On “Optimal bidding in a uniform price auction with multi-unit demand”

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Abstract

In a multiple-unit, uniform-price sealed-bid auction where price is the lowest accepted bid, we show that the ordering constraint between bids may become active. We point out that Draaisma and Noussair (1997)’s partial characterization of symmetric, strictly monotone equilibria neglects this constraint.

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1 Bid-Pooling

We study multiple-unit auctions in which $k \geq 2$ units are for sale, and bidders have symmetric independent private values. If each bidder has use for only one unit, the multiple-unit auction inherits most of the theoretical properties of the single-unit auction (?? (rjw); Branco (1996); Vickrey (1961)). papers study the bidding strategies of multiple-unit auctions when each bidder demands two units. Vickrey (1961) show the incentive-compatibility of the multiple-unit Vickrey auction. Engelbrecht-Wiggans and Kahn (1998) study the pay-your-bid discriminatory auction. Noussair (1995) and Engelbrecht-Wiggans (1999) study the uniform-price auction (UPFR auction) where price is the first-rejected (highest-rejected) bid. Draaisma and Noussair (1997) study another version of the uniform-price auction (UPLA auction) where price is the last-accepted (lowest-accepted) bid.

Engelbrecht-Wiggans and Kahn (1998) shows that in the discriminatory auction, the pooling of bids, or the submitting of identical bids when the valuation of two goods are different, can occur. However, in the UPFR auction, Engelbrecht-Wiggans (1999) shows that the first bid of each bidder is truth-telling and the second bid is shaded. Thus, pooling cannot occur in the UPFR auction. In the UPLA auction, we show by example that pooling can occur. It is contrary to the implicit assumption in Draaisma and Noussair (1997) that the bid-ordering constraint is inactive.

Draaisma and Noussair (1997) report a partial characterization of a class of symmetric, strictly monotonic equilibria to a multi-unit UPLA auction. Each bidder $i$ simultaneously submits two bids, denoted by $b^i_1$ and $b^i_2$, where $b^i_1 \geq b^i_2$. Let $v^i_1$ be the higher valued unit, and $v^i_2$ be the lower valued unit. A pure strategy is a mapping from valuations into bids $B(v_1, v_2) = (B_1(v_1, v_2), B_2(v_1, v_2))$, where $B_1(v_1, v_2) \geq B_2(v_1, v_2)$. They present the first-order necessary conditions, from which it is shown that any strictly monotone symmetric equilibrium strategy $B^*(\cdot, \cdot)$ is separable, i.e., $B^*(v_1, v_2) = (B^*_1(v_1), B^*_2(v_2))$.

Let $E\pi^i(b^i_1, b^i_2)$ be the expected profit of bidder $i$ with values $(v^i_1, v^i_2)$ and bids $(b^i_1, b^i_2)$. Draaisma and Noussair (1997) correctly show that $\frac{\partial}{\partial b^j_j} E\pi^i(b^i_1, b^i_2)$ is independent of $b^j_j$, where $j' \neq j$. However, they conclude without justification that $b^j_j = B^*_j(v^j_j)$ is independent of \hfill 2
Such a conclusion is correct in the case of unconstrained optimization, but not in a general constrained optimization. They have neglected the inequality constraint $b^i_1 \geq b^i_2$ in the derivation of the Kuhn-Tucker condition. (See Page 159, (2)). Consequently, their results (Lemma 1 and Proposition 1) are valid only when the constraint $b^i_1 \geq b^i_2$ is redundant.

2 Example

In this section we present an example in which $b^i_1 \geq b^i_2$ is not redundant, and a weakly monotone symmetric equilibrium strategy may not be separable. While this example does not disprove the Draaisma and Noussair (1997) result, we show non-separability of bids, and thus the pooling of bids can occur.

Suppose there are two symmetric bidders indexed by $i = 1, 2$. There are two units for sale. The values $(V^i_1, V^i_2)$ of bidder $i$ have continuous densities and compact supports $[\frac{1}{2}, 1]$ and $[0, \frac{1}{2}]$, respectively. Let $B^*(v_1, v_2)$ be a strictly monotone (i.e., increasing) symmetric equilibrium strategy. We first prove that in this example, there exists no strictly monotone bidding strategy that is separable.

Assume, by the way of contradiction, the separability of $B^*(v_1, v_2) = (B^*_1(v_1), B^*_2(v_2))$ as in Lemma 1 of Draaisma and Noussair (1997). Let $(D^i_1, D^i_2) = (B^*_1(V^i_1), B^*_2(V^i_2))$ be the \textit{ex ante} distribution of bids for bidder $i$. Let $H_1$ be the cumulative density function of $D^1_1$ (and also of $D^2_1$ by symmetry). Similarly, let $H_2$ be the cumulative density function of $D^2_1$ (and also of $D^2_2$). Let $h_1$ and $h_2$ be the corresponding probability density functions. It is easy to verify $B^*_1(v_1) \leq v_1$ and $B^*_2(v_2) \leq v_2$ for all $v_1$ and $v_2$, and it follows that both $D^1_1$ and $D^2_1$ have bounded supports. By the strict monotonicity of $B^*(\cdot, \cdot)$ and the continuity of $(V^i_1, V^i_2)$’s density, $H_1$ strictly increases from 0 to 1 in $[B^*_1(\frac{1}{2}), B^*_1(1)]$, while $H_2$ strictly increases from 0 to 1 in $[B^*_2(0), B^*_2(\frac{1}{2})]$.

Now we claim $B^*_1(1) = B^*_2(\frac{1}{2})$, i.e., the right endpoints of the distribution of bids $H_1$ and $H_2$ are the same. To see this, we recall the partial derivatives of bidder $i$’s expected profit
as given in expression (2) of Draaisma and Noussair (1997), for $b_1^1, b_2^1 > 0$:

$$\frac{\partial E\pi^i(b_1^1, b_2^1)}{\partial b_1^i} = (v_i^1 - b_i^1)h_2(b_1^i) - (H_2(b_1^i) - H_1(b_1^i)), \quad \text{and}$$

(1)

$$\frac{\partial E\pi^i(b_1^1, b_2^1)}{\partial b_2^i} = (v_i^2 - b_i^2)h_1(b_2^i) - 2(H_1(b_2^i)). \quad \text{(2)}$$

Let $t_1 = B_1^*(1)$ and $t_2 = B_2^*(\frac{1}{2})$. If $t_1 < t_2$, there is positive probability that $b_2 \in (t_1, t_2)$, violating the constraint $b_1 \geq b_2$ since $b_1 \leq t_1$. If $t_1 > t_2$, then for $b_1 \in (t_2, t_1)$, the first term in (1) is 0 since $h_2(b_1) = 0$. The second term is strictly negative, implying that perturbing $b_1$ downward improves the expected profit. Thus, we conclude $t_1 = t_2$. This result can be informally obtained by observing that one bidder’s first bid competes with the other bidder’s second bid.

Let $v_1 = \frac{1}{2}$ and $v_2 = \frac{1}{2}$. If we assume the strict monotonicity of $B_1^*$, we obtain

$$B_1^*(v_1) < B_1^*(1) = t_1 = t_2 = B_2^*(\frac{1}{2}),$$

violating the bid-ordering constraint $B_1^*(v_1) \geq B_2^*(v_2)$ for all $v_1 \geq v_2$. Thus, $B^*$ is not a separable function. This completes the proof.

In particular, suppose $(V_1^1, V_2^1), (V_1^2, V_2^2) \sim (U[\frac{1}{2}, 1], U[0, \frac{1}{2}])$. We show that weakly monotone symmetric functions $B_1^*$ and $B_2^*$ satisfying the first-order conditions (1) and (2) may violate the inequality constraint $b_1 \geq b_2$. Let

$$B_1^*(v_1) = \begin{cases} \frac{2}{3}v_1 + \frac{1}{24} = \frac{2}{3}(v_1 - \frac{1}{2}) + \frac{3}{8} & \text{if } v_1 \in \left[\frac{1}{2}, \frac{9}{16}\right] \\ \frac{5}{12} & \text{if } v_1 \in \left[\frac{9}{16}, 1\right]\end{cases}$$

$$B_2^*(v_2) = \begin{cases} v_2 & \text{if } v_1 \in \left[0, \frac{3}{8}\right] \\ \frac{1}{3}v_2 + \frac{1}{4} = \frac{1}{3}(v_2 - \frac{3}{8}) + \frac{3}{8} & \text{if } v_1 \in \left[\frac{3}{8}, \frac{1}{2}\right].\end{cases}$$
It follows

\[
H_1(b_1) = \begin{cases} 
0 & \text{if } b_1 \in [0, \frac{3}{8}] \\
3(b_1 - \frac{3}{8}) & \text{if } b_1 \in \left[\frac{3}{8}, \frac{5}{12}\right) \\
1 & \text{if } b_1 \in \left[\frac{5}{12}, 1\right]
\end{cases} 
\]

\[
H_2(b_2) = \begin{cases} 
2b_2 & \text{if } b_2 \in [0, \frac{3}{8}] \\
6(b_2 - \frac{3}{8}) + \frac{3}{4} & \text{if } b_2 \in \left[\frac{3}{8}, \frac{5}{12}\right] \\
1 & \text{if } b_2 \in \left[\frac{5}{12}, 1\right].
\end{cases}
\]

It is easy to verify the first-order conditions (i.e., setting (1) and (2) equal to zero) when
\[v_1 \in \left(\frac{1}{2}, \frac{9}{16}\right)\text{ and } v_2 \in \left(\frac{3}{8}, \frac{1}{2}\right)\text{ (corresponding to } b_1, b_2 \in \left(\frac{3}{8}, \frac{5}{12}\right))\]. Indeed, \(B_1^*\) and \(B_2^*\) maximize the unconstrained objective function \(E\pi_i\). However, with positive probability, the constraint \(b^1 \geq b^2\) is violated, and it follows that \(B_1^*\) and \(B_2^*\) are not separable.

We remark that \(B_1^*\) and \(B_2^*\) are neither strictly monotone nor feasible. Thus, this example is not a counter-example of the Draaisma and Noussair (1997) results. It points out that their proof is incomplete.

References