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Let $\{X(t)\}_{t\geq 0}$ be a locally bounded and infinitely divisible stochastic process, with no Gaussian component, that is self-similar with index $H>0$. Pick constants $\gamma>H$ and $c>0$. Let ν be the Lévy measure on $\mathbb{R}^{[0,\infty)}$ of X, and suppose that $R(u)\equiv \nu\big(\{y\in\mathbb{R}^{[0,\infty)}\colon \sup_{t\geq 0} y(t)/(1+ct^{\gamma})>u\}\big)$ is suitably "heavy tailed" as $u\to\infty$ (e.g., subexponential with positive decrease). For the storage process $Y(t)\equiv \sup_{s\geq t}(X(s)-X(t)-c(s-t)^{\gamma})$, we show that $\mathbf{P}\{\sup_{s\in[0,t(u)]}Y(s)>u\}\sim \mathbf{P}\{Y(\hat{t}(u))>u\}$ as $u\to\infty$, when $0\leq\hat{t}(u)\leq t(u)$ do not grow too fast with u [e.g., $t(u)=o(u^{1/\gamma})$].

1. Introduction. Let $X=\{X(t)\}_{t\geq 0}$ be an infinitely divisible (id) stochastic process, with no Gaussian component, that is self-similar with index $H>0$ (H-ss).

Given constants $c>0$ and $\gamma>H$, we consider the storage process

$$Y(t)=\sup_{s\geq t}\big(X(s)-X(t)-c(s-t)^{\gamma}\big) \qquad \text{for} \quad t\geq 0. \tag{1.1}$$

Intuitively, an H-ss process grows as t^H with time t, and so $\gamma>H$ should make Y finite valued. Nevertheless, this is not so in general (see Example 2 below). The assumptions in our theorems will, however, ensure such finiteness of Y. The reason for the name "storage process" comes from the case $\gamma=1$, with $X(t)$ denoting the total inflow into a storage facility by time t, and c the (demand) rate at which stock at the facility is depleted, then $Y(t)$ tells how much extra storage capacity one will need in the future over what is being used at time t. For an input process X with stationary increments (si), the storage process Y is stationary (if finite).

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The process Y has been used in financial applications under the name of "drawdown" [e.g., Dacorgna et al. (2001)], and is important in queueing applications, e.g., to model teletraffic, when X is Gaussian H-`sssi` (i.e., fractional Brownian motion), with $H \geq \frac{1}{2}$ and linear service $\gamma = 1$ [e.g., Norros (1994) and Piterbarg (2001)]. In this case, building on Hüsler and Piterbarg (1999), Piterbarg (2001, Theorem 5) gave a version of the remarkable property (1.3) below, that was a trigging influence for us. Recognizing this, we name that property after him. However, the Gaussian problem Piterbarg studied is very different from ours with `id` processes, and his proof, by Gaussian field theory, does not relate to non-Gaussian settings.

With ν being the Lévy measure on $\mathbb{R}^{[0,\infty)}$ of X (see Section 2.3), denote

$$\underline{R(u)} \equiv 1 \wedge \nu\left(\left\{y \in \mathbb{R}^{[0,\infty)} : \sup_{t \geq 0} \frac{y(t)}{1+ct^\gamma} > u\right\}\right) \qquad \text{for} \quad u \in \mathbb{R}. \tag{1.2}$$

In Section 3 we study the probability for overload during a time interval, $\mathbf{P}\{\sup_{s\in[0,t]} Y(s) > u\}$ as $u \to \infty$: If for each choice of a constant $t > 0$, it holds that

$$\lim_{u\to\infty} \frac{\mathbf{P}\{\sup_{s\in[0,t]} Y(s) > u\}}{\mathbf{P}\{Y(\hat{t}(u)) > u\}} = 1 \qquad \text{whenever} \quad 0 \leq \hat{t}(u) \leq t\,, \tag{1.3a}$$

then we say that the process Y has the Piterbarg property. The similar statement, for which the length $t = t(u)$ of the interval may depend on the level u,

$$\lim_{u\to\infty} \frac{\mathbf{P}\{\sup_{s\in[0,t(u)]} Y(s) > u\}}{\mathbf{P}\{Y(\hat{t}(u)) > u\}} = 1 \qquad \text{whenever} \quad 0 \leq \hat{t}(u) \leq t(u)\,, \tag{1.3b}$$

will be referred to as the generalized Piterbarg property.

One indication of the unusual behaviour of Y, is that (1.3) implies, as $u \to \infty$,

$$\mathbf{P}\left\{\bigcap_{i=1}^{n} \{Y(\hat{t}_i) > u\} \,\middle|\, \sup_{s\in[0,t]} Y(s) > u\right\} \geq 1 - \sum_{i=1}^{n}\left(1 - \frac{\mathbf{P}\{Y(\hat{t}_i) > u\}}{\mathbf{P}\{\sup_{s\in[0,t]} Y(s) > u\}}\right) \to 1\,.$$

Thus overload periods within $[0,t]$ are long enough to include any $\hat{t}_1, \ldots, \hat{t}_n \in [0,t]$.

This last conclusion leads us naturally to the question whether one can replace the minimum $\wedge_{i=1}^{n} Y(\hat{t}_i)$ taken over a finite collection of points in $[0,t]$ by the infimum over the entire interval. That is, we would like to know if

$$\lim_{u\to\infty} \frac{\mathbf{P}\{\sup_{s\in[0,t]} Y(s) > u\}}{\mathbf{P}\{\inf_{s\in[0,t]} Y(s) > u\}} = 1\text{:} \tag{1.3c}$$

This we call the strong Piterbarg property, whether or not t is a function of u.

We will make assumptions about "heavy tails" for the function R in (1.2) (e.g., subexponentiality together with positive decrease; see Section 2.1). Under additional technical assumptions on X, we establish the generalized Piterbarg property, when $t(u)$ does not grow too fast with u [e.g., $t(u) = o(u^{1/\gamma})$]. Under the same assumptions we will show that the strong Piterbarg property holds as well. Under certain weaker assumptions, we prove a weaker so called O-version of (1.3), i.e., that the probability ratios in (1.3) are bounded away from zero and infinity.

Our main "external tool" in proofs, is the work on asymptotic properties of subexponential `id` processes by Rosiński and Samorodnitsky (1993).

The Piterbarg properties are quite unusual. For example, only a degenerate α-stable or Gaussian process Y can have them. See Example 6 below.

In Section 4, we give a discussion, with examples of application, and counterexamples, for `id` H-`ss` processes X given as stochastic integrals with respect to heavy-tailed `id` random measures (see Section 2.3). This includes α-stable processes.

2. Classes of Functions and Stochastic Processes. It will be convenient to devote a separate section to describe classes of functions and stochastic processes, that feature in the rest of the article. In addition, some basic relations between these classes, and some important representation properties, are listed for easy reference.

2.1 Classes of functions. In this subsection, $f:\mathbb{R}\to(0,\infty)$ denotes a non-increasing function with $\lim_{u\to\infty} f(u)=0$, and $g:\mathbb{R}\to(0,\infty)$ a measurable function.

The function f is sub-exponential, $f\in\mathcal{S}$, if there exist independent identically distributed random variables ξ and η, such that

$$f(u)\sim \mathbf{P}\{\xi>u\} \qquad \text{and} \qquad \mathbf{P}\{\xi+\eta>u\}\sim 2\,\mathbf{P}\{\xi>u\} \qquad \text{as} \quad u\to\infty.$$

The function f is O-regularly varying, $f\in\mathbf{OR}$, if

$$\liminf_{u\to\infty} f(\lambda u)/f(u) > 0 \qquad \text{for some} \quad \lambda>1.$$

The function f has positive decrease, $f\in\mathbf{PD}$, if

$$\limsup_{u\to\infty} f(\lambda u)/f(u) < 1 \qquad \text{for some} \quad \lambda>1.$$

The function f is extended regularly varying, $f \in \mathbf{ER}$, if

$$\liminf_{u\to\infty} f(\lambda u)/f(u) \geq \lambda^{-b} \qquad \text{for} \quad \lambda \geq 1, \quad \text{for some constant} \quad b \geq 0.$$

Note: The definitions of **OR**, **PD** and **ER** are more complicated than those given above for a general non-monotone f.

The function g is regularly varying with index $\varrho \in \mathbb{R}$, $g \in \mathbf{RV}(\varrho)$, if

$$\lim_{u\to\infty} g(\lambda u)/g(u) = \lambda^{\varrho} \qquad \text{for} \quad \lambda \geq 1 \qquad (\text{or, equivalently, for } \lambda > 0).$$

Here the convergence must, in fact, be locally uniform.

Notice that the function $g \circ \log$ belongs to $\mathbf{RV}(0)$, which we denote $g \in \mathcal{L}$, if

$$\lim_{u\to\infty} g(u+\lambda)/g(u) = 1 \qquad \text{for} \quad \lambda \geq 0 \qquad (\text{or, equivalently, for } \lambda \in \mathbb{R}).$$

We have $\mathbf{ER} \cap \mathbf{PD} \subseteq \mathbf{OR} \cap \mathbf{PD} \cap \mathcal{S}$, and a monotone $f \in \bigcup_{\varrho<0} \mathbf{RV}(\varrho)$ belongs to all these classes. Further, $\mathbf{OR} \cap \mathcal{L} \subseteq \mathcal{S} \subseteq \mathcal{L}$.

The classes of functions above, and the listed relations between them, are well-known from the literature. See, e.g., Bingham et al. (1987).

2.2 Classes of stochastic processes. In the remainder of this article, $X = \{X(t)\}_{t\geq 0}$ denotes a separable stochastic process, that is continuous in probability and locally bounded (bounded on any given compact interval) a.s., and is defined on a complete probability space $(\Omega, \mathfrak{F}, \mathbf{P})$. We refer to these requirements as Condition $\mathfrak{X}$. Depending on the context, further requirements on X will be imposed later.

We write $\{\tilde{X}(t)\}_{t\geq 0} =_d X$ when the finite dimensional distributions (`fdds`) of the processes $\tilde{X}$ and X agree. For example, X is stationary, if $X(\cdot+h) =_d X$ for $h \geq 0$.

The process X has stationary increments (`si`), if

$$X(\cdot+h) - X(h) =_d X - X(0) \qquad \text{for each} \quad h \geq 0.$$

The process X is self-similar with index $H > 0$ (H-`ss`), if

$$a^{-H} X(a\,\cdot) =_d X \qquad \text{for each} \quad a > 0.$$

The process X is infinitely divisible (`id`), if to each $n \in \mathbb{N}$, there exist independent processes $\{\tilde{X}_1(t)\}_{t\geq 0}, \ldots, \{\tilde{X}_n(t)\}_{t\geq 0}$, such that

$$\tilde{X}_1 =_d \ldots =_d \tilde{X}_n \qquad \text{and} \qquad \tilde{X}_1 + \ldots + \tilde{X}_n =_d X.$$

The process X is $\underline{\alpha\text{-stable}}$, $\alpha \in (0,2]$, if for each $n \in \mathbb{N}$, there exists a constant $C_n \in \mathbb{R}$, such that, taking independent copies $\{\tilde{X}_k\}_{k=1}^n$ of X,

$$n^{-1/\alpha}(\tilde{X}_1 + \ldots + \tilde{X}_n) + C_n =_d X.$$

In particular, it turns out, the process X is Gaussian, if and only if it is 2-stable.

The process X is $\underline{\text{strictly } \alpha\text{-stable}}$, if, taking independent copies $\{\tilde{X}_k\}_{k=1}^\infty$ of X,

$$n^{-1/\alpha}(\tilde{X}_1 + \ldots + \tilde{X}_n) =_d X \qquad \text{for} \quad n \in \mathbb{N}.$$

A process X is H-ss, if and only if the Lamperti transformated process $e^{-H\cdot} X(e^{\cdot})$ is stationary [Lamperti (1962)].

α-stable processes are id. Clearly, an α-stable process X is strictly α-stable if it is symmetric α-stable ($\underline{S\alpha S}$) (α-stable with $X =_d -X$).

Of course, the classes of processes mentioned above are all quite basic, as are the indicated relations between. See, e.g., Samorodnitsky and Taqqu (1994) on further information, and for an extensive bibliography.

2.3 Representation of id processes. The fdds of an id process $\{X(t)\}_{t \in T}$, $T = [0,\infty)$, with no Gaussian component, can be described by means of a $\underline{\text{Lévy measure } \nu}$ on the cylindrical σ-algebra $\mathcal{B}$ on $\mathbb{R}^T$, and a $\underline{\text{localization}}$ parameter $\mu \in \mathbb{R}^T$:

Let π_τ be the projection of $\mathbb{R}^T$ on $\mathbb{R}^\tau$, and $\mathcal{B}_\tau$ the Borel sets in $\mathbb{R}^\tau$, for $\tau \in \mathcal{T} \equiv \{\tau \subseteq T : 1 \le \#\tau < \infty\}$. According to Maruyama (1970), a measure ν on $\mathcal{B}$ is a Lévy measure for X, if $\nu \circ \pi_\tau^{-1}$ is a Lévy measure on $\mathcal{B}_\tau$ [i.e., $1 \wedge |\cdot|^2 \in \mathbb{L}^1(\mathbb{R}^\tau, \nu \circ \pi_\tau^{-1})$] for each $\tau \in \mathcal{T}$, and there exists a $\mu \in \mathbb{R}^T$ such that

$$(2.1) \quad \mathbf{E}\{e^{i\langle \theta, X\rangle}\} = \exp\left\{ i\langle \theta, \mu\rangle + \int_{\mathbb{R}^T} \left(e^{i\langle \theta, x\rangle} - 1 - i\langle \theta, \kappa(x)\rangle\right) \nu(dx) \right\} \quad \text{for} \quad \theta \in \mathbb{R}^{(T)}.$$

Here $\mathbb{R}^{(T)} = \{x \in \mathbb{R}^T : \#\{t \in T : x(t) \neq 0\} < \infty\}$. Further, $\langle x, y\rangle = \sum_{t \in T} x(t)y(t)$ for $x \in \mathbb{R}^{(T)}$ and $y \in \mathbb{R}^T$, while $\kappa(x)(t) = x(t)\mathbf{1}_{[-1,1]}(|x(t)|)$ for $x \in \mathbb{R}^T$ and $t \in T$.

[A general id X can be represented $X =_d X_1 + X_2$, with X_1 and X_2 independent, X_1 id with no Gaussian component as in (2.1), and X_2 zero-mean Gaussian

$$\mathbf{E}\{e^{i\langle \theta, X_2\rangle}\} = \exp\left\{ -\tfrac{1}{2} \sum_{s,t \in T,\ \theta(s),\theta(t) \neq 0} \theta(s)\theta(t)\, \mathbf{E}\{X_2(s)X_2(t)\} \right\} \quad \text{for} \quad \theta \in \mathbb{R}^{(T)}.]$$

We now turn to the task of constructing and representing id processes:

Let $(S, \mathcal{S}, \lambda)$ be a σ-finite measure space, and put $\mathcal{S}_0 \equiv \{A \in \mathcal{S} : \lambda(A) < \infty\}$. An (independently scattered) id random meassure (with no Gaussian component), with control measure λ, is a map $M : \mathcal{S}_0 \to \mathbb{L}^0(\Omega, \mathcal{F})$ such that, for $A \in \mathcal{S}_0$,

$$\mathbf{E}\{e^{i\theta M(A)}\} = \exp\left\{ \int_A \left(i\theta m + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta\kappa(x) \right) \rho(\cdot, dx) \right) d\lambda \right\} \quad \text{for } \theta \in \mathbb{R}. \tag{2.2}$$

Here the localization $m \in \mathbb{L}^0(S)$ satisfies $\{\mathbf{1}_A m\}_{A \in \mathcal{S}_0} \subseteq \mathbb{L}^1(S, \lambda)$, while $\rho(s, \cdot)$ is a Lévy measure on $\mathbb{R}$ for $s \in S$, such that $\rho(\cdot, B) \in \mathbb{L}^0(S)$ for Borel sets $B \subseteq \mathbb{R}$, and

$$F(A \times \cdot) \equiv \textstyle\int_A \rho(s, \cdot)\, \lambda(ds) \qquad \text{is a Lévy measure on } \mathbb{R}, \quad \text{for each } A \in \mathcal{S}_0. \tag{2.3}$$

The stochastic integral $\int_S f\, dM$ is well-defined (in a $\mathbf{P}$-sense), for $f \in \mathbb{L}^0(S)$ with

$$\int_S \int_{\mathbb{R}} (1 \wedge |xf|^2)\, \rho(\cdot, dx)\, d\lambda \vee \int_S \left| mf + \int_{\mathbb{R}} \left(\kappa(xf) - f\kappa(x) \right) \rho(\cdot, dx) \right| d\lambda < \infty \tag{2.4}$$

[Rajput and Rosiński (1989, Section 2)]. In that case, $\int_S f\, dM$ is id, with

$$\mathbf{E}\{e^{i\theta \int_S f\, dM}\} = \exp\left\{ \int_S \left(i\theta mf + \int_{\mathbb{R}} \left(e^{i\theta xf} - 1 - i\theta\kappa(x) f \right) \rho(\cdot, dx) \right) d\lambda \right\}. \tag{2.5}$$

In the language of (2.1), the (process consisting of a) single id random variable $\int_S f\, dM$, has Lévy measure ν on $\mathbb{R}$, and localization parameter $\mu \in \mathbb{R}$, given by

$$\nu(B) = F\left(\{(s,x) \in S \times \mathbb{R} : xf(s) \in B\} \right) \quad \text{and} \quad \mu = \int_S \left(mf + \int_{\mathbb{R}} \left(\kappa(xf) - \kappa(x) f \right) \rho(\cdot, dx) \right) d\lambda.$$

In particular, e.g., by Feller (1971, p. 571), $\int_S f\, dM$ is non-negative, if and only if

$$xf(s) \geq 0 \quad \text{a.e. } (F), \qquad \textstyle\int_S \left(mf - \int_{\mathbb{R}} \kappa(x) f\, \rho(\cdot, dx) \right) d\lambda \geq 0 \tag{2.6}$$

$$\text{and} \qquad \textstyle\int_S \int_{\mathbb{R}} (1 \wedge |xf|)\, \rho(\cdot, dx)\, d\lambda < \infty.$$

Pick $f_t \in \mathbb{L}^0(S)$ satisfying (2.4) for $t \geq 0$. The process

$$X =_d \{\textstyle\int_S f_t\, dM\}_{t \geq 0} \tag{2.7}$$

is id, with Lévy measure in (2.1) given by $\nu = F \circ T_f^{-1}$, where $T_f : S \times \mathbb{R} \to \mathbb{R}^{[0,\infty)}$ is given by $T_f(s,x) = x f_{(\cdot)}(s)$. It follows that the function R in (1.2) is given by

$$R(u) = 1 \wedge \int_S \rho\left(s, \mathbb{R} \setminus \left[\frac{-u}{\sup_{t>0} f_t(s)^- / (1 + ct^\gamma)}, \frac{u}{\sup_{t>0} f_t(s)^+ / (1 + ct^\gamma)} \right] \right) d\lambda(s). \tag{2.8}$$

By (2.5), the process X in (2.7) is H-ss, if and only if

$$\int_S \left(ima^{-H} \sum_{j=1}^n \theta_j f_{at_j} + \int_{\mathbb{R}} \left(e^{ixa^{-H} \sum_{j=1}^n \theta_j f_{at_j}} - 1 - i\kappa(x) a^{-H} \sum_{j=1}^n \theta_j f_{at_j} \right) \rho(\cdot, dx) \right) d\lambda \tag{2.9}$$

does not depend on $a>0$ for any choice of $n\in\mathbb{N}$, $t_1,\ldots,t_n\geq 0$ and $\theta_1,\ldots,\theta_n\in\mathbb{R}$. Similarly, X is H-sssi, if and only if $X(0)=0$, and, with obvious notation,

$$\int_S \left(ima^{-H}\langle \theta, f_{at+h}-f_h\rangle + \int_{\mathbb{R}} \left(e^{ixa^{-H}\langle \theta, f_{at+h}-f_h\rangle} - 1 - i\kappa(x) a^{-H}\langle \theta, f_{at+h}-f_h\rangle \right) \rho(\cdot, dx) \right) d\lambda$$

does not depend on $a,h>0$ for any choice of $n\in\mathbb{N}$, $t_1,\ldots,t_n\geq 0$ and $\theta_1,\ldots,\theta_n\in\mathbb{R}$.

Notice that $X(0)=0$, if and only if $mf_0=0$ a.e. (λ) and $xf_0(s)=0$ a.e. (F).

Example 1. Define a Lévy measure μ on $\mathbb{R}$ with $\int_{-1}^1 |x|\, d\mu(x) < \infty$, by $\mu((-\infty, -x)) = r(-x)$ and $\mu((x,\infty)) = r(x)$ for $x\geq 0$ $[\mu(\{0\})=0]$, for a non-negative $r\in \mathbb{L}^0(\mathbb{R})\cap\mathbb{L}^1([-1,1])$ that is monotone and vanishes at infinity on both half-lines.

Pick an $H>0$. Let M be an id random measure on $(0,\infty)$ (equipped with the Borel σ-algebra), with Lebesgue control measure, and with

$$\rho(s,B) = Hs^{-1}\mu(s^{-H}B) \qquad \text{and} \qquad m(s) = \textstyle\int_{\mathbb{R}} \kappa(x)\, \rho(s,dx).$$

Pick $f\in\mathbb{L}^0((0,\infty))$ satisfying (2.4). Consider the id process X in (2.7), where

$$f_t(s) = f(s/t) \quad \text{if } t>0 \qquad \text{and} \qquad f_0(s)=0, \qquad \text{for } s>0. \tag{2.10}$$

This process X is H-ss, since the integral (2.9) evaluates to

$$\int_0^\infty \int_{\mathbb{R}} \left(e^{ixa^{-H}\sum_{j=1}^n \theta_j f(s/(at_j))} - 1 \right) \frac{H\mu(s^{-H}dx)}{s}\, ds = \int_0^\infty \int_{\mathbb{R}} \left(e^{i\tilde{x}\sum_{j=1}^n \theta_j f(\tilde{s}/t_j)} - 1 \right) \frac{H\mu(\tilde{s}^{-H}d\tilde{x})}{\tilde{s}}\, d\tilde{s}.$$

Moreover, we have that X is $\mathbf{P}$-continuous, from the fact that

$$\mathbf{E}\{e^{i\theta(X(t+h)-X(t))}\} = \exp\left\{ \int_0^\infty \int_{\mathbb{R}} \left(e^{i\theta x(f(s/(t+h))-f(s/t))} - 1 \right) \frac{H\mu(s^{-H}dx)}{s}\, ds \right\}.$$

If $f = \mathbf{1}_{(0,1]}$, then X has independent increments, so that $\mathbf{P}$-continuity and separability give local boundedness [e.g., Sato (1999, Theorem 11.5)].

2.4 Representation of H-ss α-stable processes. Let X be strictly α-stable H-ss, with $\alpha\in(0,2)$. In the case $\alpha=1$, assume in addition that X is $S\alpha S$.

Let $w_0 \in \mathbb{L}^0(S)$ be positive, and pick a constant $\beta \in [-1,1]$ ($\beta=0$ if $\alpha=1$). Let M be an `id` random measure (see Section 2.3), with control measure λ, and with

$$\rho(s,dx) = w_0(s)\left((1-\beta)\mathbf{1}_{(-\infty,0)}(x) + (1+\beta)\mathbf{1}_{(0,\infty)}(x)\right)|x|^{-(\alpha+1)}/(2\alpha)\,dx$$

$$\text{and} \quad m(s) = \begin{cases} \int_{\mathbb{R}} \kappa(x)\rho(s,dx) & \text{if } \alpha<1 \\ 0 & \text{if } \alpha=1. \\ \int_{\mathbb{R}} (\kappa(x)-x)\rho(s,dx) & \text{if } \alpha>1 \end{cases}$$

We say that M is a strictly α-stable random measure. [It is an exercise to deduce from (2.2), that $M(A)$ is strictly α-stable ($S\alpha S$ if $\alpha=1$), for $A \in \mathfrak{S}_0$.]

There exists $\{f_t\}_{t\geq 0} \subseteq \mathbb{L}^\alpha(S, w_0\lambda)$ [which is what (2.4) reduces to here], such that X satisfies (2.7), for some $\beta \in [-1,1]$ (e.g., $\beta=\pm 1$ works if $\alpha \neq 1$).

Now a process given by (2.7) is strictly α-stable. Denoting $x^{\langle\alpha\rangle} = |x|^\alpha \text{sign}(x)$, (2.5) shows that X is H-`ss`, if and only if $f_0=0$ a.e. (λ), and the following integrals do not depend on $a>0$ for any choice of $n \in \mathbb{N}$, $t_1,\ldots,t_n \geq 0$ and $\theta_1,\ldots,\theta_n \in \mathbb{R}$:

$$\int_S \left| a^{-H} \sum_{j=1}^n \theta_j f_{at_j} \right|^\alpha w_0\,d\lambda \qquad \text{and} \qquad \beta \int_S \left(a^{-H} \sum_{j=1}^n \theta_j f_{at_j} \right)^{\langle\alpha\rangle} w_0\,d\lambda\,.$$

Further, X is H-`sssi`, if and only if $f_0=0$ a.e. (λ), and the following integrals do not depend on $a,h>0$ for any choice of $n \in \mathbb{N}$, $t_1,\ldots,t_n \geq 0$ and $\theta_1,\ldots,\theta_n \in \mathbb{R}$:

$$\int_S \left| a^{-H} \sum_{j=1}^n \theta_j (f_{at_j+h} - f_h) \right|^\alpha w_0\,d\lambda \qquad \text{and} \qquad \beta \int_S \left(a^{-H} \sum_{j=1}^n \theta_j (f_{at_j+h} - f_h) \right)^{\langle\alpha\rangle} w_0\,d\lambda.$$

Remark. Much is known about the class of H-`sssi` α-stable processes, that is very rich for $\alpha<2$, unlike the Gaussian case. See, e.g., Samorodnitsky and Taqqu (1994), Surgailis et al. (1998), Burnecki et al. (1998), and Pipiras and Taqqu (2000a, b).

For $H \in (1/\alpha, 1]$ with $\alpha>1$, and for $H=1/\alpha>1$, it is known that (separable) H-`sssi` α-stable processes are locally bounded. For other values of H and α, local boundedness is not determined by H and α, and there exist both locally bounded and unbounded processes. Precise conditions for local boundedness are known for $\alpha<1$. See Kôno and Maejima (1991) and Samorodnitsky and Taqqu (1990, 1994).

3. Overload and the Piterbarg Properties. Here we first study the probability for overload $\mathbf{P}\{Y(t)>u\}$, and then the Piterbarg properties (1.3).

The next assumptions limit the effect of the left tail of X on the right tail of Y:

$$\limsup_{u\to\infty} \frac{\mathbf{P}\{X(1)<-\varepsilon(t(u)u^{-1/\gamma})^{-H}u^{1-H/\gamma}\}}{R(u^{1-H/\gamma})} < \infty \qquad \text{for all} \quad \varepsilon>0, \tag{3.1}$$

$$\lim_{u\to\infty} \frac{\mathbf{P}\{X(1)<-\varepsilon(t(u)u^{-1/\gamma})^{-H}v(u^{1-H/\gamma})\}}{R(u^{1-H/\gamma})} = 0 \qquad \text{for some} \quad \varepsilon>0. \tag{3.2}$$

Here (3.1) is used together with the growth condition indicated in the introduction

$$\limsup_{u\to\infty} t(u)\, u^{-1/\gamma} < \infty. \tag{3.3}$$

In (3.2), $v:\mathbb{R}\to(0,\infty)$ is a suitably selected function, with the following properties:

$$\limsup_{u\to\infty} \frac{v(u)}{u} < \infty \qquad \text{and} \qquad \lim_{u\to\infty} \frac{t(u)}{u^{1/\gamma}\,(v(u^{1-H/\gamma})/u^{1-H/\gamma})^{1/(H\wedge 1)}} = 0\,. \tag{3.4}$$

In the hypothesis of Theorem 1 below, (3.1) follows from (3.3) [by (3.11) and $R\in\mathbf{OR}$], while (3.2) follows from (3.4) [by (3.20) and $R\in\mathbf{PD}$], provided that

$$\limsup_{u\to\infty} \frac{\mathbf{P}\{X(1)<-u\}}{\mathbf{P}\{\sup_{s\in[0,1]}X(s)>u\}} < \infty\,. \tag{3.5}$$

Notice that (3.5) holds if, e.g., $X(1)$ is symmetric or non-negative.

Theorem 1. *Let X be H-*`ss` *and* `id` *with no Gaussian component, satisfying* Condition $\mathfrak{X}$. *Consider the process Y, given by* (1.1), *with $c>0$ and $\gamma>H$ constants. Suppose that the function R, given by* (1.2) [with ν given by (2.1)], *belongs to $\mathcal{S}\cap\mathbf{PD}$. Then $Y(t)<\infty$ a.s. for each $t\geq 0$, and*

$$\lim_{u\to\infty} \mathbf{P}\{Y(0)>u\} \,/\, R(u^{1-H/\gamma}) = 1\,. \tag{3.6}$$

(i) *If in addition X is* `si`, *then for $t(u)\geq 0$,*

$$\lim_{u\to\infty} \mathbf{P}\{Y(t(u))>u\} \,/\, R(u^{1-H/\gamma}) = 1\,. \tag{3.7}$$

(ii) *If in addition $R\in\mathbf{OR}$ and $t(u)\geq 0$ satisfies* (3.1) *and* (3.3), *then we have*

$$0 < \liminf_{u\to\infty} \frac{\mathbf{P}\{Y(t(u))>u\}}{R(u^{1-H/\gamma})} \leq \limsup_{u\to\infty} \frac{\mathbf{P}\{Y(t(u))>u\}}{R(u^{1-H/\gamma})} < \infty\,. \tag{3.8}$$

(iii) *If in addition there exists a function v satisfying* (3.4), *such that*

$$\lim_{u\to\infty} R(u-v(u))/R(u) = 1\,, \tag{3.9}$$

and such that (3.2) *holds, then* (3.7) *holds.*

Proof. Let $R \in \mathcal{S} \cap \mathbf{PD}$, and denote $\hat{c} = 2^\gamma / c$. For every $t \geq 0$ we have, for $u > 0$,

$$\mathbf{P}\{Y(t) > u\} \leq \mathbf{P}\Big\{X(t) < -\frac{u}{2}\Big\} + \mathbf{P}\Big\{\sup_{t \leq s < 2t} X(s) > \frac{u}{2}\Big\} + \mathbf{P}\Big\{\sup_{s \geq 2t} (X(s) - s^\gamma/\hat{c}) > \frac{u}{2}\Big\}.$$

Since X is locally bounded, the first two terms on the right go to zero as $u \to \infty$. Furthermore, we can bound from above the third term by

$$\begin{aligned} (3.10) \quad & \mathbf{P}\Big\{\sup_{0 \leq s \leq u^{1/\gamma}} X(s) > \frac{u}{2}\Big\} + \sum_{j=0}^{\infty} \mathbf{P}\Big\{\sup_{2^j u^{1/\gamma} \leq s \leq 2^{j+1} u^{1/\gamma}} X(s) > \frac{2^{\gamma j} u}{\hat{c}}\Big\} \\ & = \mathbf{P}\Big\{\sup_{s \in [0,1]} X(s) > \frac{u^{1-H/\gamma}}{2}\Big\} + \sum_{j=0}^{\infty} \mathbf{P}\Big\{\sup_{s \in [1/2,1]} X(s) > \frac{2^{(\gamma-H)j} u^{1-H/\gamma}}{\hat{c}}\Big\} \\ & \leq 2 \sum_{j=0}^{\infty} \mathbf{P}\Big\{\sup_{s \in [0,1]} \frac{X(s)}{1+cs^\gamma} > \frac{2^{(\gamma-H)j} u^{1-H/\gamma}}{(1+c)(2 \vee \hat{c})}\Big\}. \end{aligned}$$

Here $R \in \mathcal{S}$, together with Rosiński and Samorodnitsky (1993, Theorem 2.1), give

$$(3.11) \quad \limsup_{u \to \infty} \frac{1}{R(u)} \mathbf{P}\Big\{\sup_{s \in [0,1]} X(s) > (1+c)u\Big\} \leq \limsup_{u \to \infty} \frac{1}{R(u)} \mathbf{P}\Big\{\sup_{s \in [0,1]} \frac{X(s)}{1+cs^\gamma} > u\Big\} \leq 1.$$

Our $R \in \mathbf{PD}$ has an upper Matuszewska index $\mathfrak{a} < 0$ [e.g., Bingham et al. (1987, p. 71)] such that, given $-a \in (\mathfrak{a}, 0)$ and $\lambda_0 > 0$, $R(\lambda u)/R(u) \leq C\lambda^a$ for $\lambda \geq \lambda_0$ and u large enough, for some $C > 0$. Hence the right hand side of (3.10) is at most

$$2 \sum_{j=0}^{\infty} 2R\Big(\frac{2^{(\gamma-H)j} u^{1-H/\gamma}}{(1+c)(2 \vee \hat{c})}\Big) \leq 4C(1+c)^a (2 \vee \hat{c})^a \Big(\sum_{j=0}^{\infty} 2^{-a(\gamma-H)j}\Big) R(u^{1-H/\gamma})$$

for u large enough. This proves the fact that $Y(t) < \infty$ a.s.

Further, by self-similarity and Rosiński and Samorodnitsky (1993, Theorem 2.1),

$$(3.12) \quad \mathbf{P}\{Y(0) > u\} = \mathbf{P}\Big\{\sup_{s \geq 0} \frac{X(s)}{1+cs^\gamma} > u^{1-H/\gamma}\Big\} \sim R(u^{1-H/\gamma}) \qquad \text{as} \quad u \to \infty.$$

Proof of (i). For X `si`, Y is stationary, and so (3.7) is same thing as (3.6).

Proof of (ii). By (3.3), we have, for some $\theta \in (0,1)$, for all u large enough,

$$(3.13) \qquad \theta \leq \inf_{s \geq t u^{-1/\gamma}} \frac{1+c(s-tu^{-1/\gamma})^\gamma}{1+cs^\gamma} \leq \sup_{s \geq t u^{-1/\gamma}} \frac{1+c(s-tu^{-1/\gamma})^\gamma}{1+cs^\gamma} \leq 1.$$

Using self-similarity, we therefore obtain, for u large enough,

(3.14) $\mathbf{P}\{Y(t)>u\}$

$$= \mathbf{P}\left\{\sup_{s\geq tu^{-1/\gamma}} \frac{X(s)-X(tu^{-1/\gamma})}{1+c(s-tu^{-1/\gamma})^{\gamma}} > u^{1-H/\gamma}\right\} \begin{cases} \geq \mathbf{P}\left\{\sup_{s\geq tu^{-1/\gamma}} \frac{X(s)-X(tu^{-1/\gamma})}{1+cs^{\gamma}} > u^{1-H/\gamma}\right\} \\ \leq \mathbf{P}\left\{\sup_{s\geq 0} \frac{X(s)-X(tu^{-1/\gamma})}{1+cs^{\gamma}} > \theta u^{1-H/\gamma}\right\} \end{cases}.$$

Notice that, for a constant $K\geq 1$, denoting $\eta_1(u) = (tu^{-1/\gamma})^{-H}\big(1+c(Ktu^{-1/\gamma})^{\gamma}\big)$ and $\eta_2(u) = (1\vee(tu^{-1/\gamma}))^{\gamma-H}$,

(3.15) $$\mathbf{P}\left\{\sup_{s\geq tu^{-1/\gamma}} \frac{X(s)-X(tu^{-1/\gamma})}{1+cs^{\gamma}} > u^{1-H/\gamma}\right\}$$

$$\geq \mathbf{P}\left\{\sup_{s\geq Ktu^{-1/\gamma}} \frac{X(s)-X(tu^{-1/\gamma})}{1+cs^{\gamma}} > u^{1-H/\gamma}\right\}$$

$$\geq \mathbf{P}\left\{\sup_{s\geq Ktu^{-1/\gamma}} \frac{X(s)}{1+cs^{\gamma}} > 2u^{1-H/\gamma}\right\} - \mathbf{P}\left\{X(tu^{-1/\gamma}) > \big(1+c(Ktu^{-1/\gamma})^{\gamma}\big)u^{1-H/\gamma}\right\}$$

$$\geq \mathbf{P}\left\{\sup_{s\geq(1\vee Ktu^{-1/\gamma})} \frac{X(s)}{1+cs^{\gamma}} > 2u^{1-H/\gamma}\right\} - \mathbf{P}\{X(1)>\eta_1(u)u^{1-H/\gamma}\}$$

$$= \mathbf{P}\left\{\sup_{s\geq 1} \frac{X(s)}{1+c(1\vee Ktu^{-1/\gamma})^{\gamma}s^{\gamma}} > 2(1\vee Ktu^{-1/\gamma})^{-H}u^{1-H/\gamma}\right\}$$

$$- \mathbf{P}\{X(1)>\eta_1(u)u^{1-H/\gamma}\}$$

$$\geq \mathbf{P}\left\{\sup_{s\geq 1} \frac{X(s)}{1+cs^{\gamma}} > \frac{2\eta_2(u)u^{1-H/\gamma}}{K^H}\right\} - \mathbf{P}\left\{\sup_{s\geq 0} \frac{X(s)}{1+cs^{\gamma}} > \frac{\eta_1(u)u^{1-H/\gamma}}{1+c}\right\}.$$

Here we have, picking a constant $L\geq 1$,

$$\mathbf{P}\left\{\sup_{s\geq 0} \frac{X(s)}{1+cs^{\gamma}} > u\right\} \leq \mathbf{P}\left\{\sup_{s\geq L^{-1}} \frac{X(s)}{1+cs^{\gamma}} > u\right\} + \mathbf{P}\left\{\sup_{0\leq s\leq L^{-1}} \frac{X(s)}{1+cs^{\gamma}} > u\right\}$$
$$\leq \mathbf{P}\left\{\sup_{s\geq 1} \frac{X(s)}{1+cs^{\gamma}} > \frac{u}{L^{\gamma-H}}\right\} + \mathbf{P}\left\{\sup_{s\geq 0} \frac{X(s)}{1+cs^{\gamma}} > \frac{L^H u}{1+c}\right\}.$$

It follows from (3.12) and the fact that $R\in\mathbf{PD}$, that, if L is large enough, then

$$\limsup_{u\to\infty} \mathbf{P}\left\{\sup_{s\geq 0} \frac{X(s)}{1+cs^{\gamma}} > \frac{L^H u}{1+c}\right\} \Big/ \mathbf{P}\left\{\sup_{s\geq 0} \frac{X(s)}{1+cs^{\gamma}} > u\right\} < \frac{1}{2}.$$

Fixing L such that this relation holds, we get immediately

$$\mathbf{P}\left\{\sup_{s\geq 1} \frac{X(s)}{1+cs^{\gamma}} > \frac{u}{L^{\gamma-H}}\right\} \geq \frac{1}{2}\,\mathbf{P}\left\{\sup_{s\geq 0} \frac{X(s)}{1+cs^{\gamma}} > u\right\}$$

for u large enough. Therefore, by (3.12),

$$\liminf_{u\to\infty} \frac{1}{R(L^{\gamma-H}u)}\,\mathbf{P}\Big\{\sup_{s\ge 1}\frac{X(s)}{1+cs^{\gamma}} > u\Big\} \ge \frac{1}{2}\,. \tag{3.16}$$

Since $R\in\mathbf{PD}$, and $\limsup_{u\to\infty}\eta_2(u)/\eta_1(u) < \infty$, we get from (3.12) and (3.16),

(3.17)

$$\begin{aligned}\liminf_{u\to\infty} \mathbf{P}\Big\{\sup_{s\ge 1}\frac{X(s)}{1+cs^{\gamma}} > \frac{2\eta_2(u)u^{1-H/\gamma}}{K^H}\Big\}\Big/\mathbf{P}\Big\{\sup_{s\ge 0}\frac{X(s)}{1+cs^{\gamma}} > \frac{\eta_1(u)u^{1-H/\gamma}}{1+c}\Big\}\\ \ge \frac{1}{2}\liminf_{u\to\infty} R\Big(\frac{2L^{\gamma-H}\eta_2(u)u^{1-H/\gamma}}{K^H}\Big)\Big/R\Big(\frac{\eta_1(u)u^{1-H/\gamma}}{1+c}\Big) \ge 2\end{aligned}$$

for all K large enough [where $\eta_i(u)u^{1-H/\gamma}\to\infty$ for $i=1,2$]. Fixing $K\ge 1$ such that (3.17) holds, we may apply (3.16) and (3.17) on the last row of (3.15), to get

$$\begin{aligned}\liminf_{u\to\infty} \frac{1}{R(u^{1-H/\gamma})}\,\mathbf{P}\Big\{\sup_{s\ge tu^{-1/\gamma}}\frac{X(s)-X(tu^{-1/\gamma})}{1+cs^{\gamma}} > u^{1-H/\gamma}\Big\}\\ \ge \frac{1}{2}\liminf_{u\to\infty} \frac{1}{R(u^{1-H/\gamma})}\,\mathbf{P}\Big\{\sup_{s\ge 1}\frac{X(s)}{1+cs^{\gamma}} > \frac{2\eta_2(u)u^{1-H/\gamma}}{K^H}\Big\}\\ \ge \frac{1}{4}\liminf_{u\to\infty} \frac{R\big(2K^{-H}L^{\gamma-H}\eta_2(u)\,u^{1-H/\gamma}\big)}{R(u^{1-H/\gamma})} > 0\,,\end{aligned}$$

using $R\in\mathbf{OR}$ for the last inequality. By (3.14), this gives the lower bound in (3.8).

The corresponding upper bound in (3.8) follows from (3.14). This is so because

$$\begin{aligned}&\limsup_{u\to\infty} \frac{1}{R(u^{1-H/\gamma})}\,\mathbf{P}\Big\{\sup_{s\ge 0}\frac{X(s)-X(tu^{-1/\gamma})}{1+cs^{\gamma}} > \theta u^{1-H/\gamma}\Big\}\\ &\le \limsup_{u\to\infty} \frac{1}{R(u^{1-H/\gamma})}\,\mathbf{P}\Big\{\sup_{s\ge 0}\frac{X(s)}{1+cs^{\gamma}} > \tfrac{\theta}{2}u^{1-H/\gamma}\Big\} + \limsup_{u\to\infty}\frac{\mathbf{P}\big\{-X(tu^{-1/\gamma}) > \frac{\theta}{2}u^{1-H/\gamma}\big\}}{R(u^{1-H/\gamma})}\,,\end{aligned}$$

which is finite, by (3.1) and (3.12), together with self-similarity and $R\in\mathbf{OR}$.

Proof of (iii). We have

$$1-O\big((tu^{-1/\gamma})^{\gamma\wedge 1}\big) \le \inf_{s\ge tu^{-1/\gamma}}\frac{1+c(s-tu^{-1/\gamma})^{\gamma}}{1+cs^{\gamma}} \le \sup_{s\ge tu^{-1/\gamma}}\frac{1+c(s-tu^{-1/\gamma})^{\gamma}}{1+cs^{\gamma}} \le 1 \tag{3.18}$$

as $u\to\infty$. Here (3.4) shows that, with obvious notation,

$$O\big((tu^{-1/\gamma})^{\gamma\wedge 1}\big) \le o\Big(\big(v(u^{1-H/\gamma})/u^{1-H/\gamma}\big)^{(\gamma\wedge 1)/(H\wedge 1)}\Big) \le o\big(v(u^{1-H/\gamma})/u^{1-H/\gamma}\big).$$

This gives us the following version of (3.14), that for u large enough:

$$\mathbf{P}\{Y(t)>u\} \begin{cases} \ge \mathbf{P}\Big\{\sup\limits_{s\ge tu^{-1/\gamma}} \dfrac{X(s)-X(tu^{-1/\gamma})}{1+cs^{\gamma}} > u^{1-H/\gamma}\Big\} \\ \le \mathbf{P}\Big\{\sup\limits_{s\ge 0} \dfrac{X(s)-X(tu^{-1/\gamma})}{1+cs^{\gamma}} > u^{1-H/\gamma} - \varepsilon v(u^{1-H/\gamma})\Big\} \end{cases} \tag{3.19}$$

To bound the ratio in (3.7) from below, use self-similarity, (3.4) and (3.12), to get

$$\begin{aligned} \mathbf{P}\Big\{\sup_{s\in[0,tu^{-1/\gamma}]} X(s) > \varepsilon v(u^{1-H/\gamma})\Big\} &\le \mathbf{P}\Big\{\sup_{s\ge 0} \frac{X(s)}{1+cs^{\gamma}} > \frac{\varepsilon v(u^{1-H/\gamma})}{(1+c)(tu^{-1/\gamma})^{H}}\Big\} \\ &\le \mathbf{P}\Big\{\sup_{s\ge 0} \frac{X(s)}{1+cs^{\gamma}} > Ku^{1-H/\gamma}\Big\} \\ &\sim R\big(Ku^{1-H/\gamma}\big) \qquad \text{as} \quad u\to\infty, \end{aligned} \tag{3.20}$$

for any constant $K\ge 1$. Hence (3.19), together with (3.9) and (3.12), give that

$$\begin{aligned} \liminf_{u\to\infty} \frac{\mathbf{P}\{Y(t)>u\}}{R(u^{1-H/\gamma})} &\ge \liminf_{u\to\infty} \frac{1}{R(u^{1-H/\gamma})}\, \mathbf{P}\Big\{\sup_{s\ge 0} \frac{X(s)}{1+cs^{\gamma}} > u^{1-H/\gamma} - 2\varepsilon v(u^{1-H/\gamma})\Big\} \\ &\quad - \limsup_{u\to\infty} \frac{\mathbf{P}\big\{X(tu^{-1/\gamma}) > \varepsilon v(u^{1-H/\gamma})\big\}}{R(u^{1-H/\gamma})} \\ &\quad - \limsup_{u\to\infty} \frac{1}{R(u^{1-H/\gamma})}\, \mathbf{P}\Big\{\sup_{s\in[0,tu^{-1/\gamma})} X(s) > \varepsilon v(u^{1-H/\gamma})\Big\} \\ &\ge 1 - 2\limsup_{u\to\infty} R\big(Ku^{1-H/\gamma}\big)\,/\,R(u^{1-H/\gamma}) \to 1 \qquad \text{as} \quad K\to\infty, \end{aligned}$$

since $R\in\mathbf{PD}$. Of course, this establishes that

$$\liminf_{u\to\infty} \mathbf{P}\{Y(t)>u\}\,/\,R(u^{1-H/\gamma}) \ge 1.$$

On the other hand, since (3.9) and monotonicity of R give $R(u-\lambda v(u)) \sim R(u)$ for any $\lambda\in\mathbb{R}$, (3.19) together with (3.2), (3.9) and (3.12), show that

$$\begin{aligned} \limsup_{u\to\infty} \frac{\mathbf{P}\{Y(t)>u\}}{R(u^{1-H/\gamma})} &\le \limsup_{u\to\infty} \frac{1}{R(u^{1-H/\gamma})}\, \mathbf{P}\Big\{\sup_{s\ge 0} \frac{X(s)}{1+cs^{\gamma}} > u^{1-H/\gamma} - 2\varepsilon v(u^{1-H/\gamma})\Big\} \\ &\quad + \limsup_{u\to\infty} \frac{\mathbf{P}\big\{-X(1) > \varepsilon (tu^{-1/\gamma})^{-H} v(u^{1-H/\gamma})\big\}}{R(u^{1-H/\gamma})} = 1+0. \ \square \end{aligned}$$

To establish the Piterbarg property, we make use of the following assumptions:

$$\limsup_{u\to\infty} \frac{\mathbf{P}\{\inf_{s\in[0,1]} X(s) < -\varepsilon(t(u)u^{-1/\gamma})^{-H}u^{1-H/\gamma}\}}{R(u^{1-H/\gamma})} < \infty \quad \text{for all} \quad \varepsilon>0, \tag{3.21}$$

$$\lim_{u\to\infty} \frac{\mathbf{P}\{\inf_{s\in[0,1]} X(s) < -\varepsilon(t(u)u^{-1/\gamma})^{-H}v(u^{1-H/\gamma})\}}{R(u^{1-H/\gamma})} = 0 \quad \text{for some} \quad \varepsilon>0. \tag{3.22}$$

Assumption (3.21) will be used together with the growth condition (3.3), while in assumption (3.22), v is a suitably selected function that satisfies (3.4).

In the hypothesis of Theorem 2 below, (3.21) follows from (3.3) [by (3.11) and $R\in\mathbf{OR}$], while (3.22) follows from (3.4) [by (3.20) and monotonicity of R], when

$$\limsup_{u\to\infty} \frac{\mathbf{P}\{\inf_{s\in[0,1]} X(s) < -u\}}{\mathbf{P}\{\sup_{s\in[0,1]} X(s) > u\}} < \infty\,. \tag{3.23}$$

Clearly, (3.23) holds for X symmetric or non-negative. Otherwise, (3.21) and (3.22) could possibly be verified by Rosiński and Samorodnitsky (1993, Theorem 2.1), for $\inf_{s\in[0,1]} X(s)$ subexponential, or by Albin (1998, Theorem 3 and Sections 8-9).

Theorem 2. *Let X be H-*ss *and* id *with no Gaussian component, satisfying* Condition $\mathfrak{X}$. *Consider the process Y, given by* (1.1), *together with the function R, given by* (1.2), *where $c>0$ and $\gamma>H$ are constants.*

(i) *Let $R\in\mathcal{S}\cap\mathbf{OR}\cap\mathbf{PD}$. If* (3.3) *and* (3.21) *hold, we have, for $0\le\hat{t}(u)\le t(u)$,*

$$1 \le \liminf_{u\to\infty} \frac{\mathbf{P}\{\sup_{s\in[0,t(u)]} Y(s) > u\}}{\mathbf{P}\{Y(\hat{t}(u)) > u\}} \le \limsup_{u\to\infty} \frac{\mathbf{P}\{\sup_{s\in[0,t(u)]} Y(s) > u\}}{\mathbf{P}\{Y(\hat{t}(u)) > u\}} < \infty.$$

(ii) *Let $R\in\mathcal{S}\cap\mathbf{PD}$. Take $t(u)$ and v such that* (3.4), (3.9) *and* (3.22) *hold. The process Y has the strong Piterbarg property* (1.3c) *for $0\le\hat{t}(u)\le t(u)$.*

Proof of (i). It is enough to show the upper bound: Using (3.13) and (3.14), we get

(3.24)

$$\begin{aligned}\mathbf{P}\Big\{\sup_{s\in[0,t]} Y(s) > u\Big\} &= \mathbf{P}\Big\{\sup_{0\le r\le tu^{-1/\gamma}} \sup_{s\ge r} \frac{X(s)-X(r)}{1+c(s-r)^\gamma} > u^{1-H/\gamma}\Big\} \\ &\le \mathbf{P}\Big\{\sup_{s\ge 0} \frac{X(s)}{1+cs^\gamma} > \frac{\theta u^{1-H/\gamma}}{2}\Big\} + \mathbf{P}\Big\{\sup_{0\le r\le tu^{-1/\gamma}} -X(r) > \frac{\theta u^{1-H/\gamma}}{2}\Big\}\end{aligned}$$

for some $\theta \in (0,1)$. Therefore, self-similarity, (3.12), (3.21), and $R \in \mathbf{OR}$, give

$$\limsup_{u\to\infty} \frac{\mathbf{P}\{\sup_{s\in[0,t]} Y(s) > u\}}{R(u^{1-H/\gamma})} < \infty\,.$$

Now the upper bound desired follows from (3.8) [notice that (3.21) implies (3.1)].

Proof of (ii). Using (3.18) together with (3.4), as in last paragraph of the proof of part (iii) of Theorem 1, we may readily modify the estimate (3.24) to obtain

$$\begin{aligned}&\mathbf{P}\{\sup_{s\in[0,t]} Y(s) > u\}\\ &\le \mathbf{P}\left\{\sup_{s\ge 0} \frac{X(s)}{1+cs^\gamma} > u^{1-H/\gamma} - \varepsilon v(u^{1-H/\gamma})\right\} + \mathbf{P}\left\{\sup_{0\le r\le tu^{-1/\gamma}} -X(r) > \varepsilon v(u^{1-H/\gamma})\right\}.\end{aligned}$$

By application of (3.9) together with (3.12) and (3.22), this shows that

$$\limsup_{u\to\infty} \frac{\mathbf{P}\{\sup_{s\in[0,t]} Y(s) > u\}}{R(u^{1-H/\gamma})} \le 1\,.$$

On the other hand,

$$\begin{aligned}\mathbf{P}\left\{\inf_{s\in[0,t]} Y(s) > u\right\} &= \mathbf{P}\left\{\inf_{0\le r\le tu^{-1/\gamma}} \sup_{s\ge r} \frac{X(s)-X(r)}{1+c(s-r)^\gamma} > u^{1-H/\gamma}\right\}\\ &\ge \mathbf{P}\left\{\inf_{0\le r\le tu^{-1/\gamma}} \left(\sup_{s\ge r} \frac{X(s)}{1+c(s-r)^\gamma} - X(r)_+\right) > u^{1-H/\gamma}\right\}\\ &\ge \mathbf{P}\left\{\inf_{0\le r\le tu^{-1/\gamma}} \sup_{s\ge r} \frac{X(s)}{1+cs^\gamma} - \sup_{0\le r\le tu^{-1/\gamma}} X(r)_+ > u^{1-H/\gamma}\right\}\\ &= \mathbf{P}\left\{\sup_{s\ge tu^{-1/\gamma}} \frac{X(s)}{1+cs^\gamma} - \sup_{0\le r\le tu^{-1/\gamma}} X(r)_+ > u^{1-H/\gamma}\right\},\end{aligned}$$

and, as was established in the proof of Theorem 1, this gives us

$$\liminf_{u\to\infty} \frac{\mathbf{P}\{\inf_{s\in[0,t]} Y(s) > u\}}{R(u^{1-H/\gamma})} \ge 1\,.$$

Hence the strong Piterbarg property. □

Here are two easy corollaries to Theorems 1 and 2, that make use of (3.23):

Corollary 1. *Let X be H-*ss *and* id *with no Gaussian component, satisfying* Condition $\mathfrak{X}$. *Let the function R belong to $\mathcal{S}\cap\mathbf{PD}$, and assume that* (3.23) *holds.*

(i) *The Piterbarg property holds for $\gamma \le H + (H\wedge 1)$.*

(ii) *If* (3.9) *holds for* $v(u) = (1 \vee u)^{\beta}$, *for some* $\beta \in (0,1)$, *then the Piterbarg property holds for* $\gamma < H + (H \wedge 1)/(1-\beta)$. *Hence, if* (3.9) *holds for* $v(u) = (1 \vee u)^{\beta}$, *for each* $\beta \in (0,1)$, *then the Piterbarg property holds for any* $\gamma > H$.

Proof of (i). By part (ii) of Theorem 2, together with an inspection of (3.4), it is enough to exhibit a positive function v, with $\lim_{u\to\infty} v(u) = \infty$ and $\limsup_{u\to\infty} v(u)/u < \infty$, that satisfies (3.9). This is easy: Let $b_0 = 0$, $b_1 = 1$, and

$$b_{i+1} = \inf\{u \geq \max(b_i, 2i) : \inf_{x \geq u} R(x)/R(x-i) \geq 1 - 1/i\} \qquad \text{for} \quad i \geq 1 .$$

Since $R \in \mathcal{S} \subseteq \mathcal{L}$ (see Section 2.1), this is a finite increasing to infinity sequence of non-negative numbers, and we may now choose

$$v(u) = i \quad \text{if} \quad u \in [b_i, b_{i+1}), \quad \text{for} \quad i \geq 1, \qquad \text{and} \qquad v(u) = 1 \quad \text{for} \quad u < 1 .$$

Proof of (ii). Once again, the result follows from part (ii) of Theorem 2, by means of checking that (3.4) holds for $\gamma < H + (H \wedge 1)/(1-\beta)$, when $v(u) = (u \vee 1)^{\beta}$. □

In Part II of Example 5 below, we see that the Piterbarg property may be absent, when, in the notation of part (ii) of Corollary 1, $\gamma \geq H + 1/(1-\beta)$.

The next corollary is proved in the same way as Corollary 1.

Corollary 2. *Let* X *be* H-`ss` *and* `id` *with no Gaussian component, satisfying* Condition $\mathfrak{X}$. *Let the function* R *belong to* $\mathcal{S} \cap \mathbf{PD}$, *and assume that* (3.23) *holds.*

(i) *The strong Piterbarg property holds for* $t(u) \geq 0$ *such that*

$$\limsup_{u\to\infty} \frac{t(u)}{u^{1/\gamma - (1-H/\gamma)/(H\wedge 1)}} < \infty .$$

(ii) *If* (3.9) *holds for* $v(u) = (1 \vee u)^{\beta}$, *for some* $\beta \in (0,1)$, *then the strong Piterbarg property holds for* $t(u) \geq 0$ *such that*

$$\lim_{u\to\infty} \frac{t(u)}{u^{1/\gamma - (1-\beta)(1-H/\gamma)/(H\wedge 1)}} = 0 . \tag{3.25}$$

(iii) *If* (3.9) *holds for every* $v(u) = o(u)$, *then the strong Piterbarg property holds for each* $t(u) = o(u^{1/\gamma})$.

Notice that the intervals with length $t(u)$ as in (3.25), for which the strong Piterbarg property holds, do in fact shrink with u, unless $\gamma < H + (H \wedge 1)/(1-\beta)$.

4. Discussion and Examples. Here we present points of view on the results of Section 3. Examples are given, where the input process X is represented in the form of a stochastic integral, with respect to an id random measure.

We first exemplify that the storage process does not have to be finite valued, in general. (Obviously, this does not happen under the assumptions of our results.)

Example 2. Let $\{B(t)\}_{t\geq 0}$ be standard Brownian motion, and consider

$$X(t) = t^H \exp\{A^{1/2}\, e^{t^{-1}B(t)^2}\} \quad \text{for } t>0 \qquad \text{and} \qquad X(0)=0,$$

for a constant $A\in(0,H)$. By Lamperti transformation, X is H-ss. By the Law of Iterated Logarithm, X is continuous at zero, and thus continuous, with

$$\limsup_{t\to\infty} X(t)/t^\gamma = \infty \quad \text{a.s.} \qquad \text{for} \qquad \gamma\in(H, H+A).$$

Therefore, the storage process Y in (1.1) is not finite valued, for such values of γ.

4.1 id H-ss processes. In this subsection, X denotes the id H-ss process given in Example 1, which is assumed to satisfy Condition $\mathfrak{X}$. Notice that, by (2.6), X is non-negative, if f is non-negative (non-positive) and r is zero on $(-\infty,0]$ $([0,\infty))$.

Denoting, for $s>0$,

$$\sigma_+(s)^{-1} = s^H \sup_{t>0} \frac{f(s/t)^+}{1+ct^\gamma} \qquad \text{and} \qquad \sigma_-(s)^{-1} = s^H \sup_{t>0} \frac{f(s/t)^-}{1+ct^\gamma},$$

we have, by (2.8), for u large enough,

(4.1)
$$R(u) = H\int_0^\infty \mu\big(\mathbb{R}\setminus[-\sigma_-(s)u, \sigma_+(s)u]\big)\frac{ds}{s} = H\int_0^\infty \big(r(\sigma_+(s)u) + r(-\sigma_-(s)u)\big)\frac{ds}{s}.$$

Example 3. Let r be symmetric, with $r\in\mathbf{RV}(-\varrho)$ for some $\varrho>0$. By (2.2), it follows that X is symmetric. Assume that for some $\epsilon>0$

$$\int_0^\infty \big(\sigma_+(s)^{-(\varrho+\epsilon)} + \sigma_-(s)^{-(\varrho+\epsilon)}\big)\frac{ds}{s} < \infty.$$

Then by (4.1), we have

(4.2)
$$R(u) \sim H\int_0^\infty \big(\sigma_+(s)^{-\varrho} + \sigma_-(s)^{-\varrho}\big)\frac{ds}{s}\, r(u) \qquad \text{as} \quad u\to\infty.$$

Thus part (ii) of Corollary 1 shows that the Piterbarg property holds for $\gamma > H$, and part (iii) of Corollary 2 gives the strong Piterbarg property for $t(u) = o(u^{1/\gamma})$.

In Example 3, symmetry gives us (3.23), for free. Without symmetry, we may still establish the Piterbarg properties, by direct verification of (3.23) [or (3.22)].

Example 4. Take f non-negative, and not identically zero. Assume that

$$r \in \mathbf{RV}(-\varrho) \qquad \text{and} \qquad r(-\cdot) \in \mathbf{RV}(-\varrho), \qquad \text{with} \qquad \limsup_{u\to\infty} r(-u)/r(u) < \infty,$$

for some constant $\varrho > 0$. Under the same condition as in Example 3, we see that (4.2) holds with $\sigma_-(s) = 0$. Further, by (4.2) together with Rosiński and Samorodnitsky (1993, Theorem 2.1), the limit in (3.23) is

$$\left(\int_0^\infty s^{H\varrho} \sup_{r\in[s,\infty)} f(r)^\varrho \frac{ds}{s} \Big/ \int_0^\infty s^{H\varrho} \sup_{r\in[s,\infty)} f(r)^\varrho \frac{ds}{s} \right) \limsup_{u\to\infty} \frac{r(-u)}{r(u)} < \infty .$$

Hence the Piterbarg properties hold in the same way as in Example 3.

Example 5 - Part I. Pick constants $A > 0$ and $\alpha \in (0,1)$, and consider

$$(4.3) \qquad r(x) = g(x)\, \mathrm{e}^{-Ax^\alpha} \quad \text{for } x \geq 0 \qquad \text{and} \qquad r(x) = 0 \quad \text{for } x < 0,$$

where $g \in \mathbf{RV}(\varrho)$ with $\varrho \in \mathbb{R}$. Take f non-negative, so that X is non-negative.

Let $\sigma_+(s)$ take its minimal value at a unique $\hat{s} > 0$, where σ_+ is two times continuously differentiable at $\hat{s}$, with $\sigma_+''(\hat{s}) > 0$. By Taylor expansion in (4.1), we have

$$\begin{aligned} (4.4) \qquad R(u) &= Hu^{-\alpha/2} \int_{-\hat{s}u^{\alpha/2}}^\infty \frac{g\big(\sigma_+(\hat{s}+s/u^{\alpha/2})u\big)}{\hat{s}+s/u^{\alpha/2}}\, e^{-A\left(\sigma_+(\hat{s}+s/u^{\alpha/2})\right)^\alpha u^\alpha}\, ds \\ &\sim \frac{H\sigma_+(\hat{s})^\varrho}{\hat{s}}\, g(u)\, u^{-\alpha/2} \int_{\mathbb{R}} \exp\Big\{ -A\left(\sigma_+(\hat{s}) + \tfrac{1}{2}\sigma_+''(\hat{s})\, s^2/u^\alpha\right)^\alpha u^\alpha \Big\}\, ds \\ &\sim \frac{\sqrt{2\pi}\, H\sigma_+(\hat{s})^\varrho}{\hat{s}\sqrt{A\alpha\sigma_+''(\hat{s})/\sigma_+(\hat{s})^{1-\alpha}}}\, g(u) u^{-\alpha/2}\, \mathrm{e}^{-A\sigma_+(\hat{s})^\alpha u^\alpha} \qquad \text{as} \quad u \to \infty. \end{aligned}$$

Hence part (ii) of Corollaries 1 and 2 applies, for $\beta < 1-\alpha$, to give the Piterbarg property for $\gamma < H + (H \wedge 1)/\alpha$, and the strong Piterbarg property for $t \geq 0$ with

$$\lim_{u\to\infty} \frac{t(u)}{u^{1/\gamma - \mathfrak{a}(1-H/\gamma)/(H\wedge 1)}} = 0 \qquad \text{for some} \quad \mathfrak{a} > \alpha .$$

Example 5 - Part II. Here we continue the study of the case when r is given by (4.3), in the particular case when $f = \mathbf{1}_{(0,1]}$ (so that X has independent increments). We show that, in this case, the Piterbarg property is absent, if $\gamma \geq H + 1/\alpha$.

By Theorem 1 and (3.24), the Piterbarg propety is absent when

$$\limsup_{u\to\infty} \frac{1}{R(u)} \mathbf{P}\left\{ \sup_{0\leq r\leq t/u^{1/(\gamma-H)}} \sup_{s\geq r} \frac{X(s)-X(r)}{1+c(s-r)^\gamma} > u \right\} > 1 . \tag{4.5}$$

Rosiński and Samorodnitsky (1993, Theorem 2.1) does not apply here, since suprema are taken over regions that depend on u. However, the arguments for that theorem produce an asymptotic lower bound, for the probability in (4.5), which implies the following sufficient condition for (4.5):

$$\limsup_{u\to\infty} \frac{1}{R(u)} \nu\left\{ y\in\mathbb{R}^{[0,\infty)} : \sup_{0\leq r\leq t/u^{1/(\gamma-H)}} \sup_{s\geq r} \frac{y(s)-y(r)}{1+c(s-r)^\gamma} > u \right\} > 1 . \tag{4.6}$$

Denoting the numerator in (4.6) by $R_t(u)$, we have, by the inequalities $\gamma \geq H + 1/\alpha$ and $(1-x)^H \leq 1-(H\wedge 1)x$ for $x\in[0,1]$, together with (4.4), [cf. (4.1)]

$$\begin{aligned}
R_t(u) &= H\int_0^\infty \mu\left(\left\{ x>0 : y^H x \sup_{0\leq r\leq t/u^{1/(\gamma-H)}} \sup_{s\geq r} \frac{\mathbf{1}_{(0,s]}(y)-\mathbf{1}_{(0,r]}(y)}{1+c(s-t)^\gamma} > u \right\} \right) \frac{dy}{y} \\
&= H\int_0^{t/u^{1/(\gamma-H)}} r\big(y^{-H}u\big)\,\frac{dy}{y} + H\int_{t/u^{1/(\gamma-H)}}^\infty r\left(\frac{1+c(y-t/u^{1/(\gamma-H)})^\gamma}{y^H}\, u \right) \frac{dy}{y} \\
&\geq H\int_{t/u^{1/(\gamma-H)}}^\infty r\left(\sigma\big(y-t/u^{1/(\gamma-H)}\big) \left(\frac{y-t/u^{1/(\gamma-H)}}{y}\right)^H u \right) \frac{dy}{y} \\
&\geq H\int_{t/u^{1/(\gamma-H)}}^\infty r\left(\sigma\big(y-t/u^{1/(\gamma-H)}\big) \left(1-(H\wedge 1)\frac{t/u^\alpha}{y}\right) u \right) \frac{dy}{y} \\
&\sim \frac{H\sigma(\hat{s})^\varrho}{\hat{s}}\, g(u)\, u^{-\alpha/2} \int_{\mathbb{R}} \exp\left\{ -A\big(\sigma(\hat{s}) + \tfrac{1}{2}\sigma''(\hat{s})\, s^2/u^\alpha\big)^\alpha \left(u^\alpha - (H\wedge 1)\frac{t}{\hat{s}} \right) \right\} ds \\
&\sim R(u)\, \exp\big\{ A\sigma(\hat{s})^\alpha (H\wedge 1)\, t/\hat{s} \big\} \qquad \text{as} \quad u\to\infty.
\end{aligned}$$

This gives (4.6). [If nervous about this calculation, shrink the domain of integration $[t/u^{1/(\gamma-H)}, \infty)$ to $\hat{s}\pm K/u^{\alpha/2}$, and get the claim by sending $K\uparrow\infty$ at the end.]

4.2 α-stable processes. First we consider the Piterbarg properties (1.3), in the case when the process Y itself is α-stable.

Example 6. For $Y=\{Y(s)\}_{s\geq 0}$ an α-stable process, $\alpha\in(0,2]$, we may write $Y=Y_1-Y_2+\mu$, where $\mu:[0,\infty)\to\mathbb{R}$ is a suitable function, while $\{Y_1(s)\}_{s\geq 0}$ and $\{Y_2(s)\}_{s\geq 0}$ are independent α-stable processes, such that (for $j=1,2$ and $s\geq 0$)

$$\mathbf{E}\{e^{i\theta Y_j(s)}\}=\begin{cases}\exp\{-\sigma^{\alpha}_{Y_j(s)}|\theta|^{\alpha}(1-i\tan(\frac{\pi\alpha}{2})\operatorname{sign}(\theta))\} & \text{if } \alpha\neq 1\\ \exp\{-\sigma_{Y_j(s)}|\theta|(1+i\frac{2}{\pi}\operatorname{sign}(\theta)\ln(|\theta|))\} & \text{if } \alpha=1\end{cases} \quad \text{for } \theta\in\mathbb{R}.$$

Here $\sigma_{Y_j(s)}$ is the scale parameter of the α-stable random variable $Y_j(s)$. In the Gaussian case $\alpha=2$, we may take $Y_1=0$. We assume that Y satisfies Condition $\mathfrak{X}$, from which it follows that Y_1, Y_2 and μ can be taken to satisfy Condition $\mathfrak{X}$.

We are going to investigate when the Piterbarg property (1.3a) holds.

Case 1. If $Y_1\neq_d 0$, then (1.3a) holds if and only if the `fdds` of Y_1 coincide with those of a single α-stable random variable [Samorodnitsky and Taqqu (1994, Theorem 10.5.1)].

Case 2. If $Y_1=_d 0$ and $\alpha\geq 1$, then (1.3a) holds if and only if the `fdds` of Y_2 coincide with those of a single α-stable random variable, and μ is constant.

To see this, notice that σ_{Y_2} and μ must be constants on $[0,t]$ [Samorodnitsky and Taqqu (1994, Equation 1.2.11)]. Given these properties, we have

$$\begin{aligned}(4.7)\qquad \mathbf{P}\{\sup_{s\in[0,t]}Y(s)>u\} &\geq \mathbf{P}\{\{Y(r)>u\}\cup\{Y(s)>u\}\}\\ &\geq 2\,\mathbf{P}\{Y(r)>u\}-\mathbf{P}\{\tfrac{1}{2}(Y(r)+Y(s))>u\}\end{aligned}$$

for $r,s\in[0,t]$. If $\alpha>1$, then the second probability on the right is $o\big(\mathbf{P}\{Y(r)>u\}\big)$, unless $\sigma_{Y(r)+Y(s)}=\sigma_{Y(r)}+\sigma_{Y(s)}$. By Minkowski's inequality, this happens if and only if $Y(r)=Y(s)$ a.s. If $\alpha=1$, then the spectral measure Γ of $(Y(r),Y(s))$ [Samorodnitsky and Taqqu (1994, Section 2.3)] is supported on $S_2^-=\{(s_1,s_2)\in\mathbb{R}^2: s_1^2+s_2^2=1,\ s_1,s_2\leq 0\}$, and [Samorodnitsky and Taqqu (1994, Example 2.3.4)]

$$\frac{Y(r)+Y(s)}{2}\stackrel{\mathrm{d}}{=}Y(r)-\frac{2}{\pi}\left(\int_{S_2^-}\frac{s_1+s_2}{2}\ln\left|\frac{s_1+s_2}{2}\right|d\Gamma(s)-\int_{S_2^-}s_1\ln|s_1|\,d\Gamma(s)\right).$$

By a convexity argument, the term $\frac{2}{\pi}(\dots)$ on the right is strictly positive, so that the second probability on the right in (4.7) is $o\big(\mathbf{P}\{Y(r)>u\}\big)$ [Samorodnitsky and Taqqu (1994, Equation 1.2.12)], unless $Y(r)=Y(s)$ a.s.

See Talagrand (1988) and Albin (1999) on more information related to Case 2.

Case 3. If $Y_1 =_d 0$ and $\alpha<1$, then (1.3a) holds (in the sense of $0/0=1$) since Y_2 is non-negative.

Special case. If Y is $S\alpha S$, then (1.3a) holds if and only if the `fdds` of Y coincide with those of a single α-stable random variable.

Turning to (1.3b), with $t = t(u) \to \infty$ as $u \to \infty$, the above characterizations remain valid [with (1.3b) replacing (1.3a)], if appropriate global boundedness properties are imposed on Y_2 and μ in Case 1, and on μ in Case 3.

Next we consider a storage process Y, with an α-stable H-`ss` input process X.

Example 7. Let X be a strictly α-stable H-`ss` process, that satisfies Condition $\mathfrak{X}$, and is given by (2.7), where M is a strictly α-stable random measure ($S\alpha S$ if $\alpha=1$). By calculations similar to those in Examples 3 and 4, we have

$$R(u) = 1 \wedge u^{-\alpha} \int_S \left(\tfrac{1-\beta}{2} \sup_{t\geq 0} \frac{(f_t(s)^-)^\alpha}{(1+ct^\gamma)^\alpha} + \tfrac{1+\beta}{2} \sup_{t\geq 0} \frac{(f_t(s)^+)^\alpha}{(1+ct^\gamma)^\alpha} \right) w_0(s)\, d\lambda(s)\,.$$

We assume that the above integral is non-zero, so that R is not identically zero.

Provided that X satisfies (3.23), Corollary 1 now gives the Piterbarg property for any $\gamma > H$, for the storage process Y, while Corollary 2 gives the strong Piterbarg property for $t(u) = o(u^{1/\gamma})$. However, by Samorodnitsky (1988) [see also Samorodnitsky and Taqqu (1994, Theorem 10.5.1)], the limit in (3.23) is

$$\frac{\int_S \left(\frac{1+\beta}{2} \sup_{0\leq t\leq 1} (f_t^-)^\alpha + \frac{1-\beta}{2} \sup_{0\leq t\leq 1} (f_t^+)^\alpha \right) w_0\, d\lambda}{\int_S \left(\frac{1-\beta}{2} \sup_{0\leq t\leq 1} (f_t^-)^\alpha + \frac{1+\beta}{2} \sup_{0\leq t\leq 1} (f_t^+)^\alpha \right) w_0\, d\lambda}.$$

This ratio is finite, by the self-similarity of X the assumption that R is non-zero.

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