is easily verified that both \( \bar{b} \) and \( \underline{b} \) are proper bid vectors. Since \( b'' \) is the optimal bid vector given the value vector \( v'' \), we get \( \pi^n(b''|v'') \geq \pi^n(\bar{b}|v'') \) and similarly \( \pi^n(b'|v') \geq \pi^n(\underline{b}|v') \). Thus,

\[
P^n(b'') - \sum_j v''_j \cdot F_{R^n_{Q-j+1}}(b''_j) \geq P^n(\bar{b}) - \sum_j v''_j \cdot F_{R^n_{Q-j+1}}(\bar{b}_j), \quad \text{and}
\]

\[
P^n(b') - \sum_j v'_j \cdot F_{R^n_{Q-j+1}}(b'_j) \geq P^n(\underline{b}) - \sum_j v'_j \cdot F_{R^n_{Q-j+1}}(\underline{b}_j).
\]

Adding the two above inequalities yields

\[
\sum_j v''_j \cdot F_{R^n_{Q-j+1}}(b''_j) + v'_j \cdot F_{R^n_{Q-j+1}}(b'_j) \leq \sum_j v''_j \cdot F_{R^n_{Q-j+1}}(\bar{b}_j) + v'_j \cdot F_{R^n_{Q-j+1}}(\bar{b}_j). \quad (A.18)
\]
Now for any nonnegative integers \( s_1, s_2, t_1 \) and \( t_2 \) such that \( s_1 \leq s_2 \) and \( t_1 \leq t_2 \), it follows \( s_1 t_2 + s_2 t_1 \leq s_1 t_1 + s_2 t_2 \). From the choice of \( b_j = \min\{b'_j, b''_j\} \) and \( \bar{b}_j = \max\{b'_j, b''_j\} \),

\[
v_j'' \cdot F_{R_{Q-j+1}}(b''_j) + v_j' \cdot F_{R_{Q-j+1}}(b'_j) \geq v_j'' \cdot F_{R_{Q-j+1}}(\bar{b}_j) + v_j' \cdot F_{R_{Q-j+1}}(\bar{b}_j). \tag{A.19}
\]

This inequality should hold with equality because of (A.18). It follows \( F_{R_{Q-j+1}}(b''_j) = F_{R_{Q-j+1}}(\bar{b}_j) \) and \( F_{R_{Q-j+1}}(b'_j) = F_{R_{Q-j+1}}(\bar{b}_j) \). Since the support of \( R^n_{Q-j+1} \) is assumed contiguous, we get \( b''_j = \bar{b}_j = b'_j \).

We now prove Theorem 2.4. Suppose \( v' \leq v'' \), and let \( b' = \beta^n_j(v') \) and \( b'' = \beta^n_j(v'') \). Apply Lemma A.1 to \( v' - \varepsilon \mathbf{1} \) and \( v'' + \varepsilon \mathbf{1} \) where \( \varepsilon > 0 \) and \( \mathbf{1} \) is a vector consisting of 1's, and take a limit as \( \varepsilon \to 0^+ \) to obtain \( b' \leq b'' \).

Furthermore, suppose \( v'_j < v''_j \). We first prove \( b'_j < b''_j \) with the added condition \( v'_k = v''_k \) for all \( k \neq j \). Assume, by the way of contradiction, \( b'_j = b''_j \). By the separability of \( P^n \), \( \pi^n(b^n|v^n) \) is also separable in \( b^n \). From \( v'_k = v''_k \) for all \( k \neq j \), \( b' \) also maximizes \( \pi^n(\cdot|v'') \). Let \( S = \{k|b'_k = b''_j\} \) be the set of indices whose corresponding entry in \( b' \) equals \( b''_j \). Then, by the first order optimality condition,

\[
0 = \sum_{k \in S} \frac{\partial}{\partial b^n_k} \pi^n(b'|v'') = \sum_{k \in S} \frac{\partial}{\partial b^n_k} P^n(b') + v''_k \cdot f_{R^n_{Q-k+1}}(b'_k).
\]

By definition, \( b' \) maximizes \( \pi^n(\cdot|v') \). Thus, a similar equation holds:

\[
0 = \sum_{k \in S} \frac{\partial}{\partial b^n_k} \pi^n(b'|v') = \sum_{k \in S} \frac{\partial}{\partial b^n_k} P^n(b') + v'_k \cdot f_{R^n_{Q-k+1}}(b'_k).
\]

Since \( b'_j \) is in the support of \( R^n_{Q-j+1} \), \( f_{R^n_{Q-j+1}}(b'_j) \) is positive. The comparison of the above two equations imply \( v'_j = v''_j \), which is contradictory. Thus, we conclude \( b'_j \neq b''_j \), which implies \( b'_j < b''_j \).

Now, we prove \( v'_j < v''_j \) without the added condition. Define a sequence of value vectors \( v^{(0)}, v^{(1)}, \ldots, v^{(M)} \) such that \( v' = v^{(0)} \leq v^{(1)} \leq \cdots \leq v^{(M)} = v'' \), and the pair of subsequent bid vectors \( v^{(m-1)} \) and \( v^{(m)} \), \( m = 1, \ldots, M \), differ in exactly one component. It follows that there exists some \( m' \) such that \( v^{(m'-1)} < v^{(m')} \). By applying above results, we obtain \( b'_j < b''_j \).
A.2 Partial Derivatives of Bidders’ Profit Functions

For a Discriminatory auction, (2.1) and (2.2) show

\[ \frac{\partial \pi^n(b^n|v^n)}{\partial b^n_j} = (v^n_j - b^n_j) f_{Q_j+1}(b^n_j) + \overline{F}_{Q_j+1}(b^n_j). \]

For a UPFR auction, (2.3) shows

\[ \frac{\partial \pi^n(b^n|v^n)}{\partial b^n_j} = (j - 1) \cdot \{F_{Q_j+1}(b^n_j) - F_{Q_j+2}(b^n_j)\} + (j - 1) \cdot b^n_j \{f_{Q_j+1}(b^n_j) - f_{Q_j+2}(b^n_j)\} \]
\[ + (j - 1) \cdot b^n_j f_{Q_j+1}(b^n_j) - j \cdot b^n_j f_{Q_j+1}(b^n_j) + v^n_j f_{Q_j+1}(b^n_j) \]
\[ = (v^n_j - b^n_j) \cdot f_{Q_j+1}(b^n_j) + (j - 1) \cdot \{F_{Q_j+1}(b^n_j) - F_{Q_j+2}(b^n_j)\}. \]

For a UPLA auction, (2.4) shows

\[ \frac{\partial \pi^n(b^n|v^n)}{\partial b^n_j} = j \cdot \{F_{Q_j+1}(b^n_j) - F_{Q_j+2}(b^n_j)\} + j \cdot b^n_j \{f_{Q_j+1}(b^n_j) - f_{Q_j+2}(b^n_j)\} \]
\[ + (j - 1) \cdot b^n_j f_{Q_j+1}(b^n_j) - j \cdot b^n_j f_{Q_j+1}(b^n_j) + v^n_j f_{Q_j+1}(b^n_j) \]
\[ = (v^n_j - b^n_j) \cdot f_{Q_j+1}(b^n_j) + j \cdot \{F_{Q_j+1}(b^n_j) - F_{Q_j+2}(b^n_j)\}. \]

For a Vickrey auction, (2.5) shows

\[ \frac{\partial \pi^n(b^n|v^n)}{\partial b^n_j} = -b^n_j f_{Q_j+1}(b^n_j) + v^n_j f_{Q_j+1}(b^n_j) = (v^n_j - b^n_j) \cdot f_{Q_j+1}(b^n_j). \]

For a QUP auction (2.6) implies

\[ \frac{\partial P^n(b^n)}{\partial b^n_j} = (j - 1) \int_0^\infty y \cdot f_{Q_j+1}R_{Q_j+2}(b^n_j, w) dy - j \cdot b^n_j \cdot f_{Q_j+1}(b^n_j) \]
\[ = (j - 1) f_{Q_j+1}(b^n_j) \int_0^\infty y f_{Q_j+1}R_{Q_j+2}(b^n_j, w) dy - j \cdot b^n_j \cdot f_{Q_j+1}(b^n_j) \]
\[ = (j - 1) \alpha E[R^n_{Q_j+1} | R^n_{Q_j-1} = b^n_j] - j \alpha b^n_j. \]

Thus, from (2.1)

\[ \frac{\partial \pi^n(b^n|v^n)}{\partial b^n_j} = (j - 1) \alpha E[R^n_{Q_j+2} | R^n_{Q_j-1} = b^n_j] - j \alpha b^n_j + \alpha v^n_j \]

which is equivalent to (3.11).
A.3 Proof of Proposition 3.3

Let $b^n$ be a local maximizer of $\pi^n(b^n|v^n)$ subject to $b^n_1 \leq \cdots \leq b^n_Q$. Assume, by way of contradiction, that there exists $j$ such that $b^n_j < v^n_j$ for some $k \leq j$. Let $j$ be the largest such index. Then, for all $j' = k, \cdots, j$, we have $b^n_{j'} \leq b^n_j < v^n_j$, implying $b^n_{j'} - v^n_{j'} < 0$. Then, from (3.7)-(3.11), it can be seen that $\partial \pi^n(b^n|v^n)/\partial b^n_j$ is strictly positive. The choice of $j$ implies $b^n_j < v^n_j \leq v^n_{j+1} \leq b^n_{j+1}$. Thus, we can perturb $b^n_j$ upward to obtain a better feasible proper bid vector, contradicting the local optimality of $b^n$.

Now consider the Vickrey, UPFR and QUP auctions. From the first part of this proposition, we know $b^n_1 \geq v^n_1$. If $b^n_1 > v^n_1$, define an alternative proper bidding strategy $b'^n$ as the same as $b^n$ except $b'^n_1 = v^n_1$, and compare two bidding strategies $b^n$ and $b'^n$. If $R^n_Q < v^n_1$ or $R^n_Q > b^n_1$, then there is no difference in profits. However, if $R^n_Q \in [v^n_1, b^n_1]$, then the difference in profit $\pi^n(b^n|v^n)$, $R^n_Q \in [v^n_1, b^n_1]) - \pi^n(b'^n|v^n)$, $R^n_Q \in [v^n_1, b^n_1]) = b^n_1 - v^n_1$ is strictly negative, and $b^n$ does not maximize $\pi^n(\cdot|v^n)$. Therefore, $b^n_1 = v^n_1$.

A.4 Proof of Proposition 3.4

For $b > 0$, the cdf and pdf of $R^n_j$ are

$$F_{R^n_j}(b) = 1 - e^{-b/\beta} \sum_{k=0}^{j-1} \frac{(b/\beta)^k}{k!} \quad \text{and} \quad f_{R^n_j}(b) = \frac{\beta^{-j} b^{-1} e^{-b/\beta}}{(j-1)!}.$$ 

For a QUP auction, by the memoryless property, $E[R_{Q-j+1} - R_{Q-j+1} | R_{Q-j+1} = b_j] = \beta$, which implies $b_j = v_j + (j-1)\beta$.

For a Discriminatory auction,

$$\frac{F_{R^n_{Q-j+1}}(b^n_j)}{f_{R^n_{Q-j+1}}(b^n_j)} = \frac{e^{-b^n_j/\beta} \sum_{k=0}^{Q-j} (b^n_j / \beta)^k / k!}{\beta^{-(Q-j+1)} (b^n_j)^{Q-j} e^{-b^n_j/\beta} / (Q-j)!} = \sum_{k=0}^{Q-j} (b^n_j)^k \beta^{-k} \frac{Q-j)!}{k!}.$$ 

Thus,

$$b^n_j = v^n_j \frac{\beta}{(b^n_j)^{Q-j} \sum_{k=0}^{Q-j} (Q-j)! (b^n_j)^k \beta^{Q-j-k}}.$$ 

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A unique solution exists since as $b_j^n$ increases, the RHS decreases strictly whereas the LHS increases strictly. Finding $b_j^n$ involves solving a polynomial of degree $Q - j + 1$.

For a UPFR auction, 

$$
F_{Q_{j+1}}(b_j^n) - F_{Q_j} (b_j^n) = \frac{e^{-b_j^n/\beta} \cdot (b_j^n/\beta)^{Q-j+1}}{(Q-j+1)!} 
$$

Thus, $b_j^n = v_j^n + (j - 1) \cdot \frac{b_j^n}{Q-j+1}$, or $b_j^n = \frac{Q-j+1}{Q-j+2} v_j^n$.

For a UPLA auction, 

$$
F_{Q_{j+1}}(b_j^n) - F_{Q_j} (b_j^n) = \frac{e^{-b_j^n/\beta} \cdot (b_j^n/\beta)^{Q-j}}{(Q-j)!} 
$$

$$
\frac{F_{Q_{j+1}}(b_j^n) - F_{Q_j} (b_j^n)}{f_{Q_{j+1}}(b_j^n)} = \frac{e^{-b_j^n/\beta} \cdot (b_j^n/\beta)^{Q-j}/(Q-j)!}{\beta^{-(Q-j+1)}(b_j^n)^{Q-j} e^{-b_j^n/\beta}/(Q-j)!} = \beta. 
$$

Therefore, $b_j^n = v_j^n + j \beta$.

### A.5 Necessity of Conditions in Proposition 3.5

Each of two conditions in Proposition 3.5 is necessary. We illustrate this using two examples. The first example violates $\frac{\partial}{\partial b_j} \rho(j, b_j^n) \leq \frac{1}{j - 1}$. Suppose $Q = 2$ and $N = 3$. Bidder 1 has values $v_1^1 = 1$ and $v_1^2 = 2.2$. He faces two bidders, each of them having only one finite value. Bidder 2’s value is distributed $V_2^2 \sim \text{Uniform}[2, 4]$, and bidder 3’s value $V_3^3$ assumes either 3 with probability $\frac{1}{2}$ or 4 with probability $\frac{1}{2}$. By Proposition 3.3, we know the first bid of each bidder is the same as his first value. One can easily show

$$
\rho(2, b_2^1) = E[R_2^1 - R_1^1 | R_1^1 = b_2^1] = \begin{cases} 
3.5 - b_2^1 & \text{if } b_2^1 \in [2, 3) \\
4 - b_2^1 & \text{if } b_2^1 \in (3, 4].
\end{cases}
$$

We note $\rho(2, b_2^2)$ has a positive jump at $b_2^2 = 3$. Then, we get

$$
\frac{\partial}{\partial b_2} \pi^1(b_2^1) = \alpha \{ \rho(2, b_2^2) - (b_2^2 - v_2^1) \} = \begin{cases} 
\alpha(5.7 - 2b_2^1) & \text{if } b_2^1 \in [2, 3) \\
\alpha(6.2 - 2b_2^1) & \text{if } b_2^1 \in (3, 4].
\end{cases}
$$
which vanishes at $b_2^1 = 2.85$ and $b_2^3 = 3.1$. Thus there are multiple candidates for $b_2^1$. Let $b' = (1, 2.85)$ and $b'' = (1, 3.1)$. It can be shown $\pi^1(b') > \pi^1(b'')$, implying that $b''$ is not optimal.

In the second example, we show that $\frac{\partial}{\partial b} \rho(j, b) \leq 0$, a stronger condition than $\frac{\partial}{\partial b} \rho(j, b) \leq \frac{1}{j-1}$, does not ensure the bid vector obtained by (3.13) is proper. Suppose $Q = 3$ and $N = 4$. Bidder 1 has values $(v_1^1, v_1^2, v_1^3) = (1, 2, 3)$, and his competing bidders have values distributed as

$$V_1^2 = 3; \quad V_1^3 \sim U[3, 5]; \quad V_1^4 \sim U[5, 9];$$

Bidders 2, 3 and 4 have only one finite value each. We observe

$$\rho(2, b_2^1) = E[V_1^4 - V_1^3 | V_1^3 = 2] = 4,$$

is constant and thus nondecreasing in $b_2^1$. Since

$$\frac{\partial}{\partial b_2^1} \pi^1(b^1 | v^1) = \alpha\{\rho(2, b_2^1) - (b_2^1 - v_2^1)\} = \alpha\{4 - (b_2^1 - v_2^1)\},$$

$\pi_2^1(b^1)$ is maximized at $b_2^1 = 6$. Similarly, $\frac{\partial}{\partial b_3^3} \pi^1(b^3) = \alpha\{2\rho(3, b_3^3) - (b_3^3 - v_3^3)\}$ where $\rho(2, b_3^3) = E[V_1^3 - V_1^2 | V_1^2 = 3] = 1$. Hence $\frac{\partial}{\partial b_3^3} \pi^1(b^3) = \alpha\{2 - (b_3^3 - v_3^3)\}$ is decreasing in $b_3^3$, implying that $\pi_3^3(b^3)$ is maximized at $b_3^3 = 5$. Thus, $(b_1^1, b_2^1, b_3^3)$ obtained by setting (3.11) to 0 is not a proper bid vector.