The Martingale Evolution of Price Forecasts in a Supply Chain Market for Capacity: Technical Report

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Abstract

We develop a continuous time analog of the Martingale Model of Forecast Evolution (MMFE) and show that the discrete time MMFE is a special case of our model. We apply the continuous time MMFE to model the forecast evolution of the instantaneous rate of demand in a market for capacity. Using the Stochastic Maximum Principle, we show that the equilibrium market price and the optimal market production rate evolve as martingales. Finally, we study the relationship between the resolution of the demand uncertainty and the resolution of the market price uncertainty. We find that the rate of resolving price uncertainty increases as the rate of resolving demand uncertainty increases. Also, we find that the resolution of price uncertainty occurs more uniformly over time if the cost of overtime/undertime increases.

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1 Introduction

The Martingale Model of Forecast Evolution (MMFE), developed by Graves et al. (1986) and Heath and Jackson (1994), provides a framework to model evolution of forecasts in a discrete time setting. Heath and Jackson (1994) assume that forecast update in any period for demand in a future period is a normally distributed zero-mean random variable which is independent of previous forecast updates. This assumption implies that the forecasts evolve as a Martingale process, thus lending the model its name. Hausman (1969) provides the pioneering effort in the development of MMFE by suggesting that the series of ratios of successive forecasts might evolve as a quasi-Markovian system. Several research models in the supply chain management have used the MMFE to capture evolution of forecasts. (See Iida and Zipkin (2003), Lu et al. (2003), Dong and Lee (2001), Gullu (1996), Toktay and Wein (2001).) In this paper, we apply the MMFE within the context of a market for supply chain capacity.

In this paper, we develop the continuous time analog of the additive MMFE model. In our model, we assume that the forecast for the *rate of demand* at any instant $t, 0 < t < \infty$, evolves as a continuous Martingale process over $[0, t]$. The forecast at any instant s of the rate of demand for any $t \geq s$, is thus equal to the conditional expectation of the rate of demand given all the available information until s.

In the second half of the paper, we apply the continuous time MMFE model to study the evolution of the equilibrium price in a market for capacity. Our goal is to study how the resolution of demand uncertainty translates to the resolution of uncertainty of the equilibrium price. Using the Stochastic Maximum Principle, we obtain closed form solutions for optimal production and market price trajectories and show that they evolve as Martingales. We also derive the forecasting process corresponding to the equilibrium variables. This allows us to observe how the resolution in exogenous uncertainty which affects the market through the demand process, filters through to the optimal variables. We find that as the rate of resolution of demand uncertainty increases, the rate of resolution of price uncertainty also increases.

We also study the impact of supply chain cost parameters on the the rate of resolving price uncertainty. In a surprising result, we find that the resolution of uncertainty for the market price gets affected by the cost parameters. For a fixed rate of resolution of demand uncertainty, the resolution of price uncertainty occurs more uniformly over time as the cost of overtime/undertime increases with respect to the cost of holding inventory. On the other hand, if the cost of overtime/undertime reduces relative to holding/shortfall cost, then the resolution of price uncertainty is less uniform over time. In fact, the rate of resolution of uncertainty increases as the instant of the realization of the price draws closer.

Rest of the paper is structured as follows. In Section 2, we develop the continuous time analog of the additive Martingale Model of Forecast Evolution. In Section 3, we present an application of the continuous time MMFE to a market for capacity. In Section 4, we discuss the resolution of uncertainty in price and conclude in Section 5.

2 Continuous Time Martingale Model of Forecast Evolution

We develop a Weiner's process model for the continuous resolution of forecasts for the exogenous demand of a generic good in a generic market. Let $f(s,t)$ denote the forecast at time s of the rate of demand of that good at time t and, let $d_s f(s,t)$ denote the forecast update at time s. Thus $f(t,t)$ represents the actual demand at time t. Let $\mathbf{W}(\cdot)(\equiv (W_1(\cdot), W_2(\cdot), ..., W_n(\cdot))$ be an $n-$dimensional Weiner's process defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and let $\mathcal{F}_t$ be the filtration.

Assumption 2.0.1. $\mathcal{F}_t = \sigma\{\mathbf{W}(s) : 0 \leq s \leq t\}$. *Further,* $\mathcal{F}_0$ *is* $\mathcal{P}-$*complete and* $\mathcal{P}-$ *degenerate.*

Compared to the assumption on the filtration in Heath and Jackson (1994), Assumption 2.0.1 is more restrictive. However, as we shall see, the specification of filtration automatically ensures that $d_s f(s,t)$ is uncorrelated with $d_{s_1} f(s_1,t)$, for $s_1 < s$.

We assume that the forecasting process, $f(\cdot,t)$ evolves as a Martingale. Therefore, the forecast at time s of the rate of demand at t is the conditional expectation of $f(t,t)$ given the information in $\mathcal{F}_s$. We formally state the second assumption as follows.

Assumption 2.0.2. *$f(\cdot,t) : \Omega \times [0,t] \to \mathcal{R}$ is a square-integrable Martingale for any given $t \in [0,\infty)$.*

Using Theorem 4.15, Karatzas and Shreve (1991), if $f(s,t)$ is in $\mathcal{L}^2$ for every $s \leq t$ and is measurable with respect to $\mathcal{F}_s$, then there exist progressively measurable $\sigma_i \in \mathcal{L}^2, i = 1,..,n$ such that:

$$\begin{aligned} f(t,t) &= f(0,t) + \sum_{i=1}^{n} \int_0^t \sigma_i(s,t) dW_i(s) \\ &= f(0,t) + \int_0^t \sigma(s,t) \mathbf{dW}(s)^T, \end{aligned}$$

where $\sigma =(\sigma_1, \sigma_2, ..., \sigma_n)$. Hence,

$$f(s,t) := E(f(t,t)|\mathcal{F}_s) = f(0,t) + \int_0^s \sigma(u,t)\mathbf{dW}(u)^T,$$

which implies,

$$d_s f(s,t) = \sigma(s,t)\mathbf{dW}(s)^T = \sum_{i=1}^{n} \sigma_i(s,t) dW_i(s).$$

Remark 1: Observe that at this stage, unlike Heath and Jackson (1994), we do not assume that $d_s f(s,t)$ is stationary in $t - s$. We shall, however, make this assumption later when specifying $\sigma(s,t)$.

Remark 2: It is easily seen that $d_s f(s,t)$ is uncorrelated with $d_{s_1} f(s_1,t)$ for $s_1 < s$.

2.1 Additive Model

For the rest of this section, we shall assume $\sigma(\cdot, t)$ is independent of ω. Let $F(s,t)$ be the forecast at s of the cumulative demand until $t \geq s$. Thus $F(t,t)$ represents the actual cumulative demand until t. Define the realized demand over an infinitesimal interval $(t, t+dt)$ as,

$$dF(t,t) = f(t,t)dt + \sigma(t,t)\mathbf{dW}(t)^T. \tag{2.1}$$

Note that we refer the drift term in the equation for instantaneous demand as the *rate of demand*. Indeed, our definition for the rate of demand is special since rate of demand multiplied by the length of the infinitesimal interval is *not* the same as the total demand during the interval.

We assume that the diffusion coefficient $\sigma(\cdot,\cdot)$ in the definition of cumulative demand (2.1) has the same functional form as the forecast update coefficient. Indeed, this is not necessary for any of the results in this paper to hold true. We make this assumption for the following intuitive reason. At any time instant s, the forecast update for the rate of demand at $t > s$ has a variance proportional to $\sigma^2(s,t)$. Therefore, we assume that the variance of the uncertainty in demand over $(t, t+dt)$ that is resolved at the last instant be proportional to $\sigma^2(t,t)$ and hence the diffusion coefficient.

The expression for $F(s,t)$ is obtained as,

$$\begin{aligned} F(s,t) &= \int_0^s \int_0^v \sigma(u,v)\mathbf{dW}(u)^T dv + \int_0^s \sigma(v,v)\mathbf{dW}(v)^T + \int_0^t f(0,v)dv \\ &+ \int_s^t \int_0^s \sigma(u,v)\mathbf{dW}(u)^T dv. \end{aligned}$$

It is easily seen that $F(\cdot,t)$ evolves as a Martingale. We state the result formally in the following corollary.

Corollary 2.1.1. *$F(\cdot,t) : \Omega \times [0,t] \to \mathcal{R}$ is a Martingale Process.*

Proof. See the Appendix. □

In the following lemma, we present a result which will help us in comparing our model to the discrete time MMFE model.

Lemma 2.1.2. $Cov(f(s,t), f(s,t_1)) = \int_0^s \sigma(u,t)\sigma^T(u,t_1)du, s < t \leq t_1$.

Proof.

$$\begin{aligned} Cov(f(s,t), f(s,t_1)) &= E\left(\int_0^s \sigma(u,t)\mathbf{dW}(u)^T\right)\left(\int_0^s \sigma(u,t_1)\mathbf{dW}(u)^T\right) \\ &= \int_0^s \sigma(u,t)\sigma^T(u,t_1)du. \end{aligned}$$

□

So, we are led to describe the model of forecast evolution as,

$$\begin{aligned} F(s,t) &= \int_0^s \int_0^v \sigma(u,v)\mathbf{dW}(u)^T dv + \int_0^s \sigma(v,v)\mathbf{dW}(v)^T + \int_0^t f(0,v)dv \\ &+ \int_s^t \int_0^s \sigma(u,v)\mathbf{dW}(u)^T dv. \end{aligned}$$

We refer to this model as the Continuous-Time Martingale Model of Forecast Evolution (CT-MMFE).

2.2 Relationship with the Discrete Time Model

In this subsection, we establish the relationship between CTMMFE and the discrete time MMFE model. In particular, we show that the model presented in Heath and Jackson Heath and Jackson (1994) is a special case of CTMMFE. To obtain results in this section, we need the following result in order to be able to interchange the stochastic and the Lebesgue integral. Let $\mathcal{C}([0,T]\times[0,T];\mathcal{R}^n)$ be the set of all continuous functions $\Phi : [0,T]\times[0,T] \to \mathcal{R}^n$. We state the result for a one-dimensional Weiner's process only, but the result can be easily extended to an n - dimensional Weiner's process.

Proposition 2.2.1. *(A special case of Lemma 4.1, Ikeda and Watanabe (1989), pp116): Let* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ *be a filtered probability space and let* $W(\cdot)$ *be Weiner's process defined on it. Let* $Q_1 \in \mathcal{C}([0,T] \times [0,T]; \mathcal{R}^1)$*. Then,*

1.

$$\int_0^u \int_0^t Q_1(s,t) dW(s) dt = \int_0^u \int_s^u Q_1(s,t) dt dW(s), \tag{2.2}$$

2.

$$\int_0^{u_1} \int_0^{u_2} Q_1(s,t) dW(s) dt = \int_0^{u_2} \int_0^{u_1} Q_1(s,t) dt dW(s).$$

Proof. See the Appendix. □

For ease of exposition only, the lower bound of the integrals is 0 on the LHS in (2.2). The proof can be extended easily to the case when the lower bound of the integrals is strictly positive.

For any $s, t_1, t_2 \in \mathcal{N}$ such that $s + 1 \leq t_1 \leq t_2$, let δ_{s,t_1} be the forecast at s of the cumulative demand occurring in $(t_1 - 1, t_1]$, and let δ_{s,t_2} be the forecast at s of cumulative demand occurring in $(t_2 - 1, t_2]$. Therefore,

$$\begin{aligned} \delta_{s,t_1} &= F(s,t_1) - F(s,t_1 - 1) = \int_{t_1-1}^{t_1} f(0,v) dv + \int_{t_1-1}^{t_1} \int_0^s \sigma(u,v) \mathbf{dW}(u)^T dv, \\ \delta_{s,t_2} &= F(s,t_2) - F(s,t_2 - 1) = \int_{t_2-1}^{t_2} f(0,v) dv + \int_{t_2-1}^{t_2} \int_0^s \sigma(u,v) \mathbf{dW}(u)^T dv. \end{aligned}$$

The variable δ_{s,t_j} represents the forecast of demand in (discrete) time period j made at time s. In the following lemma, we obtain the covariance of forecasts for two periods in the future.

Lemma 2.2.2. *Let* $\sigma(\cdot,\cdot) \in \mathcal{C}([0,T] \times [0,T]; \mathcal{R}^n)$*. Then,*

$$Cov(\delta_{s,t_1}, \delta_{s,t_2}) = \int_0^s \left(\int_{t_1-1}^{t_1} \sigma(u,z) dz \right) \left(\int_{t_2-1}^{t_2} \sigma(u,z) dz \right)^T du.$$

Proof.

$$\begin{aligned} Cov(\delta_{s,t_1}, \delta_{s,t_2}) &= Cov\left(\int_{t_1-1}^{t_1}\int_0^s \sigma(u,z)\mathbf{dW}(u)^T dz, \int_{t_2-1}^{t_2}\int_0^s \sigma(u,z)\mathbf{dW}(u)^T dz\right) \\ &= Cov\left(\int_0^s\int_{t_1-1}^{t_1} \sigma(u,z)dz\mathbf{dW}(u)^T, \int_0^s\int_{t_2-1}^{t_2} \sigma(u,z)dz\mathbf{dW}(u)^T\right), \end{aligned}$$

where

$$\int_0^s\int_{t_1-1}^{t_1} \sigma(u,z)dz\mathbf{dW}(u)^T = \sum_{i=1}^{n}\int_0^s\int_{t_1-1}^{t_1} \sigma_i(u,z)dzdW_i(u)^T.$$

The last step follows using Proposition 2.2.1. Therefore,

$$Cov(\delta_{s,t_1}, \delta_{s,t_2}) = \int_0^s \left(\int_{t_1-1}^{t_1} \sigma(u,z)dz\right)\left(\int_{t_2-1}^{t_2} \sigma(u,z)dz\right)^T du.$$

□

The above result illustrates that the forecasts of demand in different discrete time periods have a well-defined (deterministic) correlation structure.

2.2.1 Exponential Models

In this subsection, we assume that $\sigma_i(u,t)$ is an exponential function of $t-u$. The model in this subsection is identical to the exponential model considered by Heath and Jara (2003). According to the Stone-Weierstrass Theorem, the algebra of functions $\{e^{-\lambda v} : \lambda \geq 0\}$ can approximate, as closely as desired, any continuous function defined on $[a,b]$ for any $0 \leq a < b$. Therefore, any continuous σ_i can be approximated arbitrarily closely by

$$\sigma_i(u,t) \approx \sum_{l=1}^{m} \xi_{il}\exp(\lambda_l(u-t)), \tag{2.3}$$

for suitably large m. Note that the same set of $\{\lambda_l, 1 \leq l \leq m\}$ can be used to approximate all the σ_i.

Corollary 2.2.3. *Let $\sigma_i(u,t)$ be approximated as in (2.3). Then, for $s, t_1, t_2 \in \mathcal{N}$, $s+1 \le t_1 \le t_2$:*

$$Cov(\delta_{s,t_1}, \delta_{s,t_2}) = \sum_{i=1}^{n} \sum_{1 \le l \le p \le m} \frac{\xi_{il}\xi_{ip} e^{-(\lambda_l t_1 + \lambda_p t_2)}(e^{(\lambda_l+\lambda_p)s} - 1)(1 - e^{\lambda_l})(1 - e^{\lambda_p})}{\lambda_l \lambda_p (\lambda_l + \lambda_p)}.$$

Next, we show that the forecast update over a discrete length of time can be written as a sum of normally distributed random variables. Following the same notation as in Heath and Jackson (1994), let $\varepsilon_{s,t_1} = \delta_{s,t_1} - \delta_{s-1,t_1}, s \le t_1 - 1$ be the forecast update during $(s-1, s]$ for demand in $(t_1 - 1, t_1]$ where $s + 1 \le t_1$. Then:

$$\begin{aligned}
\varepsilon_{s,t_1} &= \int_{t_1-1}^{t_1} \int_{s-1}^{s} \sigma(u,z) \mathbf{dW}(u)^T dz \\
&= \int_{s-1}^{s} \left(\int_{t_1-1}^{t_1} \sigma(u,z) dz \right) \mathbf{dW}(u)^T
\end{aligned}$$

where the last step follows using Proposition 2.2.1. Now, assuming σ_i is approximated as in (2.3):

$$\begin{aligned}
\varepsilon_{s,t_1} &= \sum_{i=1}^{n} \int_{s-1}^{s} \int_{t_1-1}^{t_1} \sum_{l=1}^{m} \xi_{il} e^{\lambda_l(u-z)} dz dW_i(u) \\
&= \sum_{i=1}^{n} \int_{s-1}^{s} \sum_{l=1}^{m} \frac{\xi_{il}}{\lambda_l} (e^{\lambda_l(u-t_1+1)} - e^{\lambda_l(u-t_1)}) dW_i(u) \\
&= \int_{s-1}^{s} \mathbf{c}_{t_1}(u) \mathbf{dW}(u)^T
\end{aligned}$$

where $\mathbf{c}_{t_1}(u)$ is a $1 \times n$ vector whose i^{th} element is given by:

$$\sum_{l=1}^{m} \frac{\xi_{il}}{\lambda_l} (\exp(\lambda_l(u - t_1 + 1)) - \exp(\lambda_l(u - t_1)))$$

which is of similar nature as in Heath and Jackson (1994). Using "time-Substitution" (McKean (1969)),

$$\varepsilon_{s,t_1} = \sum_{i=1}^{n} W_i(H_i(s,t_1))$$

where

$$H_i(s,t_1) = \int_{s-1}^{s} \left(\sum_{l=1}^{m} \frac{\xi_{il}}{\lambda_l} (\exp(\lambda_l(u - t_1 + 1)) - \exp(\lambda_l(u - t_1))) \right)^2 du,$$

implying that ε_{s,t_1} is normally distributed with mean 0 and variance $\sum_{i=1}^{n} H_i(s,t_1)$.

It is easily seen that any finite-dimensional variance-covariance matrix for discrete-time can be approximated by a suitable choice of the parameters $\{\xi_{il}, \lambda_l\}$. Following Heath and Jackson (1994), let Σ denote the variance-covariance matrix of forecast updates at time s. The following corollary provides the entries of Σ.

Corollary 2.2.4. *Let $\sigma_i(u,t)$ be approximated as in (2.3). Then, for $s, t_1, t_2 \in \mathcal{N}$, $s+1 \le t_1 \le t_2$, the covariance of forecast updates for t_1 and t_2 is equal to:*

$$Cov(\varepsilon_{s,t_1}, \varepsilon_{s,t_2}) = \sum_{i=1}^{n} \sum_{1 \le l \le p \le m} \frac{\xi_{il}\xi_{ip} e^{\lambda_l(s-t_1)+\lambda_p(s-t_2)} \begin{pmatrix} e^{-(\lambda_l+\lambda_p)} - e^{\lambda_p} + e^{-\lambda_l} \\ -e^{\lambda_l} + e^{-\lambda_p} + e^{\lambda_l+\lambda_p} \end{pmatrix}}{\lambda_l \lambda_p (\lambda_l + \lambda_p)}.$$

The above corollary illustrates that the covariance of forecast updates at $t_1, t_2 \ge s+1$ depends only on "time-to-go", that is, $t_1 - s$ and $t_2 - s$, which is in alignment with the stationary nature of approximation (2.3) for σ. The link between the CTMMFE and the discrete time MMFE of Heath and Jackson (1994) should now be clear, with the addition of assumptions of the exponential structure of $\sigma_i, i = 1 \cdots n$, and the stationarity of this structure, we can derive the discrete time MMFE as a special case of CTMMFE.

3 Application of MMFE: The Market Model for Capacity

In this section, we present an application of the continuous time MMFE model. We consider a simple, continuous-time market for the commitment of production capacity to the actual production of a single homogeneous product. The market has numerous agents of two types: owners of capacity and customers of capacity. The customers of the capacity (whom we refer to as "buyers") form an interface between the owners of the capacity (whom we refer to as "sellers") and the end-consumers of the product. We restrict our attention to interactions between sellers and buyers, and the end-consumers and their demand are an exogenous component to this model. We assume that all the agents are rational and risk-neutral but do not play any games. Because of the nature of the model, consideration of game-theoretic aspects is not likely to add much value to the model. In a finite horizon model with repeated interactions among agents, combined with inability of agents to influence prices; the likelihood of strategic behavior diminishes significantly.

In the market for capacity, the equilibrium price at every instant is determined by the balance of supply and demand for capacity. Each of the buyers determines the instantaneous rate of order placement to minimize the sum of her expected purchasing cost and the expected inventory holding/shortage cost over a finite horizon. We assume that the price paid by a buyer in the capacity market does not affect the price charged by the buyer in the consumer market. Therefore, the revenue earned by the buyer in the consumer market is not part of the objective function for the buyer.

Similarly, each seller chooses the instantaneous rate of production to minimize the expected cost of capacity-production mismatch less the revenue generated by the sale of capacity to buyers. Sellers are penalized if the rate of production is not equal to the installed capacity. Production beyond installed capacity may be carried out by overtime, extra shifts, etc. Underutilization of the capacity also incurs a penalty. The sole source of revenue for sellers is the purchases made by the buyers in the capacity market.

In the market for capacity, the function of carrying inventory/incurring backorders is implemented by buyers only: sellers do not hold inventory. The oil industry provides an example of such a market structure. Crude oil sellers (who own oil wells) can adjust oil production to handle the uncertainty in the demand of buyers. The markets for other natural resources such as metals, animal products and natural gas also fall into the same category. The phenomenon of outsourcing is also giving rise of such sellers/capacity owners who produce as per the orders placed by the customers and operate in a make-to-order fashion.

Each buyer estimates the consumer demand. We use the continuous time Martingale Model of Forecast Evolution to model the end-consumer demand faced by any buyer. Therefore, the estimate of the rate of end-consumer demand for any time instant can be decomposed into two components. The first component is deterministic and is equal to the forecast of the rate of demand at the beginning of the horizon. The second component is stochastic and comprises the forecast updates until that instant.

Any buyer's decision regarding the order quantity is dependent on future prices in the capacity market. If the buyer expects the price to be high in the future, she will place an order for a higher quantity; vice-versa when she expects the price to drop. We invoke the hypothesis of Rational Expectations (Muth Muth (1961)) in order to model how prices evolve in the capacity market. Under this approach, each agent views the evolution of prices as an exogenous stochastic process and plans his or her actions relative to a particular specification of that process. For a rational agent to adopt or select a particular specification of the price process, that specification must be consistent with market equilibrium. In particular, we assume that each agent possesses perfect information and is capable of deducing what actions all other agents would plan if those agents planned according to the same price process specification that he or she has chosen. From this, each agent could deduce whether the specified process process will result in supply-demand imbalances in the present or at any time in the future. No rational agent will adopt a price process specification that can be deduced to result in dis-equilibrium if adopted by all. Under the rational expectations hypothesis, therefore, we restrict attention to those price processes which will result in equilibrium, if adopted by all. Furthermore, no rational agent will adopt a price process specification that is different from that adopted by the majority of other agents since the realized price process will tend to follow that adopted by the majority. Based on this argument, we assume that all agents base their plans on the same price process specification and that this specification has the property that it clears the market, in the sense of aggregated planned actions of supply and demand, at all instances in the present and future. We also assume the existence of powerful, rapid, market mechanisms such as arbitrage that force the realized price process to follow the universally adopted price process specification, Clearly, this is an idealized view of a market economy but it brings the analysis of price behavior within the scope of the tools of stochastic control.

Rational Expectations theory was first proposed by John Muth (Muth (1961)) in 1961; Sargent (2003) provides a concise but rich introduction to the theory of rational expectations. This theory lends structure to the expectation formation by economic agents and its subsequent impact on decision making by the agents and on outcomes. In an uncertain environment with multiple decision makers where the outcome can not be dictated by a single agent, this theory provides a valuable foundation on which economic models can be built. An economic model based in a market setting is one example of such an application. This is the primary reason we have invoked this hypothesis to construct and analyze our model.

The theory of rational expectations has been used to analyze various economic situations, examples being the efficient markets theory of asset prices, the permanent income theory of consumption, and the price evolution of storable commodities (Sargent (2003)). We discuss the literature on commodity price evolution in detail here since the theory's models share some features with our model as well. The goal of such models is to study the impact of speculation on price stabilization of storable commodities like wheat. There are three types of agents in the market: producers, speculators, and consumers. In each period, producers bring the "harvest" to the market and sell it to speculators. The speculators decide how much of the supply to sell; the remainder is stored for the next period. The consumers of the commodity have price-dependent demand. The decision of the speculators regarding selling quantity affects prices. The speculators are the sole decision makers in this model. The uncertainty regarding the "harvest" in the future brings risk to the speculators. From the modeling perspective, understanding the formation of the expectations regarding prices in the future is critical for modeling the behavior of speculators. This is how rational expectations theory is used in these models. The pioneering effort in this area was led by Samuelson (1971). Some other articles dealing with this problem are Chambers and Bailey (1996), Deaton and Laroque (1992), and Scheinkman and Schechtman (1983). A comprehensive review is provided in Wright and Williams (1991).

Apart from the rational expectations hypothesis, our model is similar to commodity markets literature in utilizing the concept of using inventories to take advantage of prices. On the other hand, there are some differences too. While our model is based in a continuous time setting, commodity market models are based in discrete time settings. Our model differs also in terms of the objective, problem formulation, and solution techniques employed.

Another research area that shares some characteristics with our model is the Peak Load Pricing theory. The objective is to determine the capacity to be installed and the price to be charged by a social welfare maximizing decision maker for a non-storable commodity (like electricity) for which the demand is price-elastic but is non-uniform across time. A review of papers on this topic is provided in Crew et al. (1995). Our model shares the idea of using the price to shift the demand for capacity (inter-temporal nature of demand) with peak load pricing theory models. We differ in the problem formulation and the analysis approach.

Modeling of demand as a Weiner's process and a subsequent application of the Maximum Principle as a solution technique in production planning problem was first used by Sethi and Thompson (1981). They consider a linear-quadratic cost model in which the objective is to find optimal production levels to minimize cost incurred due to production and inventory levels being different from factory optimal levels for both finite and infinite horizons. Allowing production to be negative (in other words, disposal is permitted), they obtain closed form expressions for optimal production rates in feedback form. Bensoussan et al. (1984) consider a similar model for an infinite horizon in which the rate of production is constrained to be non-negative and characterize the optimal feedback solution. Fleming et al. (1987) also consider an infinite horizon production planning problem but they assume demand to evolve as a continuous time Markov chain with finite states. They consider a cost model involving convex holding/shortage and production costs and show existence of a unique optimal feedback production policy.

Finally, a detailed review of supply chain coordination models can be found in Cachon (2002). These models have focused on the design of contracts between buyers and sellers. The intertemporal nature of demand elasticity is not considered in any of the models reviewed by Cachon (2002). That is, most of the supply chain coordination models have not considered the ability of supply chain buyers to shift their demand forward or backward in time through the use of inventory or backorders. Indeed, in a B2C setting, it may be argued that consumers do not have the resources or the need to hold large amount of inventory for strategic reasons and therefore, the demand in any period depends on that period's price only. The same argument does not hold in a B2B setting because the supply chain buyer could gain substantially by shifting the time of her demand.

We are aware of only one model that considers the possibility of changing the time of buyer demand for strategic reasons. In a deterministic setting, Anand et al. (2002) discuss the notion of strategic inventory using a 2-period model in a supply chain consisting of one seller and one buyer. They show that the buyer may carry inventory from first period to the last period, only for strategic reasons. In this paper also, buyers hold inventory or incur backorders, merely to take advantage of current low prices or potentially lower prices in the future.

We are not aware of any supply chain coordination models that consider multiple supply chain interactions in a stochastic setting and this work makes a contribution in that regard as well.

3.1 Notation

There are S sellers in this market. These sellers own production facilities and accept production orders. We assume that sellers cannot change the level of capacity installed. Let C_k denote the capacity of seller k. The sellers can engage in overtime and outsourcing as well as undertime so it is not necessary for production to exactly equal capacity at any time.

Let $Y_k(t)$ denote the cumulative production by seller k through time t. We assume $Y_k(0) = 0$. Let $y_k(t)$ be the instantaneous rate of production of seller k at time t. The rate of production, $y_k(t)$, is a control variable. The relationship between $Y_k(t)$ and $y_k(t)$ is given by:

$$dY_k(t) = y_k(t)dt, t \in [0, T].$$

There are B buyers in the market who place production orders and satisfy demand from end-consumers. Let $X_j(t)$ denote the cumulative orders for production placed by buyer j through time t, and let X_{0j} denote the initial inventory of buyer j. Let $x_j(t)$ be the instantaneous order rate placed by buyer j at time t. Therefore,

$$dX_j(t) = x_j(t)dt.$$

Let $F_j(t, t)$ denote the cumulative demand from end-consumers for the sales of buyer j through time t. This demand process is exogenous to the model. Presumably, buyer j's product is sufficiently differentiated from that of other buyers' products that we can ignore competition among buyers for shares of end-consumer demand. We assume the end-consumer demand process to consist of a linear, a seasonal, a forecast update, and a Weiner's process component. We use the following model of instantaneous demand for buyer j:

$$dF_j(t,t) \equiv \left(D_j + \alpha_j \gamma \cos \gamma t + \int_0^t \sigma(u,t)\mathbf{dW}(u)^T \right) dt + \sigma(t,t)\mathbf{dW}(t)^T \tag{3.4}$$

and, hence,

$$\begin{aligned} F_j(t,t) &= D_j t + \alpha_j \sin \gamma t + \int_0^t \int_0^s \sigma(u,s)\mathbf{dW}(u)^T ds \\ &+ \int_0^t \sigma(s,s)\mathbf{dW}(s)^T. \end{aligned} \tag{3.5}$$

We assume that all buyers face the same seasonal period γ and forecast update coefficient $\sigma(\cdot,\cdot)$, but may differ in average demand rates and seasonal amplitudes. For each buyer, j, the expected average demand rate D_j will be assumed to be suitably large relative to α_j and γ to ensure that cumulative demand is non-decreasing with high-probability. In particular, we require $D_j \geq \alpha_j \gamma$. Note that we assume that the same Weiner's process drives all demand.

Let $P(t)$ denote the price of capacity, the homogeneous good, bought and sold at time t. We assume that this process is of the form:

$$P(t) = a(t) + \int_0^t \mathbf{b}(s,t)\mathbf{dW}^T(s) \tag{3.6}$$

for suitable value of $a : \mathcal{R} \to \mathcal{R}$ and $\mathbf{b} : \mathcal{R}^2 \to \mathcal{R}^n$ where the $n-$ dimensional Weiner's process $\mathbf{W}(\cdot)$ is the same process as underlies the demand model. We assume $a(\cdot)$ and $\mathbf{b}(\cdot,\cdot)$ are such that the price process is square-integrable. We formally state this assumption as follows:

Assumption 3.1.1. *$E\int_0^T P(t)^2 dt < \infty$ and $P(t)$ is $\mathcal{F}_t$-adapted.*

We are unable to prove that the equilibrium price process has this form but we can derive values of $a(\cdot)$ and $\mathbf{b}(\cdot,\cdot)$ that are consistent with equilibrium conditions and this assumption.

Provided that $a(\cdot)$ and $\mathbf{b}(\cdot,\cdot)$ satisfy certain equilibrium conditions described below, we assume that each agent in the economy plans and implements production and procurement decisions according to this specific price process. Hence, the sellers' production plans and the buyers' procurement plans will be seen to be stochastic functions of $a(\cdot)$ and $\mathbf{b}(\cdot,\cdot)$.

Let the absence of a j or k subscript indicate the total of the corresponding function over all entities in the market. For example, C is the total capacity in the market; similarly for $Y(t)$, $y(t)$, Y_0, $X(t)$, $x(t)$, X_0, and $F(t,t)$. These variables will be referred to as *market* quantities (market capacity, market orders, etc.)

A market exists for buyers to place production orders and for sellers to accept them. There is no lead time between order placement and order delivery: production is instantaneously distributed from sellers to buyers. A market price $P(t)$ per unit is paid by buyers and received by sellers for each unit of production. All agents in this market are assumed to be price-takers. That is, the number of buyers, B, and the number of sellers, S, are assumed to be large and no one buyer or seller is large enough to influence price. We consider this price to be a premium or discount from some exogenously determined price that considers such factors as unit production costs and consumer demand price sensitivity for the final good. These factors are ignored in this analysis, consistent with our assumption that the end-consumer demand process is exogenous. In this way, we focus on the role of price in managing the evolution of inventory in the economy. Since it can be either a premium or a discount, we do not constrain the sign of $P(t)$.

In the models to follow, we suppress the time argument t unless needed for clarification. In both the buyer and seller models, we assume a quadratic cost structure in order to derive explicit solutions.

Given a price process $P(\cdot)$, buyer j's problem is to choose a production order policy $x_j(\cdot)$ to minimize the total expected cost of production orders and inventory/shortfall costs. The finite horizon version of this problem with quadratic costs is:

$$
\begin{aligned}
& \min_{x_j \in \mathcal{U}_j[0,T]} E \int_0^T \left\{ \pi I_j(t)^2 + P(t)x_j(t) \right\} dt \\
& \text{s.t.} \\
dI_j(t) &= \left(x_j(t) - D_j - \alpha_j \gamma \cos \gamma t - \int_0^t \sigma(s,t) \mathbf{dW}(s)^T \right) dt \\
&- \sigma(t,t)\mathbf{dW}(t)^T, t \in [0,T] \qquad (3.7) \\
X_j(0) &= X_{0j},
\end{aligned}
$$

where $I_j(t)$ is the net inventory at time t (on hand inventory less backorders), so the objective function penalizes any deviation of net inventory from zero. The buyers can hold inventory or incur backorders, so it is not necessary for production orders to exactly equal consumer demand at any time. We assume that all the buyers start with the same level of inventory, that is, $X_{0j} = \frac{X(0)}{B}$.

In order to solve the Buyer model, we make the following assumptions on the control variable x_j and state variable I_j.

Assumption 3.1.2. *The set of controls* $\mathcal{U}_j[0,T]$ *consists of all* $x_j : [0,T] \times \Omega \to \mathcal{R}^1$ *such that* $x_j(\cdot)$ *is* $\mathcal{F}_t$*-adapted and* $E \int_0^T x_j(t)^2 dt < \infty$.

Assumption 3.1.3. *For any* $x_j^1, x_j^2 \in \mathcal{U}_j[0,T]$*, and* $\rho \in [0,1]$ *the following holds:*

$$
E\left[\int_0^T |I_j^1 + \rho I_j^2|^2 dt \right] < \infty,
$$

where I_j^1 *and* I_j^2 *are states of the system controlled by* x_j^1 *and* x_j^2*, respectively.*

Observe that price $P(\cdot)$ is treated as a random coefficient in the objective function of the Buyer model (3.7). According to Theorem 6.16, pp 49, Yong and Zhou (1999), Assumption 3.1.2 along with the linear nature of the state equation for the net inventory in the Buyer model (3.7) ensure a unique solution to the state equation. Assumption 3.1.3 is satisfied due to the square-integrability of the control variable x_j.

Similarly, each seller faces the following stochastic control problem. Given a price process $P(\cdot)$, seller k's problem is to choose a production rate policy y_k, to minimize the total expected cost of overtime/undertime less the revenue derived from production. The finite horizon version of this problem with quadratic costs is:

$$\begin{aligned} & \min_{y_k \in \mathcal{U}_k[0,T]} E \int_0^T \left\{ \kappa(C_k - y_k(t))^2 - P(t) y_k(t) \right\} dt \\ & \text{s.t.} \\ dY_k(t) &= y_k(t)dt, t \in [0,T] \qquad (3.8) \\ Y_k(0) &= Y_{0k}. \end{aligned}$$

Since sellers do not hold inventory, Y_{0k} should be equal to 0 in practice.

In order to solve the Seller model, we make the following assumptions on the control variable y_k.

Assumption 3.1.4. *The set of controls* $\mathcal{U}_k[0,T]$ *consists of all* $y_k : [0,T] \times \Omega \to \mathcal{R}^1$ *such that* $y_k(\cdot)$ *is* $\mathcal{F}_t$*-adapted and* $E\int_0^T y_k(t)^2 dt < \infty$.

Assumption 3.1.5. *For any* $y_k^1, y_k^2 \in \mathcal{U}_k[0,T]$, *and* $\rho \in [0,1]$, *the following holds:*

$$E\left[\int_0^T |2\kappa(C_k - y_k^1 - \rho y_k^2) - P|^2 dt\right] < \infty.$$

According to Theorem 6.16, pp 49, Yong and Zhou (1999), Assumption 3.1.4 along with the linearity of the state equation for the cumulative production in the Seller model (3.8) ensure a unique solution to the state equation. Assumption 3.1.5 is satisfied due to the square-integrability of P and y_k.

The above models of buyer and seller behaviors can be criticized for the symmetry of the cost functions with respect to the net inventory and capacity, respectively. In practice, backorders are penalized at a higher rate than on-hand inventory and overtime is probably more expensive than the undertime. Our focus, however, is on a higher-order behavior, that of market equilibrium price for capacity. The general trade-off considered in this paper is between the cost of overtime/undertime versus the cost of inventory and backorders, and on how a market price can serve to equilibrate this general trade-off.

Observe that we do not impose non-negativity constraints on the instantaneous rate of order process or on the instantaneous rate of production process. The imposition of those constraints would have made it impossible to obtain closed form solutions for the optimal production or the equilibrium market price. However, we assume that the value of D_j is relatively large compared to the randomness in the demand and, therefore, the probability of occurrence of negative order rate or production rate is negligible. The same assumption ensures that the demand rate is non-negative with a high probability. We discuss the implications of introducing non-negativity constraints following Proposition 3.3.5.

Observe that the costs of production (material, labor, and capital) are ignored in the Seller model (3.8): only the short term costs of production adjustment are captured. Also observe that the revenue from consumer sales are ignored in the Buyer model (3.7): only the inventory/shortfall costs are relevant. As a result, the price in this market will reflect the trade-off between the sellers' production adjustment costs and the buyers' inventory/shortfall costs.

Since all agents are assumed to be price takers, the production and production order policies $y_k(\cdot)$ and $x_j(\cdot)$ that optimize seller and buyer problems, respectively, will depend on the price process $P(\cdot)$. We assume that the market will be in equilibrium at all times. That is, the price process $P(\cdot)$ must ensure that

$$y(t) = x(t) \text{ for all } t \geq 0. \tag{3.9}$$

In equilibrium, therefore, $Y(t) = X(t) - X_0 + Y_0$.

The requirement that, in equilibrium, market demand equals market supply is, admittedly, an heroic assumption. This assumption is common in classical economic models but these models are typically discrete-time formulations. In such models, the time periods are assumed to be long enough for price adjustment mechanisms in the marketplace to react to new conditions and achieve equilibrium. There is a body of economics literature that explores these mechanisms for both the existence and stability of equilibria. Our assumption that equilibrium is achieved in continuous time begs the question of what price adjustment mechanism could bring this about. Though we do not develop the idea in this paper, we avoid the question of a specific mechanism by imagining a series of discrete time economies in which some unspecified price adjustment takes place within the periods to achieve equilibrium by the end of each period. We then imagine a convergence of these economies, with a scaling of time, to a continuous time economy of the type we have formulated here. It is the disappearance of local time, the time during which prices adjust, in the limit that creates some of the anomalies of our model from a control theory perspective. While it appears that the price process is exogenous from the planning perspective of any agent, buyer or seller, it is, in our imagined continuous-time economy, a highly tuned process sensitive to the slightest change in state. This, combined with the equally heroic assumption of rational expectations (that all agents act in accordance with a price process they all agree would achieve equilibrium), allows us to apply the mechanics of stochastic control theory to derive insights into a price process that simultaneously satisfies all first order conditions for the optimal control of agents and the condition of equilibrium in the market.

By means of (3.5), (3.7), (3.8), and (3.9), we have described a simple market for capacity in which the demand for capacity is intertemporal in nature: if capacity prices are high, buyers can defer production orders (depleting inventory or incurring shortages) while if capacity prices are low, then buyers can advance production orders (eliminating shortages or building inventory). We proceed to solve these models and to demonstrate this behavior.

Throughout this section, we make the following approximation:

$$\sigma_i(s,t) = \sum_{l=1}^{m} \xi_{il} \exp(\lambda_l(s-t)) \tag{3.10}$$

where $\sigma_i(s,t)$ is the i^{th} component of $\sigma(s,t)$.

3.2 Solution to the Market Model for Capacity

In this subsection, we present and discuss the solution to the Market model for capacity. First, we sketch the necessary and sufficient conditions for equilibrium solution to the Market model. Observe that the drift term in the state equation for the net inventory is stochastic, since we take into account the forecast updates during the horizon. To obtain necessary and sufficient conditions for optimality, we apply the version of the Stochastic Maximum Principle for a Linear Quadratic problem with random objective function and random state equation coefficients (for details, see Cadenillas and Karatzas (1995)).

3.2.1 Optimal Control of the Buyer Model

Define the Hamiltonian of the Buyer model as:

$$\begin{aligned} H_b &= -\pi I_j^2 - P x_j + p_{1,j}(x_j - D_j - \alpha_j \gamma \cos \gamma t - \int_0^t \sigma(u,t) \mathbf{dW}(u)^T) \\ &+ \mathbf{q}_{1,j} \sigma(t,t)^T. \end{aligned} \tag{3.11}$$

Observe that the coefficient of the adjoint variable $p_{1,j}$ in the Hamiltonian function is equal to the drift term in state equation for $I_j(t)$. Similarly, the coefficient of $\mathbf{q}_{1,j}$ is equal to the diffusion term in the state equation for $I_j(t)$. The adjoint variable pair $p_{1,j} : [0,T] \times \Omega \to \mathcal{R}$, and $\mathbf{q}_{1,j} : [0,T] \times \Omega \to \mathcal{R}^n$

is measurable, adapted, and is defined by the following stochastic differential equation:

$$\begin{aligned} dp_{1,j}(t) &= 2\pi I_j(t)dt + \mathbf{q}_{1,j}(t)\mathbf{dW}(t)^T, \\ p_{1,j}(T) &= 0. \end{aligned}$$

The adjoint variable $p_{1,j}$ can be interpreted as the shadow price corresponding to the net inventory "resource". At each time instant, $p_{1,j}$ is random. In the deterministic case, if the value function is sufficiently smooth, then the time rate of change of the adjoint variable $p_{1,j}$ is equal to the negative of the partial derivative of the Hamiltonian with respect to the state variable I_j. That is, $\frac{\partial p_{1,j}(t)}{\partial t} = -\frac{\partial H}{\partial I_j}$, and $\mathbf{q}_{1,j} = 0$, in the deterministic case.

The second adjoint variable vector $\mathbf{q}_{1,j}$ is not constrained to satisfy any differential equation. However, it cannot be set to zero everywhere. The boundary condition for the stochastic differential condition for $p_{1,j}$ is specified at the end of the horizon. Therefore, if $\mathbf{q}_{1,j}$ is identically set to zero, the resulting solution for $p_{1,j}$ may not be $\mathcal{F}_t$-adapted.

Using Proposition 1.2 in Cadenillas and Karatzas (1995), a necessary and sufficient condition for $\bar{x}_j$ to be optimal for the Buyer's model(3.7) is that $\forall x_j \in \mathcal{U}_j$:

$$E\left(\int_0^T (P(t) - \bar{p}_{1,j}(t))(x_j(t) - \bar{x}_j(t))\right) \geq 0, a.e.(t,\omega) \in [0,T] \times \Omega.$$

where $(\bar{p}_{1,j}, \bar{\mathbf{q}}_{1,j})$ is the adjoint variable pair that corresponds to the system controlled by $\bar{x}_j$. The above condition is satisfied if and only if

$$P(t) = \bar{p}_{1,j}(t), a.e.(t,\omega) \in [0,T] \times \Omega. \tag{3.12}$$

The last equation implies that, in equilibrium, all the buyers must have the same shadow price at all instants. Summing the last equation over all the buyers gives:

$$P = \frac{\bar{p}_1}{B}, a.e.(t,\omega) \in [0,T] \times \Omega. \tag{3.13}$$

where $\bar{p}_1 = \sum_j \bar{p}_{1,j}$.

3.2.2 Optimal Control of the Seller's Model

Similarly, we define the Hamiltonian for the seller k's model as:

$$H_s(t, Y_{k,} y_k, p_{2,k}, \mathbf{q}_{2,k}) = -\kappa(C_k - y_k)^2 + Py_k + p_{2,k} y_k.$$

The adjoint variable pair $p_{2,k} : [0,T] \times \Omega \to \mathcal{R}$, and $\mathbf{q}_{2,k} : [0,T] \times \Omega \to \mathcal{R}^n$ is measurable, adapted and is defined by the following backward stochastic differential equation:

$$\begin{aligned} dp_{2,k} &= \mathbf{q}_{2,k} \mathbf{dW}(t)^T, \\ p_{2,k}(T) &= 0. \end{aligned}$$

It is easily seen that $p_{2,k} = \mathbf{q}_{2,k} \equiv 0$. The adjoint variable $p_{2,k}$ can be interpreted as the shadow price corresponding to Y_k. Since Y_k does not appear in the objective function for the Seller model, it is appropriate that $p_{2,k}$ be uniformly zero.

According to Theorem 3.2, Cadenillas and Karatzas (1995), if the objective function is convex in the state and control variables and is allowed to be random, then $\bar{y}_k$ is an optimal control variable if and only if

$$\max_{y_k \in \mathcal{U}_k} H_s(y_k, \bar{Y}_k, \bar{p}_{2,k}, \bar{q}_{2,k}) = H_s(\bar{y}_k, \bar{Y}_k, \bar{p}_{2,k}, \bar{q}_{2,k}), a.e.(t,\omega) \in [0,T] \times \Omega,$$

where $\bar{Y}_k$ and $(\bar{p}_{2,k}, \bar{q}_{2,k})$ are the state variable and adjoint variable pair, respectively, corresponding to the system controlled by $\bar{y}_k$. The above equation yields:

$$2\kappa(\bar{y}_k - \bar{C}_k) = P, a.e.(t,\omega) \in [0,T] \times \Omega.$$

That is, in equilibrium, all sellers must have the same production-capacity mismatch. Summing the last equation over all sellers results in:

$$SP + 2\kappa(\bar{C} - \bar{y}) = \frac{S}{B}\bar{p}_1 + 2\kappa(\bar{C} - \bar{y}) = 0, a.e.(t,\omega) \in [0,T] \times \Omega \tag{3.14}$$

where the RHS is obtained by substituting for P from (3.13).

To obtain the optimal solution to the Market model, we use the equilibrium condition (3.9) to link the buyers' models and the sellers' models. We are now ready to state the necessary and sufficient conditions for the optimality of the Market model in the following proposition:

Proposition 3.2.1. *Under Assumption 3.1.1, the vector of market variables,* $(\bar{I}, \bar{y}(=\bar{x}), \bar{p}_1, \bar{q}_1)$ *is optimal if and only if it satisfies the following system of equations in equilibrium at time* $t \in [0,T]$ *:*

$$\begin{aligned}
d\bar{I}(t) &= (\bar{y}(t) - D - \alpha\gamma\cos\gamma t - B\sigma(s,t)\mathbf{dW}(s)^T)dt \\
&- B\sigma(t,t)\mathbf{dW}(t)^T, \qquad (3.15)\\
d\bar{p}_1(t) &= 2\pi\bar{I}(t)dt + \bar{\mathbf{q}}_1(t)\mathbf{dW}(t)^T, \\
\bar{p}_1(T) &= 0, \\
\bar{x}(t) &= \bar{y}(t) = C + \frac{1}{2\kappa}\frac{S}{B}\bar{p}_1(t),
\end{aligned}$$

where $\bar{\mathbf{q}}_1(t) = \sum_j \bar{\mathbf{q}}_{1,j}(t)$ *and* $C = \sum_k C_k$.

Proof. Necessity: The first equation is obtained by summing the state equation for net inventory over all buyers. Similarly, the second and third equations are obtained by adding the differential equations and the terminal conditions, respectively, for adjoint variable $p_{1,j}$ over all j. The last equation is obtained by combining (3.14) with the equilibrium condition (3.9).

Sufficiency: It is enough to find a disaggregated solution for each buyer and seller, given an aggregated solution of the above equations (3.15), that satisfies the necessary and sufficient conditions for the Buyer and Seller models. Consider the following disaggregated solution for the Buyer model,

$$(\bar{x}_j(t), \bar{I}_j(t), \bar{p}_{1,j}(t), \bar{\mathbf{q}}_{1,j}(t)) = (\frac{\bar{x} - D - \alpha\gamma\cos\gamma t}{B} + D_j + \alpha_j\gamma\cos\gamma t, \frac{\bar{I}(t)}{B}, \frac{\bar{p}_1(t)}{B}, \frac{\bar{\mathbf{q}}_1(t)}{B}),$$

and the Seller model,

$$(\bar{y}_k(t), \bar{Y}_k(t)) = (\frac{\bar{y}(t) - C}{S} + C_k, \frac{\bar{Y}(t) - Ct - Y_0}{S} + C_{0,k}t + Y_{0,k}),$$

where the equality holds componentwise. Clearly, the above solution satisfies the necessary and sufficient conditions for the Buyer and Seller models. The proof is completed by noting the uniqueness of the solution to (3.15) (see Proposition 3.3.5). □

The above result shows the market distributes the equilibrium capacity and rate of production among the sellers equally (save for the correction due to initial values). Similarly, all the buyers place orders at the same rate. This result is hardly surprising as all the sellers and buyers are identical, respectively, in cost parameters. This leads to the question of what happens when buyers and sellers are not symmetric in their cost parameters, which we discuss in the following corollary.

Corollary 3.2.2. *Suppose buyer j's cost parameter for holding inventory/incurring backorders is* π_j *and seller k's cost parameter for overtime/undertime is* κ_k. *Further, relax the assumption that* $\sigma(\cdot,\cdot)$ *is identical across all the buyers. Given an aggregated solution to the Market model,* $(\bar{I}, \bar{y}(= \bar{x}), \bar{p}_1, \bar{q}_1)$, *the following characterization of the disaggregated solution satisfies the necessary and sufficient conditions for the Buyer and Seller models, respectively. For the Buyer model,*

$$(\bar{X}_j(t), \bar{I}_j(t), \bar{p}_{1,j}(t), \bar{\mathbf{q}}_{1,j}(t)) = (\frac{\bar{X}(t) - F(t,t)}{\sum_i \frac{\pi_j}{\pi_i}} + F_j(t,t), \frac{\bar{I}(t)}{\sum_i \frac{\pi_j}{\pi_i}}, \frac{\bar{p}_1(t)}{B}, \frac{\bar{\mathbf{q}}_1(t)}{B}),$$

and the Seller model,

$$(\bar{y}_k(t), \bar{Y}_k(t)) = (\frac{\bar{y}(t) - C}{\sum_i \frac{\kappa_k}{\kappa_i}} + C_k, \frac{\bar{Y}(t) - Ct - Y_0}{\sum_i \frac{\kappa_k}{\kappa_i}} + C_k t + Y_{0,k}),$$

where the equality holds componentwise.

In the above disaggregated solution, we are able only to characterize the rate of order placement for each buyer through the cumulative order quantity X_j. We are unable to obtain the rate of order placement x_j except in a special case when the diffusion coefficient in the state equation for net inventory $\sigma_j(t,t)$ as well as market inventory at the beginning of the horizon X_0 are equal to zero. In that case,

$$\bar{x}_j(t) = \frac{\bar{x}(t) - D - \int_0^t \hat{\sigma}(u,t)\mathbf{dW}(u)^T}{\sum_i \frac{\pi_j}{\pi_i}} + D_j + \int_0^t \sigma_j(u,t)\mathbf{dW}(u)^T,$$

where $\hat{\sigma}(\cdot,\cdot) := \sum_j \sigma_j(\cdot,\cdot)$. The above two conditions do not impact the insights we obtain using the Market model. Hence, the model can be extended easily to obtain insights regarding the equilibrium price even when the agents are not identical in their cost parameter or the forecast update coefficient.

In order to solve for the optimal variables in the Market model, we define another model which we refer to as the *Integrated model*. The structure of the Integrated model is similar to a well-known problem called the Linear Regulator problem whose solution is provided by Cadenillas and Karatzas (1995). Our approach involves obtaining the necessary and sufficient conditions for optimality in the Integrated model and showing them to be equivalent to the necessary and sufficient conditions for optimality in the Market model. Thus, the optimal solution to the Integrated model can be used to obtain the optimal solution to the Market model.

As we shall see later, the Integrated model corresponds to the integrated supply chain (hence the name) in which the supply chain is wholly owned by a single agent. Thus, the owner of the supply chain not only owns the capacity but also satisfies the end-consumer demand. In the following section, we state and solve the Integrated model.

3.3 Integrated Model

The goal of the Integrated model is to choose a production policy y_ι to minimize the total expected cost of overtime/undertime and inventory/shortfall costs. The finite horizon version of this problem with quadratic costs is:

$$\min_{y_\iota \in \mathcal{U}_\iota[0,T]} E\int_0^T \left\{\kappa(C_\iota - y_\iota(t))^2 + \pi' I_\iota(t)^2\right\} dt$$

$$\begin{aligned} &\text{s.t.} \\ dI_\iota(t) &= (y_\iota(t) - D_\iota - \alpha_\iota\gamma_\iota \cos\gamma t - \int_0^t \sigma_\iota(u,t)\mathbf{dW}(u)^T)dt \\ &- \sigma_\iota(t,t)\mathbf{dW}(t)^T, t \in [0,T] \qquad (3.16) \\ I_\iota(0) &= Y_{0\iota}. \end{aligned}$$

where all the notation remains the same as before except for the distinguishing mark, subscript in, to indicate the Integrated model and the net inventory penalty cost π' where π' is defined as:

$$\pi' = \frac{S}{B}\pi.$$

As before, $\mathbf{W}(\cdot)$ is an $n-$dimensional standard Weiner's process defined on $(\Omega, \mathcal{F}, \mathcal{P})$, a complete probability space. Define $\{\mathcal{F}_t\}_{t\geq 0} = \sigma\{\mathbf{W}(s) : 0 \leq s \leq t\}$ augmented by all the $\mathcal{P}$-null sets in $\mathcal{F}$.

Assumption 3.3.1. *The set of controls $\mathcal{U}_\iota[0,T]$ consists of all $y_\iota : [0,T]\times\Omega \to \mathcal{R}^1$ such that $y_\iota(\cdot)$ is $\mathcal{F}_t$-adapted, and $E\int_0^T y_\iota(t)^2 dt < \infty$.*

Assumption 3.3.2. *For any $y_\iota^1, y_\iota^2 \in \mathcal{U}_\iota[0,T]$, and $\rho \in [0,1]$, the following holds:*

$$E\left[\int_0^T |\kappa(C_\iota - y_\iota^1 - \rho y_\iota^2) + \pi'(I_\iota^1 + \rho I_\iota^2)|^2 dt\right] < \infty,$$

where I_ι^1 and I_ι^2 are states of the systems controlled by y_ι^1 and y_ι^2, respectively.

According to Theorem 6.16, pp 49, Yong and Zhou (1999), Assumption 3.3.1 along with the linearity of the state equation for the net inventory in the Integated model (3.16) ensure a unique solution to the state equation. Assumption 3.3.2 is satisfied due to the square-integrability of y_ι.

Next, we derive the necessary and the sufficient conditions for optimality of the Integrated model.

3.3.1 Optimal Control of the Integrated Model

Define the Hamiltonian as:

$$\begin{aligned} H_\iota(y_\iota, I_\iota, p_{1,\iota}, \mathbf{q}_{1,\iota}) &= -\pi' I_\iota^2 - \kappa(C_\iota - y_\iota)^2 + \mathbf{q}_{1,\iota}\sigma_\iota(t,t) \\ &+ p_{1,\iota}\left(y_\iota - D_\iota - \alpha_\iota\gamma_\iota \cos\gamma_\iota t - \int_0^t \sigma_\iota(u,t)\mathbf{dW}(u)^T\right). \end{aligned} \tag{3.17}$$

The adjoint variable pair $p_{1,\iota} : [0,T] \times \Omega \to \mathcal{R}$, and $\mathbf{q}_{1,\iota} : [0,T] \times \Omega \to \mathcal{R}^n$ is measurable, adapted, and is defined by the following stochastic differential equation:

$$\begin{aligned} dp_{1,\iota}(t) &= 2\pi' I_\iota(t)dt + \mathbf{q}_{1,\iota}(t)\mathbf{dW}(t)^T, \\ p_{1,\iota}(T) &= 0. \end{aligned}$$

According to Theorem 3.2, Cadenillas and Karatzas (1995), if the objective function is convex in the state and control variables and is allowed to be random, then $\bar{y}_\iota$ is an optimal control variable if and only if

$$\max_{y_\iota \in \mathcal{U}_\iota} H_\iota(y_\iota, \bar{I}_\iota, \bar{p}_{1,\iota}, \bar{q}_{1,\iota}) = H_\iota(\bar{y}_\iota, \bar{I}_\iota, \bar{p}_{1,\iota}, \bar{q}_{1,\iota}), a.e.(t,\omega) \in [0,T] \times \Omega,$$

where $\bar{I}_\iota$ and $(\bar{p}_{1,\iota}, \bar{q}_{1,\iota})$ are the state variable and adjoint variable pair, respectively, corresponding to the system controlled by $\bar{y}_\iota$. Optimizing the Hamiltonian yields,

$$\bar{y}_\iota = \bar{C}_\iota + \frac{1}{2\kappa}\bar{p}_{1,\iota}, a.e.(t,\omega) \in [0,T] \times \Omega.$$

The necessary and the sufficient conditions for optimality of the Integrated model are very similar to those for the Market model. Indeed, if the optimal solution to one is known, the optimal solution to the other can be easily obtained. We formally state this in the following corollary.

Corollary 3.3.3. *Assume that* $\pi' = \frac{S}{B}\pi$ *and the cost parameter value* κ *is the same for the two models. Further assume* $C_\iota = C, D_\iota = D, \alpha_\iota = \alpha, \gamma_\iota = \gamma, Y_{0\iota} = X_0 - Y_0,$ *and* $\sigma_\iota = B\sigma$*. Then, the* *optimal vector of market variables,* $(\bar{I}, \bar{y}(=\bar{x}), \bar{p}_1, \bar{\mathbf{q}}_1)$*, and the optimal vector,* $(\bar{I}_\iota, \bar{y}_\iota, \bar{p}_{1,\iota}, \bar{\mathbf{q}}_{1,\iota})$*, of the Integrated model are related, as follows:*

$$\begin{aligned}
\bar{p}_{1,\iota} &= \frac{S}{B}\bar{p}_1, \\
\bar{\mathbf{q}}_{1,\iota} &= \frac{S}{B}\bar{\mathbf{q}}_1, \\
\bar{y}_\iota &= \bar{y}, \\
\bar{I}_\iota &= \bar{I}, a.e.(t,\omega) \in [0,T] \times \Omega.
\end{aligned}$$

Next, we obtain the solution to the Integrated model.

3.3.2 Solution to the Integrated Model

Define $y'_\iota = C_\iota - y_\iota$. With this substitution, the Integrated model becomes identical to the well-known Linear Regulator problem with a random drift term in the state equation. Cadenillas and Karatzas (1995) state the solution to the Linear Regulator problem. The optimal solution to the Linear-Quadratic problem is obtained by hypothesizing a linear relationship between the adjoint variables and the state variables. As an example, in the case of the Integrated model, this hypothesis would take the following form:

$$\begin{aligned}
\bar{p}_{1,\iota}(t) &= -Z(t)\bar{I}_\iota(t) - \varphi(t), \\
\bar{\mathbf{q}}_{1,\iota}(t) &= -\Theta(t)\bar{I}_\iota(t) + Z(t)\sigma_\iota(t,t) - \Lambda(t),
\end{aligned}$$

where $Z(\cdot), \Theta(\cdot), \varphi(\cdot)$, and $\Lambda(\cdot)$ are measurable and adapted processes. We use the solution provided by Cadenillas and Karatzas (1995) to solve for the Integrated model which we state in the following proposition.

Proposition 3.3.4. *Let $Z : [0,T] \times \Omega \to \mathcal{R}$, and $\mathbf{\Theta} : [0,T] \times \Omega \to \mathcal{R}^n$ be a pair of measurable, adapted processes that solve the backward stochastic differential equation:*

$$\begin{aligned} dZ(t) &= -\left(2\pi' - \frac{1}{2\kappa} Z(t)^2\right) dt + \mathbf{\Theta}(t)\mathbf{dW}(t), \\ Z(T) &= 0. \end{aligned} \tag{3.18}$$

Further, let $\varphi : [0,T] \times \Omega \to \mathcal{R}$, and $\mathbf{\Lambda}[0,T] \times \Omega \to \mathcal{R}^n$ be a pair of measurable, adapted processes that solve the following backward stochastic differential equation:

$$\begin{aligned} & d\varphi(t) + Z(t)(-\frac{\varphi(t)}{\kappa} + C_\iota - D_\iota - \alpha_\iota \gamma_\iota \cos \gamma_\iota t - \int_0^t \sigma_\iota(z,t)\mathbf{dW}(z))dt \\ & -\mathbf{\Theta}(t)\sigma_\iota(t,t))dt + \mathbf{\Lambda}(t)\mathbf{dW}(t) = 0, \\ & \varphi(T) = 0, \end{aligned} \tag{3.19}$$

(where $\sigma(\cdot,t)$ is given by (3.10)) such that $\varphi \in \mathcal{L}^2(0,T;\mathcal{R})$. Then the square-integrable and adapted optimal control $\bar{y}_\iota(\cdot)$ of the Integrated model (3.16) is given by:

$$\bar{y}_\iota(t) = C_\iota - \frac{1}{\kappa}\left(Z(t)\bar{I}_\iota(t) + \varphi(t)\right), t \in [0,T], \tag{3.20}$$

and the optimal adjoint processes are given by:

$$\begin{aligned} \bar{p}_{1,\iota}(t) &= -Z(t)\bar{I}_\iota(t) - \varphi(t), \\ \bar{\mathbf{q}}_{1,\iota}(t) &= Z(t)\sigma(t,t) - \mathbf{\Lambda}(t). \end{aligned}$$

Further, the optimal control is unique.

Proof. The Integrated model is a special case of the Linear-Regulator model solved by Cadenillas and Karatzas (1995), Section 3.6.1. The uniqueness of the optimal control follows from Theorem 1.6, Cadenillas and Karatzas (1995). □

The vectors $\mathbf{\Theta}(\cdot)$ and $\mathbf{\Lambda}(\cdot)$ are not required to satisfy any additional differential equations. However, their presence is necessary to ensure measurability of $Z(\cdot)$ and $\varphi(\cdot)$ with respect to $\mathcal{F}_t$.

We combine the results of the last proposition and Corollary 3.3.3 to obtain the equilibrium solution to the Market model in the following proposition.

Proposition 3.3.5. *The unique solution to (*3.15*) is given by:*

$$\begin{aligned}
\bar{y}(t) &= C - \frac{1}{\kappa}\left(Z(t)\bar{I}(t) + \varphi(t)\right), \qquad (3.21)\\
\frac{S}{B}\bar{p}_1(t) &= -Z(t)\bar{I}(t) - \varphi(t),\\
\frac{S}{B}\bar{\mathbf{q}}_1(t) &= -Z(t)\sum_{l=1}^{m}\xi_l - \mathbf{\Lambda}(t).
\end{aligned}$$

Remark 3: The expression for the optimal rate of production provides some insights regarding the behavior of the model. The rate of production at time t depends on the capacity as well as the net inventory at time t. A marginal exogenous change in the capacity is transmitted directly to the optimal rate of production. However, a marginal change in net inventory produces less effect on the rate of production as κ increases relative to π since $Z(t)/2\kappa$ (see the following proposition) decreases as κ increases for any fixed t. In other words, net inventory becomes a relatively less important factor as the cost of capacity-production mismatch increases with respect to holding/shortage cost.

In the following proposition, we obtain solutions to the stochastic differential equations (3.18)-(3.19).

Proposition 3.3.6. *i) The solution to the stochastic differential equation (*3.18*) is given by:*

$$\begin{aligned}
Z(t) &= 2\sqrt{\pi'\kappa}\left(\frac{1 - e^{2\sqrt{\frac{\pi'}{\kappa}}(t-T)}}{1 + e^{2\sqrt{\frac{\pi'}{\kappa}}(t-T)}}\right), \qquad (3.22)\\
\mathbf{\Theta}(t) &= 0
\end{aligned}$$

*ii) The solution to the stochastic differential equation (*3.19*) is given by:*

$$\varphi(t) = \varphi_1(t) + \varphi_2(t) + \varphi_3(t) \qquad (3.23)$$

where

$$\begin{aligned}
\varphi_1(t) &= \frac{2(C-D)\kappa\left(\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))-1\right)}{\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))}, \\
\varphi_2(t) &= \frac{2\alpha\gamma\kappa}{\left(1+\frac{\kappa}{\pi}\gamma^2\right)\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))}\left(\cos\gamma T-\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))\cos\gamma t\right) \\
&- \frac{2\alpha\gamma^2\kappa^{3/2}}{\left(\sqrt{\pi}\left(1+\frac{\kappa}{\pi}\gamma^2\right)\right)\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))}\sinh(\sqrt{\frac{\pi'}{\kappa}}(t-T))\sin\gamma t, \\
\varphi_3(t) &= B\sum_{i=1}^{n}\int_0^t\frac{\sqrt{\pi'\kappa}}{\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))}(A_i(u,T)-A_i(u,t))dW_i(u),
\end{aligned}$$

where

$$A_i(u,t)=\sum_{l=1}^{m}\xi_{il}\left(\frac{e^{\lambda_l(u-t)+\sqrt{\frac{\pi'}{\kappa}}(t-T)}-e^{\sqrt{\frac{\pi'}{\kappa}}(u-T)}}{\sqrt{\frac{\pi'}{\kappa}}-\lambda_l}+\frac{e^{\lambda_l(u-t)-\sqrt{\frac{\pi'}{\kappa}}(t-T)}-e^{-\sqrt{\frac{\pi'}{\kappa}}(u-T)}}{\sqrt{\frac{\pi'}{\kappa}}+\lambda_l}\right),$$

and, using the definition of $A_i(\cdot,\cdot)$,

$$\Lambda_i(t)=\frac{B\sqrt{\pi'\kappa}A_i(t,T)}{\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))}$$

where Λ_i *is the* i^{th} *component of* $\mathbf{\Lambda}$.

Proof. See the Appendix. □

The optimal solution in the feedback form stated in the Proposition 3.3.5 can be used to obtain optimal cumulative production $\bar{Y}$: multiply the equation for the optimal rate of production in the Market model (3.21) by dt on both sides and write it as

$$d\bar{Y}(t)=(C-\frac{1}{2\kappa}\left(Z(t)(\bar{Y}(t)-F(t,t)+X_0-Y_0)+\varphi(t)\right))dt,$$

by substitution of $\bar{I}(t)$ by $\bar{Y}(t)-F(t,t)+X_0-Y_0$ on RHS and of $\bar{y}(t)dt$ by $d\bar{Y}(t)$ on LHS. This stochastic differential equation can be solved using an approach similar to the *Variation of Constants* method. We state the optimal market production and equilibrium market price in the following corollary.

Corollary 3.3.7. *The optimal market cumulative production trajectory $\bar{Y}(t)$ at time $t \in [0,T]$ is given by:*

$$\begin{aligned}\bar{Y}(t) &= \cosh\Big(\sqrt{\frac{\pi'}{\kappa}}(t-T)\Big)\int_0^t \sec h\Big(\sqrt{\frac{\pi'}{\kappa}}(u-T)\Big)\left(\frac{Z(u)}{2\kappa}(F(u,u)-X_0+Y_0)+C-\frac{\varphi(u)}{2\kappa}\right)du \\ &+ Y_0\frac{\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}, \qquad (3.24)\end{aligned}$$

where

$$F(u,u) = Du + \alpha\gamma\cos\gamma u + B\sum_{i=1}^{n}\sum_{l=1}^{m}\int_0^u \frac{\xi_{il}}{\lambda_l}(1-e^{\lambda_l(s-u)}+\lambda_l)dW_i(s)$$

and $\varphi(\cdot)$ is the solution to the differential equation given by (3.23). Trivially, $\bar{I}(t) = \bar{Y}(t) - F(t,t) + X_0 - Y_0$. Further, the equilibrium market price is given by:

$$P(t) = -\frac{Z(t)\bar{I}(t)+\varphi(t)}{S}.$$

Proof. The equilibrium market price is obtained by combining (3.13) and (3.21). □

In the following corollary, we show that the equilibrium price function P can indeed be written in the form as hypothesized in (3.6). We use closed form expressions for $\bar{I}(\cdot)$ and $\varphi(\cdot)$ from Proposition 3.3.6 and Corollary 3.3.7 to obtain this result.

Corollary 3.3.8. *The expression for equilibrium price process in closed form is given by:*

$$P(t) = a(t) + \int_0^t \mathbf{b}(s,t)\mathbf{dW}(s)$$

where

$$\begin{aligned}a(t) &= -\frac{1}{S}\left(\varphi_1(t)+\varphi_2(t)+Z(t)\left(Y_0\frac{\cosh\sqrt{\frac{\pi'}{\kappa}}(t-T)}{\cosh\sqrt{\frac{\pi'}{\kappa}}(T)} - Dt - \alpha\sin\gamma t\right)\right) \\ &- \frac{Z(t)}{S}\int_0^t \frac{1}{\cosh(\sqrt{\frac{\pi'}{\kappa}}(u-T))}\left(\frac{Z(u)}{2\kappa}(Du+\alpha\sin\gamma u - X_0+Y_0)+C\right)du \\ &+ \frac{Z(t)}{S}\int_0^t \frac{1}{\cosh(\sqrt{\frac{\pi'}{\kappa}}(u-T))}\left(\frac{\varphi_1(u)+\varphi_2(u)}{2\kappa}\right)du\end{aligned}$$

where φ_1 and φ_2 are defined in Proposition 3.3.6. With $A_i(\cdot,\cdot)$ as defined in Proposition 3.3.6, the i^{th} component, $b_i(s,t)$, of function $\mathbf{b}(s,t)$ is equal to:

$$
\begin{aligned}
b_i(s,t) &= \frac{BZ(t)}{S}\sum_{l=1}^{m}\frac{\xi_{il}}{\lambda_l}(1-e^{\lambda_l(s-u)}+\lambda_l) \\
&- \frac{BZ(t)}{S}\frac{\cosh\sqrt{\frac{\pi'}{\kappa}}(t-T)}{\cosh\sqrt{\frac{\pi'}{\kappa}}(u-T)}\int_s^t\left(\frac{Z(u)}{2\kappa}\sum_{l=1}^{m}\frac{\xi_{il}}{\lambda_l}(1-e^{\lambda_l(s-u)}+\lambda_l)\right)du \\
&+ \frac{BZ(t)}{S}\frac{\cosh\sqrt{\frac{\pi'}{\kappa}}(t-T)}{\cosh\sqrt{\frac{\pi'}{\kappa}}(u-T)}\int_s^t\left(\frac{\sqrt{\pi'}(A_i(s,T)-A_i(s,u))}{2\sqrt{\kappa}\cosh\sqrt{\frac{\pi'}{\kappa}}(u-T)}\right)du \\
&+ \frac{B\sqrt{\pi'\kappa}}{S}\left(\frac{(A_i(s,T)-A_i(s,t))}{2\sqrt{\kappa}\cosh\sqrt{\frac{\pi'}{\kappa}}(t-T)}\right).
\end{aligned}
$$

Proof. See the Appendix. □

In the next section, we use the closed form optimal solution of the optimal production and equilibrium market price to show that they evolve as martingales.

4 Evolution of Market Price as a Martingale

In a way similar to the end-consumer demand forecasts, the forecasting processes of optimal production, optimal net inventory, and equilibrium market price can also be defined. Denote the forecast at s of cumulative production at t by $Y(s,t)$. Then $Y(s,t)$ can be defined as the expectation of $Y(t)$ conditional on all the information in $\mathcal{F}_s$. By definition, $Y(\cdot,t)$ is a Martingale. Define:

$$
\begin{aligned}
\varphi_3(s,t) &= E(\varphi_3(t)|\mathcal{F}_s) \\
&= B\sum_{i=1}^{n}\int_0^s\sum_{l=1}^{m}\xi_{il}\sqrt{\pi'\kappa}\left(\frac{e^{\lambda_l(u-T)}-e^{\sqrt{\frac{\pi'}{\kappa}}(t-T)+\lambda_l(u-t)}}{\sqrt{\frac{\pi'}{\kappa}}-\lambda_l}\right)dW_i(u) \\
&+ B\sum_{i=1}^{n}\int_0^s\sum_{l=1}^{m}\xi_{il}\sqrt{\pi'\kappa}\left(\frac{e^{\lambda_l(u-T)}-e^{-\sqrt{\frac{\pi'}{\kappa}}(t-T)+\lambda_l(u-t)}}{\sqrt{\frac{\pi'}{\kappa}}+\lambda_l}\right)dW_i(u).
\end{aligned}
$$

Then, it can be easily seen that:

$$
\begin{aligned}
Y(s,t) &= Y_0 \frac{\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)} \\
&+ \cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T)) \int_0^t \left(\frac{\frac{Z(z)}{2\kappa}(F(s,z) - X_0 + Y_0) + C - \frac{\varphi(s,z)}{\kappa}}{\cosh(\sqrt{\frac{\pi'}{2\kappa}}(z-T))} \right) dz.
\end{aligned}
$$

Similarly, define the forecast at s of net inventory and price at t by $I(s,t)$ and $P(s,t)$, respectively. Then:

$$
\begin{aligned}
I(s,t) &= E(I(t)|\mathcal{F}_s) \\
&= Y(s,t) - F(s,t) + X_0 - Y_0.
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
P(s,t) &= E(P(t)|\mathcal{F}_s) = -\frac{Z(t)\bar{I}(s,t) + \varphi_1(t) + \varphi_2(t) + \varphi_3(s,t)}{S} \\
&= a(t) + \int_0^s \mathbf{b}(u,t)\mathbf{dW}(u). \qquad (4.25)
\end{aligned}
$$

The forecast update at s, $d_s P(s,t)$, of the market price at t is given by:

$$
d_s P(s,t) = -\frac{Z(t) d_s \bar{I}(s,t) + d_s \varphi_3(s,t)}{S} = \mathbf{b}(s,t)\mathbf{dW}(s).
$$

where $\mathbf{b}(s,t)$ is as defined in Corollary 3.3.8.

The expression for the price forecast in a market model for capacity (4.25) is the major contribution of this paper. To gain some insight into (4.25), we explore the impact of various parameters on the quantity $d_s P(s,t)$ for a special case, $m = 1, n = 1$.

4.1 Resolution of Price Uncertainty

In this subsection, we numerically analyze the impact of the rate of resolution of the demand uncertainty and the supply chain cost parameters on the rate of resolution of price uncertainty. We consider the special case in which $m = 1$ and $n = 1$.

Recall that the forecast update at s for the rate of demand at $t, d_s f(s,t)$ is given by:

$$d_s f(s,t) = \left(\xi_{11} e^{\lambda_1 (s-t)}\right) dW_1(s).$$

The parameter λ_1 in the last equation defines the curvature of the forecast update curve of the rate of demand at t and can be interpreted as the *inverse rate* of the resolution of the uncertainty of the instantaneous rate of demand at t. A high value of λ_1 implies that most of the uncertainty in the rate of demand is resolved just before t. On the other hand, a low value of λ_1 implies that the resolution of the uncertainty in the rate of demand occurs more uniformly in the time before t.

We refer to the diffusion coefficient (that is, the coefficient of $dW_1(\cdot)$) in the expressions for the forecast updates of a variable as the forecast update coefficient for that variable. For example, the forecast update coefficient of the instantaneous rate of demand at t is equal to $\xi_{11} e^{\lambda_1 (s-t)}$ at s. The forecast update at s, of the market price at t, is given by:

$$d_s P(s,t) = -\frac{Z(t) d_s \bar{I}(s,t) + d_s \varphi_3(s,t)}{S} = b_1(s,t) dW_1(s).$$

where $b_1(s,t)$, the price forecast update coefficient, is as defined in Corollary 3.3.8. Observe the the mean of $d_s P(s,t)$ is 0 and its variance is equal to $b(s,t)^2 ds$. Therefore, the variance of cumulative forecast update until time s is equal to $\int_0^s b(u,t)^2 du$. Define the fraction of variance resolved by time s for price at time t, $FVR(s,t)$, as:

$$FVR(s,t) = \frac{\int_0^s b(u,t)^2 du}{\int_0^t b(u,t)^2 du}. \tag{4.26}$$

We use the following data for the numerical experiments:

t	T	S/B	ξ_{11}
15	30	1	1

Table 1: Data for Numerical Example

In Figures 1 and 2, we plot the forecast update coefficient of the market price over time $[0, t)$ and the fraction of variance resolved in the realization of price for different values of λ_1 and a fixed value of $\frac{\kappa}{\pi'}$ equal to 10. Figures 1 and 2 show that as the value of λ_1 increases, the resolution

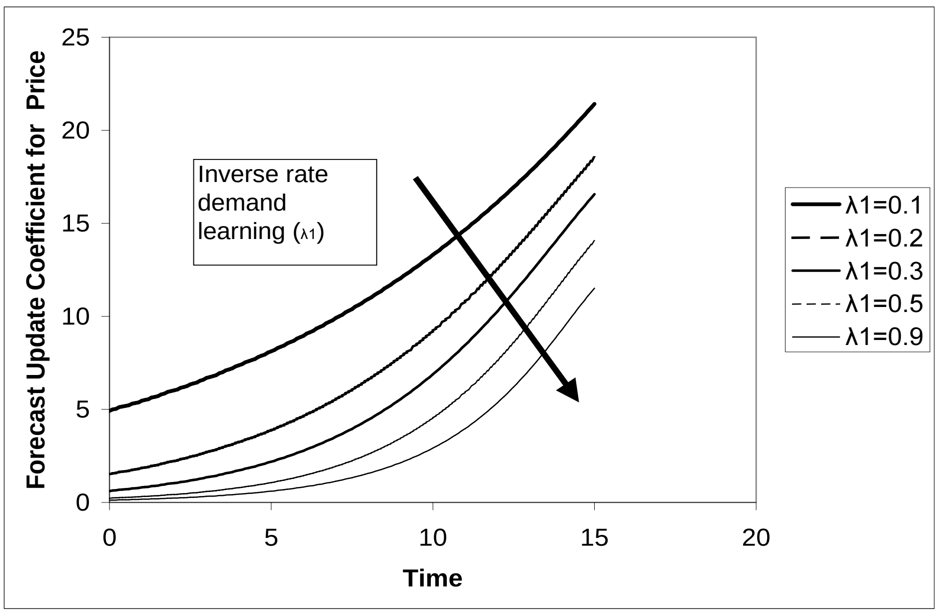

Figure 1: Evolution of Forecast Update Coefficient for Price

of price uncertainty is delayed and occurs increasingly just before t. In addition, the cumulative resolution of variance at any instant, defined by the numerator of the RHS of (4.26), reduces as λ_1 increases. This behavior is consistent with the resolution of the demand uncertainty as a function of λ_1. That is, the rate of resolving price uncertainty follows the same pattern as the rate of resolving demand uncertainty.

In Figure 3, we plot the forecast update coefficient for the market price over time in the realization of price for different values of $\frac{\kappa}{\pi'}$ and for a fixed value λ_1 equal to 1. Contrary to our own intuition, we find that the supply chain cost parameters do affect the resolution of the price uncer-

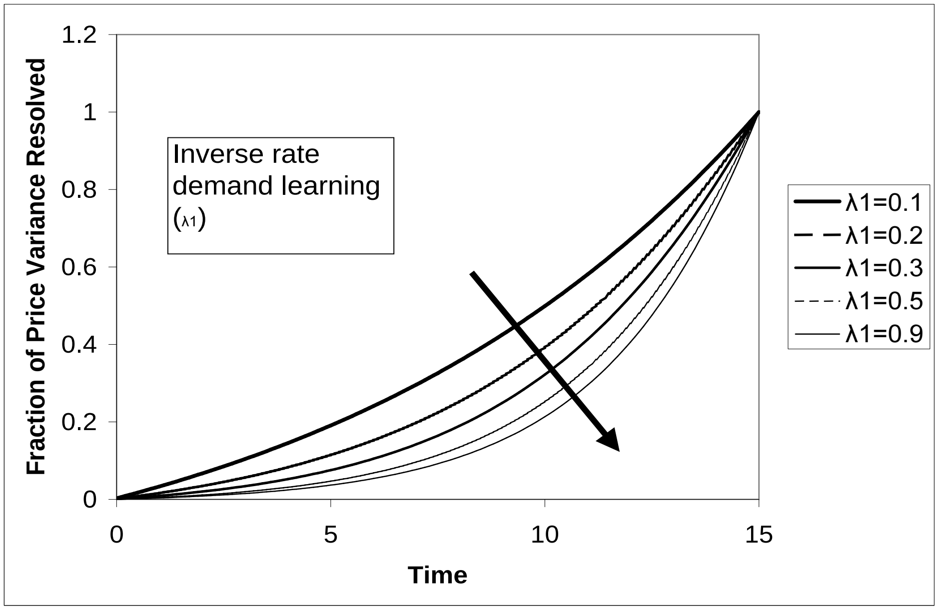

Figure 2: Fraction of Forecast Variability Resolved for Price

tainty. Figure 3 shows that as $\frac{\kappa}{\pi'}$ (that is, the relative cost of changing production with respect to holding inventory) increases, the resolution of price uncertainty occurs more uniformly in time. On the other hand, for low values of $\frac{\kappa}{\pi'}$, a greater portion of the resolution of price uncertainty occurs over a shorter duration of time before t.

The total variance resolved over $[0, t)$ (i.e. the numerator of RHS in (4.26)) may also be construed as a measure of variance of price at time t. Indeed, a price process with highly volatile forecasting process is likely be highly volatile in nature. With this understanding, we observe in Figure 3 that the forecast update coefficient for price at t increases with $\frac{\kappa}{\pi'}$ for any fixed instant $s < t$, implying that the variance of price increases with $\frac{\kappa}{\pi'}$. It follows that the market for supply chain capacity of a capital intensive firm with high $\frac{\kappa}{\pi'}$ ratio is likely to be highly volatile. On the

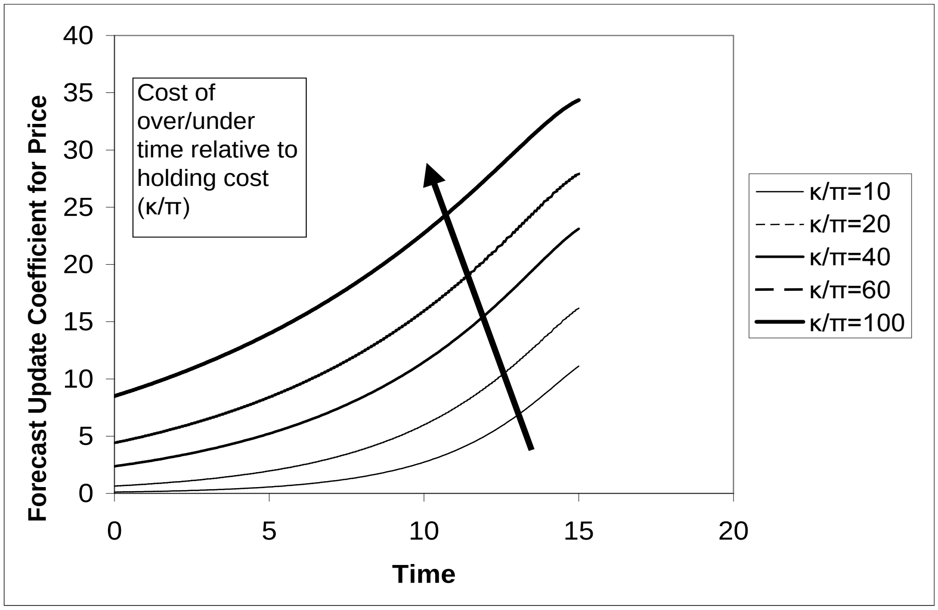

Figure 3: Evolution of Price Forecast Update Coefficient for Different Cost Parameters

other hand, the price volatility is likely to be low in a capacity market for which the relative cost of production-capacity mismatch is low compared to the holding cost.

This finding, that the cost structure of the supply chain has a direct impact on the ability of participants to forecast the price of capacity, is a new result in supply chain analysis.

Observations from Figure 3 provide a glimpse of how the market for capacity as a system handles the exogenous uncertainty which is realized through the end-consumer demand. The market has two variables at its disposal to tackle the exogenous uncertainty: rate of production and inventory. In a perfect market, either the sellers may update the rate of production keeping pace with the uncertainty as it unfolds itself or, buyers may hold (incur) excessive inventory (backorders). A combination of the two is also possible. If the market handles uncertainty through the inventory/backorders then the buyers must be given incentives in the form of the price discounts/premiums which increases the variance of price. The relative values of the cost parameters κ and π determine the share of each variable in handling uncertainty.

Indeed, observations from Figure 3 provide evidence for the above surmise. When the cost of overtime/undertime κ is relatively high compared to the holding cost π, varying the rate of production is expensive and the market handles the exogenous uncertainty through inventory/backorders. This results in increased price variance. Using the same assertion, the involvement of the rate of production increases as the cost parameter κ comes relatively closer to π.

5 Conclusion

In this paper, we have generalized the existing discrete time MMFE model to a continuous-time model. We also considered a market for production capacity in which the buyers of production capacity take into account the forecast updates in the future while making production decisions. We obtained expressions for forecasts of the market price and the optimal production rate and analyzed the impact of the rate of resolving demand uncertainty and of the supply chain cost parameters on the rate of resolving price uncertainty. We found, surprisingly, that uncertainty in the price of capacity is resolved later in a market characterized by low production capacity mismatch costs, relative to holding/backorder costs.

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6 Appendix

6.1 Proof of Corollary 2.1.1

$$
\begin{aligned}
F(s,t) &= F(s,s) + \int_s^t f(s,v)dv \\
&= F(s,s) + \int_s^t f(0,v)dv + \int_s^t \int_0^s d_u f(u,v)dv \\
&= \int_0^s f(v,v)dv + \int_0^s \sigma(v,v)\mathbf{dW}(v)^T + \int_s^t f(0,v)dv \\
&+ \int_s^t \int_0^s d_u f(u,v)dv,
\end{aligned}
$$

where the last step follows by noting that

$$
F(s,s) = F(0,0) + \int_0^s f(v,v)dv + \int_0^s \sigma(v,v)\mathbf{dW}(v)^T,
$$

and by assuming $F(0,0) = 0$.

$$
\begin{aligned}
F(s,t) &= \int_0^s \left(f(0,v) + \int_0^v d_u f(u,v) \right) dv + \int_0^s \sigma(v,v)\mathbf{dW}(v)^T \\
&+ \int_s^t f(0,v)dv + \int_s^t \int_0^s \sigma(u,v)\mathbf{dW}(u)^T dv \\
&= \int_0^s \int_0^v \sigma(u,v)\mathbf{dW}(u)^T dv + \int_0^s \sigma(v,v)\mathbf{dW}(v)^T + \int_0^t f(0,v)dv \\
&+ \int_s^t \int_0^s \sigma(u,v)\mathbf{dW}(u)^T dv.
\end{aligned}
$$

Therefore,

$$
F(t,t) = \int_0^t \int_0^v \sigma(u,v)\mathbf{dW}(u)^T dv + \int_0^t \sigma(v,v)\mathbf{dW}(v)^T + \int_0^t f(0,v)dv.
$$

Now,

$$\begin{aligned}
E(F(t,t)|\mathcal{F}_s) &= E\left(\left(\int_0^t \int_0^v \sigma(u,v)\mathbf{dW}(u)^T dv + \int_0^t \sigma(v,v)\mathbf{dW}(v)^T\right) |\mathcal{F}_s\right) \\
&+ \int_0^t f(0,v)dv \\
&= E\left(\int_0^s \int_0^v \sigma(u,v)\mathbf{dW}(u)^T dv|\mathcal{F}_s\right) \\
&+ \int_s^t \int_0^v \left(\sigma(u,v)\mathbf{dW}(u)^T dv|\mathcal{F}_s\right) \\
&+ \int_0^s \sigma(v,v)\mathbf{dW}(v)^T + \int_0^t f(0,v)dv \\
&= \int_0^s \int_0^v \sigma(u,v)\mathbf{dW}(u)^T dv + \int_s^t \int_0^s \sigma(u,v)\mathbf{dW}(u)^T dv \\
&+ \int_0^s \sigma(v,v)\mathbf{dW}(v)^T + \int_0^t f(0,v)dv \\
&= F(s,t).
\end{aligned}$$

6.2 Proof of Proposition 2.2.1

Our goal is to show that the four conditions specified in the proof of Lemma 4.1 in Ikeda and Watanabe Ikeda and Watanabe (1989) are satisfied when the integrand has the form stated in Proposition 2.2.1. Let $\Psi(s,t,\omega) = Q_1(s,t)1_{[s,u]}(t)$. As a result, we can write the LHS of (2.2), equivalently, as:

$$\int_0^u \int_0^t Q_1(s,t)dW(s)dt = \int_{\mathcal{R}^1} \int_0^t Q_1(s,t)1_{[s,u]}(t)dW(s)dt$$

Note that we have specified ω as an argument to Ψ even though the integrand is independent of ω. This is done so that we can apply the four conditions specified in Lemma 4.1 in Ikeda and Watanabe Ikeda and Watanabe (1989) directly.

Condition 1: *$((s,\omega),t) \in ([0,\infty),\Omega)\times\mathcal{R}^1 \to \Psi(s,t,\omega)$ is $\mathcal{S}\times\mathcal{B}(\mathcal{R}^1)$ - measurable where $\mathcal{S}$ is the smallest $\sigma-$field on $[0,\infty)\times\Omega$ s.t. all left continuous $\mathcal{F}_s-$ adapted processes $Z : [0,\infty)\times\Omega \to Z_s(\omega)$ are measurable.*

Consider the following functions:

$$\begin{aligned} Z_n(s,t) &= Z(s,\frac{i}{n}u), 0 < s \le t, 0 \le \frac{i-1}{n}u < t \le \frac{i}{n}u, i \in \{1,..,n\} \\ Z_n(0,0) &= Z(0,0) \\ &= 0; s > t, t \in (u,\infty) \cup (-\infty,0], \end{aligned}$$

where

$$Z(s,\frac{i}{n}u) = Q_1(s,\frac{i}{n}u).$$

Clearly, $Z_n(s,\cdot)$ is left continuous for each fixed s. Consider any set $A \in \mathcal{B}(\mathcal{R}^1)$. For any $t \in B_i = (\frac{i}{n}u, \frac{i+1}{n}u]$ for any $i < n$, $Z_n^{-1}(\cdot,t)(A) =: C_{s,i} \in \mathcal{S}$. By definition of product spaces, $(C_{s,i}, B_i) \in \mathcal{S}\times\mathcal{B}(\mathcal{R}^1)$. Therefore $Z_n^{-1}(\cdot,\cdot)(A) = \cup_i(C_{s,i}, B_i) \in \mathcal{S}\times\mathcal{B}(\mathcal{R}^1)$ proving the measurability of $Z_n(\cdot,\cdot)$ with respect to $\mathcal{S}\times\mathcal{B}(\mathcal{R}^1)$. Now, due to the left continuity of $Z_n(s,t)$ in t, $Z_n(s,t) \to Z(s,t)$, we have that $Z(s,t) \in \mathcal{S}\times\mathcal{B}(\mathcal{R}^1)$.

Condition 2: *There exists a non-negative Borel-measurable function $f(t)$ such that*

$$|\Psi(s,t,\omega)| \le f(t)$$

for every s,t,ω. This follows immediately from the continuity of $Q_1(\cdot,\cdot)$.

Condition 3: $(t,\omega) \to \int_0^{t_1} \Psi(s,t,\omega)dW(s,\omega)$ *is $\mathcal{B}(\mathcal{R}^1) \times \mathcal{F}-$measurable for each $t_1 \ge 0$.*

To see this,

$$\begin{aligned} \int_0^{t_1} \Psi(s,t,\omega)dW(s,\omega) &= \int_0^{t_1} Q_1(s,t)1_{(0,u]}(t)1_{(0,t]}(s)dW(s,\omega) \\ &= 1_{(0,u]}(t)\int_0^{t_1\wedge t} Q_1(s,t)dW(s,\omega) \\ &= 1_{(0,u]}(t)W(H(t,t_1\wedge t),\omega) \end{aligned}$$

where

$$H(t,\cdot) = \int_0^{\cdot} Q_1^2(s,t)ds.$$

The last step follows from "time substitution" (McKean McKean (1969)). For any ω, define:

$$
\begin{aligned}
Z_n(t,\omega) &= W(H(\frac{p}{n}u, t_1 \wedge \frac{p}{n}u),\omega), p \in \{1,.,n\}, t \in (\frac{p-1}{n}u, \frac{p}{n}u] \\
&= 0, \text{ otherwise.}
\end{aligned}
$$

For each n, and for any $\omega, t \to Z_n(t,\omega)$ is left continuous. Further, for any t, $Z_n(t,\omega)$ is measurable with respect to $\mathcal{F}_{H(t_1 \wedge \frac{p}{n}u)} \subset \mathcal{F}$. In a way similar to that used to establish Condition 1, it can be shown that

$$
Z_n(t,\omega) \in \mathcal{B}(\mathcal{R}^1) \times \mathcal{F}.
$$

Due to the left continuity of $Z_n(t,\omega)$ in t,

$$
\lim_{n\to\infty} Z_n(t,\omega) = Z(t,\omega) = B(H(t, t_1 \wedge t),\omega)1_{(0,u]}(t),
$$

which will also be measurable with respect to $\mathcal{B}(\mathcal{R}^1) \times \mathcal{F}$.

Condition 4: $\int_{\mathcal{R}^1} f(t)dt < \infty$.

Taking $f(t) := 1_{[s,u](t)} \sup_{0<s<t\leq u} |Q_1(s,t)|$, we have

$$
\int_{\mathcal{R}^1} f(t)dt \leq u(\sup_{0<s<t\leq u} |Q_1(s,t)|) < \infty.
$$

6.3 Proof of Proposition 3.3.6

The differential equation involving $Z(\cdot)$ does not have any stochastic terms other than $\mathbf{\Lambda}(t)\mathbf{dW}(t)$. Setting $\mathbf{\Lambda}(\cdot)$ to zero results in a standard one-dimensional Riccatti differential equation which is easily solvable using standard techniques.

We follow closely the approach outlined in Yong and Zhou (1999) for the solution of the following backward stochastic differential equation. We drop the subscripts from $C_\iota, D_\iota, \alpha_\iota,$ and γ_ι for clarity.

Also, we substitute $\sigma_\iota(\cdot,\cdot)$ by $B\sigma(\cdot,\cdot)$.

$$\begin{aligned} d\varphi(t) &+ Z(t)(-\frac{\varphi(t)}{2\kappa} + C - D - \alpha\gamma\cos\gamma t - B\int_0^t \sigma(u,t)\mathbf{dW}(u))dt \\ &+ \mathbf{\Lambda}(t)\mathbf{dW}(t) = 0 \\ & \varphi(T) = 0 \end{aligned}$$

where

$$Z(t) = 2\sqrt{\pi'\kappa}\left(\frac{1 - e^{2\sqrt{\frac{\pi'}{\kappa}}(t-T)}}{1 + e^{2\sqrt{\frac{\pi'}{\kappa}}(t-T)}}\right).$$

Consider the following stochastic differential equation:

$$\begin{aligned} d\chi(t) + \left(\frac{1}{2\kappa}Z(t)\chi(t)\right)dt &= 0, \\ \chi(0) &= 1. \end{aligned}$$

The solution to the differential equation for $\chi(\cdot)$ is given by:

$$\begin{aligned} c_1 e^{\int -\sqrt{\frac{\pi'}{\kappa}}\left(\frac{1-e^{2\sqrt{\frac{\pi'}{\kappa}}(t-T)}}{1+e^{2\sqrt{\frac{\pi'}{\kappa}}(t-T)}}\right)dt} &= c_1 e^{\int \sqrt{\frac{\pi'}{\kappa}}\tanh(\sqrt{\frac{\pi'}{\kappa}}(t-T))dt} \\ &= c_1 \cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T)) \end{aligned}$$

where, using the boundary condition, c_1 is determined to be:

$$c_1 = \frac{1}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}.$$

Applying Ito's formula to $\chi(t)\varphi(t)$:

$$\begin{aligned} d[\chi(t)\varphi(t)] &= -\chi(t)Z(t)(C - D - \alpha\gamma\cos\gamma t - B\int_0^t \sigma(u,t)\mathbf{dW}(u))dt \\ &- \chi(t)\mathbf{\Lambda}(t)\mathbf{dW}(t). \end{aligned}$$

Integrating both sides:

$$
\begin{aligned}
\chi(t)\varphi(t) - \chi(T)\varphi(T) &= \int_t^T \chi(s)Z(s)\left(C - D - \alpha\gamma\cos\gamma s - B\sigma(u,s)\mathbf{dW}(u)\right)ds \\
&+ \int_t^T \chi(s)\mathbf{\Lambda}(s)\mathbf{dW}(s) \\
\chi(t)\varphi(t) &= \theta + B\int_0^t \chi(s)Z(s)\int_0^s \sigma(u,s)\mathbf{dW}(u)ds \\
&+ \int_t^T \chi(s)Z(s)(C - D - \alpha\gamma\cos\gamma s)ds + \int_t^T \chi(s)\mathbf{\Lambda}(s)\mathbf{dW}(s)
\end{aligned}
$$

where

$$
\theta = -B\int_0^T \chi(s)Z(s)\int_0^s \sigma(u,s)\mathbf{dW}(u)ds.
$$

Taking the conditional expectation with respect to $\mathcal{F}_t$:

$$
\begin{aligned}
E(\chi(t)\varphi(t)|\mathcal{F}_t) &= E(\theta|\mathcal{F}_t) + B\int_0^t \chi(s)Z(s)\int_0^s \sigma(u,s)\mathbf{dW}(u)ds \\
&+ \int_t^T \chi(s)Z(s)(C - D - \alpha\gamma\cos\gamma s)ds \\
\chi(t)\varphi(t) &= -E\left(\left(B\int_0^T \chi(s)Z(s)\int_0^s \sigma(u,s)\mathbf{dW}(u)ds\right)|\mathcal{F}_t\right) \\
&+ \int_t^T \chi(s)Z(s)(C - D - \alpha\gamma\cos\gamma s) \\
&+ B\int_0^t \chi(s)Z(s)\int_0^s \sigma(u,s)\mathbf{dW}(u)ds.
\end{aligned}
$$

Now, simplifying the RHS:

$$
\begin{aligned}
\varphi_1'(t) &= \int_t^T \chi(s)Z(s)(C-D)ds = -\int_t^T \frac{2\sqrt{\pi'\kappa}\sinh(\sqrt{\frac{\pi'}{\kappa}}(s-T))\,(C-D)}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}ds \\
&= \frac{2(C-D)\kappa}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}\left(\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T)) - 1\right).
\end{aligned}
$$

Similarly,

$$\begin{aligned}
\varphi_2'(t) &= -\int_t^T \chi(s)Z(s)\alpha\gamma\cos\gamma s ds \\
&= \int_t^T \frac{2\sqrt{\pi'\kappa}}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}\left(\sinh(\sqrt{\frac{\pi'}{\kappa}}(s-T))\right)\alpha\gamma\cos\gamma s ds \\
&= \frac{2\alpha\gamma\kappa}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}\left(\cos\gamma T - \cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))\cos\gamma t\right) \\
&- \frac{2\alpha\gamma^2\kappa^{3/2}}{\sqrt{\pi'}\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}\left(\sinh(\sqrt{\frac{\pi'}{\kappa}}(t-T))\sin\gamma t\right) - \frac{\kappa}{\pi}\gamma^2\varphi_2'(t) \\
\varphi_2'(t) &= \frac{2\alpha\gamma}{1+\frac{\kappa}{\pi}\gamma^2}\left(\frac{\kappa\cos\gamma T - \kappa\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))\cos\gamma t}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}\right) \\
&- \frac{2\alpha\gamma^2}{(1+\frac{\kappa}{\pi}\gamma^2)\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}\left(\frac{\kappa^{3/2}}{\pi^{1/2}}\sinh(\sqrt{\frac{\pi'}{\kappa}}(t-T))\sin\gamma t\right)
\end{aligned}$$

and finally:

$$\begin{aligned}
\varphi_3''(t) &= -B\int_0^t \frac{\sqrt{\pi'\kappa}}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}\left(e^{\sqrt{\frac{\pi'}{\kappa}}(s-T)} - e^{-\sqrt{\frac{\pi'}{\kappa}}(s-T)}\right)\int_0^s \sigma(u,s)\mathbf{dW}(u)ds \\
&= -B\int_0^t\int_u^t \frac{\sqrt{\pi'\kappa}}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}\left(e^{\sqrt{\frac{\pi'}{\kappa}}(s-T)} - e^{-\sqrt{\frac{\pi'}{\kappa}}(s-T)}\right)\sigma(u,s)ds\mathbf{dW}(u)
\end{aligned}$$

using Proposition 2.2.1. Assuming σ is approximated by (3.10),

$$\varphi_3''(t) = -B\sum_{i=1}^n \int_0^t \frac{\sqrt{\pi'\kappa}}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)} A_i(u,t)dW_i(u)$$

where

$$A_i(u,t) = \sum_{l=1}^m \xi_{il}\left(\frac{e^{\lambda_l(u-t)+\sqrt{\frac{\pi'}{\kappa}}(t-T)} - e^{\sqrt{\frac{\pi'}{\kappa}}(u-T)}}{\sqrt{\frac{\pi'}{\kappa}}-\lambda_l} + \frac{e^{\lambda_l(u-t)-\sqrt{\frac{\pi'}{\kappa}}(t-T)} - e^{-\sqrt{\frac{\pi'}{\kappa}}(u-T)}}{\sqrt{\frac{\pi'}{\kappa}}+\lambda_l}\right). \quad (6.27)$$

Similarly,

$$\begin{aligned}
\theta &= B\int_0^T \frac{\sqrt{\pi'\kappa}}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}\left(e^{\sqrt{\frac{\pi'}{\kappa}}(s-T)} - e^{-\sqrt{\frac{\pi'}{\kappa}}(s-T)}\right)\int_0^s \sigma(u,s)\mathbf{dW}(u)ds \\
&= B\int_0^T\int_u^T \frac{\sqrt{\pi'\kappa}}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}\left(e^{\sqrt{\frac{\pi'}{\kappa}}(s-T)} - e^{-\sqrt{\frac{\pi'}{\kappa}}(s-T)}\right)\sigma(u,s)ds\mathbf{dW}(u) \\
&= B\sum_{i=1}^n \int_0^T \frac{\sqrt{\pi'\kappa}}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)} A_i(u,T)dW_i(u),
\end{aligned}$$

where $A_i(u,\cdot)$ is defined in (6.27).

Now, for $t<T$,

$$E(\theta|\mathcal{F}_t) = B\sum_{i=1}^{n}\int_0^t \frac{\sqrt{\pi'\kappa}}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}A_i(u,T)dW_i(u).$$

Therefore,

$$\begin{aligned}\varphi_3''(t) \quad &+ \quad E(\theta|\mathcal{F}_t)\\ &= \quad B\sum_{i=1}^{n}\int_0^t \frac{\sqrt{\pi'\kappa}}{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}(A_i(u,T)-A_i(u,t))dW_i(u)\\ &=: \quad \varphi_3'(t)\end{aligned}$$

which is measurable with respect to $\mathcal{F}_t$. Therefore,

$$\varphi(t) = \frac{\cosh(\sqrt{\frac{\pi'}{\kappa}}T)}{\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T)}(\varphi_1'(t)+\varphi_2'(t)+\varphi_3'(t)).$$

Using equation (2.25), pp 352, Yong and Zhou (1999):

$$\Lambda_i(t) = \frac{B\sqrt{\pi'\kappa}A_i(t,T)}{\cosh(\sqrt{\frac{\pi'}{\kappa}}(t-T))}.$$

6.4 Proof of Corollary 3.3.7

Let $\psi(t)$ be such that

$$d(\psi(t)\bar{Y}(t)) = \psi(t)(C+\frac{1}{2\kappa}(Z(t)(F(t,t)-X_0+Y_0)-\varphi(t)))dt. \tag{6.28}$$

The LHS of the above equation is equal to,

$$d(\psi(t)\bar{Y}(t)) = \psi(t)d\bar{Y}(t)+d(\psi(t))\bar{Y}(t),$$

which implies that $\psi(t)$ must satisfy the following differential equation,

$$\frac{d\psi(t)}{\psi(t)} \quad = \quad \frac{Z(t)}{2\kappa}dt, \psi(0)=1.$$

Solving for $\psi(\cdot)$ in the above equation,

$$\psi(t) \quad = \quad \frac{\cosh\sqrt{\frac{\pi'}{\kappa}}T}{\cosh\sqrt{\frac{\pi'}{\kappa}}(t-T)}.$$

Integrating (6.28) on both sides,

$$\begin{aligned}
\psi(t)\bar{Y}(t) - \psi(0)\bar{Y}(0) &= \int_0^t \psi(u)(C + \frac{1}{2\kappa}\left(Z(u)(F(u,u) - X_0 + Y_0) - \varphi(u)\right))du \\
\bar{Y}(t) &= \cosh\sqrt{\frac{\pi'}{\kappa}}(t-T)\int_0^t \frac{C + \frac{1}{2\kappa}\left(Z(u)(F(u,u) - X_0 + Y_0) - \varphi(u)\right)}{\cosh\sqrt{\frac{\pi'}{\kappa}}(u-T)}du \\
&+ \frac{\cosh\sqrt{\frac{\pi'}{\kappa}}(t-T)}{\cosh\sqrt{\frac{\pi'}{\kappa}}T}Y_0
\end{aligned}$$

6.5 Proof of Corollary 3.3.8

Using Proposition 3.3.7,

$$P(t) = \frac{-Z(t)(\bar{Y}(t) - F(t,t) + X_0 - Y_0) - \phi_1(t) - \phi_2(t) - \phi_3(t)}{S}.$$

The deterministic component of the RHS is equal to,

$$a(t) = \frac{-Z(t)\left(\frac{\cosh\sqrt{\frac{\pi'}{\kappa}}(t-T)}{\cosh\sqrt{\frac{\pi'}{\kappa}}T}Y_0 - Dt - \alpha\sin\gamma t + X_0 - Y_0\right) - \phi_1(t) - \phi_2(t)}{S}.$$

The remaining terms are equal to

$$\begin{aligned}
& \frac{BZ(t)}{S}\sum_{i=1}^{n}\sum_{l=1}^{m}\int_0^t \frac{\xi_{il}}{\lambda_l}(1 - e^{\lambda_l(s-u)} + \lambda_l)dW_i(s) \\
- \quad & \frac{BZ(t)}{S}\sum_{i=1}^{n}\int_0^t \frac{\cosh\sqrt{\frac{\pi'}{\kappa}}(t-T)}{\cosh\sqrt{\frac{\pi'}{\kappa}}(u-T)}\int_0^u \left(\frac{Z(u)}{2\kappa}\sum_{l=1}^{m}\frac{\xi_{il}}{\lambda_l}(1 - e^{\lambda_l(s-u)} + \lambda_l)\right)dW_i(s)du \\
+ \quad & \frac{BZ(t)}{S}\sum_{i=1}^{n}\int_0^t \frac{\cosh\sqrt{\frac{\pi'}{\kappa}}(t-T)}{\cosh\sqrt{\frac{\pi'}{\kappa}}(u-T)}\int_0^u \left(\frac{\sqrt{\pi'}(A_i(s,T) - A_i(s,u))}{2\sqrt{\kappa}\cosh\sqrt{\frac{\pi'}{\kappa}}(u-T)}\right)dW_i(s)du \\
- \quad & \frac{B\sqrt{\pi'\kappa}}{S}\sum_{i=1}^{n}\int_0^t \left(\frac{(A_i(s,T) - A_i(s,t))}{2\sqrt{\kappa}\cosh\sqrt{\frac{\pi'}{\kappa}}(t-T)}\right)dW_i(s).
\end{aligned}$$

where $A_i(\cdot,\cdot)$ is defined in Proposition 3.3.6. Applying Proposition 2.2.1 to the second and third terms to interchange the Lebesegue and stochastic integrals in the above expression,

$$
\begin{aligned}
&\frac{BZ(t)}{S}\sum_{i=1}^{n}\sum_{l=1}^{m}\int_0^t \frac{\xi_{il}}{\lambda_l}(1-e^{\lambda_l(s-u)}+\lambda_l)dW_i(s) \\
-\quad&\frac{BZ(t)}{S}\sum_{i=1}^{n}\int_0^t \frac{\cosh\sqrt{\frac{\pi'}{\kappa}}(t-T)}{\cosh\sqrt{\frac{\pi'}{\kappa}}(u-T)}\int_s^t\left(\frac{Z(u)}{2\kappa}\sum_{l=1}^{m}\frac{\xi_{il}}{\lambda_l}(1-e^{\lambda_l(s-u)}+\lambda_l)\right)dudW_i(s) \\
+\quad&\frac{BZ(t)}{S}\sum_{i=1}^{n}\int_0^t \frac{\cosh\sqrt{\frac{\pi'}{\kappa}}(t-T)}{\cosh\sqrt{\frac{\pi'}{\kappa}}(u-T)}\int_s^t\left(\frac{\sqrt{\pi'}(A_i(s,T)-A_i(s,u))}{2\sqrt{\kappa}\cosh\sqrt{\frac{\pi'}{\kappa}}(u-T)}\right)dudW_i(s) \\
-\quad&\frac{B\sqrt{\pi'\kappa}}{S}\sum_{i=1}^{n}\int_0^t\left(\frac{(A_i(s,T)-A_i(s,t))}{2\sqrt{\kappa}\cosh\sqrt{\frac{\pi'}{\kappa}}(t-T)}\right)dW_i(s).
\end{aligned}
$$

If the above expression were to be represented as $\sum_{i=1}^{n} b_i(s,t)dW_i(s)$, then $b_i(s,t)$ will be equal to,

$$
\begin{aligned}
&\frac{BZ(t)}{S}\sum_{l=1}^{m}\frac{\xi_{il}}{\lambda_l}(1-e^{\lambda_l(s-u)}+\lambda_l) \\
-\quad&\frac{BZ(t)}{S}\frac{\cosh\sqrt{\frac{\pi'}{\kappa}}(t-T)}{\cosh\sqrt{\frac{\pi'}{\kappa}}(u-T)}\int_s^t\left(\frac{Z(u)}{2\kappa}\sum_{l=1}^{m}\frac{\xi_{il}}{\lambda_l}(1-e^{\lambda_l(s-u)}+\lambda_l)\right)du \\
+\quad&\frac{BZ(t)}{S}\frac{\cosh\sqrt{\frac{\pi'}{\kappa}}(t-T)}{\cosh\sqrt{\frac{\pi'}{\kappa}}(u-T)}\int_s^t\left(\frac{\sqrt{\pi'}(A_i(s,T)-A_i(s,u))}{2\sqrt{\kappa}\cosh\sqrt{\frac{\pi'}{\kappa}}(u-T)}\right)du \\
-\quad&\frac{B\sqrt{\pi'\kappa}}{S}\left(\frac{(A_i(s,T)-A_i(s,t))}{2\sqrt{\kappa}\cosh\sqrt{\frac{\pi'}{\kappa}}(t-T)}\right).
\end{aligned}
$$