SMALL AND LARGE TIME SCALE ANALYSIS
OF A NETWORK TRAFFIC MODEL

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ABSTRACT. Recent empirical studies of the internet and WAN traffic data have observed multi-fractal behavior at time scales below a few hundred milliseconds. There have been some attempts to model this phenomenon, but there is no model to connect the small time scale behavior with behavior observed at large time scales of bigger than a few hundred milliseconds. There have been separate analyses of models for high speed data transmissions, which show that appropriate approximations to large time scale behavior of cumulative traffic are either fractional Brownian motion or stable Lévy motion, depending on the input rates assumed. This paper tries to bridge this gap and develops and analyzes a model offering an explanation of both the small and large time scale behavior of a network traffic model based on the infinite source Poisson model. Previous studies of this model have usually assumed that transmission rates are constant and deterministic. We consider a non-constant, multifractal, random transmission rate at the user level which results in cumulative traffic exhibiting multifractal behavior on small time scales and self-similar behavior on large time scales. Also, we model the file size and the transmission rate, which are more natural objects than the usually modeled transmission time and rate.

1. INTRODUCTION

Traditionally, teletraffic modeling used transmission times with finite variance. With the advent of high speed network traffic, concepts like heavy tails, long-range dependence and self-similarity gained acceptance. Leland et al. [20] gave the first empirical evidence in the context of LAN traffic. By now, the inadequacies of the finite variance model and short range dependence have been well-documented (cf. [4, 33, 42]).

A useful model for analyzing internet traffic is the M/G/∞ input model (cf. [12, 13, 15, 16, 25, 30, 32]). This model assumes an infinite number of sources capable of transmitting data. These sources start transmitting at times \( \{\Gamma_k\} \); we assume \( \{\Gamma_k\} \) is a sequence strictly increasing to \( \infty \). For large time scale results, we need to put some exponential assumption on the inter-initiation times \( \{\Gamma_k - \Gamma_{k-1}\} \), which we shall discuss later in detail. Each transmission consists of a file of size \( J_k \) and a transmission schedule \( \{A_k(t), t \geq 0\} \), both chosen at random according to some distribution to be specified. We assume \( J_k \) are positive and \( A_k(t) \) denotes the cumulative amount of data transmitted in time \( t \) after the transmission has begun. \( A_k \) is a non-decreasing càdlàg function starting from 0 and increasing to \( \infty \), which vanishes on the negative half-line. The quantity of interest is the traffic process which results from aggregating cumulative traffic from all sources in \( [0,t] \), and which is defined as

\[
X(t) = \sum_{k=1}^{\infty} A_k(t - \Gamma_k) \wedge J_k.
\]

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To date, most analyses studied the process $X$ at large time scales and assumed the transmission schedule to be a constant, non-random process and hence without loss of generality to be the identity function. Mikosch et al. considered this case in [25] and showed that possible limits depend on the input rate and may be either fractional Brownian motion (fBm) with long range dependent Gaussian increments or a stable Lévy motion with stationary independent increments but infinite variance. However, the assumption of unit transmission rate is highly arbitrary and restrictive. There have been efforts by several authors to remove this deficiency. Konstantopoulo and Lin [17] first introduced a non-constant but deterministic transmission schedule. Under this assumption and a constant input rate, they showed a Lévy limit for the cumulative traffic process at a large time scale. Resnick and van den Berg [32] showed the limit to hold in Skorohod's [37] $M_1$ topology as well. Maulik et al. [24] considered the case where the transmission schedule was linear with randomly chosen slope. However, none of these efforts could capture the complete essence of a random transmission schedule process. Also, all the above attempts modeled the transmission by choosing a transmission schedule and the length of transmission, which is not very realistic. In this paper, we try to address these two issues among others. We model by considering a random, time-dependent transmission schedule $A_k$ and choosing the file size $J_k$ at the beginning of each transmission. The length of transmission $L_k$ is obtained as a function of these two random quantities.

Both fBm and stable Lévy motion possess self-similarity, which is consistent with the macroscopic analysis of the network traffic data at a time scale of a few hundred milliseconds or larger. However, these models were posed without considering the complicated multifractal behavior of the WAN traffic observed at fine time scales below a few hundred milliseconds. Paxson and Floyd [26] observed the limitations of the usual model in their study. Later Riedi and Lévy Véhel [35] and Mannsalo and Norros [22] analyzed different WAN traces to empirically observe the multifractal behavior of ATM WAN traces. These observations stimulated researchers to look for a model which could explain both the microscopic as well as the macroscopic behaviors. In [7, 6, 10, 11, 18, 36], attempts were made to consider a conservative cascade model as the transmission schedule. We explain the fine time scale behavior by assuming individual transmission schedules exhibit multifractality. This results in multifractal behavior for the cumulative traffic process at the microscopic level and still gives a stable Lévy motion as the macroscopic approximation.

Thus our model offers an explanation of both the micro and macroscopic behavior of the cumulative traffic process and suggests that the multifractal behavior of the cumulative traffic process results from similar behavior of the individual input processes. This suggests empirical studies should examine the behavior of individual, user level input processes.

This paper is organized as follows: Section 2 collects the notations used in the paper and defines the important variables. In Section 3, we review the basic concepts of the multifractal spectrum as required in our discussion. Section 4 is a quick review of the space $\mathbb{D}(0, \infty)$ endowed with Skorohod's [37] $M_1$ topology. We state the model in detail and the main results at the micro and macro levels in Section 5, and discuss the conditions under which the main results hold. Section 6 considers the multifractal analysis, whereas Section 7 proves the approximation for large time scales.

2. Notations

We need to introduce the following notations for our discussion. For a non-decreasing function $x$, we define its left continuous inverse as

\[
    x^+(t) = \inf\{u : x(u) \geq t\}
\]
and its right continuous inverse as

\[ x^\rightarrow(t) = \inf\{u : x(u) > t\} \]

For a non-negative random variable \( U \), we denote its distribution function by \( F_U \), i.e., \( F_U(u) = P[U \leq u] \). Let \( \tilde{F}_U(u) = 1 - F_U(u) \). We define the quantile function \( \tilde{b}_U \) as

\[ \tilde{b}_U(T) = \inf\left\{ u : \tilde{F}_U(u) \leq \frac{1}{T} \right\} = \left( \frac{1}{F_U} \right)^{(T)}. \]

Recall that a function \( \phi \) is regularly varying of index \( \alpha \) and is denoted by \( \phi \in RV_\alpha \) (cf. Section 0.4 of [33]), if for all \( u > 0 \),

\[ \lim_{t \to \infty} \frac{\phi(tu)}{\phi(t)} = u^\alpha. \]

We say that \( U \) has a tail of index \( \alpha_U \) \( > 0 \), if \( \tilde{F}_U \in RV_{\alpha_U} \). In such a case (cf. Proposition 0.8(v) of [33]), \( \tilde{b}_U \in RV_{\alpha_U^{-1}} \) and also we have

\[ \lim_{T \to \infty} TP[U_1 > \tilde{b}_U(T)u] = u^{-\alpha_U}. \]

Conversely, if (2.3) holds, then \( \tilde{b}_U \in RV_{\alpha_U^{-1}} \) and \( \tilde{F}_U \in RV_{-\alpha_U} \). In either of these cases, we can choose a strictly increasing, absolutely continuous function \( b_U \), such that \( \tilde{b}_U \sim b_U \), i.e.,

\[ \lim_{T \to \infty} \frac{\tilde{b}_U(T)}{b_U(T)} = 1 \]

(cf. Proposition 0.8(vii) of [33]). We can further say that

\[ \lim_{T \to \infty} TP[U_1 > b_U(T)u] = u^{-\alpha_U}. \]

Recall that \( \{J_k\} \) and \( \{A_k\} \) were introduced in Section 1. We define the transmission length of \( k \)-th source as

\[ L_k = \inf\{t : A_k(t) \geq J_k\} = A_k^+(J_k). \]

Then \( L_k \) are i.i.d. Let \( F_L \) be the marginal distribution of \( L_1 \). We have

\[ F_L(x) = P[L_1 \leq x] = P[A_1(x) \geq J_1] \] by right continuity of paths of \( A_1 \)

and

\[ \tilde{F}_L(x) = P[L_1 > x] = P[A_1(x) < J_1] \]

3. Hölder Exponent and Multifractal Spectrum

We first recall the definition of the Hölder exponent of a function. In the literature, two types of Hölder exponents have been considered, namely, the one based on exponential growth rate and the one based on polynomial approximation. In this paper, we shall emphasize the first one.

**Definition 3.1.** The Hölder exponent based on exponential growth rate of the function \( x \) at \( t \) is defined as

\[ h_x(t) := \liminf_{\varepsilon \downarrow 0} \frac{\log \sup_{u, |u-t| \leq \varepsilon} |x(u) - x(t)|}{\log \varepsilon}. \]

The Hölder exponent of the sum of two functions satisfies the following inequality.
Proposition 3.1. For two functions \( x \) and \( y \), we have
\[
(3.2) \quad h_{x+y}(t) \geq h_x(t) \land h_y(t).
\]
Furthermore, equality holds if \( h_x(t) \neq h_y(t) \).

Proof. By the triangle inequality, \(|x + y|(u) - (x + y)(t)| \leq |x(u) - x(t)| + |y(u) - y(t)|\) and hence
\[
\sup_{u:|u-t| \leq \varepsilon} |x + y|(u) - (x + y)(t)| \leq \sup_{u:|u-t| \leq \varepsilon} |x(u) - x(t)| + \sup_{u:|u-t| \leq \varepsilon} |y(u) - y(t)|.
\]
Since \( \log \varepsilon < 0 \), we have
\[
h_{x+y}(t) = \liminf_{\varepsilon \downarrow 0} \frac{\log \sup_{u:|u-t| \leq \varepsilon} |x + y|(u) - (x + y)(t)|}{\log \varepsilon} \\
\geq \liminf_{\varepsilon \downarrow 0} \frac{\log \left( \sup_{u:|u-t| \leq \varepsilon} |x(u) - x(t)| + \sup_{u:|u-t| \leq \varepsilon} |y(u) - y(t)| \right)}{\log \varepsilon} \\
(3.3) \quad = \liminf_{\varepsilon \downarrow 0} \frac{\log \sup_{u:|u-t| \leq \varepsilon} |x(u) - x(t)|}{\log \varepsilon} \wedge \liminf_{\varepsilon \downarrow 0} \frac{\log \sup_{u:|u-t| \leq \varepsilon} |y(u) - y(t)|}{\log \varepsilon} = h_x(t) \land h_y(t).
\]
Here is the explanation for the equality in (3.3): For any sequence, for which the limit for the quotient on the left side of (3.3) exists, choose a further subsequence such that limits for both the quotients on the right side of (3.3) also exist, and hence are greater than or equal to \( h_x(t) \) and \( h_y(t) \) respectively. Then the limit for the quotient on the left side will be equal to the minimum of the limits for \( x \) and \( y \), and hence greater than or equal to \( h_x(t) \land h_y(t) \). For the reverse inequality, assume without loss of generality that \( h_x(t) \leq h_y(t) \). Fix \( \varepsilon > 0 \). Choose a sequence such that the limit for the quotient on the right side corresponding to \( x \) is less than \( h_x(t) + \varepsilon \). Choose a further subsequence such that the limit for the quotient corresponding to \( y \) exists. Then the limit for the quotient on the left side along this subsequence is the minimum of the limits for the quotients corresponding to \( x \) and \( y \) along it, and hence is smaller than \( h_x(t) + \varepsilon \). Thus, the left side of (3.3) is smaller than \( h_x(t) + \varepsilon \), for all \( \varepsilon > 0 \). Hence the left side of (3.3) is smaller than or equal to \( h_x(t) = h_x(t) \land h_y(t) \).

Now assume \( h_x(t) \neq h_y(t) \) and without loss of generality assume \( h_x(t) < h_y(t) \). Now, observe \( h_{-y}(t) = h_y(t) \). Since \( x = (x+y) + (-y) \), we have, by previous discussion, \( h_x(t) \geq h_{x+y}(t) \land h_{-y}(t) = h_{x+y}(t) \land h_y(t) \). But since, by assumption, \( h_x(t) < h_y(t) \), we must have \( h_x(t) \land h_y(t) = h_x(t) \geq h_{x+y}(t) \). Thus, \( h_x(t) \land h_y(t) = h_{x+y}(t) \). \( \Box \)

Remark 3.1. A strict inequality may hold in (3.2) if \( h_x(t) = h_y(t) \). For example, consider \( x(t) = t^2 - t \) and \( y(t) = t^2 + t \). Then \( (x+y)(t) = 2t^2 \) and \( h_x(0) = h_y(0) = 1 \), but \( h_{x+y}(0) = 2 \).

For non-decreasing functions, the definition of \( h_x(t) \) in (3.1) simplifies to
\[
(3.4) \quad h_x(t) = \liminf_{\varepsilon \downarrow 0} \frac{\log((x(t+\varepsilon) - x(t)) \wedge (x(t) - x(t-\varepsilon)))}{\log \varepsilon} = \liminf_{\varepsilon \downarrow 0} \frac{\log(x(t+\varepsilon) - x(t-\varepsilon))}{\log \varepsilon}
\]

The above definition (3.4) helps us to conclude an equality in (3.2) if \( x \) and \( y \) are non-decreasing.

Proposition 3.2. If \( x \) and \( y \) are two non-decreasing functions, then \( h_{x+y}(t) = h_x(t) \land h_y(t) \).

Proof. In the case of non-decreasing functions, we use the representation (3.4) and then
\[
h_{x+y}(t) = \liminf_{\varepsilon \downarrow 0} \frac{\log((x+y)(t+\varepsilon) - (x+y)(t-\varepsilon))}{\log \varepsilon}
\]
\[
= \liminf_{\varepsilon \to 0} \frac{\log[(x(t + \varepsilon) - x(t - \varepsilon)) + (y(t + \varepsilon) - y(t - \varepsilon))]}{\log \varepsilon} \\
= \liminf_{\varepsilon \to 0} \frac{\log(x(t + \varepsilon) - x(t - \varepsilon))}{\log \varepsilon} \wedge \liminf_{\varepsilon \to 0} \frac{\log(y(t + \varepsilon) - y(t - \varepsilon))}{\log \varepsilon} = h_x(t) \wedge h_y(t).
\]

The penultimate equality holds due to an argument similar to that for (3.3). \qed

As mentioned earlier, there is another Hölder exponent based on polynomial approximation. We discuss this exponent in the following definitions.

**Definition 3.2.** We say a function \(x\) belongs to the class \(C_h\) at \(t\), and write \(x \in C_h(t)\), if there exists a polynomial \(P\) of degree at most \(h\), and \(\varepsilon > 0\), and \(C > 0\), such that
\[
|x(u) - P(u)| < C|u - t|^h, \text{ for all } u \text{ with } |u - t| < \varepsilon.
\]

**Definition 3.3.** The Hölder exponent based on polynomial approximation of the function \(x\) at the point \(t\) is defined as
\[
H_x(t) := \sup\{h \geq 0 : x \in C_h(t)\}.
\]

In general, we may only conclude that the Hölder exponent based on polynomial approximation for the sum of two functions is greater than or equal to the minimum of that of each individual one. The equality may fail to hold even if the functions are increasing, as given in the following example.

**Example 3.1.** Let us consider the functions
\[
x(t) = t + t^{(1,5)} \text{ and } y(t) = t - t^{(1,5)}, \text{ for } t \in (-\frac{1}{3}, \frac{1}{3}),
\]
where \(t^{(u)} = \text{sgn}(t)|t|^u\). It is easy to check that \(t^{(u)}\) is differentiable for \(u \geq 1\) and the derivative is \(u|t|^{u-1}\). Thus we have \(x'(t) = 1 + 1.5\sqrt{|t|} > 0\) and hence is increasing. Also \(y'(t) = 1 - 1.5\sqrt{|t|} > 0\) for \(t \in (-\frac{1}{3}, \frac{1}{3})\) and hence is increasing.

Observe \((x + y)(t) = 2t\) and clearly, \(H_{x+y} \equiv \infty\). On the other hand,
\[
|x(t) - t| = |t^{(1,5)}| = |t|^{1.5}
\]
and hence \(H_x(0) \geq 1.5\). Now, let \(0 < h < H_x(0)\) and \(P\) be the corresponding approximating polynomial. Clearly, \(P(0) = x(0) = 0\) and if the coefficient of the linear term is 1, then \(P(t) = t^2Q(t)\), where \(Q\) is a polynomial, and hence \(|f(t) - P(t)| = |t^{1.5} + t^2Q(t)| = O(t^{1.5})\) as \(t \to 0\). Thus \(h \leq 1.5\). If the coefficient of the linear term is not 1, then \(P(t) = t + tQ(t)\), for some polynomial \(Q\), and hence \(|f(t) - P(t)| = |t^{1.5} + tQ(t)| = O(t)\) as \(t \to 0\). Thus \(h \leq 1\). So we conclude \(H_x(0) = 1.5\). Similarly, it can be checked \(H_y(0) = 1.5\). Thus, we have \(H_{x+y}(0) = \infty > 1.5 = H_x(0) \wedge H_y(0)\).

This lack of equality makes the Hölder exponent based on polynomial approximation more difficult to analyze and we shall not emphasize it. Still it is worth mentioning that the following relation holds between two Hölder exponents.

**Proposition 3.3.** We have the following relation between two Hölder exponents:
\[
h_x(t) \leq H_x(t).
\]
Further, if \(h_x(t) \notin \mathbb{N}\), then \(h_x(t) = H_x(t)\).

**Proof.** See Lemma 2.3 of [34] and the discussion preceding it. \qed

We end the section by defining the multifractal spectrum of the Hölder exponent.
Definition 3.4. The multifractal spectrum of the function \( x \) for the Hölder exponent based on exponential growth rate is
\[
d_x(a) = \dim \{ t > 0 : h_x(t) = a \}, \quad a \in [0, \infty),
\]
where for a set \( \Lambda \), \( \dim(\Lambda) \) is the Hausdorff dimension (cf. Chapter 2 of [5]) of \( \Lambda \).

4. The Space \( \mathbb{D} \) and the \( M_1 \) Topology

In this section, we review the space \( \mathbb{D} \) endowed with the \( M_1 \) topology. Throughout this paper, we shall consider the space \( \mathbb{D} \) to be endowed with the \( M_1 \) topology, unless otherwise mentioned. A good reference for the topics considered in this section is the forthcoming book by Whitt [41]. We define \( \mathbb{D} \) to be the set of all càdlàg functions on \([0, \infty)\). We denote \( \mathbb{D}_T \) to be the set of all càdlàg functions on \([0, T]\). We first define the \( M_1 \) topology on \( \mathbb{D}_T \).

To define the metric, we need to define the completed graph \( \Gamma_x \) of a function \( x \in \mathbb{D}_T \), which is the set
\[
\Gamma_x = \{ (t, z) \in [0, T] \times \mathbb{R} : z = \alpha x(t-) + (1 - \alpha) x(t) \text{ for some } \alpha \in [0, 1] \}.
\]
\( \Gamma_x \) is a connected set in \( \mathbb{R}^2 \) obtained by connecting \((t, x(t-))\) and \((t, x(t))\). We define the natural order on the points of \( \Gamma_x \) as follows: for \((t_1, z_1), (t_2, z_2) \in \Gamma_x \), we say \((t_1, z_1) < (t_2, z_2)\) if \(t_1 < t_2\) or \(t_1 = t_2\) and \(|z_1 - x(t_1)| > |z_2 - x(t_2)|\), i.e., we rank the points in an increasing order as we traverse the completed graph from the left end \((0, x(0))\) to the right end \((T, x(T))\).

A parametric representation of the completed graph \( \Gamma_x \) is a continuous non-decreasing (according to the above-defined order) function \((r(\cdot), u(\cdot))\) mapping \([0, 1]\) onto \( \Gamma_x \). Let \( \Pi_x \) be the collection of all parametric representations of \( \Gamma_x \). Then the metric giving the \( M_1 \) topology on \( \mathbb{D}_T \) is given by
\[
d_T(x_1, x_2) = \inf_{(r_j, u_j) \in \Pi_{x_j}} \| r_1 - r_2 \| \vee \| u_1 - u_2 \|,
\]
where for a function \( f : [0, 1] \to \mathbb{R} \) we define
\[
\| f \| = \sup_{t \in [0, 1]} |f(t)|.
\]
Finally we define the \( M_1 \) metric on \( \mathbb{D} \) as:
\[
d(x_1, x_2) = \int_0^\infty e^{-t} \| d_t(x_1, x_2) \wedge 1 \| dt.
\]

We also denote the space of non-decreasing and unbounded càdlàg functions by \( \mathbb{D}_T \) and endow it with the relative topology of \( M_1 \) topology on \( \mathbb{D} \). We use the following results on the \( M_1 \) topology in the sequel.

Theorem 4.1. The relative topology of \( \mathbb{D}_T \) is the topology given by pointwise convergence on a dense subset of \([0, \infty)\) including 0.

See Corollary 12.5.1 of [41] for a proof.

Corollary 4.1. If \( X_k, k \geq 0 \) are random processes taking values in the space \( \mathbb{D}_T \) with the \( M_1 \) topology, then \( X_k \Rightarrow X_0 \) if \( X_k \overset{fd}{\Rightarrow} X_0 \), where the above convergence is in the sense of weak convergence of finite dimensional distributions on \([0, \infty)\).

Theorem 4.2. The evaluation map \( \pi : [0, \infty) \times \mathbb{D} \to \mathbb{R} \) defined by \( \pi(t, x) = x(t) \) is continuous at \((t, x)\) iff \( x \) is continuous at \( t \).

See Lemma 12.5.1 of [41] for a proof.
Corollary 4.2. The projection map $\pi_{t_1,\ldots,t_k}$ defined by $\pi_{t_1,\ldots,t_k}(x) = (x(t_1),\ldots,x(t_k))$ is continuous at $x$ if $x$ is continuous at $(t_1,\ldots,t_k)$.

Theorem 4.3. The right continuous inverse map $\Phi$ defined by $\Phi(x) = x^\rightarrow$, where $x^\rightarrow$ is as defined in (2.2), is continuous on $\mathbb{D}_r$.

The proof follows trivially from the theorem in Section 2 of [40].

As we have seen in the definition (2.5) of $L_k$, we shall have opportunity to use the left continuous right limit, i.e., cagràd functions as well. Let us denote the space of càglàd functions on $[0,\infty)$ by $\mathbb{D}$. We endow this space with $M_1$ topology as well, which is defined similarly through the completed graph and increasing parameterization. Notice that for a function $x$, the left continuous inverse $x^-$, defined by (2.1) and the right continuous inverse $x^\rightarrow$ differ only by the value at the jump points of $x$. Thus they have the same completed graphs. Hence, for a sequence of functions $\{x_n\}_{n \geq 0}$, we have $x_n^- \rightarrow x^-$ in the $M_1$ topology iff $x_n^\rightarrow \rightarrow x^\rightarrow$ in the $M_1$ topology. So, we have the following corollary to Theorem 4.3, where $\mathbb{D}_r$ is the space of unbounded increasing càglàd functions:

Corollary 4.3. The left continuous inverse map $\tilde{\Phi}$ defined by $\tilde{\Phi}(x) = x^-$ is continuous on $\mathbb{D}_r$.

The following result extends Lamperti’s theorem (cf. Theorem 2 of [19] and Section 2 of [3]) where the convergence is extended to happen in $\mathbb{D}$ with $M_1$ topology.

Theorem 4.4. Suppose $Z$ is a process taking values in $\mathbb{D}$ endowed with the $M_1$ topology. If for some function $\sigma$ and some process $\zeta$ with values in $\mathbb{D}$, which is proper, i.e., $\zeta(t)$ has non-degenerate distribution for all $t > 0$, we have

$$\frac{Z(T)}{\sigma(T)} \Rightarrow \zeta(\cdot),$$

then there exists $H > 0$, such that $\sigma \in RV_H$ and $\zeta$ is $H$-self-similar ($H$-ss). Also, $\zeta$ is continuous in probability.

The proof of Theorem 4.4 is along the same lines of [3], since the proofs in section 2 of [3] depends only on finite dimensional convergence on a dense set.

Remark 4.1. Note that Corollary 4.2 implies finite dimensional convergence on a dense set of points $C_\zeta$, which is the set of points where $\zeta$ is continuous in probability, unlike in Lamperti’s theorem, where the finite dimensional convergence takes place on the entire half line $[0,\infty)$. However, Theorem 4.4 allows us to conclude the convergence in finite dimensional distributions, since $\zeta$ is continuous in probability.

5. Model Specification

In this section, we state the assumptions of the model. First we recall the basic assumptions stated in Section 1.

1. We denote the time when $k$-th transmission begins by $\Gamma_k$. $\{\Gamma_k\}$ is a sequence strictly increasing to $\infty$.
2. The size of the file transmitted is $J_k$ and we assume $J_k > 0$.
3. The transmission schedule is denoted by $A_k(\cdot)$, where $A_k(t)$ denotes the amount of data transmitted in time $t$ after the $k$th transmission has begun. It is a non-decreasing càdlàg function starting at 0 and increasing to $\infty$, which vanishes on the negative real axis.

5.1. Small time scale behavior. To study the behavior of the cumulative traffic process $X(\cdot)$ for small time scales, we need to make the following further minimal assumptions on the transmission schedule $\{A_k\}$:

1. We assume $\{A_k\}$ are identically distributed and have stationary increments.
(5) The multifractal spectrum of $A_k(\cdot)$ is not degenerated to a single point, which ensures that we consider processes with paths that show real multifractal behavior.

(6) The multifractal spectrum of $A_k(\cdot)$ restricted to any (non-random) interval is non-random.

Remark 5.1. If $A_k$ is, for example, an increasing Lévy process, then, restricted to any interval, it has a non-random multifractal spectrum for the Hölder exponent based on exponential growth rate. (Jaffard [14] shows that the multifractal spectrum of a Lévy process restricted to $[0, 1]$ for the Hölder exponent based on polynomial approximation is non-random. The same proof works for any interval. By Proposition 3.3, an upper bound for the Hölder exponent $H$ is also an upper bound for the Hölder exponent $H$. Also the lower bounds for $H$ in Proposition 2 of Jaffard’s work [14] works for $h$ since the approximating polynomial for those lower bounds are constant.)

The following theorem summarizes the small time scale behavior of the cumulative traffic process.

Theorem 5.1. If the assumptions (1) – (6) hold, then with probability 1, $d_X = d_{A_1}$.

So the aggregate traffic process $X(\cdot)$ inherits the multifractal structure of the individual transmission schedules. This implies that a possible explanation for observed multifractality in aggregate traffic is intermittency of individual transmissions, presumably caused by blocking and congestion. See [23, 29].

5.2. Large time scale behavior. For multifractal analysis, we study the process path by path. However, for large time scales, we need to make additional distributional assumptions, which we summarize as follows:

(7) We assume $\{\Gamma_k\}$ forms a homogeneous Poisson process with intensity parameter $\lambda$.

(8) We also assume $\{(A_k, J_k) : k \geq 1\}$ are i.i.d. and independent of $\{\Gamma_k\}$.

(9) Let us define

$$\tilde{A}_1^{(T)}(\cdot) = A_1(T \cdot)\text{.}$$

We assume there exists a regularly varying function $\tilde{\sigma}$ of index $H$ and a proper random process $\chi$ with stationary increments, taking values in $\mathbb{D}$, such that for each fixed $\varepsilon > 0$,

$$\frac{1}{F_j(\tilde{\sigma}(T))} \mathbb{P} \left[ \frac{J_1}{\tilde{\sigma}(T)} > \varepsilon, \frac{\tilde{A}_1^{(T)}}{\tilde{\sigma}(T)} \in \cdot \right] \xrightarrow{w} \varepsilon^{-a_j} \mathbb{P}[\chi \in \cdot]$$

on $\mathbb{D}$, where the above convergence is in the sense of weak convergence of finite measures.

Remark 5.2. Observe that by (3), we have, for $t > 0$, $\tilde{A}_1^{(T)}(t)$ goes to $\infty$ with probability 1, as $T \to \infty$. Thus, we must have $\tilde{\sigma}(T) \to \infty$, as otherwise $\tilde{A}_1^{(T)}(\cdot) / \tilde{\sigma}(T)$ goes to a function identically equal to $\infty$ with probability 1, which contradicts (5.1). Hence, we also have $H \geq 0$.

Remark 5.3. Let $\Lambda$ be a Borel subset of $\mathbb{D}$, such that $\mathbb{P}[\chi \in \partial \Lambda] = 0$, where $\partial \Lambda$ is the boundary of $\Lambda$. Then, the convergence

$$\frac{1}{F_j(\tilde{\sigma}(T))} \mathbb{P} \left[ \frac{J_1}{\tilde{\sigma}(T)} > \varepsilon, \frac{\tilde{A}_1^{(T)}}{\tilde{\sigma}(T)} \in \Lambda \right] \xrightarrow{w} \varepsilon^{-a_j} \mathbb{P}[\chi \in \Lambda]$$

is locally uniform in $\varepsilon \in (0, \infty]$, since the converging functions are monotone in $\varepsilon$ and the limit is continuous in $\varepsilon$ with a finite limit at $\infty$ (cf. Section 0.1 of [33]).

Remark 5.4. Note that, in assumption (4), we have only assumed that the marginal distribution of $A_1$ has stationary increments, which is not enough to conclude that $\chi$ has stationary increments. So we include that as part of the assumption (9).
(10) For all \( \gamma > 0 \), assume
\[
\lim_{\epsilon \downarrow 0} \limsup_{T \to \infty} \frac{1}{F_J(\tilde{\sigma}(T))} P \left[ \frac{J_1}{\tilde{\sigma}(T)} \leq \epsilon, \frac{L_1}{T} > \gamma \right] = 0.
\]
Recall that \( L_1 \) is defined in (2.5).

(11) We finally assume that
\[
E \left[ \chi(1)^{-\alpha_J} \right] < \infty.
\]

Remark 5.5. Our assumptions are not restrictive but do require \( A_1(0) = 0 \). This might be an impediment to the modeling of the download of cached files where one might prefer to allow \( P[A_1(0) > 0] > 0 \).

The following theorem summarizes the large time scale behavior of the input process.

Theorem 5.2. Suppose the assumptions (1) – (3) and (7) – (11) hold and define
\[
Y_T(t) = \frac{X(Tt) - XTt E(J_1)}{b_J(T)}.
\]
Then we have
\[
Y_T \overset{\text{fd}}{\to} Z_{\alpha_J},
\]
where \( Z_{\alpha} \) is mean 0, skewness 1, \( \alpha \)-stable Lévy motion with scale parameter \( \left( \frac{\alpha}{\alpha} \right)^\frac{1}{\alpha} \), and
\[
C_{\alpha} = \frac{1 - \alpha}{\Gamma(2 - \alpha) \cos \left( \frac{\pi \alpha}{2} \right)}.
\]

We now amplify the implications of the assumptions (7) – (11).

Proposition 5.1. If (5.1) holds, then \( \chi \) is \( H \)-self-similar.

Proof. First observe that from (5.1), by considering the set \( D \), we have
\[
\lim_{T \to \infty} \frac{F_J(\tilde{\sigma}(T)\epsilon)}{F_J(\tilde{\sigma}(T))} = \epsilon^{-\alpha_J}
\]
and by Remark 5.3 the convergence is locally uniform in \( \epsilon \in (0, \infty) \). Thus we further have
\[
\frac{F_J(\tilde{\sigma}(Ts))}{F_J(\tilde{\sigma}(T))} \to s^{-H\alpha_J},
\]
since \( \tilde{\sigma} \in RV_H \).

Using (5.1) and Corollary 4.2, we have
\[
\frac{1}{F_J(\tilde{\sigma}(T))} P \left[ \frac{J_1}{\tilde{\sigma}(T)} > \epsilon, \frac{A_1(Tu)}{\tilde{\sigma}(T)} \in A \right] \overset{w}{\to} \epsilon^{-\alpha_J} P[\chi(u) \in \cdot]
\]
for all \( u \) such that \( \chi \) is continuous in probability at \( u \). By Remark 5.3, for every fixed Borel subset \( \Lambda \) of \( \mathbb{R} \) with \( P[\chi(u) \in \partial \Lambda] = 0 \), the convergence in (5.6)
\[
\frac{1}{F_J(\tilde{\sigma}(T))} P \left[ \frac{J_1}{\tilde{\sigma}(T)} > \epsilon, \frac{A_1(Tu)}{\tilde{\sigma}(T)} \in \Lambda \right] \overset{w}{\to} \epsilon^{-\alpha_J} P[\chi(u) \in \Lambda]
\]
is locally convergent in \( \epsilon \in (0, \infty) \). Let \( s, t \geq 0 \) be such that \( \chi \) is continuous in probability at \( t \) and \( st \). Then, from (5.5), (5.6) and local uniform convergence, we have
\[
\frac{1}{F_J(\tilde{\sigma}(Ts))} P \left[ \frac{J_1}{\tilde{\sigma}(Ts)} > 1, \frac{A_1(Tst)}{\tilde{\sigma}(Ts)} \in \cdot \right] \overset{w}{\to} P[\chi(t) \in \cdot]
\]
and
\[
\frac{1}{F_J(\hat{\sigma}(T_s))} P \left[ \frac{J_1}{\hat{\sigma}(T_s)} > 1, \frac{A_1(T_{st})}{\hat{\sigma}(T)} \in \cdot \right] = \frac{\hat{F}_J(\hat{\sigma}(T_s))}{F_J(\hat{\sigma}(T_s))} \frac{1}{F_J(\hat{\sigma}(T))} P \left[ \frac{J_1}{\hat{\sigma}(T)} > \hat{\sigma}(T_s), \frac{A_1(T_{st})}{\hat{\sigma}(T)} \in \cdot \right] \to s^{H_{\sigma}}_H \sigma \P(\chi(st) \in \cdot) = \P(\chi(st) \in \cdot),
\]
where we also use the fact that \( \hat{\sigma} \in RV_H \). Now define the distribution functions
\[
G_T(x) = \frac{1}{F_J(\hat{\sigma}(T_s))} P \left[ \frac{J_1}{\hat{\sigma}(T_s)} > 1, A_1(T_{st}) \leq x \right].
\]
Then from (5.7) and (5.8), we get
\[
G_T(\hat{\sigma}(T_s) \cdot) \to \P(\chi(t) \leq \cdot)
\]
and
\[
G_T(\hat{\sigma}(T) \cdot) \to \P(\chi(st) \leq \cdot)
\]
Then by the Convergence of Types Theorem (cf. Proposition 0.3 of [33]), we have
\[
\frac{\hat{\sigma}(T_s)}{\hat{\sigma}(T)} \to C
\]
and
\[
\P(\chi(t) \leq x) = \P(\chi(st) \leq Cx).
\]
However, \( \hat{\sigma} \) being a regularly varying function of index \( H \), we must have \( C = s^{H} \). Thus, we have
\[
\chi(st) \stackrel{d}{=} s^{H} \chi(t), \text{ for all } s, t \geq 0, \text{ such that } \chi \text{ is continuous in probability at } st \text{ and } t.
\]
Since the points, where \( \chi \) is continuous in probability, are dense in \([0, \infty)\) (cf. pg. 138 of [1]) and \( \chi \) has càdlàg paths, we can conclude that \( \chi(st) \stackrel{d}{=} s^{H} \chi(t) \), for all \( s, t \geq 0 \). Similarly, using Corollary 4.2 and the multivariate analog of the Convergence of Types Theorem (cf. pg. 28 of [9]), we have that \( (\chi(st_1), \ldots, \chi(st_k)) \stackrel{d}{=} s^{H}(\chi(t_1), \ldots, \chi(t_k)) \) for \( t_1, \ldots, t_k \in [0, \infty) \). So we conclude that \( \chi \) is \( H \)-ss.

**Remark 5.6.** From the assumption (9), we know that \( \chi \) has stationary increments. Thus \( \chi \) is an \( H \)-self-affine (\( H \)-sa) process, i.e., an \( H \)-ss process with stationary increments. Since \( \chi \) is càdlàg (and hence has a measurable version, cf. Theorem 2.6 of [2]) and non-degenerate, using Lemma 1.2 and Theorem 1.3 of [38] we also have \( H > 0 \).

Further observe that \( \mathbb{D} \) is a closed subset of \( \mathbb{D} \) with \( M_1 \) topology. Since by assumption \( A_1 \) has almost surely non-decreasing paths, the finite measures on the left side of (5.1) have support \( \mathbb{D}_1 \). Thus from the convergence in (5.1) with \( \varepsilon = 1 \), we conclude that the distribution of \( \chi \) is supported on \( \mathbb{D}_1 \) and so \( \chi \) has almost surely non-decreasing paths.

Since \( \chi \) is proper, we have \( \P(\chi \equiv 0) = 0 \), and since \( \chi \) is a non-decreasing, \( H \)-sa process, we have, from Theorem 2.1 of [39], that \( H \geq 1 \). Also, in case \( H = 1 \), we have \( \chi(t) \equiv t \chi(1) \) almost surely, which is the case of random but time-invariant transmission rate. Such case has been considered in [24], though under a slightly different set of hypotheses. Levy, Pipiras and Taqqu [21, 27, 28] also considered a similar case for superposition of on-off processes in context of renewal-reward processes. However, since in the case \( H = 1 \), the paths are almost everywhere linear and hence non-fractal, they are excluded from the current discussion and we assume \( H > 1 \). For the case \( H > 1 \), we have, from Theorem 3.1 of [39], that \( E[\chi(p)] = \infty \) for \( p \geq \frac{1}{H} \), i.e., \( \chi(1) \) has infinite mean.
Remark 5.7. Since we know from Remark 5.6 that $H > 0$, we can replace $\tilde{\sigma}$ in (5.1) with a strictly increasing and continuous regularly varying function $\sigma \sim \tilde{\sigma}$ (cf. Proposition 0.8(vii) of [33]).

In the following lemma, we study the tail behavior of $J_1$.

Lemma 5.1. Under the assumption (9), $J_1$ has a tail of index of $\alpha_J$.

Proof. We have already seen that the limit in (5.5) converges locally uniformly in $\varepsilon \in (0, \infty]$. So, we can have, using the fact from Remark 5.7 that $\sigma \sim \tilde{\sigma}$,

$$\lim_{T \to \infty} \frac{\tilde{F}_j(\sigma(T)\varepsilon)}{\tilde{F}_j(\tilde{\sigma}(T))} = \varepsilon^{-\alpha_J}.$$  

Then evaluating at $\varepsilon = 1$, we have

$$\tilde{F}_j \circ \sigma \sim \tilde{F}_j \circ \tilde{\sigma}$$

and hence we have

$$\lim_{T \to \infty} \frac{\tilde{F}_j(\sigma(T))}{\tilde{F}_j(\tilde{\sigma}(T))} = \varepsilon^{-\alpha_J}.$$  

Now, from Remarks 5.2 and 5.7, we know that $\sigma(T)$ increases to $\infty$ continuously. Hence we conclude that $\tilde{F}_j \in RV_{-\alpha_J}$, i.e., $J_1$ has a tail of index $\alpha_J$. \hfill \Box

Remark 5.8. Since $J_1$ has a tail of index $\alpha_J$, following the discussion in Section 2, we may choose a continuous, strictly increasing function $b_J$, which is regularly varying of index $\alpha_J^{-1}$, such that for all $w > 0$, we have

$$\lim_{T \to \infty} TP[J_1 > b_J(T)w] = w^{-\alpha_J}.$$  

Remark 5.9. Using (5.9), we can replace $\tilde{F}_j(\tilde{\sigma}(T))$ in (5.1) with $\tilde{F}_j(\sigma(T))$. Using Remark 5.3 we can also replace $\frac{\tilde{A}_j}{\sigma(T)}$ by $\frac{\tilde{A}_1}{\tilde{\sigma}(T)}$, to obtain

$$\frac{1}{\tilde{F}_j(\sigma(T))} P \left[ \frac{J_1}{\sigma(T)} > \varepsilon, \frac{\tilde{A}_1}{\tilde{\sigma}(T)} \in \cdot \right] \overset{w}{\sim} \varepsilon^{-\alpha_J} P[\chi \in \cdot]$$

on $\mathbb{D}$. For $\mathbf{t} = (t_1, \ldots, t_k)$ and $\mathbf{x} = (x_1, \ldots, x_k)$, let us define

$$G^{(T)}_t(\mathbf{x}) = \frac{1}{\tilde{F}_j(\sigma(T))} P \left[ \frac{J_1}{\sigma(T)} > \varepsilon, \frac{\tilde{A}_1(t_1)}{\tilde{\sigma}(T)} \leq x_1, \ldots, \frac{\tilde{A}_1(t_k)}{\tilde{\sigma}(T)} \leq x_k \right]$$

and

$$G^{(0)}_t(\mathbf{x}) = \varepsilon^{-\alpha_J} P[\chi(t_1) \leq x_1, \ldots, \chi(t_k) \leq x_k].$$

Since $\chi$ is $H$-ss, it is continuous in probability, and so the convergence in (5.11) implies, by Corollary 4.2, that

$$\lim_{T \to \infty} G^{(T)}_t(\mathbf{x}) = G^{(0)}_t(\mathbf{x})$$

for all $\mathbf{t}$ and continuity points $\mathbf{x}$ of $G^{(0)}_t(\cdot)$. It is also easily checked that for all continuity points $\mathbf{x}$ of $G^{(0)}_t(\cdot)$, if $(x_{T,1}, \ldots, x_{T,k}) := \mathbf{x}_T \to \mathbf{x}$, then

$$\lim_{T \to \infty} G^{(T)}_t(\mathbf{x}_T) = G^{(0)}_t(\mathbf{x}).$$
Thus, we get, as $T \to \infty$,
\[
\frac{1}{F_J(\sigma(T))} \mathbb{P} \left[ \frac{J_1}{\sigma(T)} > \varepsilon, \frac{A_1^{(T)} (t_1)}{\sigma(T)} \leq x_1, \ldots, \frac{A_1^{(T)} (t_k)}{\sigma(T)} \leq x_k \right] = G_t^{(T)} \left( \frac{\tilde{\sigma}(T)}{\sigma(T)} \mathbf{x} \right) \to G_t^{(0)}(\mathbf{x})
\]
for all $t$ and for all continuity points $\mathbf{x}$ of $G_t^{(0)}(\cdot)$. Then, by Corollary 4.1, we conclude that
\[
(5.12) \quad \frac{1}{F_J(\sigma(T))} \mathbb{P} \left[ \frac{J_1}{\sigma(T)} > \varepsilon, \frac{A_1^{(T)}}{\sigma(T)} \in \cdot \right] \xrightarrow{\varepsilon \to 0} e^{-\alpha J} \mathbb{P} [\chi \in \cdot]
\]
on $\mathbb{D}$.

Remark 5.10. Using (5.9), we can replace $\tilde{F}_J(\tilde{\sigma}(T))$ in (5.2) with $\tilde{F}_J(\sigma(T))$ to get
\[
\lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{1}{F_J(\sigma(T))} \mathbb{P} \left[ \frac{J_1}{\sigma(T)} \leq \varepsilon, \frac{L_1}{T} > \gamma \right] = 0.
\]

Also, since $\sigma \sim \tilde{\sigma}$, we have,
\[
\lim_{T \to \infty} \frac{1}{F_J(\sigma(T))} \mathbb{P} \left[ \frac{J_1}{\sigma(T)} \leq \varepsilon, \frac{L_1}{T} > \gamma \right] = \limsup_{T \to \infty} \frac{1}{F_J(\sigma(T))} \mathbb{P} \left[ \frac{J_1}{\tilde{\sigma}(T)} \leq \sigma(T), \frac{L_1}{T} > \gamma \right]
\]
\[
\leq \limsup_{T \to \infty} \frac{1}{F_J(\sigma(T))} \mathbb{P} \left[ \frac{J_1}{\sigma(T)} \leq 2 \varepsilon, \frac{L_1}{T} > \gamma \right].
\]

Hence, we get, for all $\gamma > 0$,
\[
(5.13) \quad \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{1}{F_J(\sigma(T))} \mathbb{P} \left[ \frac{J_1}{\sigma(T)} \leq \varepsilon, \frac{L_1}{T} > \gamma \right] = 0.
\]

Recall that $\sigma$ is a regularly varying function of index $H$. We know that $b_J$ is also a regularly varying function of index $\alpha_J^{-1}$. So $\sigma^{-1} \circ b_J$ is a regularly varying function of index $(H \alpha_J)^{-1}$ continuously increasing to $\infty$.

Then taking the limits in (5.12) and (5.13) along $\sigma^{-1}(b_J(T))$, we get the following equivalent formulations of the assumptions (9) and (10):

(9') Define
\[
A_1^{(T)}(\cdot) = A_1(\sigma^{-1}(b_J(T)) \cdot).
\]

There exists a regularly varying function $\sigma$ of index $H$ and a $\mathbb{D}$-valued random process $\chi$ with stationary increments which is also proper, such that for each fixed $\varepsilon > 0$,
\[
(5.14) \quad \mathbb{T} \mathbb{P} \left[ \frac{J_1}{b_J(T)} > \varepsilon, \frac{A_1^{(T)}}{b_J(T)} \in \cdot \right] \xrightarrow{\varepsilon \to 0} e^{-\alpha J} \mathbb{P} [\chi \in \cdot]
\]
on $\mathbb{D}$.

(10') For all $\gamma > 0$, assume
\[
(5.15) \quad \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \mathbb{T} \mathbb{P} \left[ \frac{J_1}{b_J(T)} \leq \varepsilon, \frac{L_1}{\sigma^{-1}(b_J(T))} > \gamma \right] = 0.
\]

Henceforth we shall use the assumptions (9) and (10) interchangeably with the assumptions (9') and (10').

The following lemma shows that the assumption (9') is not vacuous.

Lemma 5.2. If $J_1$ and $A_1$ are independent, then (5.14) holds iff

(i) $J_1$ has tail of index $\alpha_J$ and
(ii) $A_1$ belongs to the domain of a random càdlàg process $\chi$, which is proper, i.e., there exists a function $\sigma$ such that
\[
\frac{A_1(T)}{\sigma(T)} \xrightarrow{\text{fd}} \chi(\cdot).
\]

Remark 5.11. If $A_1$ belongs to the domain of a càdlàg process, then, by Theorem 2 of [19], we must necessarily have $\sigma \in RV_H$ and $\chi$ is $H$-ss.

Proof of Lemma 5.2. Observe that (5.14) reads for $\varepsilon = 1$:
\[
TP \left[ \frac{J_1}{b_J(T)} > 1, \frac{A_1(T)}{b_J(T)} \in \cdot \right] = TP \left[ \frac{J_1}{b_J(T)} > 1 \right] P \left[ \frac{A_1(T)}{b_J(T)} \in \cdot \right] \xrightarrow{\text{w}} P[\chi \in \cdot]
\]
on $\mathbb{D}$. We have seen in Remark 5.8 that, if (5.14) and hence (5.1) holds, then (i) holds and, hence in particular, $TP \left[ \frac{J_1}{b_J(T)} > 1 \right] \rightarrow 1$. Thus, if $J_1$ and $A_1$ are indeed independent, from (5.16), we have
\[
\frac{A_1(\sigma^+(b_J(T)))}{b_J(T)} \Rightarrow \chi
\]
on $\mathbb{D}$. Taking the above limit along $(\bar{F}_J(\sigma(T)))^{-1}$ instead of $T$ and using finite dimensional convergence and locally uniform convergence arguments as in Remark 5.9, we get
\[
\frac{A_1(T)}{\sigma(T)} \Rightarrow \chi(\cdot)
\]
on $\mathbb{D}$. Also, since $\chi$ is $H$-ss (from Proposition 5.1), $\chi$ has no fixed point of discontinuity. Since the projection map $\pi_{t_1,t_2,\ldots,t_k}(x) = (x(t_1), x(t_2), \ldots, x(t_k))$ is continuous at continuity points $t_1, t_2, \ldots, t_k$ of $\chi$, by the Continuous Mapping Theorem, we have the required finite dimensional convergence. This completes the proof of “only if” part.

Conversely, assume (i) and (ii) hold. From (ii), we can conclude that
\[
\frac{A_1(\sigma^+(b_J(T)))}{b_J(T)} \xrightarrow{\text{fd}} \chi
\]
and then using Corollary 4.1, we have
\[
\frac{A_1(T)}{b_J(T)} \Rightarrow \chi.
\]
Also, since $J_1$ has tail of index $\alpha_J$, (5.10) holds and then using independence of $J_1$ and $A_1$, we conclude that (5.14) holds. This completes the proof of “if” part.

The following proposition gives a set of sufficient conditions for conditions (5.14), (5.15) and (5.3) to hold.

**Proposition 5.2.** Assume that
(i) $J_1$ and $A_1$ are independent.
(ii) $J_1$ has a tail of index $\alpha_J$.
(iii) $A_1$ is itself a proper $H$-ss process.
(iv) $E[A_1(1)^{-\rho}] < \infty$ for some $\rho > \alpha_J$.

Then (5.14), (5.15) and (5.3) hold.
Proof. It is trivial to check that a $H$-ss process $A_1$ is in the domain of attraction of itself corresponding to $\sigma(T) = T^H$. Thus Lemma 5.2 shows that, under independence of $J_1$ and $A_1$, (5.14) holds for the above $\sigma(T)$. Also (5.3), which states that $A_1(1)^{-1}$ has $\alpha_J$-th moment finite, holds since we have assumed the existence of an even higher moment in (iv). So we need to check (5.15) only. Note that $\sigma^{-1}(T) = T^{1/H}$. Also, by self-similarity, $E[A_1(\gamma)^{-\rho}] < \infty$.

Observe that,

$$TP \left[ \frac{J_1}{J_1(b_j(T))} \leq \epsilon, \frac{A_1^{-1}(J_1)}{\sigma^{-1}(J_1)} > \gamma \right] = TP \left[ \frac{A_1\left(\frac{b_j(T)}{b_j(T)} \right)}{b_j(T)} < \frac{J_1}{b_j(T)} \leq \epsilon \right]$$

$$= TP \left[ \frac{A_1(\gamma)}{b_j(T)} < \frac{J_1}{b_j(T)} \leq \epsilon \right] \text{ by self-similarity}$$

$$= T \int_0^\epsilon P[A_1(\gamma) < s] F_j(b_j(T)ds)$$

$$= T \int_0^\epsilon P \left[ A_1(\gamma)^{-1} > s^{-1} \right] F_j(b_j(T)ds)$$

$$\leq E[A_1(\gamma)^{-\rho}] T \int_0^\epsilon s^\rho F_j(b_j(T)ds)$$

$$= E[A_1(\gamma)^{-\rho}] \frac{T}{b_j(T)^\rho} \int_0^{b_j(T)\epsilon} s^\rho F_j(ds)$$

$$\sim E[A_1(\gamma)^{-\rho}] \frac{T}{b_j(T)^\rho} \frac{\alpha_J}{\rho - \alpha_J} (b_j(T)\epsilon)^{\rho - \alpha_J} F_j(b_j(T)\epsilon)$$

$$\to E[A_1(\gamma)^{-\rho}] \frac{\alpha_J}{\rho - \alpha_J} e^{\rho - \alpha_J},$$

where the limit in (5.17) holds by Lemma on pages 578-579 of [8] as $\rho > \alpha_J$. Also, $\rho > \alpha_J$ again implies (5.15) holds. \qed

Remark 5.12. If we assume that $A_1$ is a $H$-ss process with stationary, independent increments, then $A_1(1)$ is a positive stable random variable of index $\frac{1}{H}$ and hence has a density which decays exponentially near 0 (cf. Theorem 2.5.2 of [43]) and so has all negative moments finite. Thus it satisfies the conditions of Proposition 5.2. Also Remark 5.1 shows that a Lévy process satisfies the conditions for the multifractal analysis. Thus a $\frac{1}{H}$-stable Lévy process satisfies the requirements of the transmission schedule.

6. Multifractal Analysis

In this section, we prove Theorem 5.1. We shall only use the assumptions made on the paths of the transmission schedule.

By the stationary increment property, the transmission schedule $A_1$ has same multifractal spectrum $d$ restricted to any interval of length $l$, since

$$\{A_1(t) : 0 \leq t < l\} \overset{d}{=} \{A_1(t) - A_1(a) : a \leq t < a + l\}$$
and the multifractal spectrum \( d \) of \( \{ A_1(t) : a \leq t < a+l \} \) is the same as that of \( \{ A_1(t) - A_1(a) : a \leq t < a+l \} \). Thus for any \( l \), the multifractal spectra \( d \) of \( A_1 \) restricted to the intervals \([il, (i+1)l]\) for \( i \geq 0 \) are non-random and the same. So for all \( l > 0 \), there is a probability 1 set, on which, for all \( i \geq 0 \), the multifractal spectra \( d \) of \( A_1 \) restricted to the intervals \([il, (i+1)l]\) are the same and independent of \( \omega \). Also, these spectra are the same as \( d_{A_1} \) a.e., since \( d_{A_1} \) is the supremum of the spectra \( d \) of \( A_1 \) restricted to countable partitioning intervals. Thus, combining all these facts, we may conclude the following lemma:

**Lemma 6.1.** If \( A_1 \) is a transmission schedule satisfying the conditions of the model, then there is a probability 1 set, on which for all \( i, n \in \mathbb{N} \), the multifractal spectrum of \( A_1 \) based on exponential growth rate restricted to the intervals \([\frac{i-1}{n}, \frac{i}{n}]\) is the same as that of \( d_{A_1} \) and independent of \( \omega \).

In the following lemma, we calculate the Hölder exponent of the input process, \( h_X \), bearing in mind Proposition 3.2.

**Lemma 6.2.** If \( t \) is not the time of birth or death of a session, then \( h_X(t) = \bigwedge_{k} h_{A_k}(t - \Gamma_k) \), where the minimum is taken over the indices corresponding to which sessions are active. (We use the convention that the minimum over an empty set is \( \infty \).)

**Proof.** First observe that, since \( X \) and the \( A_k \)'s are non-decreasing, we can use the definition in (3.4). Also observe that, till time \( t \), only a finite number of sessions have started, since \( \Gamma_k \to \infty \). Thus, at time \( t \), only a finite number of sessions, say \( n \), are active. Since \( t \) is not a birth or death time of a session, there exists \( \delta > 0 \) such that, only these \( n \) sessions transmit in the interval \((t-\delta, t+\delta)\). These transmitting sessions contribute terms of the form \( A_k(u - \Gamma_k) \wedge J_k = A_k(u - \Gamma_k) \), \( u \in (t-\delta, t+\delta) \) to the sum \( X(.) \). The non-transmitting sessions contribute terms of the form \( A_k(u - \Gamma_k) \wedge J_k \), which are constant at 0 or \( J_k \) for \( u \in (t-\delta, t+\delta) \) and in either case, the Hölder exponent will be \( \infty \) at \( t \). If there are no active sessions, then \( X \) itself is constant and \( h_X(t) = \infty \), the same as the minimum of the empty set. Otherwise, only the active sessions contribute to the calculation of the liminf in (3.4) and if \( i_1, \ldots, i_n \) are the indices of the active sessions, we have

\[
h_X(t) = \liminf_{\varepsilon \downarrow 0} \frac{\log \sum_{k=1}^{n} (A_{i_k}(t - \Gamma_{i_k} + \varepsilon) - A_{i_k}(t - \Gamma_{i_k} - \varepsilon))}{\log \varepsilon} = \bigwedge_{k=1}^{n} \liminf_{\varepsilon \downarrow 0} \frac{\log (A_{i_k}(t - \Gamma_{i_k} + \varepsilon) - A_{i_k}(t - \Gamma_{i_k} - \varepsilon))}{\log \varepsilon} = \bigwedge_{k=1}^{n} h_{A_{i_k}}(t - \Gamma_{i_k}).
\]

The penultimate equality holds due to an argument similar to that for (3.3). \( \square \)

Now we are ready to prove Theorem 5.1.

**Proof of Theorem 5.1.** Define for \( a < \infty \),

\[
E_a^X = \{ t > 0 : h_X(t) = a \}.
\]

Let \( E_a^* \) be the set of points of \( E_a^X \) except birth and death points of the sessions. Now observe that \( E_a^* \) differs from \( E_a \) only by countably many points and thus they have same Hausdorff dimension.

Now, suppose \( t \in E_a^* \). Then by Lemma 6.2, we have

\[
h_X(t) = \bigwedge_{k} h_{A_k}(t - \Gamma_k) = a,
\]

where the minimum is taken over the indices corresponding to the active sessions. If there are no active sessions, then \( X \) is constant in a neighborhood of \( t \) and hence has Hölder exponent \( \infty \) at \( t \),
contradicting the assumption that \( a < \infty \). Thus, there are a positive and finite number of sessions running at time \( t \) and there exists some \( k \) such that \( h_{A_k}(t - \Gamma_k) = a \), that is, \( t \in E_a^{A_k} + \Gamma_k \). So,

\[
d_X(a) \leq \sup_{k} \dim(E_a^{A_k} + \Gamma_k) = \sup_{k} d_{A_k}(a),
\]

where the last equality holds since Hausdorff dimension is translation-invariant. However, since the \( A_k \)'s are identically distributed and \( d_{A_k}(a) \) is non-random, we have \( d_{A_k}(a) \) non-random and constant (independent of both \( \omega \) and \( k \)). Thus,

\[
d_X(a) \leq d_{A_1}(a) \text{ a.e.}
\]

To prove the other inequality, consider the interval \( I = [\Gamma_1, (\Gamma_1 + \inf\{t : A_1(t) \geq J_1 \}) \wedge \Gamma_2) \). Since \( \Gamma_k \)'s are strictly increasing, \( A_1(0+) = 0 < J_1 \) a.e., we have that the interval \( I \) is almost surely non-empty. Also, only the first session is active on the interval \( I \). Thus, \( X(t) = A_1(t - \Gamma_1) \) and \( h_X(t) = h_{A_1}(t - \Gamma_1) \) on \( I \). Then,

\[
E_a^X \supset (E_a^{A_1} + \Gamma_1) \cap I = (E_a^{A_1} \cap I') + \Gamma_1,
\]

where \( I' = [0, \inf\{t : A_1(t) \geq J_1 \} \wedge (\Gamma_2 - \Gamma_1)] \). Thus,

\[
\dim(E_a^X) \geq \dim((E_a^{A_1} \cap I') + \Gamma_1) = \dim(E_a^{A_1} \cap I'),
\]

by the translation-invariance of Hausdorff dimension. We already know \( \inf\{t : A_1(t) \geq J_1 \} \wedge (\Gamma_2 - \Gamma_1) > 0 \) a.e. Then choose a probability 1 set on which the conclusions of Lemma 6.1 are satisfied and \( \inf\{t : A_1(t) \geq J_1 \} \wedge (\Gamma_2 - \Gamma_1) > 0 \) holds. For any \( \omega \) in this set, choose \( n \in \mathbb{N} \), such that,

\[
\inf\{t : A_1(t) \geq J_1 \} \wedge (\Gamma_2 - \Gamma_1) > \frac{1}{n}.
\]

Then we have, for that \( \omega \),

\[
d_X(a) \geq \dim(E_a^{A_1} \cap I') \geq \dim(E_a^{A_1} \cap [0, \frac{1}{n}]) = d_{A_1}(a),
\]

where the last equality holds by Lemma 6.1. Thus we have with probability 1, \( d_X \equiv d_{A_1} \). \( \square \)

7. **Large Time Scale Behavior**

7.1. **Poisson process representation.** We consider the following Poisson point process to facilitate the analysis:

\[
M = \sum_{k=1}^{\infty} \delta(\Gamma_k, A_k, J_k),
\]

which has mean measure \( \lambda d\gamma \times P[A_1 \in da, J_1 \in dj] \) on \((0, \infty) \times \mathbb{D} \times (0, \infty)\). The random variable \( X(T) \) is a function of \( M \) restricted to \( \mathcal{R}_T = \{ (\gamma, a, j) \in (0, \infty) \times \mathbb{D} \times (0, \infty) : \gamma < T \} \) and more precisely,

\[
X(T) = \sum_{k=1}^{\infty} [J_k \wedge A_k(T - \Gamma_k)] 1_{\mathcal{R}_T}(\Gamma_k, A_k, J_k).
\]

It helps to split \( \mathcal{R}_T \) into two disjoint sets

\[
\mathcal{R}_T^{(1)} = \{ (\gamma, a, j) \in (0, \infty) \times \mathbb{D} \times (0, \infty) : \gamma < T, j \leq a(T - \gamma) \}
\]

and

\[
\mathcal{R}_T^{(2)} = \{ (\gamma, a, j) \in (0, \infty) \times \mathbb{D} \times (0, \infty) : \gamma < T, j > a(T - \gamma) \}.
\]
\( \mathcal{R}_T^{(1)} \) and \( \mathcal{R}_T^{(1)} \) correspond to the regions where transmission has ended or is continuing respectively, by time \( T \). Correspondingly, the input process \( X \) breaks into two sums:

\[
X_1(T) = \sum_{k=1}^{\infty} J_k \mathbf{1}_{\mathcal{R}_T^{(1)}}(\Gamma_k, A_k, J_k)
\]

and

\[
X_2(T) = \sum_{k=1}^{\infty} A_k(T - \Gamma_k) \mathbf{1}_{\mathcal{R}_T^{(2)}}(\Gamma_k, A_k, J_k).
\]

Since \( X_i(T), i = 1, 2 \) are functions of \( M|_{\mathcal{R}_T^{(i)}} \), \( i = 1, 2 \) respectively with \( \mathcal{R}_T^{(1)} \cap \mathcal{R}_T^{(2)} = \emptyset \), we have \( X_1(T) \) and \( X_2(T) \) are independent.

Next we analyze the part restricted to \( \mathcal{R}_T^{(1)} \), where the transmission has ended. This part will contribute towards the limiting behavior, the other part due to \( \mathcal{R}_T^{(2)} \) will be probabilistically negligible.

Observe,

\[
E \left[ M \left( \mathcal{R}_T^{(1)} \right) \right] = \lambda \int_{\gamma=0}^{T} P[J_1 \leq A_1(T - \gamma)] \, d\gamma
\]

\[
= \lambda \int_{\gamma=0}^{T} P[J_1 \leq A_1(\gamma)] \, d\gamma
\]

\[
= \lambda \int_{\gamma=0}^{T} F_L(\gamma) \, d\gamma \quad \text{by (2.6)}
\]

\[
= : \lambda \hat{F}_L(T).
\]

Thus \( E \left[ M \left( \mathcal{R}_T^{(1)} \right) \right] \) is finite for all \( T \) and \( E \left[ M \left( \mathcal{R}_T^{(1)} \right) \right] \sim \lambda T \), as \( \hat{F}_L(T) \sim T \) and therefore \( M|_{\mathcal{R}_T^{(1)}} \) has the following representation:

\[
M|_{\mathcal{R}_T^{(1)}} \overset{d}{=} \sum_{k=1}^{P(T)} \mathcal{E} \left( \tau_k^{(T)}, S_k^{(T)}, W_k^{(T)} \right),
\]

where \( P(T) \) is a Poisson random variable with mean \( \lambda \hat{F}_L(T) \) independent of i.i.d. random vectors

\[
\left( \tau_k^{(T)}, S_k^{(T)}, W_k^{(T)} \right) \sim \frac{1}{\hat{F}_L(T)} \, d\gamma P[A \in da, J \in dj] \bigg|_{\mathcal{R}_T^{(1)}}.
\]

Then, we have

\[
X_1(T) \overset{d}{=} \sum_{k=1}^{P(T)} W_k^{(T)}.
\]
7.2. Tail behavior and moment conditions. Now we study the tail behavior of $W_1^T$.

**Proposition 7.1.** For the above representation (7.3), we have

$$\lim_{T \to \infty} TP \left[ W_1^{(T)} > b_J(T)w \right] = w^{-\alpha_J}.$$  

**Proof.** First observe that,

$$TP[W_1^{(T)} > b_J(T)w] = \frac{T}{F_L(T)} \int_0^T P[b_J(T)w < J_1 \leq A_1(T - \gamma)] d\gamma$$

(7.6)

$$= \frac{T}{F_L(T)} \int_0^T \gamma = 0 P[b_J(T)w < J_1 \leq A_1(\gamma)] d\gamma$$

(7.7)

$$\sim \int_{\gamma = 0}^T P[b_J(T)w < J_1 \leq A_1(\gamma)] d\gamma.$$

Thus, we have

$$\lim_{T \to \infty} \sup_T TP \left[ W_1^{(T)} > b_J(T)w \right] = w^{-\alpha_J}.$$  

(7.8)

Till now we have not used any assumption about the joint distribution of $(J_1, A_1)$. For the lower bound, we need to use (5.14) of the assumption (9'). Then, we have, from (7.7)

$$TP[W_1^{(T)} > b_J(T)w] \sim \int_{\gamma = 0}^T P[b_J(T)w < J_1 \leq A_1(\gamma)] d\gamma$$

$$= \frac{\sigma^{\leftarrow}(b_J(T))}{T} \int_{\gamma = 0}^T \frac{T}{F_L(T)} P[b_J(T)w < J_1 \leq A_1(\sigma^{\leftarrow}(b_J(T))\gamma)] d\gamma$$

(7.9)

$$\geq \frac{\sigma^{\leftarrow}(b_J(T))}{T} \int_{\gamma = N}^T \frac{T}{F_L(T)} P[b_J(T)w < J_1 \leq A_1(\sigma^{\leftarrow}(b_J(T))N)] d\gamma$$

$$= \left(1 - N \frac{\sigma^{\leftarrow}(b_J(T))}{T}\right) TP \left[w < \frac{J_1}{b_J(T)} \leq \frac{A_1(\sigma^{\leftarrow}(b_J(T))N)}{b_J(T)}\right]$$

(7.10)

$$\sim TP \left[w < \frac{J_1}{b_J(T)} \leq \frac{A_1(\sigma^{\leftarrow}(b_J(T))N)}{b_J(T)}\right]$$

(7.11)

$$\geq TP \left[w < \frac{J_1}{b_J(T)} \leq K, \frac{A_1(\sigma^{\leftarrow}(b_J(T))N)}{b_J(T)} \geq K\right]$$

(7.12)

$$\Rightarrow (w^{-\alpha_J} - K^{-\alpha_J}) P[\chi(N) \geq K]$$

(7.13)

$$= (w^{-\alpha_J} - K^{-\alpha_J}) P[\chi(1) \geq KN^{-H}].$$
The inequality in (7.9) holds for any $N$ and all large enough $T$. The asymptotic equivalence in (7.10) holds since $\frac{a_{\alpha,j}(b_j(T))}{T} \to 0$ as the function is regularly varying of index $\frac{1}{\alpha_j} - 1 < 0$. The inequality holds for all $K > w$. The convergence in (7.12) holds for all continuity points $K$ of the distribution of $\chi(N)$ due to (5.14). Finally, (7.13) holds due to the $H$-self-similarity of $\chi$. Thus, for all $N$ and for all $K > w$, which are continuity points of the distribution of $\chi(N)$ for all $N \in \mathbb{N}$, we have

$$\lim \inf_{T \to \infty} TP[W_1^{(T)} > Mb_j(T)] \geq (w^{-\alpha_j} - K^{-\alpha_j}) P[\chi(1) \geq KN^{-H}].$$

Since $\chi$ is proper and hence $P[\chi = 0] = 0$, we have, by Theorem 2.4(a) of [38], $P[\chi(1) > 0] = 1$. Then first letting $N \to \infty$ through natural numbers and then letting $K \to \infty$ through the continuity points of the distribution of $\chi(N)$, for all $N \in \mathbb{N}$, we get

$$\lim \inf_{T \to \infty} TP[W_1^{(T)} > Mb_j(T)] = w^{-\alpha_j}.$$ 

Combining this with (7.8), we have,

$$\lim_{T \to \infty} TP[W_1^{(T)} > Mb_j(T)] = w^{-\alpha_j}.$$ 

\[\square\]

Next we need to check a few moment conditions, which are summarized in the following lemmas.

**Lemma 7.1.** For the above representation (7.3), we have

$$(7.14) \quad \lim_{K \to \infty} \sup_{T} \text{TE} \left[ \frac{W_1^{(T)}}{b_j(T)} \frac{1}{\left[ W_1^{(T)} > K \right]} \right] = 0.$$ 

**Proof.** We have

$$(7.15) \quad \text{TE} \left[ \frac{W_1^{(T)}}{b_j(T)} \frac{1}{\left[ W_1^{(T)} > K \right]} \right] = KTP \left[ W_1^{(T)} > Kb_j(T) \right] + \int_{w=K}^{\infty} TP \left[ W_1^{(T)} > b_j(T)w \right] dw$$

First we consider the second term on the right side of (7.15). Note that

$$\int_{w=K}^{\infty} TP[W_1^{(T)} > b_j(T)w] dw = \frac{T}{F_L(T)} \int_{w=K}^{\infty} \int_{\gamma=0}^{T} P[b_j(T)w < J_1 \leq A_1(\gamma)] d\gamma dw \quad \text{by (7.6)}$$

$$\leq \frac{T^2}{F_L(T)} \int_{w=K}^{\infty} P[J_1 > b_j(T)w] dw$$

$$\sim 1 \cdot \frac{T}{b_j(T)} \frac{Kb_j(T) \ P[J_1 > Kb_j(T)]}{\alpha_j - 1} \quad \text{by Karamata’s theorem}$$

$$(7.16) \quad \sim \frac{1}{\alpha_j - 1} K^{1-\alpha_j}.$$ 

Also, for the first term of the right side of (7.15), we have, from (7.5),

$$(7.17) \quad \lim_{T \to \infty} KTP \left[ W_1^{(T)} > Kb_j(T) \right] = K^{1-\alpha_j}.$$
Thus, adding (7.17) and (7.16), and using (7.15), we have,

\[
\limsup_{T \to \infty} T \mathbb{E} \left[ W_1^{(T)} \right] \text{pr} \left[ \frac{w(T)}{b_j(T)} > K \right] \leq \frac{\alpha_j}{\alpha_j - 1} K^{1-\alpha_j}
\]

and, since \( \alpha_j > 1 \), we have (7.14).

\[\Box\]

**Lemma 7.2.** For the above representation (7.3), we have for all \( K > 0 \),

(7.18) \[
\limsup_{T \to \infty} T \mathbb{V} \left[ W_1^{(T)} \right] \text{pr} \left[ \frac{w(T)}{b_j(T)} \leq K \right] < \infty
\]

and

(7.19) \[
\lim_{\varepsilon \to 0} \limsup_{T \to \infty} T \mathbb{V} \left[ W_1^{(T)} \right] \text{pr} \left[ \frac{w(T)}{b_j(T)} \leq \varepsilon \right] = 0
\]

Proof. Using (7.6) and Karamata’s theorem, we have

\[
T \mathbb{V} \left[ W_1^{(T)} \right] \text{pr} \left[ \frac{w(T)}{b_j(T)} \leq K \right] \leq T \mathbb{E} \left[ \left( \frac{W_1^{(T)}}{b_j(T)} \right)^2 \right] \text{pr} \left[ \frac{w(T)}{b_j(T)} \leq K \right]
\]

\[
= \int_{w=0}^{K} 2wT \mathbb{P} \left[ w < \frac{W_1^{(T)}}{b_j(T)} \leq K \right] dw
\]

\[
= \frac{T}{F_L(T)} \int_{w=0}^{K} \int_{\gamma=0}^{T} 2w \mathbb{P} \left[ J_1 \leq A_1(\gamma), w < \frac{J_1}{b_j(T)} \leq K \right] d\gamma dw
\]

\[
\leq \frac{T}{F_L(T)} \int_{w=0}^{K} 2wT \mathbb{P} \left[ w < \frac{J_1}{b_j(T)} \leq K \right] dw
\]

\[
\sim 2T \int_{w=0}^{b_j(T)K} wT \mathbb{P} \left[ J_1 > wb_j(T) \right] dw - TK^2 \mathbb{P} \left[ J_1 > b_j(T)K \right]
\]

\[
\sim 2T \int_{w=0}^{b_j(T)K} wT \mathbb{P} \left[ J_1 > w \right] dw - K^{2-\alpha_j}
\]

\[
\sim 2TK^2 \mathbb{P} \left[ J_1 > b_j(T)K \right] - K^{2-\alpha_j}
\]

\[
\sim \frac{2}{2-\alpha_j} K^{2-\alpha_j} - K^{2-\alpha_j} = \frac{\alpha_j}{2-\alpha_j} K^{2-\alpha_j} < \infty
\]

and hence we have (7.18). Also, since \( \alpha_j < 2 \), we also have (7.19).
**Lemma 7.3.** For the above representation (7.3), we have

\[(7.20) \quad \lim_{T \to \infty} E \left[ \frac{W_1^{(T)}}{b_J(T)} \right] = 0. \]

**Proof.** Using (7.6), we have

\[
E \left[ \frac{W_1^{(T)}}{b_J(T)} \right] = \int_{w=0}^{\infty} P \left[ W_1^{(T)} > b_J(T)w \right] dw
\]

\[
= \frac{1}{F_L(T)} \int_{w=0}^{\infty} \int_{\gamma=0}^{T} P \left[ b_J(T)w < J_1 \leq A_1(\gamma) \right] d\gamma dw
\]

\[
\leq \frac{T}{F_L(T)} \int_{w=0}^{\infty} P \left[ b_J(T)w < J_1 \right] dw \sim \frac{E[J_1]}{b_J(T)} \to 0.
\]

\[\square\]

### 7.3. One-dimensional convergence.

Now we are ready to prove Theorem 5.2 for the process \(X_1\) in the one-dimensional case, although with a different centering.

**Lemma 7.4.** Under assumptions and notations used in Theorem 5.2, we have

\[
\frac{\sum_{k=1}^{P(T)} W_k^{(T)} - P(T) E \left[ W_1^{(T)} \right]}{b_J(T)} \Rightarrow Z_{\alpha_1}(1),
\]

where \(P(T)\) and \(W_k^{(T)}\) are defined in (7.3).

**Proof.** Define

\[
R_T(t) := \sum_{k=1}^{[Tt]} \left\{ \frac{W_k^{(T)}}{b_J(T)} - E \left[ \frac{W_1^{(T)}}{b_J(T)} \right] \right\}.
\]

Then, using (7.5), (7.14) (7.18), (7.19) and (7.20), we have, as in Section 2 of [31], that \(R_T \Rightarrow \Xi_{\alpha_1}(1)\) in \(\mathbb{D}\) endowed with Skorohod's \(J_1\) topology, where \(\Xi_\alpha\) is \(\alpha\)-stable Lévy motion with mean 0, skewness parameter 1 and scale parameter \(C_{\alpha}^{-\frac{1}{\alpha}}\).

Since \(P(T)\) has a Poisson distribution with mean \(\lambda \tilde{F}_L(T)\) and \(\tilde{F}_L(T) \sim T \to \infty\), we have

\[
\frac{P(T)}{T} \to \lambda.
\]

By independence of \(R_T\) and \(P(T)\), we have

\[(7.21) \quad \left( R_T, \frac{P(T)}{T} \right) \Rightarrow (\Xi_{\alpha_1}, \lambda) \text{ in } \mathbb{D} \times [0, \infty).\]

Now, we know from Theorem 4.2 that the function \(\pi : \mathbb{D} \times [0, \infty) \to [0, \infty)\) defined by

\[(7.22) \quad \pi(x, t) = x(t)\]
is continuous at \((x, t)\) iff \(x\) is continuous at \(t\). Also, since we know from Theorem 4.4 \(\Xi_{\alpha, J}\) has no fixed point of discontinuity, we have that \(\Xi_{\alpha, J}\) is continuous at \(\lambda\) with probability 1. Thus, using the Continuous Mapping Theorem on the convergence in (7.21), we get

\[
\frac{\sum_{k=1}^{P(\tau)} W_k^{(T)} - P(T) \mathbb{E}[W_1^{(T)}]}{b_J(T)} = \mathbb{E}_{\alpha, J} \left( \frac{P(T)}{T} \right) \Rightarrow \Xi_{\alpha, J}(\lambda) = Z_{\alpha, J}(1) \text{ in } \mathbb{R}
\]

\(\square\)

Next we change the centering to the one suggested in Theorem 5.2. We use the assumptions (9), (10) and (11) to study the tail behavior of \(L_1\), which we then use to change the centering.

**Proposition 7.2.** Under the assumptions \((9'), (10')\) and (11), stated in (5.14), (5.15) and (5.3), \(L_1\) has a tail of index \(H_{\alpha, J}\).

**Proof.** We consider the function \(\Psi : (0, \infty) \times \mathbb{D} \to (0, \infty) \times \mathbb{D}\) by

\[
\Psi(t, x) = (t, x^{\leftarrow}).
\]

By Corollary 4.3, \(\Psi\) is continuous at \((t, x) \in (0, \infty) \times \mathbb{D}\). Also the function \(\pi\), defined by (7.22), is continuous at \((t, x)\), if \(x\) is continuous at \(t\). Now \(\chi\) is supported on \(\mathbb{D}_+\) and does not have any fixed point of discontinuity. Thus the map \(\pi \circ \Psi(t, x) = x^{\leftarrow}(t)\) is continuous with probability 1. Hence by the Continuous Mapping Theorem, we have, for all \(\varepsilon > 0\),

\[
\mathbb{P} \left[ \frac{A^{\leftarrow}_{t} \left( J_1 \frac{1}{\sigma^{\leftarrow}(b_J(T))} \right)}{\sigma^{\leftarrow}(b_J(T))} \in \cdot \right] \xrightarrow{\mathcal{W}} \int_{\{t \geq x^{\leftarrow}(t) \geq \gamma\}} \nu_{\alpha, J}^{\leftarrow}(dt) \mathbb{P}[\chi \in dx].
\]

Hence for all \(\varepsilon > 0\) and \(\gamma > 0\) such that \(\{\gamma\}\) has zero limit measure, we have

\[
\mathbb{P} \left[ \frac{A^{\leftarrow}_{t} \left( J_1 \frac{1}{\sigma^{\leftarrow}(b_J(T))} \right)}{\sigma^{\leftarrow}(b_J(T))} > \gamma \right] \to \int_{x^{\leftarrow}(t) > \gamma} \nu_{\alpha, J}^{\leftarrow}(dt) \mathbb{P}[\chi \in dx]
\]

or,

\[
\mathbb{P} \left[ \frac{A^{\leftarrow}_{t} \left( J_1 \frac{1}{b_J(T)} \right)}{\sigma^{\leftarrow}(b_J(T))} > \gamma, \frac{J_1}{b_J(T)} > \varepsilon \right] \to \int (\varepsilon \vee x(\gamma))^{-\alpha} \mathbb{P}[\chi \in dx]
\]

or,

\[
\mathbb{P} \left[ \frac{L_1}{\sigma^{\leftarrow}(b_J(T))} > \gamma, \frac{J_1}{b_J(T)} > \varepsilon \right] \to \mathbb{E} \left[ (\varepsilon \vee \chi(\gamma))^{-\alpha} \right],
\]

Finally, letting \(\varepsilon \downarrow 0\), using the Monotone Convergence Theorem, we have

\[
\lim_{\varepsilon \downarrow 0} \lim_{T \to \infty} \mathbb{P} \left[ \frac{L_1}{\sigma^\leftarrow(b_J(T))} > \gamma, \frac{J_1}{b_J(T)} > \varepsilon \right] = \mathbb{E} \left[ \chi(\gamma)^{-\alpha} \right] = \gamma^{-H_{\alpha, J}} \mathbb{E} \left[ \chi(1)^{-\alpha} \right]
\]

which is finite due to (5.3). Also from (5.15), we have that

\[
\lim_{\varepsilon \downarrow 0} \limsup_{T \to \infty} \mathbb{P} \left[ \frac{L_1}{\sigma^\leftarrow(b_J(T))} > \gamma, \frac{J_1}{b_J(T)} \leq \varepsilon \right] = 0.
\]

Thus, given \(\delta > 0\), we can choose \(\varepsilon > 0\), such that the following happen:

\[
\gamma^{-H_{\alpha, J}} \mathbb{E} \left[ \chi(1)^{-\alpha} \right] - \delta < \lim_{T \to \infty} \mathbb{P} \left[ \frac{L_1}{\sigma^\leftarrow(b_J(T))} > \gamma, \frac{J_1}{b_J(T)} > \varepsilon \right] \leq \gamma^{-H_{\alpha, J}} \mathbb{E} \left[ \chi(1)^{-\alpha} \right],
\]
0 \leq \limsup_{T \to \infty} TP \left[ \frac{L_1}{\sigma^\gamma(b_j(T))} > \gamma, \frac{J_1}{b_j(T)} \leq \varepsilon \right] < \frac{\delta}{2}.

Thus, there exists $T_0$ such that for all $T > T_0$, we have
\[
\gamma^{-H_\alpha J} E \left[ \chi(1)^{-\alpha J} \right] - \delta < TP \left[ \frac{L_1}{\sigma^\gamma(b_j(T))} > \gamma, \frac{J_1}{b_j(T)} \geq \varepsilon \right] < \gamma^{-H_\alpha J} E \left[ \chi(1)^{-\alpha J} \right] + \frac{\delta}{2}.
\]
Thus adding, we can conclude that for all $T > T_0$,
\[
\gamma^{-H_\alpha J} E \left[ \chi(1)^{-\alpha J} \right] - \delta < TP \left[ \frac{L_1}{\sigma^\gamma(b_j(T))} > \gamma \right] < \gamma^{-H_\alpha J} E \left[ \chi(1)^{-\alpha J} \right] + \delta.
\]
Hence, we have
\[
\lim_{T \to \infty} TP \left[ \frac{L_1}{\sigma^\gamma(b_j(T))} > \gamma \right] = \gamma^{-H_\alpha J} E \left[ \chi(1)^{-\alpha J} \right],
\]
which in turn implies that $L_1$ has a tail of index $H_\alpha J$. \hfill \Box

Now, we use Proposition 7.2 to obtain the required centering.

**Lemma 7.5.** Under assumptions and notations as used in Theorem 5.2, we have
\[
\frac{X_1(T) - \lambda T \mu_J}{b_j(T)} \Rightarrow Z_\alpha J(1),
\]
where $\mu_J = E(J_1)$.

**Proof.** Due to (7.4) and Lemma 7.4, it is enough to prove that
\[
P(T) \left[ \frac{W_1(T)}{b_j(T)} \right] - \lambda T \mu_J \left[ \frac{W_1(T)}{b_j(T)} \right] \to 0.
\]
We rewrite the left side of (7.23) as
\[
P(T) \left[ \frac{W_1(T)}{b_j(T)} \right] - \lambda T \mu_J \left[ \frac{W_1(T)}{b_j(T)} \right] = \frac{P(T) - \lambda \hat{F}_L(T)}{\sqrt{\lambda \hat{F}_L(T)}} \sqrt{\lambda \hat{F}_L(T)} \left[ \frac{W_1(T)}{b_j(T)} \right] - \frac{\lambda}{b_j(T)} \left[ \hat{F}_L(T) \right] \left( \mu_J \right) \left[ W_1(T) \right].
\]
Now, we observe that, using (7.6)
\[
\sqrt{\lambda \hat{F}_L(T)} \left[ \frac{W_1(T)}{b_j(T)} \right] = \sqrt{\lambda \hat{F}_L(T)} \int_{w=0}^{\infty} P[W_1(T) > b_j(T)w] \, dw
\]
\[
= \sqrt{\frac{\lambda}{F_L(T)}} \int_{w=0}^{\infty} \int_0^T P[b_j(T)w < J_1 \leq A_1(\gamma)] \, d\gamma \, dw
\]
\[
\leq \sqrt{\frac{\lambda}{F_L(T)}} \int_{w=0}^{\infty} P[J_1 > b_j(T)w] \, dw
\]
\[
\frac{\sqrt{T}}{b_J(T)} \sim \sqrt{\lambda F_L(T)} \mathbb{E}[J_1] \to 0,
\]

since \( \frac{\sqrt{T}}{b_J(T)} \in RV_{\frac{1}{2}} \) and \( \alpha_J < 2 \). Also, \( P(T) \) having a Poisson distribution with mean \( \lambda \hat{F}_L(T) \sim \lambda T \to \infty \), we have, from the central limit theorem, that \( \frac{P(T) - \lambda \hat{F}_L(T)}{\sqrt{\lambda \hat{F}_L(T)}} \) converges weakly to a standard normal distribution and hence is bounded in probability. Thus by Slutsky’s theorem the first term on the right side of (7.24) goes to 0 in probability, i.e.,

\[
(7.25) \quad \frac{P(T) - \lambda \hat{F}_L(T)}{\sqrt{\lambda \hat{F}_L(T)}} \sqrt{\lambda \hat{F}_L(T)} \mathbb{E} \left[ \frac{W_1(T)}{b_J(T)} \right] \rightarrow 0.
\]

Now we consider the second term on the right side of (7.24). Observe that

\[
T \mu_J = \int_{j=0}^{\infty} \int_{\gamma=0}^{T} P[J_1 > j] d\gamma dj
\]

and

\[
\hat{F}_L(T) \mathbb{E} \left[ W_1(T) \right] = \hat{F}_L(T) \int_{j=0}^{\infty} P \left[ W_1(T) > j \right] dj = \int_{j=0}^{T} \int_{\gamma=0}^{\infty} P[j < J_1 \leq A_1(\gamma)] d\gamma dj \quad \text{by (7.6)}.
\]

Thus, we have,

\[
(7.26) \quad T \mu_J - \hat{F}_L(T) \mathbb{E} \left[ W_1(T) \right] = \int_{j=0}^{\infty} \int_{\gamma=0}^{T} P[J_1 > j, J_1 > A_1(\gamma)] d\gamma dj
\]

\[
= \int_{j=0}^{T} \int_{\gamma=0}^{\infty} P[J_1 > j, L_1 > \gamma] d\gamma dj
\]

\[
\leq \int_{\gamma=0}^{T} \left[ \gamma \hat{F}_L(\gamma) + \int_{j=\gamma}^{\infty} \hat{F}_J(j) dj \right] d\gamma.
\]

Hence,

\[
\frac{\lambda}{b_J(T)} \left( T \mu_J - \hat{F}_L(T) \mathbb{E} \left[ W_1(T) \right] \right) = \frac{\lambda}{b_J(T)} \int_{\gamma=0}^{T} \left[ \gamma \hat{F}_L(\gamma) + \int_{j=\gamma}^{\infty} \hat{F}_J(j) dj \right] d\gamma.
\]

Now, \( \hat{F}_J \in RV_{\alpha_J} \) with \( 1 < \alpha_J < 2 \), and, therefore, using Karamata’s theorem twice in succession, we obtain that

\[
\frac{\lambda}{b_J(T)} \int_{\gamma=0}^{T} \int_{j=\gamma}^{\infty} \hat{F}_J(j) dj \sim \frac{\lambda}{(2 - \alpha_J)(\alpha_J - 1)} \frac{T^2 \hat{F}_J(T)}{b_J(T)} \in RV_{2-\alpha_J} \frac{1}{\alpha_J}
\]
and because \(2 - \alpha_J - \frac{1}{\alpha_J} = -\frac{(\alpha_J-1)^2}{\alpha_J} < 0\), we have
\[
\frac{\lambda}{b_J(T)} \int_0^T \int \tilde{F}_J(j) \, dj \to 0.
\]

Also, since \(\tilde{F}_L \in RV_{-H\alpha_J}\), if \(H \alpha_J < 2\), using Karamata’s theorem again, we have
\[
\frac{\lambda}{b_J(T)} \int_0^T \gamma \tilde{F}_L(\gamma) \, d\gamma \sim \frac{\lambda}{2 - \alpha_J} \frac{T^2 \tilde{F}_L(T)}{b_J(T)} \in RV_{2 - H\alpha_J - \frac{1}{\alpha_J}},
\]
and because \(2 - H\alpha_J - \frac{1}{\alpha_J} \leq 2 - \alpha_J - \frac{1}{\alpha_J} < 0\) (recall \(H \geq 1\) necessarily), we again have
\[
\frac{\lambda}{b_J(T)} \int_0^T \gamma \tilde{F}_L(\gamma) \, d\gamma \to 0.
\]

If \(H\alpha_J > 2\), then we know that \(L_1\) has finite second moment and from \(b_J(T) \to \infty\), we conclude that
\[
\frac{\lambda}{b_J(T)} \int_0^T \gamma \tilde{F}_L(\gamma) \, d\gamma \to 0 \cdot \int_0^T \gamma \tilde{F}_L(\gamma) \, d\gamma = 0 \cdot \frac{1}{2} \mathbb{E}[L_1^2] = 0.
\]

Thus,
\[
(7.27) \quad \frac{\lambda}{b_J(T)} \left( T \mu_J - \tilde{F}_L(T) \mathbb{E} \left[ W_1^{(T)} \right] \right) \to 0,
\]

which along with (7.25) gives (7.23) and hence completes the proof of Lemma 7.5.

Now, we are ready to complete the proof of one-dimensional convergence provided we show the negligibility of the part due to \(X_2\), which we do in the next lemma.

**Lemma 7.6.** Under assumptions and notations as used in Theorem 5.2, we have
\[
(7.28) \quad \frac{X_2(T)}{b_J(T)} \to 0,
\]
where \(X_2\) is defined by (7.2).

**Proof.** First observe that,
\[
(7.29) \quad X_2(T) = \iint \int_{\mathcal{R}_T^{(2)}} a(T - \gamma) M(d\gamma, da, dj) \leq \iint \int_{\mathcal{R}_T^{(2)}} j M(d\gamma, da, dj),
\]
since \(a(T - \gamma) < j\) on \(\mathcal{R}_T^{(2)}\). Also, since \(M\) is a Poisson point process with mean measure \(\lambda d\gamma \times \mathbb{P}[A_1 \in da, J_1 \in dj]\), we have that
\[
\mathbb{E} \left[ \iint \int_{\mathcal{R}_T^{(2)}} j M(d\gamma, da, dj) \right] = \lambda \iint \int_{\mathcal{R}_T^{(2)}} j \, d\gamma \, \mathbb{P}[A_1 \in da, J_1 \in dj]
\]
\[
\begin{align*}
&= \lambda \int_{\gamma=0}^{T} \int_{j > \alpha(t - \gamma)} \int_{j > \alpha(T - \gamma)} j \, d\gamma \, P[A_1 \in da, J_1 \in dj] \\
&= \lambda \int_{\gamma=0}^{T} \int_{j > \alpha(t - \gamma)} \int_{j > \alpha(T - \gamma)} dx \, d\gamma \, P[A_1 \in da, J_1 \in dj] \\
&= \lambda \int_{\gamma=0}^{T} \int_{j > x} \int_{j > \alpha(T - \gamma)} P[A_1 \in da, J_1 \in dj] \, dx \, d\gamma \\
&= \lambda \int_{\gamma=0}^{T} \int_{j > 0} P[J_1 > x, J_1 > A_1(T - \gamma)] \, dx \, d\gamma \\
&= \lambda \int_{\gamma=0}^{T} \int_{j > 0} P[J_1 > j, J_1 > A_1(\gamma)] \, dj \, d\gamma
\end{align*}
\]

Then we have,

\[
E \left[ \frac{1}{b_j(T)} \int_{\mathcal{R}_T^{(2)}} j \, M(\gamma, da, dj) \right] = \frac{\lambda}{b_j(T)} \int_{\gamma=0}^{T} \int_{j > 0} P[J_1 > j, J_1 > A_1(\gamma)] \, d\gamma \, dj
\]

\[
= \frac{\lambda}{b_j(T)} \left( T \mu_J - \hat{F}_L(T) E \left[ W_1(T) \right] \right)
\]

by (7.26)

\[
\rightarrow 0
\]

by (7.27)

Thus, we have,

\[
(7.30) \quad \frac{1}{b_j(T)} \int_{\mathcal{R}_T^{(2)}} j \, M(\gamma, da, dj) \stackrel{p}{\rightarrow} 0.
\]

Hence, combining (7.29) and (7.30), we conclude that

\[
\frac{X_2(T)}{b_j(T)} \leq \frac{1}{b_j(T)} \int_{\mathcal{R}_T^{(2)}} j \, M(\gamma, da, dj) \stackrel{p}{\rightarrow} 0.
\]

Finally, we complete the proof of Theorem 5.2 in the one-dimensional case.

**Proof of Theorem 5.2.** (for the one-dimensional case) Recall from Lemma 7.5 that

\[
\frac{X_1(T) - XT \mu_J}{b_j(T)} \Rightarrow Z_{\alpha_J}(1),
\]

and from Lemma 7.6 that

\[
\frac{X_2(T)}{b_j(T)} \stackrel{p}{\rightarrow} 0.
\]
Adding them we get,

$$
\frac{X(T) - \lambda T \mu_J}{b_J(T)} \Rightarrow Z_{\alpha_J}(1).
$$

Thus, since $b_J \in RV_{\alpha_J}$, we have, for all $t > 0$,

$$
(7.31) \quad Y_T(t) = \frac{X(T - \lambda T \mu_J)}{b_J(T)} = \frac{b_J(T - \lambda T \mu_J)}{b_J(T)} \Rightarrow t^{\alpha_J} Z_{\alpha_J}(1) = Z_{\alpha_J}(t).
$$

\[ \square \]

7.4. Finite dimensional convergence. We complete the proof of Theorem 5.2 by showing finite dimensional convergence.

Proof of Theorem 5.2. Let $0 < s < t$. Observe that

$$
X_1(Tt) - X_1(Ts) = \iint j M(d\gamma, da, dj)
$$

is independent of

$$
X_1(Tu) = \iint j M(d\gamma, da, dj) \quad \forall u \leq s,
$$

since they are the functions of Poisson point process restricted to disjoint sets. Hence $X_1(T\cdot)$ has independent increments. Also, let us define

$$
B_T(s, t) = \iiint \mathbb{1}_{T \gamma < \gamma \leq T_i} j M(d\gamma, da, dj) = \iint \mathbb{1}_{T_i, u}(T, 0, 0) j M(d\gamma, da, dj)
$$

and

$$
C_T(s, t) = \iint \mathbb{1}_{a(T_t - \gamma) \leq a(Tt - \gamma)} j M(d\gamma, da, dj).
$$

Observe that $X_1(Tt) - X_1(Ts) = B_T(s, t) + C_T(s, t)$. Now, setting $N(\cdot) = M(\cdot + (Ts, 0, 0))$, we get

$$
B_T(s, t) = \iint j N(d\gamma, da, dj) \overset{a}{\overset{d}{\Rightarrow}} \iint j M(d\gamma, da, dj) = X_1(T(t - s)),
$$

where the equality in distribution follows from the fact that, by invariance of Lebesgue measure under translation, $M$ and $N$ have same mean measure and hence the same distribution. So, by (7.31), we have,

$$
(7.32) \quad \frac{B_T(s, t) - \lambda T(t - s) \mu_J}{b_J(T)} \Rightarrow Z_{\alpha_J}(t - s) \overset{d}{=} Z_{\alpha_J}(t) - Z_{\alpha_J}(s).
$$

Also, we have,

$$
C_T(s, t) \leq \iint \mathbb{1}_{T_i} j M(d\gamma, da, dj)
$$
and (7.30) implies
\[
C_T(s,t) \leq \frac{1}{b_j(T)} \int_{\mathcal{R}_T^{(2)}} j M(d\gamma, da, dj) = \frac{b_j(T_s)}{b_j(T)} \frac{1}{b_j(T_s)} \int_{\mathcal{R}_T^{(2)}} j M(d\gamma, da, dj) \xrightarrow{P} 0.
\]

Then adding (7.32) and (7.33), we get
\[
Y_1^{(T)}(t) - Y_1^{(T)}(s) \Rightarrow Z_{\alpha,j}(t) - Z_{\alpha,j}(s),
\]
where
\[
Y_1^{(T)}(t) = \frac{X_1(T_t) - \lambda T_t \mu_j}{b_j(T)}.
\]

By the independent increment property of \( Y_1^{(T)} \) and \( Z_{\alpha,j} \), coordinatewise convergence of increments implies joint convergence of increments. Thus,
\[
Y_1^{(T)} \xrightarrow{fdl} Z_{\alpha,j}.
\]

Also, by (7.28),
\[
\frac{X_2(T_t)}{b_j(T)} = \frac{X_2(T_t)}{b_j(T)} \cdot \frac{b_j(T_t)}{b_j(T)} \xrightarrow{P} 0.
\]

Thus, we have, for all \( 0 \leq t_1 < \cdots < t_k \),
\[
\frac{X_2(T_{t_1}), \ldots, X_2(T_{t_k})}{b_j(T)} \xrightarrow{P} 0.
\]
Adding, (7.34) and (7.35), we get
\[
Y^{(T)} \xrightarrow{fdl} Z_{\alpha,j}.
\]

\[\square\]

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