Harris irreducibility of iterates of iid random maps on $R^+$

K. B. Athreya
Cornell University

Abstract

Let $S = (0, L), L \leq \infty, \Theta = (0, K), K \leq \infty$ and $f(x, \theta) = \theta h(x)$ be a map from $SX\Theta \rightarrow S$ where $h$ is a strictly positive continuous function. Let $\{\theta_i\}_{i \geq 1}$ be iid $\Theta$ valued r.v. and $X_0$ an independent $S$ valued r.v. Let $\{X_n\}$ be a Markov chain defined by the random iteration scheme $X_{n+1} = \theta_{n+1} h(X_n)$. In this paper a set of sufficient conditions are provided for this chain to be Harris irreducible. Application to random unimodal maps including logistic maps is also given.

AMS (2000) Classification Primary 60J05, 92D25, Secondary 60F05

Keywords and phrases: Harris irreducibility, iteration of iid random maps, Markov chains, S-unimodal, stable periodic orbits, logistic maps.

Research supported in part by AFOSR, IISI F 49620-01-1-0076.

K. B. Athreya
School of ORIE
Rhodes Hall
Cornell University
Ithaca, NY 14853
email: athreya@orie.cornell.edu
Harris irreducibility of iterates of iid random maps on $R^+$

K. B. Athreya
Cornell University

1 Introduction

Let $S = (0, L), L \leq \infty, \Theta = (0, K), K \leq \infty, f(x, \theta) = \theta h(x)$ be a map from $S \times \Theta \rightarrow S$ where $h$ is a strictly positive continuous function. Let $\{\theta_i\}_{i \geq 1}$ be iid $\Theta$ valued r.v. with distribution $Q(\cdot)$ and $X_0$ an independent $S$ valued r.v. Let $\{X_n\}_{n \geq 0}$ be defined by the random iteration scheme.

$$X_{n+1} \equiv \theta_{n+1} h(X_n), \quad n \geq 0 \quad (1)$$

Then $\{X_n\}$ is an $S$-valued Markov chain with transition probability function

$$P(x, A) \equiv P \left(\theta_1 \in \frac{A}{h(x)}\right) \quad (2)$$

This class of Markov chains arises in a natural way in ecology and economics. (See Athreya (2002), and the references therein, Majumdar & Mitra (2000).) They include random logistic maps, random Ricker maps, random Hassel maps etc. The existence of nontrivial stationary measures for such chains has been investigated by a number of authors. (See Athreya (2002), Athreya and Dai (2000), Bhattacharya and Rao (1993), Bhattacharya and Waymire (2002), Dai (2000), Gyllenberg et al (1994), Vellekoop and Hognas (1997).)

In this note a set of sufficient conditions is provided for the Harris irreducibility of such chains. It is known that for Harris irreducible chains a stationary distribution, if one exists, is necessarily unique. For the case of random logistic maps Athreya and Dai (2002) recently provided an example of nonuniqueness where the distribution of $\theta_1$ is not smooth and in fact, is supported by two values. This suggested that if the distribution of $\theta_1$ is not too discrete then the chain should be Harris and hence the stationary measure be unique. This note is a substantiation of this.

2 Summary of results

In the following $\{X_n\}$ is as in (1). Our first result is a local irreducibility theorem.

Theorem 1 Suppose:
\[ i) \exists \alpha \in \Theta \text{ and } \delta > 0 \text{ and a strictly positive Borel measurable function } \psi \text{ from} \]
\[ J = (\alpha - \delta, \alpha + \delta) \subset \Theta \text{ to } (0, \infty) \exists \text{ for all Borel sets } B \subset J, \]
\[ Q(B) \equiv P(\theta_1 \in B) \geq \int_B \psi(\theta) \, m(d\theta) \tag{3} \]
where \( m(\cdot) \) is Lebesgue measure.

\[ ii) \exists p \in S \exists f(p, \alpha) \equiv \alpha h(p) = p \tag{4} \]

Then, \( \exists \eta > 0 \exists x \in I = (p - \eta, p + \eta), A \subset I, m(A) > 0 \]
\[ \implies P(x, A) \equiv P(X_1 \in A | X_0 = x) > 0 \tag{5} \]

The next result is a global irreducibility result.

**Theorem 2** Suppose, in addition to \( i) \) and \( ii) \) of Theorem 1,

\[ iii) \forall x \in S, \exists \text{ a finite set } \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \text{ contained in the support of } Q \exists Y_n \in I \]
where \( Y_0 = x, Y_{j+1} = f(Y_j, \alpha_{j+1}) \quad j = 0, 1, \ldots, n - 1. \]

Then \( \{X_n\} \) in Harris irreducible with reference measure
\[ \phi(\cdot) \equiv m(\cdot \cap I) \tag{6} \]

The next result is a generalization of Theorem 2.

**Theorem 3** Suppose \( i) \) of Theorem 1 holds and that \( ii) \) is replaced by \( \ddot{i}i) \exists p \in S \]
and \( m \geq 1 \exists \)
\[ f^{(m)}(p, \alpha) = p \tag{7} \]
Then, \( \exists \eta > 0 \exists x \in I = (p - \eta, p + \eta), A \subset I, m(A) > 0 \implies \]
\[ P(X_m \in A | X_0 = x) > 0 \tag{8} \]

If, in addition, \( iii) \) of Theorem 2 holds for this \( I \) then \( \{X_n\} \) is Harris irreducible with reference measure \( \phi \) in (6).

**Corollary 1** (Random logistic maps)

Let \( S = (0, 1), \Theta = (0, 4), h(x) = x(1 - x). \) Let the distribution of \( \theta_1 \) have an absolutely continuous component with a strictly positive density on some open interval \( J \subset (1, 3). \) Then \( \{X_n\} \) is Harris irreducible.
The next result exploits the fact that a sufficient condition for (iii) of Theorem 2 to hold in the case of when \( h(\cdot) \) is \( S \)-unimodal on \([0, 1]\) is for \( (p, \alpha) \) to be such that \( p \) is a stable periodic point for \( f(\cdot, \alpha) \equiv \alpha h(\cdot) \).

**Theorem 4** Let \( S = [0, 1] \), \( \Theta = [0, L] \), \( f(x, \theta) = \theta h(x) \). Suppose:

i) \( h(\cdot) \) is \( S \)-unimodal (defined below)

ii) \( \exists (p, \alpha) \in S \times \Theta \) s.t. \( p \neq 0, \alpha \neq 0 \), \( p \) is a stable periodic point for \( f(\cdot, \alpha) \)

iii) \( \exists \delta > 0 \) and a strictly positive Borel function \( \psi : J \equiv (\alpha - \delta, \alpha + \delta) \rightarrow (0, \infty) \) \( \exists \) for all Borel sets \( B \subset J \)

\[
Q(B) \equiv P(\theta_1 \in B) \geq \int_{B} \psi(\theta)m(d\theta)
\] (9)

where \( m(\cdot) \) is Lebesgue measure.

Then \( \{X_n\} \) is Harris irreducible.

A special case of the above is the random logistic maps.

**Theorem 5** Let \( S = [0, 1], \Theta = [0, 4], \)

\[
X_{n+1} = \theta_{n+1}X_n(1 - X_n), n \geq 0
\] (10)

with \( \{\theta_n\}_{n \geq 1} \) begin i.i.d. \( \theta \) valued rv and \( X_0 \) an independent \( S \)-valued rv. Suppose \( \exists \) an open interval \( J \subset [0, 4] \) and a Borel function \( \psi : J \rightarrow (0, \infty) \) such that for all Borel sets \( B \subset J \)

\[
Q(B) \equiv P(\theta_1 \in B) \geq \int_{B} \psi(\theta)m(d\theta)
\] (11)

where \( m(\cdot) \) is Lebesgue measure.

If \( J \cap (1, 4) = \emptyset \) then assume, in addition that there exists a \( \beta > 1 \) in the support of \( Q(\cdot) \) such that \( f(\cdot, \beta) \) admits a stable periodic point \( p \) in \((0, 1)\).

Then \( \{X_n\} \) is Harris irreducible.

**Remark 1** If \( P(\theta_1 \leq 1) = 1 \) then it can be shown that \( X_n \downarrow 0 \text{ w.p.l} \) and hence is not Harris irreducible in \((0, 1)\).

Another set of sufficient conditions for (iii) of Theorem 2 to hold is provided below.
Theorem 6 Suppose, the hypotheses of Theorem 3 hold. Suppose, in addition, that
\[ \exists \beta \in \text{support of } Q \ni m\{x : \omega_\beta(x) \cap I \neq \emptyset\} = 1. \]
where \( \omega_\beta(x) \) is the limit point set of the orbit 0\_x of \( x \) under the map \( f(\cdot, \beta) \).

Then \( \{X_n\} \) is Harris irreducible.

Theorem 7 Suppose \( \{X_n\} \) defined by (1) satisfies:

a) it is Harris irreducible

b) the distribution of \( \theta_1 \) has a nonzero absolutely continuous component and
c) the function \( h \) satisfies \( A \subset (0, L), m(A) = 0 \implies m(h^{-1}(A) \cap (0, L)) = 0 \)
d) for some initial distribution \( X_0 \) the sequence of occupation measures,

\[ \mu_n(\cdot) = \frac{1}{n} \sum_{i=0}^{n-1} P(X_j \in \cdot) \quad (12) \]

has a vague limit point \( \nu \) such that \( \nu(0, L) > 0 \).

Then, \( \exists \) a probability distribution \( \pi \) such that \( \pi(0, L) = 1, \pi \) is absolutely continuous, and \( \pi \) is the unique stationary distribution in \( (0, 1) \) for the Markov chain \( \{X_n\} \)

Remark 2 In (d) we do not need tightness of \( \{\mu_n\} \).

Theorem 8 Under the hypothesis of Theorem 5 and the additional condition \( E|n\theta_1 > 0, E|n(4 - \theta_1)| < \infty \), there is a unique absolutely continuous stationary probability distribution \( \pi \) in \( (0, 1) \) for the Markov chain \( \{X_n\} \) defined by (10).

3 The proofs

Proof of Theorem 1

For any \( \eta > 0, \{x\} \) and \( A \subset I \equiv (p - \eta, p + \eta) \),

\[ \sup \frac{A}{xg(x)} \leq \frac{p + \eta}{\inf \{h(x) : x \in I\}} \equiv a(\eta) \]

and

\[ \inf \frac{A}{xg(x)} > \frac{p - \eta}{\sup \{h(x) : x \in I\}} \equiv b(\eta) \]
Since \( h(\cdot) \) is continuous at \( p \) and by (ii) \( ah(p) = p \) it follows that there exists \( \eta > 0 \) such that \( a(\eta) < \alpha + \delta \) and \( b(\eta) > \alpha - \delta \) where \( \delta \) is as in (i).

Now, \( x \in I, A \subset I, m(A) > 0 \implies P(f(x, \theta_1) \in A) = P\left( \theta_1 \in \frac{A}{h(\xi)} \right) \) which is \( > 0 \) since \( \frac{A}{h(\xi)} \subset J \) and \( m(A) > 0 \) implies \( m\left( \frac{A}{h(\xi)} \right) > 0 \) and (3) holds.

\[ \square \]

**Proof of Theorem 2**

Fix \( x \in S \) and \( A \subset I \) with \( m(A) > 0 \). By continuity of \( f(\cdot, \cdot) \) and (iii) \( \forall x \in S, \exists \varepsilon > 0 \) \( \exists \{ \alpha_i \}_{i=1}^n \) as in (iii) \( |\theta_i - \alpha_i| < \varepsilon \), \( i = 1, 2, ..., n \), \( X_0 = x, X_{j+1} = f(X_j, \theta_{j+1}), j = 0, 1, ..., n - 1 \implies X_n \in I \).

Thus, \( P_x(X_n \in I) \geq \prod_{i=1}^n P(|\theta_i - \alpha_i| < \varepsilon) > 0. \)

Now, if \( A \subset I \) and \( m(A) > 0 \), then by the Markov property, \( P_x(X_{n+1} \in A) \)

\[ \geq E_x(P_{X_n}(X_1 \in A) : X_n \in I). \]

By Theorem 1, the integrand in the above expectation is \( > 0 \) and it has just been shown that \( P_x(X_n \in I) > 0 \).

So \( P_x(X_{n+1} \in A) > 0. \)

\[ \square \]

**Remark 3** It may be noted that by Theorem 1 the chain is aperiodic as well.

**Proof of Theorem 3**

Let \( \delta \) be as in (i). Choose \( \eta_1 > 0 \) small such that

\[ \alpha - \delta < \left( \frac{p - \eta_1}{p + \alpha \eta_1} \right) \alpha < \left( \frac{p + \eta_1}{p - \alpha \eta_1} \right) \alpha < \alpha + \delta \]

Let \( x_j = f^{(j)}(x, \alpha), \quad j = 0, 1, 2, ..., m - 1 \)

Now (ii)' implies that \( f(x_{m-1}, \alpha) = p \), ie \( ah(x_{m-1}) = p \).

By continuity of \( h \), \( \forall \eta_1 > 0, \exists \eta_2 > 0 \implies |y - x_{m-1}| < \eta_2 \implies |h(y) - \frac{y}{\alpha}| < \eta_1. \)

Next, by continuity of \( f(\cdot, \cdot) \), \( \forall \eta_2 > 0, \exists \eta_3 > 0 \implies |\theta_i - \alpha| < \eta_3, \quad i = 1, 2, ..., m - 1, \quad |x - p| < \eta_3, \quad X_0 = x \implies |X_{m-1} - x_{m-1}| < \eta_2 \), where \( X_{j+1} = f(X_j, \theta_{j+1}), \quad j = 0, 1, 2... \)
Let $C$ be the event $\{|\theta_i - \alpha| < \eta_3, \ i = 1, 2, ..., m - 1\}$ and $\eta = \eta_3 \wedge \eta_1$.

Then on the event $C$, $|x - p| < \eta$, $A \subset I \equiv (p - \eta, p + \eta)$ implies

$$\frac{A}{h(X_{m-1})} \subset \left( \frac{p - \eta_1}{\frac{p}{\alpha} + \eta_1}, \frac{p + \eta_1}{\frac{p}{\alpha} - \eta_1} \right)$$

$$\subset (\alpha - \delta, \alpha + \delta), \text{ by choice of } \eta_1.$$ 

Also $m(\cdot)$ being Lebesgue measure,

$$m(A) > 0 \implies m\left( \frac{A}{h(X_{m-1})} \right) > 0.$$ 

Now

$$P_x(X_m \in A) \geq P_x((X_m \in A) \cap C)$$

$$\geq E_x(P(X_m \in A|X_{m-1}) : C) \text{ by Markov property}$$

Since $\theta_n$ is independent of $\{X_j\}_{j \leq m-1}, P(X_m \in A|X_{m-1}) = P\left( \theta_n \in \frac{A}{h(X_{m-1})}, X_{m-1} \right)$ which is $> 0$ on the event $C$. Finally, $P_x(C) \geq (P(|\theta_i - \alpha| < \eta))^{m-1} > 0$ by (i).

Now the proof can be completed as in that of Theorem 2.

$\square$

**Remark 4** It may be noted that the period of the chain is a divisor of $n_0$.

Proof of Corollary 1 It is known (de Melo and van Strien (1993)) that for any $\alpha \in J \subset (1, 3), p = 1 - \frac{1}{\alpha}$ in a stable fixed point of the map $f(\cdot, \alpha) \equiv \alpha x(1 - x)$ and that for all $0 < x < 1$, $f^n(x, \alpha) \rightarrow p$. So condition (iii) of Theorem 2 holds and the result now follows from Theorem 3.

For a related result see Dai (2000).

For the proof of Theorem 4 the definition of $S$-unimodal maps is needed.

**Definition** A map $f : [0, 1] \rightarrow [0, 1]$ is called $S$-unimodal if

1. $f$ is three times continuously differentiable
2. $f$ is unimodal with a mode at $c$ in $(0, 1)$ such that $f''(c) < 0$ and $f$ is strictly increasing in $(0, c)$ and strictly decreasing in $(c, 1)$
3. $f(0) = f(1) = 0$

4. The Schwartzian derivative of $f$

$$(Sf)(x) \equiv \begin{cases} \frac{f''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2, & \text{if } f'(x) \neq 0 \\ -\infty & \text{if } f'(x) = 0 \end{cases}$$

is $< 0$ for all $0 < x < 1$.

Examples of $S$-unimodal maps are $f(x) = x(1 - x), f(x) = x^2 \sin \pi x$

**Proof of Theorem 4**

It is known (see Guckenheimer (1979), de Melo and van Strien (1993)) that for the $S$-unimodal maps $f(\cdot, \alpha)$ with a stable periodic point $p, m(K) = 1$ where

$$K = \{x = 0 < x < 1, w_\alpha(x) = \gamma(p)\} = 1$$

where $w_\alpha(x)$ is the limit point set of the orbit of $x$ under $f(\cdot, \alpha)$ and $\gamma(p)$ is the orbit set of the periodic point $p$, i.e., the set $\{f^{(j)}(p, \alpha)\}_{j=0}^{n-1}$ and $m(\cdot)$ is Lebesgue measure. Let $I$ be as in Theorem 1 whose existence is valid under the present hypotheses. Thus, for $\forall x \in K, f^{(n)}(x, \alpha) \in I$ for some large $n$. Since $\alpha \in J \subset$ support of $Q$, and $f$ is jointly continuous,

$$P_x(X_n \in I \text{ for some } n \geq 1) > 0 \text{ for } \forall x \in K. \quad (13)$$

Next, we show that for any $0 < x < 1$

$$P_x(X_1 \in K) > 0. \quad (14)$$

Now $0 < x < 1$ implies $0 < \alpha h(x) < 1$ and since $m(K) = 1$,

$$m\left(\frac{K}{h(x)} \cap J\right) \geq m([0, \alpha] \cap J) = \delta > 0.$$ 

By (9), $P(\theta_1 \in \frac{K}{h(x)} \cap J) > 0$.

Thus $P_x(X_1 \in K) = P(\theta_1 \in \frac{K}{h(x)}) \geq P(\theta_1 \in \frac{K}{h(x)} \cap J) > 0$ establishing (14).

Clearly (13) and (14) show that for any $0 < x < 1$, $P_x(X_n \in I \text{ for some } n \geq 1) > 0$.

This with (8) from Theorem 3 completes the proof.

$\Box$
Proof of Theorem 5

It is easy to check that \( h(x) = x(1 - x) \) has the S-unimodal property.

Case 1 \( J \cap (1, \lambda_{MF}) \neq \emptyset \) where \( \lambda_{MF} = 3 \cdot 5699 \ldots \) is the Myrberg-Feigenbaum point. It is known (de Melo and van Strien (1993)) that there exists an \( \alpha \) in \( J \) and \( 0 < p < 1 \) such that \( p \) is a stable periodic point for \( f(x, \alpha) = \alpha x(1-x) \). Hence Theorem 4 applies.

Case 2 \( J \cap (\lambda_{MF}, 4) \neq \emptyset \). By a result of Graczyk and Swiatek (1997) asserting that the set of all \( \alpha \) in \( (0, 4) \) such that \( f(\cdot, \alpha) \) has a stable periodic orbit is dense, there is \( \alpha \) in \( J \) and \( 0 < p < 1 \) such that \( p \) is a stable periodic point for \( f(x, \alpha) = \alpha x(1-x) \) and again Theorem 4 applies. (This argument works for Case 1 as well.)

Case 3 \( J \cap (1, 4) = \emptyset \) but \( J \cap (0, 1) \neq \emptyset \). Let \( J \cap (0, 1) = (\alpha - \delta, \alpha + \delta) \) for some \( 0 < \alpha < 1 \) and \( \delta > 0 \). By hypothesis \( \exists \beta > 1 \) in the support of \( Q \) such that \( f(\cdot, \beta) \) has a stable fixed point \( p \) in \( (0, 1) \). By Guckeheimer’s (1979) theorem for every \( \varepsilon \) > 0, \( m(K_\beta) = 1 \) where \( K_\beta \equiv \{ x : \omega_\beta(x) \cap (p_-, p_+ \in) \} \).

Again, as in the proof of Theorem 4, for any \( 0 < x < 1 \),

\[
P_x(X_n \in (p-, p+ \in) \text{ for some } n \geq 1) > 0 \tag{15}\]

Since \( h(p) = p(1-p) > 0 \), there exists an \( \eta > 0 \) such that

\[
\alpha - \delta < \frac{h(p)\alpha - \eta}{h(p) + \eta} < \frac{h(p)\alpha + \eta}{h(p) - \eta} < \alpha + \delta
\]

For this \( \eta > 0 \), there exists an \( \varepsilon > 0 \) such that

\[
|x - p| < \varepsilon \implies |h(x) - h(p)| < \eta
\]

Let \( I \equiv (h(p)\alpha - \eta, h(p)\alpha + \eta) \). Since \( \beta h(p) = p \), and \( \beta > 1, I \subset (0,1) \) for \( \eta \) small. Then, for any \( |x - p| < \varepsilon, A \subset I \)

\[
\frac{A}{h(x)} \subset (\alpha - \delta, \alpha + \delta)
\]

Thus, for \( |x - p| < \varepsilon \) and \( A \subset I \) with \( m(A) > 0 \),

\[
P_x(X_1 \in A) = P(\theta_1 \in \frac{A}{h(x)}) > 0, \tag{16}\]

since \( m\left(\frac{A}{h(x)}\right) > 0 \), and (11) holds. Combining (15) and (16) shows that for any \( 0 < x < 1 \), and any \( A \subset I \) with \( m(A) > 0, P_x(X_n \in A \text{ for some } n \geq 1) > 0 \). That is, \( \{X_n\} \) is Harris reducible with reference measure \( \phi(\cdot) = m(\cdot \cap I) \).

\[\square\]
Proof of Theorem 6

Let \( K_\beta = \{ x : \omega_\beta(x) \cap I \neq \phi \} \). Then, as in the proof of Theorem 4, then since \( \beta \in \text{supp} \, Q \), for \( \forall x \in K_\beta \)
\[
P_x(X_n \in I \text{ for some } n \geq 1) > 0 \tag{17}
\]
Also for any \( 0 < x < L \),
\[
P_x(X_1 \in K_\beta) = P \left( \theta_1 \in \frac{K_\beta}{h(x)} \right)
= P \left( \theta_1 \in \frac{K_\beta}{h(x)} \cap J \right)
\]
Since \( m(K_\beta^c) = 0 \) and for any \( 0 < x < L \), it holds that \( Kh(x) \leq L \), it follows that
\[
\frac{K_\beta}{h(x)} \supset \frac{K}{L} K_\beta.
\]
Since \( K > \alpha + \delta \) and \( m(K_\beta) = L \), we get \( m(\frac{K_\beta}{L} K_\beta \cap J) = 2\delta > 0 \).
This, in turn, implies that \( P(\theta_1 \in \frac{K_\beta}{h(x)} \cap J) > 0 \), ie
\[
P_x(X_1 \in K_\beta) > 0. \tag{18}
\]
Now (17) and (18) imply that for any \( 0 < x < 1 \)
\[
P_x(X_n \in I \text{ for some } n \geq 1) > 0.
\]
Now (8) of Theorem 3 completes the proof.

Proof of Theorem 7

Since \( h(x) \) is continuous, the Markov chain \( \{ X_n \} \) has the Feller property. Since \( S = (0, L) \) admits an “approximate identity,” ie, a sequence of functions \( g_r(\cdot) \) such that for each \( r, g_r(\cdot) \) is continuous with compact support in \((0, L), 0 < g_r(\cdot) < 1, g_r(x) \uparrow 1 \) for all \( 0 < x < L \), and since the chain \( \{ X_n \} \) is Feller with state space \( S \), it follows from Athreya (2002) Theorem 4, that any vague limit point of \( \mu_{nx} \) is stationary for \( \{ X_n \} \).

Since \( \mu(0, L) > 0 \), the measure \( \pi(\cdot) \equiv \frac{\mu(\cdot)}{\mu(0, L)} \) is a well defined probability measure on \((0, L) \) that is stationary for \( \{ X_n \} \).

Next, since by (a), \( \{ X_n \} \) is Harris irreducible on \((0, L), \pi \) is unique.
Now suppose $X_0 \sim \pi$. Then $X_1 = \theta_1 h(X_0)$ also has distribution $\pi$. Let $r$ and $s$ denote respectively the weight of the absolutely component of $\pi$ and $Q$, the distribution of $\theta_1$.

By (c), the distribution of $h(X_0)$ has also an absolutely continuous component with weight at least $r$. By independence of $\theta_1$ and $X_0$, it follows that $(1-r) \leq (1-s)(1-r)$.

Since $s > 0$ by (b), it follows that $r = 1$.

\[\square\]

**Proof of Theorem 8**

It is known from Athreya and Dai (2000) that under the extra hypothesis of Theorem 8 condition (d) of Theorem 7 holds. Also the function $h(x) = x(1-x)$ satisfies condition (c) of Theorem 7 in $(0, 1)$. From Theorem 5, condition (a) follows. Condition (b) is part of the hypothesis. So Theorem 8 follows from Theorem 7.

\[\square\]

**Remark 5** Athreya (2002) has extended the sufficiency conditions of Athreya and Dai (2000) for the existence of a vague limit point $\mu$ of $\{\mu_n\}$ with $\mu(0, L) > 0$ for random logistic maps to a general class of maps on $(0, \infty)$. So it is possible to extend Theorem 8 to that context as well.

Acknowledgement The author wishes to thank Professor John Guckenheimer for some useful discussions.
References:


