EXTREME VALUE THEORY, ERGODIC THEORY, AND THE BOUNDARY BETWEEN SHORT MEMORY AND LONG MEMORY FOR STATIONARY STABLE PROCESSES

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ABSTRACT. We study the partial maxima of stationary α-stable processes. We relate their asymptotic behavior to the ergodic theoretical properties of the flow. We observe a sharp change in the asymptotic behavior of the sequence of partial maxima as flow changes from being dissipative to being conservative, and argue that this may indicate a change from a short memory process to a long memory process.

1. INTRODUCTION

Let $\mathbf{X} = (X_0, X_1, \dots)$ be a stationary symmetric α-stable (SαS) process, $0 < \alpha < 2$. How does one decide whether or not the process has long range dependence?

Recall that the assumption of α-stability means that for every $n \geq 1$, $\mathbf{X}^{(1)} + \ldots + \mathbf{X}^{(n)} \stackrel{\mathrm{d}}{=} n^{1/\alpha}\mathbf{X}$, where $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \dots$ are iid copies of $\mathbf{X}$. Therefore, one may think of stationary SαS processes as heavy tailed analogs of stationary zero mean Gaussian processes which result if one sets $\alpha = 2$ in the definition of α-stability. See Samorodnitsky and Taqqu (1994) more information on α-stable processes.

A very attractive feature of stationary Gaussian processes is that (apart from the variance, a global scale constant) the entire distribution of a process is determined by the correlation function $\rho_n = \mathrm{Corr}(X_0, X_n)$, $n = 0, 1, 2, \dots$. In particular, the correlation function tells everything about length of the memory in the process. A common definition of long range dependence says that

$$\text{the process has} \begin{cases} \text{short memory} & \text{if } \sum_{n=0}^{\infty} |\rho_n| < \infty \\ \text{long memory} & \text{if } \sum_{n=0}^{\infty} |\rho_n| = \infty \end{cases} . \tag{1.1}$$

See, for example, Cox (1984).

Let $S_n = X_0 + \ldots + X_{n-1}$, $n = 1, 2, \dots$ be the partial sum process based on a stationary process $\mathbf{X}$. For Gaussian processes summability of correlations means that the variance of S_n grows at most linearly fast (and, apart from arguably degenerate cases, the variance grows at

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exactly linear rate). On the other hand, if the correlations are not summable, then the variance of S_n typically grows faster than linearly fast, and the precise rate of growth depends on further properties of the correlation function. Therefore, the distributional rate of growth of S_n changes from typically $n^{1/2}$ to something larger as correlations stop being summable.

Since α-stable random variables with $0 < \alpha < 2$ have infinite second moment, one cannot use correlations to tell when a stationary α-stable process has long range dependence. Covariance-like functions have been tried (see e.g. Astrauskas et al. (1991)), but their usefulness seems to be limited. In fact, even for stationary processes with a finite second moment the definition (1.1) of long range dependence based on the true correlation function is of uncertain value unless the process is a Gaussian process, or very close to being one.

A different approach appears to be called for. Suppose that $(\mathcal{P}_\theta, \theta \in \Theta)$ is a family of laws of a stationary stochastic process $(X_0, X_1, X_2, \dots)$, where Θ is some parameter space. Assume that the marginal laws of the process do not change much as θ varies (perhaps, the marginal laws remain constant, or only the global scale changes, if we are considering, say, Gaussian or SαS processes). Suppose we are given a functional of interest R, a (measurable) functional on $\mathbb{R}^\infty$. The behavior of this functional is different, in general, under different laws $\mathcal{P}_\theta$. Suppose that there is a partition of the parameter space Θ into two parts, Θ_0 and Θ_1, such that the behavior of the functional changes dramatically as one crosses the boundary between Θ_0 and Θ_1. Then it may make sense to talk about that boundary as a boundary between short range dependence and long range dependence. That is, the change from short memory to long memory occurs as a phase transition.

Existence of such boundaries has been observed before. For stationary zero mean Gaussian processes parameterized by the common variance, and by the correlation function, such a phase transition occurs when the correlations stop being summable. The functional of interest here is the sequence of the partial sums, and its distributional rate of growth changes significantly at the boundary. The rate of growth of the partial sums may change its order of magnitude whether or not the second moment is finite. This has been observed many times on the example of the increments of self-similar processes with stationary increments. In the Gaussian case the family of such processes are fractional Brownian motions, parameter H of self-similarity has to be in the interval $(0,1)$, and the partial sums of the increment process (the so-called fractional Gaussian noise) increase at the rate higher than $n^{1/2}$ when $H > 1/2$. Hence $H = 1/2$ is considered to be the boundary between short and long memory for fractional Gaussian noise. See e.g. Mandelbrot (1975) and Mandelbrot and Taqqu (1979), as well as a more recent discussion in Beran (1994). A

similar phenomenon occurs for the increments of self-similar α-stable processes with stationary increments, $1 < \alpha < 2$, that have infinite variance. Here the the range of parameter H of self-similarity is still $(0,1)$, and the boundary where the partial sums of the increment process start increasing at the rate higher than the iid case (i.e. faster than $n^{1/\alpha}$) is that of $H = 1/\alpha$. See e.g. Samorodnitsky and Taqqu (1994). No such boundary is possible if we consider the increments of self-similar α-stable processes with stationary increments and $0 < \alpha \leq 1$. Does that mean that long range dependence is impossible if $0 < \alpha \leq 1$?

The present paper uncovers a different boundary between short and long memory for stationary α-stable processes. We believe that this is a very fundamental boundary, and it is based on ergodic-theoretical properties of nonsingular flows underlying such processes. Specifically, we concentrate on the partial maxima sequence

$$M_n = \max\Big(|X_0|, |X_1|, \dots, |X_{n-1}|\Big), \; n = 1, 2, \dots \tag{1.2}$$

and its distributional rate of growth. We will see that the parameter space consists of two parts, in one of which the partial maxima grow at the rate $n^{1/\alpha}$, which is the rate at which partial maxima of iid α-stable random variables grow, while in the other part of the parameter space the partial maxima grow at a strictly slower rate. Moreover, in the latter part of the parameter space the actual rate of growth may depend on the choice of the parameters. This boundary is present for all $0 < \alpha < 2$.

This paper is organized as follows. In Section 2 we provide a background on intergral representations of stationary SαS processes, and elements of the theory developed by Jan Rosiński relating such intergral representations and ergodic theory. In Section 3 we discuss the behavior of a certain deterministic sequence controlling the rate of growth of the partial maxima. In Section 4 we prove the main result, Theorem 4.1, dealing with the asymptotic behavior of the sequence partial maxima. Section 5 discusses a number of examples illustrating the results of Section 4, A brief conclusion discussing what the results of this paper tell us about short and long memory for stationary stable processes is in Section 6.

2. Ergodic theory and representations of stationary α-stable processes

Throughout this section $\mathbf{X} = (X_0, X_1, \dots)$ is a stationary symmetric α-stable process, $0 < \alpha < 2$. Every (not necessarily stationary) SαS process has an integral representation

$$X_n = \int_E f_n(x)\, M(dx)\,, \quad n = 0, 1, 2, \dots\,, \tag{2.1}$$

where M is a symmetric α-stable random measure on a measurable space $(E, \mathcal{E})$ with a σ-finite control measure m, while $f_n \in L^\alpha(m, \mathcal{E})$ for all n. See Chapter 3 in Samorodnitsky and Taqqu (1994) on α-stable random measures and integrals with respect to these measures, and Chapter 13 there on integral representations as above as well as on the history of such representations.

When a process $\mathbf{X}$ is stationary, the integral representation can be selected to be of a particular form, according to a theory developed in a series of papers by Jan Rosiński; see e.g. Rosiński (1995), and various facts presented below can be found in that paper. Specifically, a stationary SαS process has an integral representation of the form (2.1) with

$$f_n(x) = a_n(x) \left(\frac{dm \circ \phi^n}{dm}(x) \right)^{1/\alpha} f \circ \phi^n(x) \;\; x \in E \,, \tag{2.2}$$

for $n = 0, 1, 2, \ldots$, where $\phi : E \to E$ is a measurable non-singular map (i.e. a one-to-one map with both ϕ and ϕ^{-1} measurable, mapping the control measure m into an equivalent measure),

$$a_n(x) = \prod_{j=0}^{n-1} u \circ \phi^j(x), \,, \;\; x \in E \,,$$

for $n = 0, 1, 2, \ldots$, with $u : E \to \{-1, 1\}$ a measurable function and $f \in L^\alpha(m, \mathcal{E})$.

That is, the process $\mathbf{X}$ is determined by a single function $f \in L^\alpha(m, \mathcal{E})$, a *cocycle* $(a_n \,, n = 0, 1, 2, \ldots)$ and a *flow* $(\phi^n \,, n = 0, 1, 2, \ldots)$. This triple (taken together with the space $(E, \mathcal{E}, m)$ on which it lives) can, therefore, be taken as a parameterization of stationary SαS processes. When working with this parameterization, the task is to relate the ergodic-theoretic properties of the flow to the probabilistic properties of the stable process. For us, the property of interest is the length of memory of a stationary stable process.

Let $E = C \cup D$ be the Hopf decomposition of E with respect to the flow. That is, C and D are measurable ϕ-invariant sets, such that the flow is conservative on C and dissipative on D; see Krengel (1985) for the various ergodic theoretical notions and facts we use in this paper. Writing

$$\begin{aligned} X_n &= \int_C a_n(x) \left(\frac{dm \circ \phi^n}{dm}(x) \right)^{1/\alpha} f \circ \phi^n(x) \; M(dx) + \int_D a_n(x) \left(\frac{dm \circ \phi^n}{dm}(x) \right)^{1/\alpha} f \circ \phi^n(x) \; M(dx) \\ &:= X_n^C + X_n^D \,, \;\; n = 0, 1, 2, \ldots \,, \end{aligned} \tag{2.3}$$

leads to a unique in law decomposition of a stationary symmetric α-stable process into a sum of two independent such processes, one of which is generated by a conservative flow, and the other by a dissipative flow.

Intuitively, one expects stable processes generated by conservative flows to have a longer memory than those generated by dissipative flows, simply because a conservative flow "tends to keep coming back", and so the same values of the random measure M contribute to observations X_n

far separated in time. Consider, for example, stationary processes generated a dissipative flow. Such a process has a *mixed moving average* representation of the form

$$X_n = \int_W \int_{\mathbb{R}} f(v, x-n)\, M(dv, dx)\,, \quad n = 0, 1, 2, \dots\,, \tag{2.4}$$

with M a symmetric α-stable random measure on a product measurable space $(W \times \mathbb{R}, \mathcal{W} \times \mathcal{B})$ with control measure $m = \nu \times \mathrm{Leb}$, where ν is a σ-finite measure on $(W, \mathcal{W})$, and $f \in L^\alpha(m, \mathcal{W} \times \mathcal{B})$. Such processes are always mixing irrespectively of what the kernel f in either (2.2) or (2.4) is (or what the cocycle in (2.2) is). See Surgailis et al. (1993). On the other hand, stable processes generated by conservative flows are often not even ergodic. For example, if a conservative flow is measure preserving and the expected return time to any set of a finite positive measure is finite, then the stable process is not ergodic (see Rosiński and Samorodnitsky (1996)). Another example demonstrating that conservative flows tend to lead to a longer memory (in the case $1 < \alpha < 2$) can be found in Mikosch and Samorodnitsky (2000) who studied ruin probabilities.

3. The sequence (b_n)

Let $\mathbf{X} = (X_0, X_1, \dots)$ be a stationary symmetric α-stable process, $0 < \alpha < 2$. We assume that the process is given in an integral representation of the form (2.1), where (f_n) is of the form (2.2). It turns out that, to a large extent, the asymptotic behavior of the maximal functional M_n in (1.2) is related to the quantity defined below. Let

$$b_n = \left(\int_E \max_{j=0,\dots,n-1} |f_j(x)|^\alpha\, m(dx) \right)^{1/\alpha}, n = 1, 2, \dots\,. \tag{3.1}$$

In fact, to a certain extent b_n controls "the size" of M_n even without the assumption of stationarity of the process. Indeed, for any $0 < p < \alpha$ there are constants $c_{\alpha,p}$, $C_{\alpha,p} \in (0, \infty)$ such that for $1 < \alpha < 2$

$$c_{\alpha,p} \le \frac{1}{b_n} (E M_n^p)^{1/p} \le C_{\alpha,p} (\log n)^{1/\alpha'}\,, \tag{3.2}$$

where α' is the conjugate of α in $1/\alpha + 1/\alpha' = 1$, while for $\alpha = 1$

$$c_{1,p} \le \frac{1}{b_n} (E M_n^p)^{1/p} \le C_{1,p}\, L_2 n\,, \tag{3.3}$$

where $L_2 n = \max(1, \log\log n)$. Finally, for $0 < \alpha < 1$,

$$c_{\alpha,p} \le \frac{1}{b_n} (E M_n^p)^{1/p} \le C_{\alpha,p}\,. \tag{3.4}$$

See Theorem 2.1 in Marcus (1984).

We will see that for stationary SαS processes the sequence (b_n) tells us even more about the sequence (M_n) of the partial maxima. Note, however, that the sequence (b_n) is not completely

determined by the process, and may depend on a particular integral representation. Nevertheless, certain important features of this sequence are determined by the flow in (2.2) underlying the process. In particular, the next result shows that the sequence b_n grows at slower rate for processes generated by a conservative flow than for processes generated by a dissipative flow.

Theorem 3.1. *Let (f_n) be given by (2.2).*

(i) If the flow (ϕ^n) is conservative, then

$$n^{-1/\alpha} b_n \to 0 \quad \text{as } n \to \infty \tag{3.5}$$

(ii) If the flow is dissipative, and the process is given in the mixed moving average form (2.4), then

$$\lim_{n\to\infty} n^{-1/\alpha} b_n = \left(\int_W \int_0^1 g(v,x)^\alpha \, \nu(dv)\, dx \right)^{1/\alpha} \in (0,\infty)\,, \tag{3.6}$$

where

$$g(v,x) = \sup_{k=0,\pm1,\pm2,\ldots} |f(v,x+k)| \quad \text{for } v \in W \text{ and } x \in (0,1)\,. \tag{3.7}$$

Proof. (i) Suppose first that the flow (ϕ^n) is control measure m preserving, and that $f = \mathbf{1}_A$ for some $A \in \mathcal{E}$ with $0 < m(A) < \infty$. In that case the Radon-Nykodim derivative in (2.2) disappears, and we get

$$b_n^\alpha = m\left(\bigcup_{k=0}^{n-1} \phi^{-k}(A) \right), \; n = 1, 2, \ldots\,. \tag{3.8}$$

We use a construction used to prove the Kac recurrence theorem; see e.g. Theorem 3.6 in Krengel (1985). Let

$$\tau_A(x) = \inf\{k \geq 1 : \phi^k(x) \in A\} \quad \text{for } x \in A$$

be the first recurrence time to A, let for $k = 1, 2, \ldots$

$$R_k = \{x \in A : \tau_A(x) = k\}$$

and

$$A_k = \{x \in A^c : \phi^k(x) \in A,\, \phi^j(x) \notin A,\, j = 1, \ldots, k-1\}$$

be, correspondingly, the set of points in A returning to A after k steps, and the set of points outside of A entering A for the first time after k steps. Let also $A_0 = A$. Note that

$$m\left(\bigcup_{k=0}^{n-1} \phi^{-k}(A) \right) = \sum_{k=0}^{n-1} m(A_k)\,. \tag{3.9}$$

Furthermore, by the measure-preserving property of the flow,

$$m(A_k) = \sum_{j=k+1}^{\infty} m(R_j), \ k = 0, 1, 2, \dots . \tag{3.10}$$

Using (3.10) with $k = 0$ shows that $m(A_k)$ is represented as the tail of a convergent series and, hence, $m(A_k) \to 0$ as $k \to \infty$. Therefore, by (3.9) we conclude that

$$\frac{1}{n} m \left(\bigcup_{k=0}^{n-1} \phi^{-k}(A) \right) \to 0 \ \text{ as } n \to \infty,$$

which establishes (3.5) for indicator functions in the measure preserving case.

To establish (3.5) in the general case (i.e. not necessarily measure preserving conservative flow and a general kernel f), denote

$$w_n(x) = \frac{dm \circ \phi^n}{dm}(x) \, , \ n = 0, 1, 2, \dots \, , \ x \in E \, .$$

Then

$$\phi_*(x, y) = \left(\phi(x), \frac{y}{w_1(x)} \right) \, , \ x \in E \, , \ y > 0$$

defines a conservative flow on $\big(E \times (0, \infty), \mathcal{E} \times \mathcal{B}, m \times \text{Leb}\big)$ that preserves the measure $m \times \text{Leb}$; see Maharam (1964).

Let

$$A = \{(x, y) \in E \times (0, \infty) : 0 < y \le |f(x)|^\alpha\} \, .$$

Note that

$$m \times \text{Leb}(A) = \int_E |f(x)|^\alpha \, m(dx) \in (0, \infty) \, .$$

Furthermore,

$$\begin{aligned} m \times \text{Leb} \left(\bigcup_{k=0}^{n-1} \phi_*^{-k}(A) \right) &= \int_E \int_0^\infty \mathbf{1}_{\cup_{k=0}^{n-1} \phi_*^{-k}(A)}(x, y) \, m(dx) \, dy \\ &= \int_E \int_0^\infty \max_{k=0,\dots,n-1} \mathbf{1}_{\phi_*^{-k}(A)}(x, y) \, m(dx) \, dy \qquad (3.11) \\ &= \int_E \int_0^\infty \max_{k=0,\dots,n-1} \mathbf{1} \left(0 < y \le w_k(x) |f \circ \phi^k(x)|^\alpha \right) \, m(dx) \, dy \\ &= \int_E \max_{k=0,\dots,n-1} w_k(x) |f \circ \phi^k(x)|^\alpha \, m(dx) = \int_E \max_{k=0,\dots,n-1} |f_k(x)|^\alpha \, m(dx) = b_n^\alpha \, . \end{aligned}$$

Since the left hand side of (3.11) is $o(n)$ by the already considered case of measure preserving flows and indicator functions, this establishes (3.5) in full generality.

(ii) We start with the case where f has a compact support, that is

$$f(v, x) \mathbf{1}_{W \times [-m, m]^c}(v, x) \equiv 0 \ \text{ for some } m = 1, 2, \dots . \tag{3.12}$$

In that case for $n \geq 2m+1$ we have

$$b_n^\alpha = \int_W \int_{-m-n+1}^{m} \max_{k=0,\dots,n-1} |f(v,x+k)|^\alpha \, \nu(dv)\, dx$$

$$= \sum_{j=m-n+1}^{-m} \int_W \int_j^{j+1} \max_{k=0,\dots,n-1} |f(v,x+k)|^\alpha \, \nu(dv)\, dx$$

$$+ \sum_{j=-m-n+1}^{m-n} \int_W \int_j^{j+1} \max_{k=0,\dots,n-1} |f(v,x+k)|^\alpha \, \nu(dv)\, dx$$

$$+ \sum_{j=-m+1}^{m-1} \int_W \int_j^{j+1} \max_{k=0,\dots,n-1} |f(v,x+k)|^\alpha \, \nu(dv)\, dx := T_n + R_n^{(1)} + R_n^{(2)} \, .$$

Observe that for each $j = m-n+1, \dots, m$ and each $x \in [j, j+1]$

$$\max_{k=0,\dots,n-1} |f(v,x+k)| = g(v, x-j) \, ,$$

while

$$\max_{k=0,\dots,n-1} |f(v,x+k)| \leq g(v, x-j)$$

for other values of j. Therefore,

$$T_n = (n-2m) \int_W \int_0^1 g(v,x)^\alpha \, \nu(dv)\, dx \, ,$$

while

$$|R_n^{(i)}| \leq 2m \int_W \int_0^1 g(v,x)^\alpha \, \nu(dv)\, dx \quad \text{for } i = 1, 2.$$

Therefore (3.6) in the case of a compact supported f follows. In the general case, given $\epsilon > 0$ choose a compact supported f_ϵ such that $|f_\epsilon(v,x)| \leq |f(v,x)|$ for all v, x, and

$$\int_W \int_{I\!R} |f(v,x)|^\alpha \, \nu(dv)\, dx - \int_W \int_{I\!R} |f_\epsilon(v,x)|^\alpha \, \nu(dv)\, dx \leq \epsilon \, .$$

Let

$$g_\epsilon(v,x) = \sup_{k=0,\pm1,\pm2,\dots} |f_\epsilon(v,x+k)| \, ,$$

$v \in W$, $x \in (0,1)$. Then

$$0 \leq \int_W \int_0^1 g(v,x)^\alpha \, \nu(dv)\, dx - \int_W \int_0^1 g_\epsilon(v,x)^\alpha \, \nu(dv)\, dx$$

$$\leq \int_W \int_0^1 \sup_{k=0,\pm1,\pm2,\dots} \Big(|f(v,x)|^\alpha - |f_\epsilon(v,x+k)|^\alpha \Big) \, \nu(dv)\, dx$$

$$\leq \int_W \int_0^1 \sum_{k-\infty}^{\infty} \Big(|f(v,x)|^\alpha - |f_\epsilon(v,x+k)|^\alpha \Big) \, \nu(dv)\, dx$$

$$= \int_W \int_{I\!R} |f(v,x)|^\alpha \, \nu(dv)\, dx - \int_W \int_{I\!R} |f_\epsilon(v,x)|^\alpha \, \nu(dv)\, dx \leq \epsilon \, .$$

Therefore,

$$\left|\frac{1}{n}b_n^\alpha - \int_W\int_0^1 g(v,x)^\alpha\,\nu(dv)\,dx\right|$$
$$\le \frac{1}{n}\left|\int_W\int_{-\infty}^\infty \max_{k=0,\dots,n-1}|f(v,x+k)|^\alpha\,\nu(dv)\,dx - \int_W\int_{-\infty}^\infty \max_{k=0,\dots,n-1}|f_\epsilon(v,x+k)|^\alpha\,\nu(dv)\,dx\right|$$
$$+\left|\frac{1}{n}\int_W\int_{-\infty}^\infty \max_{k=0,\dots,n-1}|f_\epsilon(v,x+k)|^\alpha\,\nu(dv)\,dx - \int_W\int_0^1 g_\epsilon(v,x)^\alpha\,\nu(dv)\,dx\right|$$
$$+\left|\int_W\int_0^1 g_\epsilon(v,x)^\alpha\,\nu(dv)\,dx - \int_W\int_0^1 g(v,x)^\alpha\,\nu(dv)\,dx\right| := T_n^{(1)}+T_n^{(2)}+T_n^{(3)}\,.$$

By the above, $T_n^{(3)}\le\epsilon$, and the same argument shows that $T_n^{(1)}\le\epsilon$ as well. Furthermore, by the already considered compact support case, $T_n^{(2)}\to 0$ as $n\to\infty$. Hence

$$\limsup_{n\to\infty}\left|\frac{1}{n}b_n^\alpha - \int_W\int_0^1 g(v,x)^\alpha\,\nu(dv)\,dx\right|\le 2\epsilon\,,$$

and, since $\epsilon>0$ is arbitrary, the proof of (3.6) is complete. □

4. Maxima of stationary stable processes

In this section we investigate the rate of growth of the sequence (M_n) of partial maxima of a stationary SαS process, $0<\alpha<2$. We will see, in particular, that if such a process has a non-zero component $\mathbf{X}^C$ in (2.3) generated by a dissipative flow then the partial maxima grow at the rate of $n^{1/\alpha}$, while if the process is generated by a conservative flow then the partial maxima grow at a slower rate. The following is the main theorem of this paper.

Theorem 4.1. *Let $\mathbf{X}=(X_0,X_1,\dots)$ be a stationary SαS process, $0<\alpha<2$, with integral representation (2.1), and functions (f_n) given by (2.2).*

(i) Suppose that $\mathbf{X}$ is not generated by a conservative flow (i.e. that the component $\mathbf{X}^C$ in (2.3) generated by a dissipative flow is non-zero). Then

$$\frac{1}{n^{1/\alpha}}M_n \Rightarrow C_\alpha^{1/\alpha}\,K_X\,Z_\alpha \tag{4.1}$$

weakly as $n\to\infty$, where

$$K_X=\left(\int_W\int_0^1 g(v,x)^\alpha\,\nu(dv)\,dx\right)^{1/\alpha}$$

and g given by (3.7) for any representation of $\mathbf{X}^C$ in the mixed moving average form (2.4). Furthermore,

$$C_\alpha=\left(\int_0^\infty x^{-\alpha}\sin x\,dx\right)^{-1}=\begin{cases}\frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)} & \text{if } \alpha\ne 1\\ 2/\pi & \text{if } \alpha=1\end{cases}\,, \tag{4.2}$$

and Z_α is the standard Frechét-type extreme value random variable with the distribution

$$P\big(Z_\alpha \le z\big) = e^{-z^{-\alpha}}, \ z > 0\,.$$

(ii) Suppose that $\mathbf{X}$ is generated by a conservative flow. Then

$$\frac{1}{n^{1/\alpha}} M_n \to 0 \tag{4.3}$$

in probability as $n \to \infty$. Furthermore, with b_n given by (3.1),

$$\left(\frac{1}{c_n} M_n\right) \quad \text{is not tight for any positive sequence } c_n = o(b_n)\ , \tag{4.4}$$

while

$$\left(\frac{1}{b_n f_n} M_n\right) \quad \text{is tight, where} \quad f_n = \begin{cases} (\log n)^{1/\alpha'} & \text{if } 1 < \alpha < 2 \\ L_2 n & \text{if } \alpha = 1 \\ 1 & \text{if } 0 < \alpha < 1 \end{cases}\ . \tag{4.5}$$

If for some $\theta > 0$ and $c > 0$

$$b_n \ge c n^\theta \quad \text{for all } n \ge 1 \tag{4.6}$$

then (4.5) holds with $f_n \equiv 1$ for all $0 < \alpha < 2$.

Furthermore, for $n = 1, 2, \ldots$ let η_n be a probability measure on $\big(E, \mathcal{E}\big)$ with

$$\frac{d\eta_n}{dm}(x) = b_n^{-\alpha} \max_{j=0,\ldots,n-1} |f_j(x)|^\alpha\,,\ x \in E\,, \tag{4.7}$$

and let $U_j^{(n)}$, $j = 1, 2$ be independent E-valued random variables with common law η_n. Suppose that (4.6) holds and that, in addition, for any $\epsilon > 0$

$$P\left(\text{for some } k = 0, 1, \ldots, n-1,\ \frac{\left|f_k\left(U_j^{(n)}\right)\right|}{\max_{i=0,\ldots,n-1}\left|f_i\left(U_j^{(n)}\right)\right|} > \epsilon,\ j = 1, 2\right) \to 0 \tag{4.8}$$

as $n \to \infty$. Then

$$\frac{1}{b_n} M_n \Rightarrow C_\alpha^{1/\alpha}\, Z_\alpha \tag{4.9}$$

weakly as $n \to \infty$.

Remark 4.2. Here are some sufficient conditions for (4.8). If

$$\lim_{n\to\infty} \frac{b_n}{n^{1/2\alpha}} = \infty\,, \tag{4.10}$$

then (4.8) holds. Indeed, let r_n denote the probability in the left hand side of (4.8). Clearly,

$$r_n \le \sum_{k=0}^{n-1} \left(P\left(\frac{\left|f_k\left(U_1^{(n)}\right)\right|}{\max_{i=0,\ldots,n-1}\left|f_i\left(U_1^{(n)}\right)\right|} > \epsilon\right)\right)^2\ .$$

Furthermore, for every $k = 0, 1, \dots, n-1$

$$P\left(\frac{\left|f_k\left(U_1^{(n)}\right)\right|}{\max_{i=0,\dots,n-1}\left|f_i\left(U_1^{(n)}\right)\right|} > \epsilon\right)$$

$$= b_n^{-\alpha}\int_E \mathbf{1}\left(\frac{|f_k(x)|}{\max_{i=0,\dots,n-1}|f_i(x)|} > \epsilon\right) \max_{i=0,\dots,n-1}|f_i(x)|^\alpha \, m(dx)$$

$$\le \epsilon^{-\alpha} b_n^{-\alpha}\int_E |f_k(x)|^\alpha \, m(dx)\,,$$

and (4.8) follows from (4.10) since, by the stationarity, the last integral does not depend on k.

Alternatively, assume that m is a finite measure, the flow is measure m preserving, the sequence $\left(b_n^{-\alpha}\max_{j=0,\dots,n-1}|f_j(x)|^\alpha\right)$, $n = 1, 2, \dots$ is uniformly integrable with respect to m, and for every $\epsilon > 0$

$$\lim_{n\to\infty} n^{1/2}\, m\Big\{x \in E : |f(x)| > \epsilon b_n\Big\} = 0\,, \tag{4.11}$$

where f is the kernel in (2.2). Then (4.8) holds.

Indeed, let $||m||$ be the total mass of m. Given a $\delta > 0$ select $M > 0$ such that

$$\int_E \mathbf{1}\left(\max_{i=0,\dots,n-1}|f_i(x)|^\alpha > M\, b_n^\alpha\right) \max_{i=0,\dots,n-1}|f_i(x)|^\alpha \, m(dx) \le \delta\, b_n^\alpha$$

for all $n \ge 1$. We have

$$r_n \le 4\delta$$

$$+P\left(\max_{i=0,\dots,n-1}\left|f_i\left(U_j^{(n)}\right)\right| \le M\, b_n \ \text{ and for some } k = 0, 1, \dots, n-1, |f_k(x)| > \epsilon\,\delta\,||m||^{-1}\, b_n\right)$$

$$\le 4\delta + M^{2\alpha} n\left(\int_E \mathbf{1}\left(|f(x)| > \epsilon\,\delta\,||m||^{-1}\, b_n\right) m(dx)\right)^2 .$$

Therefore, using (4.11) we obtain

$$\limsup_{n\to\infty} r_n \le 4\delta\,,$$

and (4.8) follows by letting $\delta \to 0$.

Proof of Theorem 4.1. We use a series representation of the random vector $(X_0, X_1, \dots, X_{n-1})$ of the form

$$X_k = b_n\, C_\alpha^{1/\alpha}\sum_{j=1}^\infty \varepsilon_j\, \Gamma_j^{-1/\alpha}\, \frac{f_k\left(U_j^{(n)}\right)}{\max_{i=0,\dots,n-1}\left|f_i\left(U_j^{(n)}\right)\right|}\,,\ k = 0, 1, \dots, n-1\,, \tag{4.12}$$

where C_α is given by (4.2), $\varepsilon_1, \varepsilon_2, \dots$ are iid Rademacher random variables (symmetric ± 1-valued random variables), $\Gamma_1, \Gamma_2, \dots$ is a sequence of the arrival times of a unit rate Poisson process on $(0, \infty)$, and $\left(U_j^{(n)}\right)$ are iid E-valued random variables with common law given by (4.7). All three

sequences are independent. See Section 3.10 in Samorodnitsky and Taqqu (1994). Of course, the representation in (4.12) is in law.

We start with observing that (4.5) follows from (3.2), (3.3) and (3.4) regardless of the properties of the flow. To check (4.4), we have, using the series representation (4.12) and symmetry, for any $x > 0$

$$P\left(\frac{1}{c_n}M_n > x\right) \geq \frac{1}{2}P\left(\max_{k=0,\dots,n-1} b_n\, C_\alpha^{1/\alpha}\left|\varepsilon_1\,\Gamma_1^{-1/\alpha}\,\frac{\left|f_k\left(U_1^{(n)}\right)\right|}{\max_{i=0,\dots,n-1}\left|f_i\left(U_1^{(n)}\right)\right|}\right| > c_n\, x\right)$$

$$= \frac{1}{2}P\left(\Gamma_1^{-1/\alpha} > C_\alpha^{-1/\alpha}\frac{c_n}{b_n}\,x\right) \to \frac{1}{2}$$

as $n \to \infty$. Hence lack of tightness.

Suppose now that (4.6) holds. Let $K = 1, 2, \dots$ be such that

$$\alpha\,(K+1)\theta > 1\,. \tag{4.13}$$

We claim that, in this case, for all $\epsilon > 0$ satisfying

$$0 < \epsilon < \frac{1}{K} \tag{4.14}$$

we have

$$P\left(\max_{k=0,\dots,n-1}|X_k| > \lambda\, b_n,\, \Gamma_1^{-1/\alpha} \leq \epsilon\lambda\right) \to 0 \quad \text{as } n \to \infty \tag{4.15}$$

for all $\lambda > 0$. Indeed, choose

$$\frac{1}{\theta} < p < \alpha(K+1)\,. \tag{4.16}$$

Notice that the probability in the left hand side of (4.15) is bounded from above by

$$\sum_{k=0}^{n-1} P\left(|X_k| > \lambda\, b_n,\, \Gamma_j^{-1/\alpha}\,\frac{\left|f_k\left(U_j^{(n)}\right)\right|}{\max_{i=0,\dots,n-1}\left|f_i\left(U_j^{(n)}\right)\right|} \leq \epsilon\lambda \;\text{ for all } j = 1, 2, \dots\right)\,.$$

For f in (2.2) let

$$||f||_\alpha = \left(\int_E |f(x)|^\alpha\, m(dx)\right)^{1/\alpha}\,,$$

and notice that for any $k = 0, 1 \dots, n-1$ the points

$$b_n\,\varepsilon_j\,\Gamma_j^{-1/\alpha}\,\frac{f_k\left(U_j^{(n)}\right)}{\max_{i=0,\dots,n-1}\left|f_i\left(U_j^{(n)}\right)\right|}\,,\; j = 1, 2, \dots$$

represent a symmetric Poisson random measure on $I\!R$ whose mean measure assigns a mass of $x^{-\alpha}||f||_\alpha^\alpha/2$ to the set (x, ∞) for every $x > 0$. Since the same random measure can be represented

by the points

$$\varepsilon_j\,\Gamma_j^{-1/\alpha}||f||\,,\ j=1,2,\dots\,,$$

we conclude that the probability in (4.15) is bounded from above by

$$\begin{aligned} &nP\left(C_\alpha^{1/\alpha}\sum_{j=1}^{\infty}\varepsilon_j\,\Gamma_j^{-1/\alpha}>\lambda\,||f||^{-1}\,b_n,\,\Gamma_j^{-1/\alpha}\le\epsilon\lambda\,||f||^{-1}\,b_n\right)\\ &\le nP\left(C_\alpha^{1/\alpha}\sum_{j=K+1}^{\infty}\varepsilon_j\,\Gamma_j^{-1/\alpha}>\lambda\left(1-\epsilon K\right)||f||^{-1}\,b_n\right)\\ &\le n\,b_n^{-p}\frac{||f||^pE\left|C_\alpha^{1/\alpha}\sum_{j=K+1}^{\infty}\varepsilon_j\,\Gamma_j^{-1/\alpha}\right|^p}{\lambda^p\left(1-\epsilon K\right)^p}\,. \end{aligned}$$

By the choice of p in (4.16) the latter expression goes to 0 as $n\to\infty$ and, hence, (4.15) follows.

Fix $\epsilon>0$ satisfying (4.14). Given $\delta>0$ choose $\lambda>0$ such that $P(\Gamma_1^{-1/\alpha}>\epsilon\lambda)\le\delta/2$, choose n_0 such that

$$P\left(\max_{k=0,\dots,n-1}|X_k|>\lambda\,b_n,\,\Gamma_1^{-1/\alpha}\le\epsilon\lambda\right)\le\frac{\delta}{2}\ \text{ for } n>n_0$$

and $\lambda'\ge\lambda$ such that

$$P\left(M_k>\lambda'\,b_k\right)\le\delta\ \text{ for } k=1,\dots n_0.$$

Then

$$P\left(\frac{1}{b_n}M_n>\lambda'\right)\le\delta$$

for all $n\ge1$, and so (4.5) holds with $f_n\equiv1$.

Now, suppose that $\mathbf{X}$ is generated by a conservative flow. Let $\mathbf{Y}$ be a stationary SαS process independent of $\mathbf{X}$, also given by an integral representation of the form (2.1), say,

$$Y_n=\int_{E'}g_n(x)\;M'(dx)\,,\quad n=0,1,2,\dots\,,$$

where M' is a SαS random measure with control measure m', independent of M in the integral representation of $\mathbf{X}$, with the functions g_n also given in the form (2.2), with some non-singular conservative flow ϕ' on E', and such that

$$b_n^Y=\left(\int_{E'}\max_{j=0,\dots,n-1}|g_j(x)|^\alpha\,m'(dx)\right)^{1/\alpha}\ ,n=1,2,\dots$$

satisfies (4.6) for some $\theta>0$. Processes $\mathbf{Y}$ with the above properties exist; see the examples in the next section. However the above step may require enlarging the probability space we are working with. Let $\mathbf{Z}=\mathbf{X}+\mathbf{Y}$. Then $\mathbf{Z}$ is a stationary SαS process generated by a conservative flow. We use its natural integral representation on $E\cup E'$ with the naturally defined flow on that space. Let b_n^Z be the corresponding quantity in (3.1) defined for the process $\mathbf{Z}$. Note that

$b_n^Z \geq b_n^Y$ for all n, hence the process $\mathbf{Z}$ satisfies (4.6) as well. By the already proven part of the theorem, the sequence $(b_n^Z)^{-1} \max_{k=0,\dots,n-1} |Z_k|$, $n = 1, 2, \dots$ is tight. Since for any $x > 0$ and $n = 1, 2, \dots$

$$P\left(\max_{k=0,\dots,n-1} |Z_k| > x\right) \geq \frac{1}{2} P\left(\max_{k=0,\dots,n-1} |X_k| > x\right)$$

by the symmetry of $\mathbf{Y}$, we conclude that the sequence $(b_n^Z)^{-1} \max_{k=0,\dots,n-1} |X_k|$, $n = 1, 2, \dots$ is tight as well.

However, the process $\mathbf{Z}$ is generated by a conservative flow and, hence, by Theorem 3.1, $b_n^Z = o(n^{1/\alpha})$. Therefore, (4.3) follows.

Suppose now that (4.8) holds. Then for every $1 \leq j_1 < j_2$ and $\epsilon > 0$

$$P\left(\text{for some } k = 0, 1, \dots, n-1, \quad \Gamma_{j_i}^{-1/\alpha} \frac{\left|f_k\left(U_{j_i}^{(n)}\right)\right|}{\max_{m=0,\dots,n-1}\left|f_m\left(U_{j_i}^{(n)}\right)\right|} > \epsilon, \ i = 1, 2\right)$$

$$\leq P\left(\Gamma_1 \leq \tau\right) + P\left(\text{for some } k = 0, 1, \dots, n-1, \ \frac{\left|f_k\left(U_j^{(n)}\right)\right|}{\max_{m=0,\dots,n-1}\left|f_m\left(U_j^{(n)}\right)\right|} > \epsilon\, \tau^{1/\alpha}, \ j = 1, 2\right)$$

for any $\tau > 0$. Letting first $n \to \infty$ and then $\tau \to 0$ shows that for every $1 \leq j_1 < j_2$ and $\epsilon > 0$

(4.17)

$$\lim_{n\to\infty} P\left(\text{for some } k = 0, 1, \dots, n-1, \quad \Gamma_{j_i}^{-1/\alpha} \frac{\left|f_k\left(U_{j_i}^{(n)}\right)\right|}{\max_{m=0,\dots,n-1}\left|f_m\left(U_{j_i}^{(n)}\right)\right|} > \epsilon, \ i = 1, 2\right) = 0\,.$$

Observe, further, that for any $\epsilon > 0$

$$P\left(\text{for some } k = 0, 1, \dots, n-1, \ \Gamma_j^{-1/\alpha} \frac{\left|f_k\left(U_j^{(n)}\right)\right|}{\max_{m=0,\dots,n-1}\left|f_m\left(U_j^{(n)}\right)\right|} > \epsilon \ \text{ for at least 2 different } j\right)$$

$$:= \varphi_n^{(1)}(\epsilon) \ \leq P\left(\Gamma_J^{-1/\alpha} > \epsilon\right)$$

$$+ \sum_{j_1=1}^{J-1} \sum_{j_2=j_1+1}^{J-1} P\left(\text{for some } k = 0, 1, \dots, n-1, \quad \Gamma_{j_i}^{-1/\alpha} \frac{\left|f_k\left(U_{j_i}^{(n)}\right)\right|}{\max_{m=0,\dots,n-1}\left|f_m\left(U_{j_i}^{(n)}\right)\right|} > \epsilon, \ i = 1, 2\right)$$

for any $J = 1, 2 \dots$. Letting $n \to \infty$ and using (4.17), and then letting $J \to \infty$ shows that for every $\epsilon > 0$

$$\lim_{n\to\infty} \varphi_n^{(1)}(\epsilon) = 0\,. \tag{4.18}$$

Suppose now that both (4.6) and (4.8) hold. Let K be as in (4.13). Let $\epsilon > 0$ and $0 < \delta < 1$ satisfy

$$0 < \epsilon < \frac{\delta}{K}\,. \tag{4.19}$$

For any $\lambda > 0$ we have

$$P\left(\frac{1}{b_n}M_n > \lambda\right) \le P\left(C_\alpha^{1/\alpha}\Gamma_1^{-1/\alpha} > \lambda(1-\delta)\right) + \varphi_n^{(1)}\left(C_\alpha^{-1/\alpha}\epsilon\,\lambda\right) \tag{4.20}$$

$$+P\left(\max_{k=0,\dots,n-1}\left|\sum_{j=1}^{\infty}\varepsilon_j\,\Gamma_j^{-1/\alpha}\frac{f_k\left(U_j^{(n)}\right)}{\max_{i=0,\dots,n-1}\left|f_i\left(U_j^{(n)}\right)\right|}\right| > C_\alpha^{-1/\alpha}\lambda,\ \Gamma_1^{-1/\alpha} \le C_\alpha^{-1/\alpha}\lambda(1-\delta),\ \text{and}\right.$$

$$\left.\text{for each } k=0,1,\dots,\ \Gamma_j^{-1/\alpha}\frac{\left|f_k\left(U_j^{(n)}\right)\right|}{\max_{i=0,\dots,n-1}\left|f_i\left(U_j^{(n)}\right)\right|} > C_\alpha^{-1/\alpha}\epsilon\,\lambda \ \text{ for at most one } j=1,2,\dots\right)$$

$$:= P\left(C_\alpha^{1/\alpha}\Gamma_1^{-1/\alpha} > \lambda(1-\delta)\right) + \varphi_n^{(1)}\left(C_\alpha^{-1/\alpha}\epsilon\,\lambda\right) + \varphi_n^{(2)}(\epsilon,\delta).$$

Proceeding similarly to the argument used in proving (4.15) we have

$$\varphi_n^{(2)}(\epsilon,\delta) \le \sum_{k=0}^{n-1} P\left(\left|\sum_{j=1}^{\infty}\varepsilon_j\,\Gamma_j^{-1/\alpha}\frac{f_k\left(U_j^{(n)}\right)}{\max_{i=0,\dots,n-1}\left|f_i\left(U_j^{(n)}\right)\right|}\right| > C_\alpha^{-1/\alpha}\lambda,\right. \tag{4.21}$$

$$\Gamma_j^{-1/\alpha}\frac{\left|f_k\left(U_j^{(n)}\right)\right|}{\max_{i=0,\dots,n-1}\left|f_i\left(U_j^{(n)}\right)\right|} \le C_\alpha^{-1/\alpha}\lambda(1-\delta) \ \text{ for each } j=1,2,\dots$$

$$\left.\text{and } \ \Gamma_j^{-1/\alpha}\frac{\left|f_k\left(U_j^{(n)}\right)\right|}{\max_{i=0,\dots,n-1}\left|f_i\left(U_j^{(n)}\right)\right|} > C_\alpha^{-1/\alpha}\epsilon\,\lambda \ \text{ for at most one } j=1,2,\dots\right)$$

$$= nP\left(\left|\sum_{j=1}^{\infty}\varepsilon_j\,\Gamma_j^{-1/\alpha}\right| > C_\alpha^{-1/\alpha}\lambda\,||f||^{-1}\,b_n,\right.$$

$$\left.\Gamma_1^{-1/\alpha} \le C_\alpha^{-1/\alpha}\lambda(1-\delta)\,||f||^{-1}\,b_n,\ \text{ and }\ \Gamma_2^{-1/\alpha} \le C_\alpha^{-1/\alpha}\epsilon\,\lambda\,||f||^{-1}\,b_n\right)$$

and the latter expression goes to zero as $n \to \infty$ by the choice of ϵ and δ, as in the proof of (4.15).

We conclude by (4.18) and (4.21) that for any $0 < \delta < 1$

$$\limsup_{n\to\infty} P\left(\frac{1}{b_n}M_n > \lambda\right) \le P\left(C_\alpha^{1/\alpha}\Gamma_1^{-1/\alpha} > \lambda(1-\delta)\right)$$
$$= 1 - \exp\left\{-C_\alpha\,\lambda^{-\alpha}(1-\delta)^{-\alpha}\right\}\ ,$$

and letting $\delta \to 0$ we obtain

$$\limsup_{n\to\infty} P\left(\frac{1}{b_n}M_n > \lambda\right) \le 1 - \exp\left\{-C_\alpha\,\lambda^{-\alpha}\right\}\ . \tag{4.22}$$

In the opposite direction, the argument is similar. For any ϵ and $\delta > 0$ satisfying (4.19) we have

$$P\left(\frac{1}{b_n}M_n > \lambda\right) \ge P\left(C_\alpha^{1/\alpha}\Gamma_1^{-1/\alpha} > \lambda(1+\delta)\right) - \varphi_n^{(1)}\left(C_\alpha^{-1/\alpha}\epsilon\,\lambda\right)$$

$$-P\left(\max_{k=0,\dots,n-1}\left|\sum_{j=1}^{\infty}\varepsilon_j\,\Gamma_j^{-1/\alpha}\frac{f_k\left(U_j^{(n)}\right)}{\max_{i=0,\dots,n-1}\left|f_i\left(U_j^{(n)}\right)\right|}\right|\le C_\alpha^{-1/\alpha}\lambda,\ \Gamma_1^{-1/\alpha}>C_\alpha^{-1/\alpha}\lambda(1+\delta),\ \text{and}\right.$$

$$\left.\text{for each } k=0,1,\dots,\ \Gamma_j^{-1/\alpha}\frac{\left|f_k\left(U_j^{(n)}\right)\right|}{\max_{i=0,\dots,n-1}\left|f_i\left(U_j^{(n)}\right)\right|}>C_\alpha^{-1/\alpha}\epsilon\,\lambda\ \text{ for at most one } j=1,2,\dots\right)$$

$$:=P\left(C_\alpha^{1/\alpha}\Gamma_1^{-1/\alpha}>\lambda(1+\delta)\right)-\varphi_n^{(1)}\left(C_\alpha^{-1/\alpha}\epsilon\,\lambda\right)-\varphi_n^{(3)}(\epsilon).$$

As above, the choice of ϵ and δ gives us

$$\lim_{n\to\infty}\varphi_n^{(3)}(\epsilon,\delta)=0\,, \tag{4.23}$$

and so we conclude by (4.18) and (4.23) that for any $\delta>0$

$$\begin{aligned}\liminf_{n\to\infty}P\left(\frac{1}{b_n}M_n>\lambda\right)&\ge P\left(C_\alpha^{1/\alpha}\Gamma_1^{-1/\alpha}>\lambda(1+\delta)\right)\\&=1-\exp\left\{-C_\alpha\,\lambda^{-\alpha}(1+\delta)^{-\alpha}\right\}\,,\end{aligned}$$

and letting $\delta\to 0$ we obtain a lower bound matching (4.22). This proves (4.9).

If $\mathbf{X}$ is not generated by a conservative flow, then it follows from Theorem 3.1 that

$$n^{-1/\alpha}b_n\to K_X\quad\text{as } n\to\infty.$$

In particular, both conditions (4.6) and (4.8) are satisfied (see Remark 4.2). Therefore, (4.1) follows from the already proven (4.9), and the proof of all parts of the theorem is complete. □

Remark 4.3. It is, of course, clear that the statement of part (ii) of Theorem 4.1 remains valid with any integral representation of the process, not necessarily with functions of the form (2.2).

Remark 4.4. The assumptions (4.6) and (4.8) mean, intuitively, that one and only one Poisson jump in the series representation (4.12) significantly contributes to the value of M_n for large n. Because of that an extreme value distribution arises as a limit. Either of these two assumptions may fail, as will be seen from the examples in the next section. Even though a complete limit theory in such cases is unavailable at the moment, limiting distributions are likely to depend on the number of Poisson jumps that contribute significantly to the value of the maximum. In particular, the limiting distribution is not, in general, an extreme value distribution. See Example 5.1.

Remark 4.5. Given a stationary SαS process $\mathbf{X}$, let

$$M_n^{(0)}=\max\Big(X_0,X_1,\dots,X_{n-1}\Big),\ n=1,2,\dots\,. \tag{4.24}$$

The same argument as that used in proof of Theorem 4.1 shows that, if $\mathbf{X}$ is not generated by a conservative flow, then

$$\frac{1}{n^{1/\alpha}} M_n^{(0)} \Rightarrow C_\alpha^{1/\alpha}\, K_X^{(0)}\, Z_\alpha \tag{4.25}$$

weakly as $n \to \infty$, where

$$K_X^{(0)} = \left(\frac{1}{2} \int_W \int_0^1 g_+^{(0)}(v,x)^\alpha\, \nu(dv)\, dx + \frac{1}{2} \int_W \int_0^1 g_-^{(0)}(v,x)^\alpha\, \nu(dv)\, dx \right)^{1/\alpha},$$

with

$$g_+^{(0)}(v,x) = \sup_{k=0,\pm1,\pm2,\dots} f(v,x+k)_+, \quad g_-^{(0)}(v,x) = \sup_{k=0,\pm1,\pm2,\dots} f(v,x+k)_-$$

$v \in W$, $x \in (0,1)$, for any representation of $\mathbf{X}^C$ in the mixed moving average form (2.4). Here a_+ and a_- are, correspondingly, the positive and negative parts of a real number a. In the particular case of linear α-stable process this result is in Theorem 3.8.3 in Leadbetter et al. (1983).

It is, of course, clear that if $\mathbf{X}$ is generated by a conservative flow, then $n^{-1/\alpha} M_n^{(0)} \to 0$ in probability as $n \to \infty$.

Integral representations of the type (2.1) with the functions (f_n) given in the form (2.2) exist not only for symmetric α-stable processes, but also for all strictly α-stable processes with $\alpha \neq 1$ (in this case the random measure M may not be symmetric, and will, in general, have a non-constant skewness intensity). See Rosiński (1994). The following result is a straightforward consequence of Theorem 4.1.

Theorem 4.6. *Let $\mathbf{X} = (X_0, X_1, \dots)$ be a stationary strictly α-stable process, $0 < \alpha < 2$, $\alpha \neq 1$, with integral representation (2.1), and functions (f_n) given by (2.2).*

(i) Suppose that $\mathbf{X}$ is not generated by a conservative flow. Then (4.1) holds.

(ii) Suppose that $\mathbf{X}$ is generated by a conservative flow. Then (4.3) holds.

Proof. In the case $0 < \alpha < 1$ a strictly α-stable process has a series representation of the form (4.12) (but with (ε_j) dependent on $\big(U_j^{(n)}\big)$ and not necessarily symmetric). The entire argument of Theorem 4.1 used to prove (4.1) and (4.3) goes through in this case. Hence, we only need to consider the case $1 < \alpha < 2$. In that case there is a positive constant $\tau > 0$ such that

$$\min\Big(P(X_1 > 0),\, P(X_1 < 0)\Big) \geq \tau\,. \tag{4.26}$$

See Theorem 4 in Zolotarev (1957).

Let $\mathbf{Y}$ be an independent copy of $\mathbf{X}$. Then $\mathbf{Z} = \mathbf{X} - \mathbf{Y}$ is a stationary SαS process that has an integral representation (2.1) with the same functions (f_n) as $\mathbf{X}$ does, but where the random measure M is now replaced by a symmetric α-stable random measure M' whose control measure

is given by $m' = 2m$; see Samorodnitsky and Taqqu (1994). If $\left(b_n^Z\right)$ is the sequence (3.1) defined for $\mathbf{Z}$ using this integral representation, we have

$$b_n^Z = 2^{1/\alpha} b_n \;\; n = 1, 2, \ldots . \tag{4.27}$$

Suppose that $\mathbf{X}$ is generated by a conservative flow, then so is $\mathbf{Z}$. We have by (4.26), for any $\lambda > 0$

$$P\left(\max_{k=0,\ldots,n-1} |Z_n| > \lambda\right) \geq \tau P\left(\max_{k=0,\ldots,n-1} |X_n| > \lambda\right) .$$

Therefore, (4.3) for $\mathbf{X}$ follows from (4.3) for $\mathbf{Z}$ proven in Theorem 4.1.

If $\mathbf{X}$ is not generated by a conservative flow, then neither is $\mathbf{Z}$. We have by Theorem 4.1 and (4.27),

$$\frac{1}{n^{1/\alpha}} \max_{k=0,\ldots,n-1} |Z_n| \Rightarrow 2^{1/\alpha} C_\alpha^{1/\alpha} \, K_X \, Z_\alpha \tag{4.28}$$

weakly as $n \to \infty$.

Let

$$K_X(n) = \min\{k = 0, 1, \ldots, n-1 : |X_k| = M_n\}$$

and similarly with $K_Y(n)$. For every $\lambda > 0$ and $\epsilon > 0$ we have by the stationarity and independence

$$\begin{aligned} P\left(\max_{k=0,\ldots,n-1} |Z_n| \leq \lambda\, n^{1/\alpha}\right) &= P\left(\max_{k=0,\ldots,n-1} |X_n - Y_n| \leq \lambda\, n^{1/\alpha}\right) \\ \leq P\left(\max_{k=0,\ldots,n-1} |X_n| \leq \lambda\,(1+\epsilon)\, n^{1/\alpha}\right) &\; P\left(\max_{k=0,\ldots,n-1} |Y_n| \leq \lambda\,(1+\epsilon)\, n^{1/\alpha}\right) \\ &+ P\left(\left|Y_{K_X(n)}\right| > \epsilon\, n^{1/\alpha}, \;\text{ or }\; \left|X_{K_Y(n)}\right| > \epsilon\, n^{1/\alpha}\right) \\ \leq \left(P\left(\max_{k=0,\ldots,n-1} |X_n| \leq \lambda\,(1+\epsilon)\, n^{1/\alpha}\right)\right)^2 &+ 2\, P\left(|X_1| > \epsilon\, n^{1/\alpha}\right) . \end{aligned}$$

Therefore, using (4.28), we have, for any $\lambda > 0$,

$$\liminf_{n\to\infty} P\left(\max_{k=0,\ldots,n-1} |X_n| \leq \lambda\, n^{1/\alpha}\right) \geq \exp\left\{-C_\alpha\, \lambda^{-\alpha} (1+\epsilon)^\alpha\right\}$$

for any $\epsilon > 0$. Letting $\epsilon \to 0$ we obtain

$$\liminf_{n\to\infty} P\left(\max_{k=0,\ldots,n-1} |X_n| \leq \lambda\, n^{1/\alpha}\right) \geq \exp\left\{-C_\alpha\, \lambda^{-\alpha}\right\} \tag{4.29}$$

for any $\lambda > 0$.

In the other direction, for every $\lambda > 0$ and $0 < \epsilon < 1$

$$\begin{aligned} P\left(\max_{k=0,\ldots,n-1} |Z_n| \leq \lambda\, n^{1/\alpha}\right) &\geq \left(P\left(\max_{k=0,\ldots,n-1} |X_n| \leq \lambda\,(1-\epsilon)\, n^{1/\alpha}\right)\right)^2 \\ &- P\left(\text{for some } k = 0, 1, \ldots, n-1,\; |X_k| > \epsilon\, n^{1/\alpha} \;\text{ and }\; |Y_k| > \epsilon\, n^{1/\alpha}\right) . \end{aligned}$$

Arguing as in the proof of Theorem 4.1 we see, by stationarity and independence, that

$$P\left(\text{for some } k=0,1,\dots,n-1,\ |X_k|>\epsilon\, n^{1/\alpha} \ \text{ and } \ |Y_k|>\epsilon\, n^{1/\alpha}\right)$$
$$\le n\left(P\left(|X_1|>\epsilon\, n^{1/\alpha}\right)\right)^2 \to 0$$

as $n\to\infty$. Once again, using (4.28), we have, for any $\lambda>0$,

$$\limsup_{n\to\infty} P\left(\max_{k=0,\dots,n-1}|X_n|\le \lambda\, n^{1/\alpha}\right)\le \exp\left\{-C_\alpha\,\lambda^{-\alpha}(1-\epsilon)^\alpha\right\}$$

for any $0<\epsilon<1$. Letting $\epsilon\to 0$ we obtain an upper bound matching (4.29) and, hence, complete the proof. □

Remark 4.7. We see immediately from the proof of Theorem 4.6 that the statement (4.9) extends to the skewed case as well if (4.10) holds.

5. Examples

The results of the previous section describe completely the limiting behavior of the partial maxima of stationary SαS processes not generated by a conservative flow. The picture for processes generated by conservative flows is less complete, and in this section we consider several examples of such processes to illustrate what may happen.

Our first example shows that, in general, the partial maximum M_n does not have an extreme value limit.

Example 5.1. Let $Z_0, Z_1, \dots$ be iid standard normal random variables, independent of a positive $\alpha/2$-stable random variable A with Laplace transform $Ee^{-\theta A}=e^{-\theta^{\alpha/2}}$, $\theta\ge 0$. Then $X_n=A^{1/2}Z_n$, $n=0,1,2,\dots$ is a stationary SαS process, the simplest type of *sub-Gaussian* SαS processes; see Section 3.7 in Samorodnitsky and Taqqu (1994). This process has an integral representation of the form

$$X_n=(d_\alpha)^{-1}\int_{\mathbb{R}^{\mathbb{Z}}} g_n\, M(d\mathbf{g}),\ n=0,1,2,\dots, \tag{5.1}$$

where $d_\alpha=\sqrt{2}\left(E|Z_0|^\alpha\right)^{1/\alpha}$, and M a SαS random measure on $\mathbb{R}^{\mathbb{Z}}$ whose control measure m is a probability measure under which the projections g_k, $k\in\mathbb{Z}$ are iid standard normal random variables. With ϕ being the usual left shift operator on $\mathbb{R}^{\mathbb{Z}}$, this is a representation with the functions given in the form (2.2), with $a_n\equiv 1$ and a measure preserving flow. The flow is, obviously, conservative, as is any measure preserving flow on a finite measure space. As usual, the representation (5.1) is in law.

Since

$$b_n^\alpha = (d_\alpha)^{-\alpha} E \max_{k=0,1,\dots,n-1} |Z_k|^\alpha \sim (d_\alpha)^{-\alpha} \Big(2\log n\Big)^{\alpha/2}$$

as $n \to \infty$, we see that the assumption (4.6) in Theorem 4.1 fails. Since, furthermore,

$$\frac{1}{\big(2\log n\big)^{1/2}} \max_{k=0,1,\dots,n-1} |Z_k| \to 1 \quad \text{with probability 1}$$

as $n \to \infty$, we see directly that

$$\frac{1}{b_n} M_n \Rightarrow (d_\alpha)^{-1} A^{1/2}\,,$$

a non-extreme value limit, even though the partial maximum M_n does grow at the rate b_n.

Our next example shows that, if $1 \le \alpha < 2$ and without the assumption (4.6), the sequence of the partial maxima of the process may grow faster than the sequence (b_n), i.e. (4.5) may not hold with $f_n \equiv 1$.

Example 5.2. Let $\mathbf{X}$ be given by (5.1) with $d_\alpha = 1$, where this time the control measure m of the SαS random measure M is a probability measure under which the projections g_k, $k = \dots, -1, 0, 1, 2, \dots$ are iid Rademacher random variables. We have, once again, a conservative measure preserving flow, and $b_n \equiv 1$. Let $1 \le \alpha < 2$. We claim that (4.5) with $f_n \equiv 1$ does not hold. Indeed, if it did, then the SαS process $\mathbf{X}$ would be a.s. bounded, and then for any $0 < p < \alpha$ we would have

$$E M_\infty^p := E \sup_{k=0,1,\dots} |X_k|^p < \infty$$

(see e.g. Araujo and Giné (1980)), which would contradict (3.2) in the case $1 < \alpha < 2$ or (3.3) in the case $\alpha = 1$.

The next example exhibits a variety of rates at which the sequence (b_n) and the partial maxima of the processes can grow. We look at a class of stationary SαS processes generated by null recurrent Markov chains, introduced in Rosiński and Samorodnitsky (1996) and studied in more details in Resnick et al. (2000) and Mikosch and Samorodnitsky (2000).

Example 5.3. We start with an irreducible null-recurrent Markov chain on $\mathbb{Z}$ with law $P_i(\cdot)$, $i \in \mathbb{Z}$ and transition probabilities (p_{jk}). Let $\pi = (\pi_i)_{i\in\mathbb{Z}}$ be the σ-finite invariant measure corresponding to the family (P_i) satisfying $\pi_0 = 1$. Let P_i^* be the bilateral extension of P_i to $E = \mathbb{Z}^{\mathbb{Z}}$; that is, under P_i^*, $x_0 = i$, $(x_0, x_1, \dots)$ is a Markov chain with transition probabilities (p_{jk}) and $(x_0, x_{-1}, \dots)$ is a Markov chain with transition probabilities $\big(\pi_k\, p_{kj}/\pi_j\big)$. Define a σ-finite measure m on E by

$$m(\cdot) = \sum_{i=-\infty}^{\infty} \pi_i\, P_i^*(\cdot)\,,$$

and observe that m is invariant under the left shift operator ϕ, and the latter generates a conservative flow (see Harris and Robbins (1953)).

Let $\mathbf{X}$ be a stationary SαS process defined by the integral representation (2.1), with M being a SαS random measure with control measure m,

$$f_n(x) = f \circ \phi^n(x) \;\; x \in E\,, \;\; n = 0, 1, 2, \dots\,,$$

with

$$f(x) = \mathbf{1}_0\,(x_0)\,, \;\; x = \big(\dots, x_{-1}, x_0, x_1, x_2, \dots\big) \in E\,.$$

Because of the null recurrence of the Markov chain, this is a mixing stationary process (unlike, say, the processes in Examples 5.1 and 5.2); see Rosiński and Samorodnitsky (1996).

Let d be the period of the Markov chain. Assume that

$$P_0\,(x_{nd} = 0) = n^{-\gamma} L(n) \;\; \text{as } n \to \infty \tag{5.2}$$

for some $\gamma \in (0,1)$ and a slowly varying function L. Let

$$\tau = \tau(x) = \inf\{n \geq 1 : \, x_n = 0\}$$

be the first return time to 0. Notice that

$$b_n^\alpha = \sum_{i=-\infty}^{\infty} \pi_i\, P_i\,(x_k = 0 \;\; \text{for some} \;\; k = 0, \dots .n-1) = P_0(\tau \geq n) + \sum_{i=-\infty}^{\infty} \pi_i\, P_i(\tau \leq n-1)$$

$$= P_0(\tau \geq n) + m(\tau \leq n-1) \;\sim\; \frac{d^{1-\gamma}}{\Gamma(1+\gamma)\,\Gamma(1-\gamma)}\, n^\gamma\,(L_n)^{-1}$$

as $n \to \infty$ by Remark 3.1 and Lemma 3.3 in Resnick et al. (2000). In particular, it follows from Theorem 4.1 and Remark 4.2 that

$$\frac{L(n)^{1/\alpha}}{n^{\gamma/\alpha}}\, M_n \Rightarrow \left(\frac{C_\alpha\, d^{1-\gamma}}{\Gamma(1+\gamma)\,\Gamma(1-\gamma)}\right)^{1/\alpha} Z_\alpha \tag{5.3}$$

as $n \to \infty$ if $\gamma > 1/2$.

What happens if $\gamma \leq 1/2$? Let $Y_0^{(i)}, Y_1^{(i)}, \dots,\, i = 1, 2$ be two independent Markov chains with the same transition probabilities (p_{jk}) as before. Then

$$Y_n^* = \left(Y_n^{(1)}, Y_n^{(2)}\right), \;\; n = 0, 1, 2 \dots$$

is a Markov chain with state space $\mathbb{Z}^2$ and transition probabilities

$$p^*_{(j_1,j_2),(k_1,k_2)} = p_{j_1,k_1} p_{j_2,k_2}\,, \;\; j_1, j_2, k_1, k_2 \in \mathbb{Z}\,.$$

Let

$$\tau^* = \inf\{n \geq 1 : \, Y_n^* = (0,0)\}$$

be the first time the new Markov chain returns to $(0,0)$.

Let r_n be the probability in the left hand side of (4.8). For any $0 < \epsilon < 1$ we have

$$r_n = b_n^{-2\alpha} \left[P_{(0,0)}\left(\tau^* \geq n\right) + \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \pi_j \, \pi_j \, P_{(i,j)}\left(\tau^* \leq n-1\right) \right] .$$

Notice that $\pi^*_{ij} = \pi_i \pi_j$ for $(i,j) \in \mathbb{Z}^2$ is an invariant measure for $\left(Y^*_n\right)$, and let m^* be the shift-invariant measure defined on $\left(\mathbb{Z}^2\right)^{\mathbb{Z}}$ using p^* and π^* in the same way as measure m was defined above using p and π. Then

$$r_n = b_n^{-2\alpha} \left[P_{(0,0)}\left(\tau^* \geq n\right) + m^*\left(\tau^* \leq n-1\right) \right] .$$

It follows from (5.2) that

$$P^*_{(0,0)}\left(Y^*_{nd} = (0,0)\right) = \left(P_0\left(x_{nd} = 0\right)\right)^2 = n^{-2\gamma} \left(L(n)\right)^2 \quad \text{as } n \to \infty .$$

Therefore, for $\gamma \leq 1/2$ we may, once again, appeal to Remark 3.1 and Lemma 3.3 in Resnick et al. (2000) to conclude that

$$m^*\left(\tau^* \leq n-1\right) \sim \frac{d^{1-2\gamma}}{\Gamma(1+2\gamma)\,\Gamma(1-2\gamma)} \, n^{2\gamma} \left(L(n)\right)^{-2}$$

as $n \to \infty$ and, hence,

$$r_n \to d^{-1} \, \frac{\Gamma(1+\gamma)^2 \, \Gamma(1-\gamma)^2}{\Gamma(1+2\gamma)\,\Gamma(1-2\gamma)}$$

as $n \to \infty$. In particular, the condition (4.8) holds in the case $\gamma = 1/2$, and so (5.3) holds in this case as well.

On the other hand, (4.8) fails in the case $0 < \gamma < 1/2$, and we conjecture that (5.3) does not hold either (compare with Remark 4.4).

In the previous example we saw classes of stationary SαS processes generated by conservative flows satisfying (4.6) and (4.8) and, hence, also (4.9). For these processes b_n is regularly varying at infinity with exponent $\gamma \geq 1/(2\alpha)$. The following example exhibits stationary SαS processes generated by conservative flows and also satisfying both (4.6) and (4.8), for which b_n can be regularly varying at infinity with any exponent $0 < \gamma < 1/\alpha$.

Example 5.4. Once again, let $\mathbf{X}$ be given by (5.1) with $d_\alpha = 1$, but now the control measure m of the SαS random measure M is a probability measure under which the projections g_k, $k = \ldots, -1, 0, 1, 2, \ldots$ are iid positive Pareto random variables with

$$m(g_0 > x) = x^{-\theta} \quad \text{for } x \geq 1$$

for some $\theta > \alpha$. Note that for every $0 < p < \theta$

$$\int_E \max_{k=0,1,\ldots,n-1} g_k \, m(d\mathbf{g}) \sim c_{p,\theta} \, n^{p/\theta} \quad \text{as } n \to \infty \tag{5.4}$$

for some finite positive constant $c_{p,\theta}$. Using (5.4) with $p = \alpha$ shows that

$$b_n \sim c_{\alpha,\theta}^{1/\alpha}\, n^{1/\theta} \quad \text{as } n \to \infty\,,$$

and so (4.6) holds. Using, furthermore, (5.4) with $\alpha < p < \theta$ shows uniformly integrability of the sequence $\big(b_n^{-\alpha} \max_{j=0,\dots,n-1} |g_j(g)|^\alpha\big)$, $n = 1, 2, \dots$ with respect to m. Since (4.11) is obvious in this case, we conclude that (4.8) holds as well.

That is, the convergence to extreme value distribution (4.9) holds here in all cases, and the exponent of regular variation $1/\theta$ of b_n spans here the entire range $(0, 1/\alpha)$.

All of the examples above emphasize the message of Theorem 4.1: while the partial maxima grow at the rate of $n^{1/\alpha}$ for stationary SαS processes with a non-degenerate part generated by a dissipative flow, the rate of growth is strictly slower if the process is generated by a conservative flow. In all these examples the actual rate rate of growth of b_n and, hence, of the partial maximum was determined by the properties of the conservative flow. Our final example exhibits a new class of stationary SαS processes generated by a conservative flow, in which the actual rate of growth of the partial maxima is determined by the kernel f in the representation (2.2).

Example 5.5. Let $E = (0, 1]$, m the Lebesgue measure on E, and define $\phi : E \to E$ by $\phi(x) = \{2x\}$, where $\{a\}$ is the fractional part of a real number a. Obviously, $(\phi^n,\, n = 0, 1, 2, \dots)$ is a conservative measure m preserving flow (but it is not invertible). This flow is refered to as the *dyadic transformation*; see Example 1.6 in Billingsley (1965).

Let $h_1, h_2, \dots$ be a sequence of nonnegative numbers such that

$$\sum_{k=1}^{\infty} h_k^\alpha\, 2^{-k} < \infty\,, \tag{5.5}$$

and suppose that the limit

$$\theta := \lim_{k\to\infty} \frac{\log_2 h_k}{k} \in [0, 1/\alpha] \tag{5.6}$$

exists.

For $x = (x_1, x_2, \dots) \in (0, 1]$ given in its binary expansion define

$$K(x) = \inf\{j \geq 1 : x_j = 1\}\,, \tag{5.7}$$

and let $\mathbf{X}$ be a stationary SαS process defined by (2.1) with $f_n(x) = f \circ \phi^n(x)$, $x \in E$, where we choose

$$f(x) = h_{K(x)},\ x \in E\,. \tag{5.8}$$

Of course we do not need to worry about the fact that neither K nor f are defined for binary rational numbers.

Denoting

$$R_n(x) = \max_{j=1,\dots,n} \sup\{m \geq 1 : x_j = x_{j+1} = \dots = x_{j+m-1} = 0\} \tag{5.9}$$

to be the longest run of zeroes starting in $\{1, \dots n\}$, $n = 1, 2, \dots$, we see that

$$b_n^\alpha = \int_0^1 h^\alpha_{R_n(x)}\, dx, \;\; n = 1, 2, \dots . \tag{5.10}$$

We claim that

$$\lim_{k\to\infty} \frac{\log_2 b_k}{\log_2 k} = \theta\,. \tag{5.11}$$

Suppose that $\theta < 1/\alpha$. Then for every $\epsilon \in (0, 1/\alpha - \theta)$ we have

$$b_n^\alpha \leq c_\epsilon \int_0^1 2^{\alpha(\theta+\epsilon)\, R_n(x)}\, dx, \;\; n = 1, 2, \dots$$

for some finite positive constant c_ϵ. An elementary Bernoulli trials computation shows that for any $0 < \gamma < 1$

$$\lim_{n\to\infty} \frac{\log_2 \int_0^1 2^{\gamma\, R_n(x)}\, dx}{n} = \gamma\,. \tag{5.12}$$

Hence

$$\limsup_{k\to\infty} \frac{\log_2 b_k}{\log_2 k} \leq \theta + \epsilon\,,$$

and letting $\epsilon \to 0$ we see that, if $0 \leq \theta < 1/\alpha$, then

$$\limsup_{k\to\infty} \frac{\log_2 b_k}{\log_2 k} \leq \theta\,. \tag{5.13}$$

Of course, the fact that (5.13) also holds for $\theta = 1/\alpha$ is trivial.

Similarly, if $\theta > 0$, then for every $\epsilon \in (0, \theta)$ we have

$$b_n^\alpha \geq c_\epsilon \int_0^1 2^{\alpha(\theta-\epsilon)\, R_n(x)}\, dx, \;\; n = 1, 2, \dots$$

for some finite positive constant c_ϵ, and using, once again, (5.12) and letting $\epsilon \to 0$ we obtain

$$\liminf_{k\to\infty} \frac{\log_2 b_k}{\log_2 k} \geq \theta \tag{5.14}$$

for $\theta > 0$. Since (5.14) is trivial for $\theta = 0$, we obtain (5.11) from (5.13) and (5.14).

One can easily see from Theorem 4.1 and a slight modification of the proof of (4.4) that (5.11) implies that

$$\frac{\log_2 M_n}{\log_2 n} \to \theta \;\; \text{in probability}$$

as $n \to \infty$.

6. Conclusions

We have seen that the sequence of partial maxima of a stationary SαS process generated by a dissipative flow grows always at the rate of $n^{1/\alpha}$ irrespectively of the further properties of the flow or of the kernel in the integral representation of the process. On the other hand, the sequence of partial maxima of stationary SαS processes generated by a conservative flow grows always at the rate slower than $n^{1/\alpha}$. However, the actual rate of growth here depends on the further properties of the flow and on the kernel. This is an example of a phase transition we associate with a passage between short memory and long memory. In this respect it makes sense to call stationary SαS processes generated by dissipative flows short range dependent, and those generated by conservative flows long range dependent. It is important to note this phenomenon exists for all $0 < \alpha < 2$.

Examples of phase transitions of this kind for stationary SαS processes have been observed before; see for example the discussion of ruin probabilities in Mikosch and Samorodnitsky (2000) or change in the rate of growth of partial sums in the case of increment processes of self-similar α-stable processes with stationary increments; see Samorodnitsky and Taqqu (1994). In the previous cases the functionals for which the phase transitions were observed, were based on partial sums of a process, which resulted in the phase transition being observed only in the range $1 < \alpha < 2$. In the present paper the phase transition occurs for all $0 < \alpha < 2$. It is interesting to mention that, unlike in the case of partial sums, maxima of long range dependent processes grow slower than those of short range dependent processes. This results in the fact that the maxima of a sum of two independent processes, one of which is generated by a dissipative flow, and the other by a conservative flow, will grow at the rate dictated by the dissipative part. In this sense, the long memory is being hidden in such a sum.

It is also important to mention that, while stationary SαS processes generated by conservative flows considered in Mikosch and Samorodnitsky (2000), were also in the long range dependent part of the parameter space, some processes generated by dissipative flows were also long range dependent in that case.

This suggests that there may be multiple phase transitions indicating long range dependence, and the complete structure of those is yet to be discovered.

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