Using Auctions for Procurement

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Abstract

Due to the advancement in Interenet, auctions are increasingly used not only in distribution but also in procurement in supply chain. We study a reverse auction in which a buyer procures a single unit of good or service from one of many competing sellers through auctions. Sellers have i.i.d. costs. We show that the buyer should prefer the second-price auction to the first-price auction. In the first-price reverse auction, there are multiple symmetric equilibrium bidding strategies. The symmetric bidding strategy in the second-price auction has a unique equilibrium, at which the expected cost to the buyer is less than or equal to the expected cost at any of the multiple equilibria of the first-price auction. If the first-price auction is used, the buyer should use a reserve price to eliminate the multiplicity of the symmetric equilibrium bidding strategies, and should select it so as to strike an optimal balance between the expected price and shortage cost. These findings contrast with the results found in forward auctions, which is shown by a classical auction theory to have the uniqueness of symmetric equilibrium bidding strategy in the first-price auction.

1 Introduction

With the rise of Internet and electronic commerce, auctions are increasingly used to determine prices and to allocate resources. Auctions can be administered fairly and efficiently online, eliminating costly negotiations due to their clearly defined rules. The use of auctions is not limited to the transfer of goods among end-users (as in the case of art and antiques), but also includes supply chain management. Auctions are used to distribute goods from upper-echelon producers to lower-echelon dealers and consumers. For nearly a century, the floral industry has been running auctions, participation of which is now available through the Internet (e.g. BloomAuction.com in Canada). Electronics manufacturers and distributors increasingly rely on auction sites such as eBay.com to liquidate their products.

Auctions are also increasingly used in the procurement side of a supply chain. For example, Covisint and Fast Buyer, business-to-business solution and product providers for the automobile industry founded by OEM's, provide online auction services. These auctions can be originated not only by a seller (in a forward auction), but also a buyer (in a reverse auction). In a reverse auction, a buyer sets up an auction, specifies an auction type, and notifies qualified suppliers to submit bids. Another application of auctions in procurement is the U. S. Government who uses reverse auctions to award contracts among competing bidders.

Although reverse auctions are commonly used, surprisingly little amount has been written about them in auction literature. The auction literature is almost entirely written for a forward auction, and asserts that a reverse auction has equivalent properties. The equivalence does indeed exist, but it is a mathematical one. In a reverse auction, a bidder is a seller and prefers any price greater than his cost to losing his bid; in a forward auction, a bidder is a buyer, and wants to pay less than his value of the object. The range of acceptable prices to a bidder is unbounded in a reverse auction, but is bounded in a forward auction. This difference is due to the existence of an implicit reserve price of zero in a forward auction, since both values and bids have to be nonnegative. In a reverse auction, there is no fixed natural reserve price, and bids can be arbitrarily high. The buyer is contractually bound to pay the price determined by the price-determining mechanism and the set of bids.

This assymmetry results in significant supply chain implications, particularly in the design of sealed-bid reverse auctions. Only a handful of papers (e.g. Carey (1993)) have been written about reverse auctions. In reverse (forward) auctions, the seller (buyers) with the lowest (highest) bid wins the object. There are two common methods of determining the price of the object. In the first-price reverse (forward) auction, the price is the same as the winning bid; in the second-price reverse (forward) auction, the price is given by the lowest (highest) losing bid. The buyer in a reverse auction often has the power to design the auction mechanism. Suppose that the U. S. government sets up an auction to procure a certain service. It is reasonable to assume that each bidder willing to provide this service has an independent and identically distributed cost. Then, is the risk-neural expected cost to the government irrelevant to the type of a sealed-bid reverse auction chosen? Does setting buyer's reserve price have any impact? Suppose now that an automobile plant wants to buy a new press. Which type of auction should be used to attain the lowest expected cost? These are some of the questions addressed in this paper.

This paper makes the following strategic implications for the buyer who wants to procure using an auction. First, when there is no reserve price, the buyer should prefer the second-price auction over the first-price auction because the second-price auction admits only one symmetric equilibrium for bidding strategies. Thus, in the second-price auction, the behavior of bidders is more predictable and stable. Furthermore, the (unique) expected payment by the buyer in the second-price auction is less than or equal to any of the multiple equilibria for the first-price auction. Second, if a first-price reverse auction is used, the buyer should set a reserve price. The existence of the reserve price eliminates the multiplicity of bidding strategies. Third, in reverse auctions, the reserve price should be determined to strike the optimal balance between the expected price and the shortfall probability. Depending on the shortage cost, setting the reserve price below the buyer's shortage cost may prove to be optimal.

In this paper, we study single-unit single-period sealed-bid reverse auctions in which bidders are symmetric and have independently and identically distributed private costs. In particular, we examine first-price and second-price reverse auctions. For a first-price reverse auction, we derive symmetric bidding strategies by the sellers and the buyer's expected payment, and examine the impact of the seller's reserve price. The first-price reverse auction is compared with the second-price reverse auction.

The main contribution of this paper is to articulate how designing a first-price auction for procurement is different from designing it for distribution. In each of the first-price and second-price forward auctions, Milgrom and Weber (1982) show that there is a unique symmetric equilibrium. The much-celebrated Revenue Equivalence Theorem due to Vickrey (1961) and its generalizations due to Myerson (1981) and Riley and Samuelson (1981) imply that the expected revenue to the seller is the same in the first-price and second-price forward auctions. Based on literature, the buyer may be tempted to draw an analogous result in setting up a reverse auction. Klemperer (2002) warns that poorly understood economic theory may find inappropriate applications and yield unexpected results. We show that in the reverse first-price auction, in general, there are multiple symmetric Nash equilibria for bidding strategies. [1] Each of these bidding strategies corresponds to a distinct expected payment by the buyer. The first-price reverse auction bidding strategy, corresponding to the lowest payment by the buyer, has the same expected payment as the (unique) bidding strategy of the second-price reverse auction. In the second-price reverse auction, by comparison,

[1] The multiplicity of equilibrium bidding strategies in first-price reverse auction exists because the sellers' bids have an unbounded support. One may argue that it is not a reasonable model since there is a finite amount of money in the world. This has merit if one adopts a purely mathematical point of view grounded in the auction theory. However, we believe that considering an unbounded support might be useful for the following reasons: First, the value of the object is often very small compared to the amount of money the buyer possesses. In this situation the seller might perceive that the buyer has an essentially infinite amount of money and take action according to this perception. Second, distributions with an unbounded support such as an exponential distribution are commonly used in operations literature. Third, in real auctions (such as the electricity market), we observe very large bids, which cannot otherwise be adequately explained.

a simple extension of Vickrey (1961) shows an analogous result in the first-price auction that bidding one's own cost is a dominant strategy of every seller.

The second contribution is to study the impact of the buyer's reserve price in eliminating the multiplicity of bidding strategies, as well as maximizing the buyer's cost. Such benefits are consistent with a recent trend in the automobile industry. More buyers are setting reserve prices when they originate auctions, following Covisint's recommendation. Furthermore, we remove a widely-used restriction in classical auction theory, which, in a reverse auction, translates to an assumption that the penalty cost to the buyer for failed procurement falls within the support of sellers' costs. In the supply chain context, the buyer's shortage cost is often much more expensive than the cost of production. We show that even if the penalty cost is higher than the highest possible cost by the seller, it may still be profitable for the buyer to set the reserve price less than the maximum cost.

The third contribution is to allow the distribution of sellers' costs to have a support that is not bounded above, as in an exponential distribution. Such a general modeling assumption is very common in operations research and operations management literature. Yet, auction literature has almost always assumed an upper bound for distributions of costs. (In forward auctions, this assumption translates to the existence of a lower bound on buyers' values, which is reasonable for the sale of an object.) Since the support of costs is not bounded above, neither the support of bids nor the support of the payment by the bidder is bounded above. It is the first paper, to our knowledge, that introduces such generalizations to the auction theory.

In Section 2, we develop theoretical properties of the first-price reverse auction, including bidding strategies, expected buyer's payment, and optimal reserve price of the buyer. These findings are illustrated using an exponential cost distribution in Section 3, and a uniform cost distribution in Section 4.

2 Reverse Auction

2.1 Model

In a reverse auction, the buyer wants to procure an indivisible object from one of $N \geq 2$ sellers. Seller $i = 1, \cdots, N$ has a nonnegative private cost c_i of production, having a p.d.f. f_i, c.d.f. F_i, and a continuous support. Denote the smallest of the infima of these supports by $\underline{c}$, which is nonnegative. Seller i submits a nonnegative bid b_i, and the buyer buys the object from the highest-bidding seller. The price equals the lowest bid in the first-price reverse auction, and equals the second-lowest bid in the second-price reverse auction.

We study the bidding behavior of the seller i as a function of his cost c_i given the distribution of others' bids. The seller i's problem is to choose his bid b_i as to maximize expected profit, which is the product of the probability that he wins the bid and the conditional expected price minus the cost of production given that he has won the unit. Alternatively, since the probability p_i that seller i submits the winning bid is a nondecreasing function of his bid b_i, it is convenient to consider bid $b_i(p_i)$ as a function of probability p_i. (We define $b_i(p_i)$ the smallest bid such that the probability that i wins the bid is at least p_i.) The expected profit associated with seller i's choice of p_i is $e_i(p_i) - c_i \cdot p_i$ where $e_i(p_i)$ is the expected payment to i. (In the first-price reverse auction, $e_i(p_i) = b_i(p_i) \cdot p_i - c_i \cdot p_i$.) Let $p_i^*(c_i)$ minimize the expected profit function $e_i(p_i) - c_i \cdot p_i$.

If e_i is differentiable, the first-order condition of the expected profit yields

$$e_i'(p_i^*(c_i)) = c_i. \tag{2.1}$$

In general, either (2.1) or a limit argument shows

$$e_i(p_i^*(c_i)) = e_i(p_i^*(\underline{c})) + \int_{\underline{c}}^{c_i} u \; dp_i^*(u). \tag{2.2}$$

This is a mathematical expression of the Revenue Equivalence Theorem. Now, (2.2) depends on the auction mechanism used only through p_i^* and $e_i(p_i^*(\underline{c}))$. A reverse auction is *efficient* if the seller with the lowest cost wins the bid. All efficient reverse auctions have a common p_i^*.

Thus, all efficient symmetric bidding strategies with the same expected payment $e_i(p_i^*(\underline{c}))$ given that seller i's cost is $\underline{c}$, have the same expected payment by the buyer.

In the second-price reverse auction, bidding one's cost is the dominant strategy. Thus, $e_i(p_i^*(\underline{c}))$ equals to the second lowest cost of sellers given that seller i's cost is at the lowest possible value $\underline{c}$. This value of $e_i(p_i^*(\underline{c}))$ specifies the unique solution to (2.2).

2.2 First-Price Reverse Auction and Bidding Strategy

This section computes differentiable bidding strategies in the first-price reverse auction, and shows that the symmetric bidding strategy is not unique in general.

In the first-price reverse auction, we have $e_i(p_i) = b_i(p_i) \cdot p_i$, and $e_i'(p_i) = b_i(p_i) + b_i'(p_i) \cdot p_i$. It follows from (2.1),

$$\frac{c_i - b_i(p_i^*(c_i))}{p_i^*(c_i)} = \frac{d}{dp_i^*(c_i)} b_i(p_i^*(c_i)) = \frac{\frac{d}{dc_i} b_i(p_i^*(c_i))}{\frac{d}{dc_i} p_i^*(c_i)}.$$

Now we switch our notation so that we decide on bids $b_i(c_i)$, the probability $p_i(b_i(c_i))$ of winning depends on the bidding function $b_i(c_i)$. Thus,

$$b_i(c_i) \quad = \quad c_i - \frac{p_i(b_i(c_i))}{\frac{d}{dc_i} p_i(b_i(c_i))} \cdot \frac{d}{dc_i} b_i(c_i). \tag{2.3}$$

If the bidding function is increasing, then as the cost c_i increases, the probability $p_i(b_i(c_i))$ of winning weakly decreases. Thus, bid-inflation occurs; i.e., $b_i(c_i) \geq c_i$.

Now we find a symmetric bidding strategy $\beta(\cdot)$ that maps each seller's cost to his bid. (We drop subscript in the bidding strategy $\beta(\cdot)$ because sellers employ a symmetric strategy.) We assume that the private costs of production by each seller has a common distribution with a finite mean. Let f and F denote the p.d.f. and c.d.f of the cost distribution. It can be shown that the symmetric bidding strategy $\beta(\cdot)$ is strictly increasing if it is continuous. (See Appendix A.) From the strict monotonicity of $\beta(\cdot)$, the probability that a bidder wins the unit is the probability that his cost is lower than the cost of any other bidder. Thus, the probability $p_i(b_i(c_i)) = p(c_i)$ of winning is a function of cost c_i only, and this function $p(\cdot)$ is common for all bidders. Equation (2.3) becomes

$$b_i(c_i) \quad = \quad c_i - \frac{p_i(c_i)}{\frac{d}{dc_i}p_i(c_i)} \cdot \frac{d}{dc_i}b_i(c_i).$$

Since there are $N-1$ sellers other than i, we have

$$\begin{aligned} p_i(c_i) &= (1-F(c))^{N-1} \\ \frac{d}{dc_i}p_i(c_i) &= -(N-1)\cdot f(c)\cdot(1-F(c))^{N-2}, \end{aligned}$$

which implies

$$\beta(c) = c + \frac{\beta'(c)}{(N-1)} \cdot \frac{1-F(c)}{f(c)}. \tag{2.4}$$

We now solve this differential equation. Because $\underline{c}$ is the infimum of the support of costs, the solution to the above differential equation (2.4) is given by

$$\beta(c) = c - e^{A(c)} \int_{\underline{c}}^{c} e^{-A(u)}du + Ce^{A(c)}$$

where C is some scalar, and

$$A(c) = \int_{\underline{c}}^{c} (N-1) \cdot \frac{f(u)}{1-F(u)}du = (1-N)\log(1-F(c)).$$

Thus,

$$\beta(c) = c - (1-F(c))^{1-N} \int_{\underline{c}}^{c} (1-F(u))^{N-1}du + C(1-F(c))^{1-N}. \tag{2.5}$$

This is a necessary condition for a symmetric bidding strategy. Since $A(\cdot)$ is strictly increasing in cost c within the support of costs, the second term in (2.5) is strictly decreasing.

There is no incentive for bidders to win the auction at the price below their productions costs. It follows $\beta(c) \geq c$ for all possible c, which implies from (2.5) that C has to big enough to ensure $\int_{\underline{c}}^{c} e^{-A(u)}du \leq C$ for all c in the support of costs. This condition is satisfied if and only if

$$\int_{\underline{c}}^{\infty} (1-F(u))^{N-1}du \leq C. \tag{2.6}$$

If the cost distribution has a bounded support, then the integral is proper, and taken over the entire support. It can be shown based on the finite mean of the cost distribution that the improper integral in the above expression is finite.

We denote by $\beta_C(\cdot)$ the bidding strategy given by (2.5) and scalar C. If the bidding strategy $\beta_{C'}(\cdot)$ satisfies $\beta_{C'}(c) \geq c$ and is strictly increasing, then $\beta_{C''}(\cdot)$ also satisfies $\beta_{C''}(c) \geq c$ and is strictly increasing for all $C'' \geq C'$. Thus, if a symmetric bidding strategy exists, then there are multiple symmetric strategies in general. It is clear that higher bids lead to a higher payment by the buyer, and thus the bidding strategy with the minimal C is preferred by the buyer. The minimal C is obtained by replacing the inequality in (2.6) with equality. It can be shown that the symmetric bidding strategy corresponding to this minimal C in the first-price reverse auction yields the lowest possible expected payment by the buyer, which equals the expected payment in the second-price reverse auction.

We remark the second factor of the integrand of $A(c)$ the density of the hazard rate of the cost distribution.

2.3 Buyer's Reserve Price in the First-Price Reverse Auction

This section shows that in the first-price reverse auction, the buyer's reserve price eliminates the multiplicity of symmetric equilibrium bidding strategies. Furthermore, if there is no reserve price, the unique expected payment by the buyer in the second-price auction is less than or equal to any of the multiple equilibria for the first-price auction. It also shows what the buyer should set her reserve price.

Suppose the buyer sets a reserve price R beyond which sellers are not allowed to bid, and R is in the support of sellers' costs. An analysis similar to the previous section shows that any symmetric bidding strategy in the first-price reverse auction must satisfy (2.5) where C satisfies following two conditions. The first condition is $\beta_C(R) = R$. The second condition is (2.6) where the integral is taken in the support of costs no more than R; i.e. $\beta_C(c) \geq c$ for all $c \in [\underline{c}, R]$. Therefore, the choice of R determines the unique symmetric bidding strategy by specifying the constant $C = C(R)$ of integration in (2.5). It follows

$$C(R) = \int_{\underline{c}}^{R} (1 - F(u))^{N-1} du,$$

and

$$\beta_{C(R)}(c) = c + (1 - F(c))^{1-N} \int_{c}^{R} (1 - F(u))^{N-1} du. \tag{2.7}$$

We observe two properties. First, the second term in the right side of (2.7) decreases in c for a fixed reserve price R. The amount of bid-inflation decreases in the cost of production. Second, a seller with a fixed cost c will increase his bid as the buyer's reserve price R increases.

If a seller's cost is greater than R, then this seller has no incentive to participate in the auction. Suppose seller i's cost c_i equals R. Then, the expected payment received by i is the product of his bid, which is $\beta(R) = R$, and the probability that he wins the bid, which is the probability that every other bidder's cost is greater than R. We compare this with the second-price reverse auction, in which the expected payment received by i is also the reserve price R multiplied by the probability that he has the lowest cost. Thus, $e_i(p_i^*(R))$ is the same in both the first-price and the second-price reverse auctions. It follows from (2.2) that the bid-your-cost strategy of the second-price reverse auction and the bidding strategy $\beta_{C(R)}(\cdot)$ of the first-price reverse auction yield the same expected payment by the buyer, which is a familiar conclusion of the Revenue Equivalence Theorem.

In the first-price reverse auction, as R increases, the corresponding $C(S)$ also increases. Suppose that there exists at least one symmetric bidding strategy for the first-price reverse auction when there is no reserve price. We have shown that the limit of $C(R)$ as R approaches infinity is finite and defined, and we denote it by $C(\infty)$. This is the minimal C discussed at the end of the previous section.

If every seller's cost is greater than R, then the buyer cannot purchase any unit. It occurs with probability $(1 - F(R))^N$. Let L be the penalty cost to the buyer if she fails to procure the unit. The buyer's expected cost is the expected price of the object plus expected penalty cost:

$$\int_{\underline{c}}^{R} \beta_{C(R)}(c) f^{(N)}(c) dc + L(1 - F(R))^N, \tag{2.8}$$

where $f^{(N)}(c) = Nf(c)(1-F(c))^{N-1}$ is the density of the minimum of N costs of all bidders having identical and independent distributions. The first term of (2.8) depends on C in the bidding behavior (2.5), which in turn is a function of the reserve price R. Thus, the buyer's problem is to set the cost of R as to minimize his expected cost (2.8).

3 Exponential Distribution of Costs

This section uses an exponential distribution of costs to illustrate a case where the supports for the seller's cost distribution and the buyer's payment are not bounded.

Suppose the cost c_i of each seller i is distributed as an independent and identical exponential distribution with a rate parameter $\lambda > 0$, i.e. $f(x) = \lambda e^{-\lambda x} 1_{[x \geq 0]}$. Then, the support of the cost distributions is unbounded.

Lemma 3.1. *In first-price auction, if costs are distributed as an exponential distribution with rate parameter λ, then the symmetric bidding strategy is*

$$\beta(c) = c + \frac{1}{\lambda(N-1)} + C_1 e^{\lambda(N-1)c}, \tag{3.9}$$

where $C_1 \geq 0$ specifies symmetric bidding functions which strictly increases and satisfies $\beta(c) \geq c$. Furthermore, the expected price is

$$\frac{1}{\lambda N} + \frac{1}{\lambda(N-1)} + \frac{C_1 N}{\lambda}$$

which is at least the expected price of the unique bidding strategy of the second-price auction.

Proof. From (2.5),

$$\begin{aligned}
\beta(c) &= c - e^{\lambda(N-1)c} \int_0^c e^{-\lambda(N-1)u} du + C_o e^{\lambda(N-1)c} \\
&= c - \frac{e^{\lambda(N-1)c}}{\lambda(N-1)} (1 - e^{-\lambda(N-1)c}) + C_o e^{\lambda(N-1)c} \\
&= c + \frac{1}{\lambda(N-1)} + C_1 e^{\lambda(N-1)c}
\end{aligned}$$

for some scalar C_o and $C_1 = C_o - \frac{1}{\lambda(N-1)}$. Condition (2.6) implies that $C_o \geq \int_0^\infty e^{-A(u)}du = \frac{1}{\lambda(N-1)}$, and thus $C_1 \geq 0$, which is required since $e^{\lambda(N-1)c}$ and its derivative can be arbitrarily large for a large c.

For $C_1 \geq 0$, the bidding strategy $\beta(\cdot)$ is strictly increasing. The minimum of N exponential distributions with rate parameter λ is distributed as an exponential distribution with rate parameter λN. Denote the density of this distribution by $f^{(N)}(c) = \lambda N e^{-\lambda N c}$. The expected payment by the auctioneer is

$$\begin{aligned}
\int_0^\infty \beta(c) f^{(N)}(c)dc &= \int_0^\infty c f^{(N)}(c)dc + \frac{1}{\lambda(N-1)} + C_1 \int_0^\infty e^{\lambda(N-1)c} f^{(N)}(c)dc \\
&= \frac{1}{\lambda N} + \frac{1}{\lambda(N-1)} + C_1 N \int_0^\infty \lambda e^{-\lambda c} dc \\
&= \frac{1}{\lambda}(\frac{1}{N} + \frac{1}{N-1}) + C_1 \frac{N}{\lambda}.
\end{aligned}$$

We compare this to the expected payment in the second-price auction. The expected cost of the minimum of N exponential distributions with rat parameter is $\frac{1}{\lambda N}$, and by memoryless property, the expected cost of the gap between the first and the second order statistic is $\frac{1}{\lambda(N-1)}$. Thus the expected price in the second-price auction is the expectation of the second order statistics, which is $\frac{1}{\lambda N} + \frac{1}{\lambda(N-1)}$. □

Figure 1 illustrates some multiple symmetric bidding strategies.

Now, suppose the buyer sets a reserve price $R > 0$. A seller does not participate in the auction if his cost is greater than R, and setting $\beta(R) = R$ specifies $C_1 = -\frac{\exp(-\lambda(N-1)R)}{\lambda(N-1)}$ in (3.9). The probability that the buyer will fail to purchase any unit is $(1 - F(R))^N = e^{-\lambda N R}$, and the expected total cost by the buyer is

$$\begin{aligned}
&\int_0^R \beta(c) f^{(N)}(c)dc + L(1 - F(R))^N \\
&= \int_0^R c f^{(N)}(c)dc + \frac{1 - e^{-\lambda N R}}{\lambda(N-1)} + C_1 \int_0^R e^{\lambda(N-1)c} f^{(N)}(c)dc + L e^{-\lambda N R} \\
&= \int_0^R \lambda N c e^{-\lambda N c} dc + \frac{1 - e^{-\lambda N R}}{\lambda(N-1)} + C_1 N \int_0^R \lambda e^{-\lambda c} dc + L e^{-\lambda N R}.
\end{aligned}$$

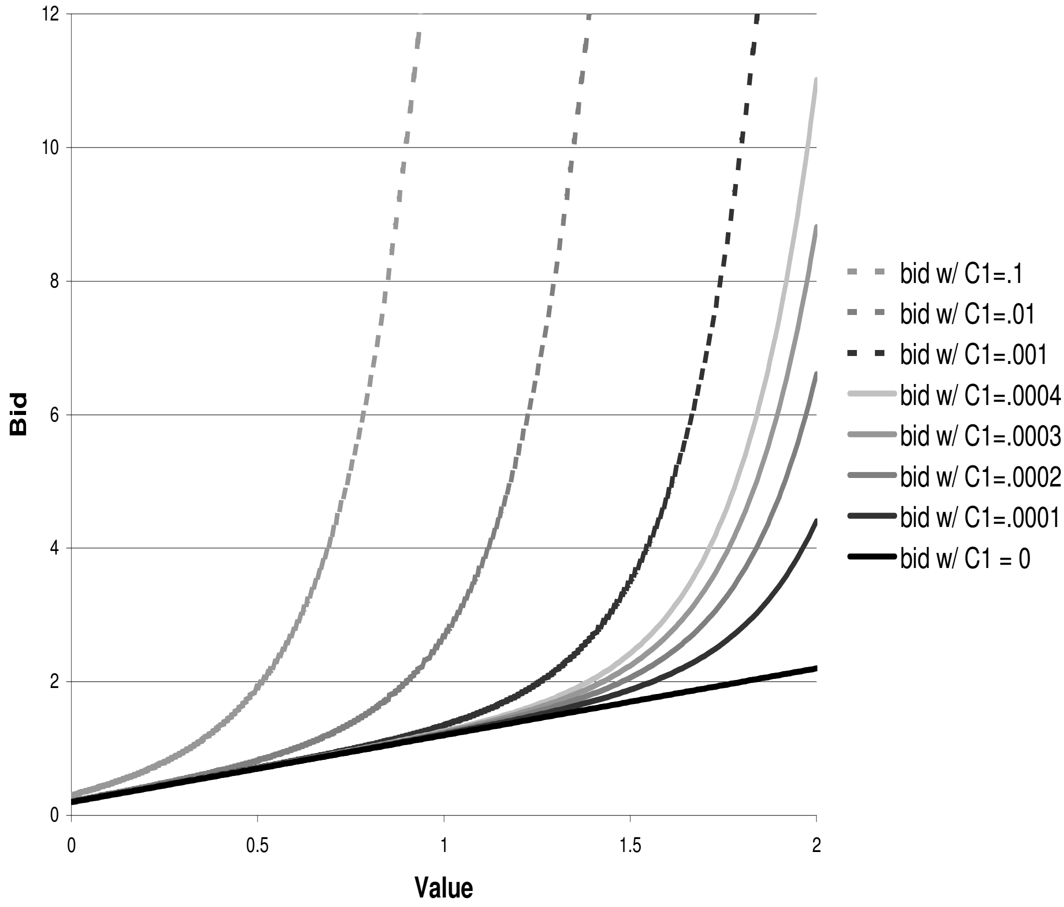

Figure 1: Multiple Symmetric Bidding Functions. There are 6 sellers bidding for sale. Costs are distributed as an i.i.d exponential distribution with rate parameter 1. See (3.9).

Since $\int_0^R \lambda N c e^{-\lambda N c} dc = [-ce^{-\lambda N c} - \frac{e^{-\lambda N c}}{\lambda N}]_{c=0}^R = \frac{1}{\lambda N}(1 - e^{-\lambda N R}) - Re^{-\lambda N R}$, the above expression becomes

$$\frac{1}{\lambda N}(1 - e^{-\lambda N R}) - Re^{-\lambda N R} + \frac{1 - e^{-\lambda N R}}{\lambda(N-1)} - \frac{e^{-\lambda(N-1)R}}{\lambda(N-1)} N(1 - e^{-\lambda R}) + Le^{-\lambda N R}.$$

We want to find the optimal reserve price R for the buyer. Differentiation of the above expected total cost yields

$$Ne^{-\lambda(N-1)R} \cdot [\lambda R + e^{\lambda R} - \lambda L - 1].$$

Since ${}^{-\lambda(N-1)R}$ is strictly positive, and $\lambda R + e^{\lambda R}$ is strictly increasing in R, the expected payment is quasi-convex in R. The optimal choice of R satisfies

$$\lambda R + e^{\lambda R} = \lambda L + 1. \tag{3.10}$$

With respect to R, the right-side is a constant no less than 1, and the left-side increases strictly from 1 to infinity as R goes from 0 to infinity. Thus, there must be a unique optimal R satisfying (3.10). Equation (3.10) is equivalent to $\lambda(L - R) = e^{\lambda R} - 1$, which is strictly positive for all $R > 0$. It follows that $R < L$: the optimal reserve price for the buyer is strictly less than his shortfall penalty cost. This finding is consistent with the optimal reserve price result (e.g. McAfee and McMillan (1987)) where the support of costs is restricted to be bounded.

Figure 2 demonstrates the relationship between the buyer's reserve price, and the probability that she will win the auction. When the reserve price is low, the probability of procuring the object is low. As the buyer increases the reserve price, the probability of procurement also rises.

The buyer pays the price of the object if purchase is made, and incurs a penalty cost if she fails to procure an object. In Figure 3, the expected total cost of the buyer, consisting of the purchase cost and the penalty cost, is given as a function of the buyer's reserve price, where $N = 6$, $\lambda = 1$ and $L = 0.6$. We see that the expected total cost is minimized when when the probability of purchase is 0.813, which occurs from Figure 2, when the reserve price is $R = 0.28$. We can verify that these parameters satisfy the optimality condition (3.10).

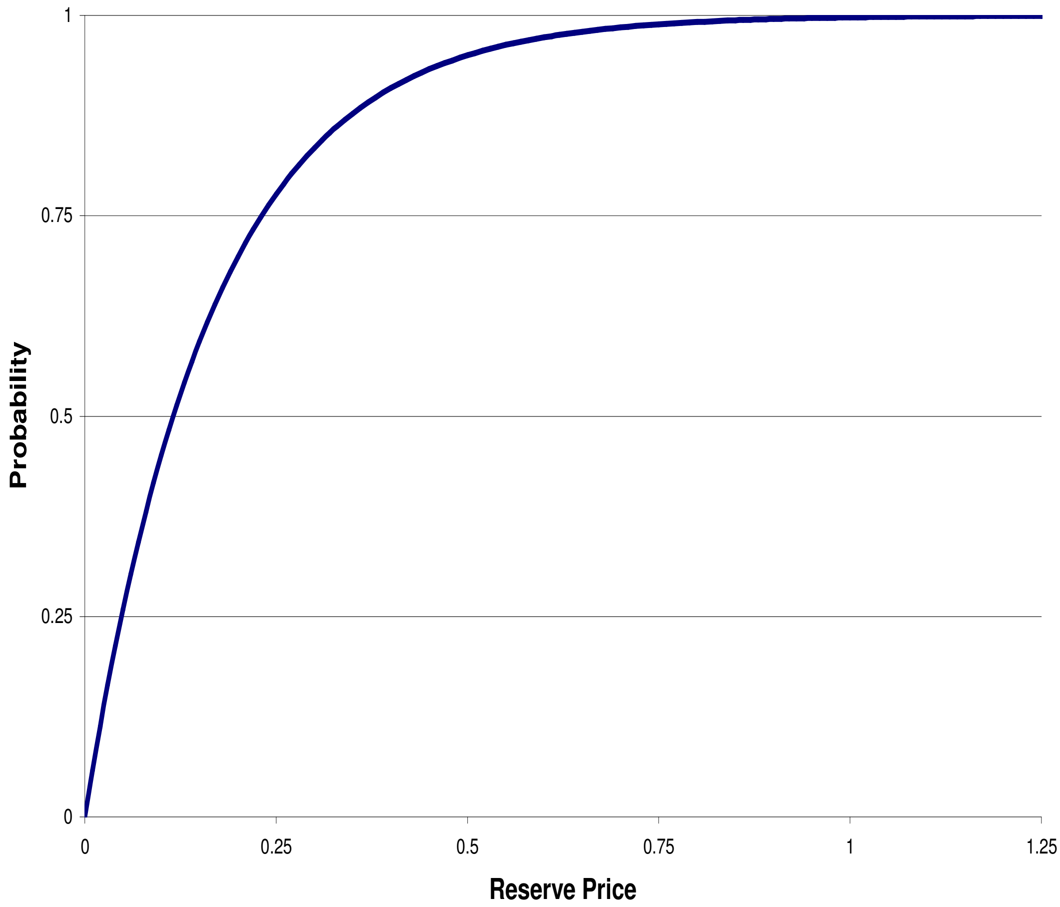

Figure 2: Probability of Purchasing vs. Buyer's Reserve Price. There are 6 sellers bidding for sale. Costs are distributed as an i.i.d exponential distribution with rate parameter 1. The lower left corner corresponds to a low reserve price, and the upper corner corresponds to a high reserve price.

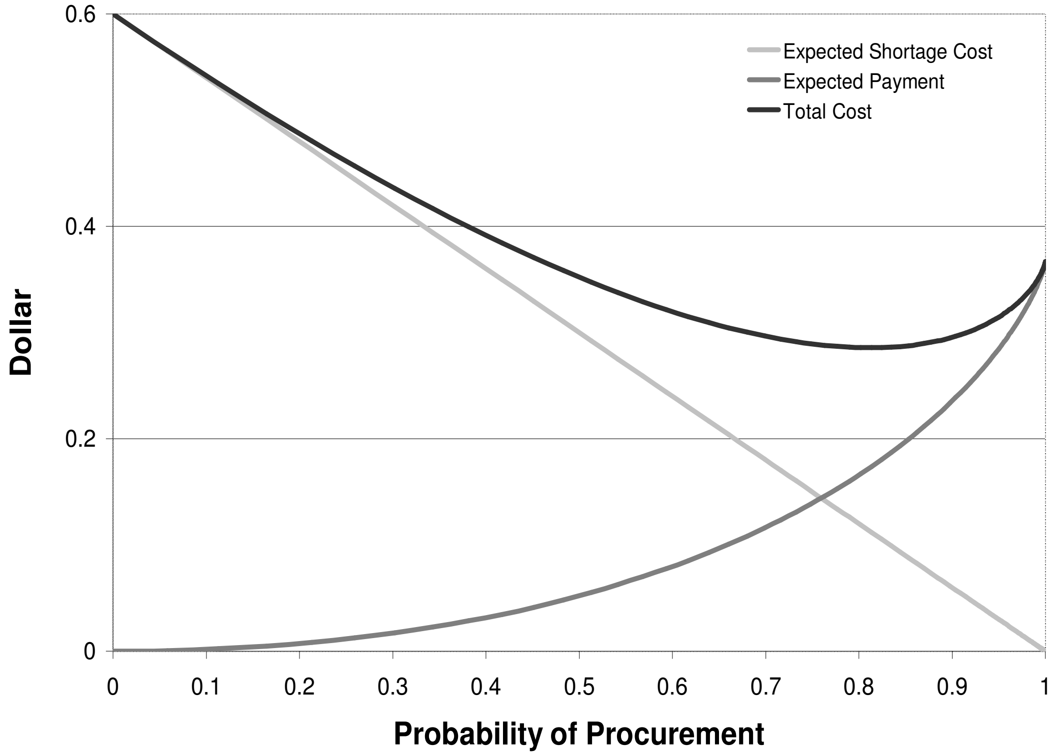

Figure 3: Probability of Purchasing vs. Expected Total Cost to Buyer. There are 6 sellers bidding for sale. Costs are distributed as an i.i.d exponential distribution with rate parameter 1. The penalty cost for failed purchase is 0.60 dollars. The expected cost to the buyer consists of the expected payment and expected penalty cost.

4 Uniform Distribution of Costs

This section assumes that the seller's cost has a uniform distribution. It shows that the buyer's shortfall penalty cost is higher than the highest possible cost by the seller, and thus the buyer may set the reserve price less than the maximum cost of the seller, achieving less than the globally optimal allocation of the object.

Suppose the cost c_i of each bidder i is distributed as a uniform $[0,1)$ distribution, i.e. $f(c) = 1_{[0\le c<1]}$. The support of cost distributions is bounded.

Lemma 4.1. *Using the uniform* $[0,1)$ *distribution of costs, symmetric bidding strategies are given by*

$$\beta(c) = \frac{1+(N-1)c}{N} + C_1(1-c)^{1-N},$$

for some scalar $C_o \ge \int_0^1 e^{-A(u)}du = \frac{1}{N}$ *and* $C_1 = C_o - \frac{1}{(N-1)} \ge 0$. *The expected price is*

$$\frac{2}{N+1} + C_1 N$$

which is at least the expected price of the unique bidding strategy in the second price auction.

Proof. For $c \in (0,1)$,

$$\begin{aligned}\beta(c) &= c - (1-c)^{-(N-1)}\int_0^c (1-u)^{(N-1)}du + C_o(1-c)^N \\ &= \frac{(N-1)c}{N} + \frac{1}{N} - \frac{1}{N}(1-c)^{1-N} + C_o(1-c)^N \\ &= \frac{(N-1)c}{N} + \frac{1}{N} + C_1(1-c)^N\end{aligned}$$

for some scalar $C_o \ge \int_0^1 e^{-A(u)}du = \frac{1}{N}$ and $C_1 = C_o - \frac{1}{N} \ge 0$. Since $(1-c)^{1-N}$ tends to positive infinity as c approaches 1 from left, we require that C_1 is nonnegative for $\beta(\cdot)$ to be proper. The expected payment made by the buyer is

$$\begin{aligned}&\int_0^1 \frac{1+(N-1)c}{N} f^{(N)}(c)dc + \int_0^1 C_1(1-c)^{1-N} f^{(N)}(c)dc \\ &= \frac{1}{N}\int_0^1 f^{(N)}(c)dc + \frac{N-1}{N}\int_0^1 cf^{(N)}(c)dc + C_1\int_0^1 (1-c)^{1-N}\cdot N(1-c)^{N-1}dc \\ &= \frac{1}{N}(1) + \frac{N-1}{N}(\frac{1}{N+1}) + C_1 N = \frac{2}{N+1} + C_1 N.\end{aligned}$$

We recall that the expected payment of the second price auction, in which bidding at cost is the unique symmetric equilibrium, is the expected cost of the second order statistic. Assuming the uniform [0,1) distribution of costs, the expected payment is $\frac{2}{N+1}$, the expected cost of the second order statistics of N uniform [0,1) distributions. □

If there is a reserve price $R \in [1, \infty)$, then the symmetric Nash-equilibrium corresponding to $C_1 = 0$ is unique. The bidding strategy and the expected price do not depend on the exact cost of the reserve price as long as it is at least 1. Therefore, the existence of an arbitrarily high reserve cost eliminates the multiplicity of bidding strategies.

If $R \in (0, 1)$, then it follows from $\beta(R) = R$ that $C_1 = -\frac{1}{N}(1-R)^N$. The expected total cost to the buyer is the sum of the expected payment and the expected penalty:

$$\begin{aligned}&\int_0^R \beta(c) f^{(N)}(c)dc + L(1-F(R))^N \\ =\ & \frac{1}{N}\int_0^R f^{(N)}(c)dc + \frac{N-1}{N}\int_0^R cf^{(N)}(c)dc + C_1 N\int_0^R dc + L(1-R)^N.\end{aligned}$$

Since $f^{(N)}(c) = Nc(1-c)^{N-1}$, we have $\int_0^R f^{(N)}(c)dc = 1-(1-R)^N$, and $\int_0^R cf^{(N)}(c)dc = [-c(1-c)^N - \frac{1}{N+1}(1-c)^{N+1}]_{c=0}^R = -R(1-R)^N - \frac{1}{N+1}(1-R)^{N+1} + \frac{1}{N+1}$. Thus, the above equations becomes

$$\frac{[1-(1-R)^N]}{N} + \frac{N-1}{N}[-R(1-R)^N - \frac{(1-R)^{N+1}}{N+1} + \frac{1}{N+1}] - (1-R)^N R + L(1-R)^N.$$

It can be shown that the derivative of the expected total cost is

$$(1-R)^{N-1}[2NR - (2+LN)].$$

Thus, the expected total cost is quasi-convex, and is minimized when the buyer sets

$$R = \min\{\frac{LN+2}{2N}, 1\}.$$

We observe that R is at most 1 if the shortfall penalty cost L is at most $\frac{2(N-1)}{N}$. Even if the shortfall penalty cost is greater than the supremum of the support of costs (e.g. $L \in [1, \frac{2(N-1)}{N})$, we see that it may still be optimal for the buyer to set a reserve price somewhere in the interior of the support of costs.

A Appendix

Proposition A.1. *In the forward first-price auction, suppose that the supports of the distribution of costs and the distribution of corresponding bids mapped by a symmetric bidding strategy β are continuous. Then, β is strictly increasing.*

Proof. Suppose $\beta(c') = \beta(c'')$ for some $c' < c''$. In (2.3), the left side expression as well as the second term in the right side depends on c only through $\beta(c)$. By taking the difference of $\beta(c')$ and $\beta(c'')$, we get $c' = c''$ implying the injectivity of β.

Suppose, by way of contradiction, that $\beta(\cdot)$ is not increasing. Then, there must exist c' and c'' such that $c' < c''$ and $\beta(c') > \beta(c'')$. Since $\beta(c'')$ maximizes the expected profit of the bidder given his cost c'', it follows that

$$(\beta(c'') - c'') \cdot P[\beta(c'') \text{ is the highest bid}] \geq (\beta(c') - c'') \cdot P[\beta(c') \text{ is the highest bid}].$$

From $\beta(c'') > \beta(c')$ and the continuity of the support of bids, the probability that $\beta(c'')$ is the highest bid is strictly greater than the probability that $\beta(c')$ is the highest bid. Thus,

$$(c'' - c') \cdot P[\beta(c'') \text{ is the highest bid}] < (c'' - c') \cdot P[\beta(c') \text{ is the highest bid}].$$

By adding two above inequalities, we get

$$(\beta(c'') - c') \cdot P[\beta(c'') \text{ is the highest bid}] > (\beta(c') - c') \cdot P[\beta(c') \text{ is the highest bid}]$$

contradicting the optimality of $\beta(c'')$. We conclude that β is strictly increasing. ☐

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