An analysis of the Longstaff-Schwartz algorithm
for American option pricing

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Abstract

Recently, F.A. Longstaff and E.S. Schwartz proposed a Monte-Carlo method for the
computation of American option prices, based on least squares regression. Under fairly
general conditions, we prove the almost sure convergence of the algorithm. We also deter-
mine the rate of convergence and further prove that the normalized error is asymptotically
Gaussian.

KEY WORDS: American options, optimal stopping, Monte-Carlo methods, least squares
regression.

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1 Introduction

The computation of American option prices is a challenging problem, especially when several
underlying assets are involved. The mathematical problem to solve is an optimal stopping
problem. In classical diffusion models, this problem is associated with a variational inequality,
for which, in higher dimensions, classical PDE methods are ineffective.

Recently, various authors introduced numerical methods based on Monte-Carlo tech-
niques (see, among others, [1, 2, 3, 5]). The starting point of these methods is to replace the

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time interval of exercise dates by a finite subset. This amounts to approximating the American option by a so called *Bermuda* option. The solution of the corresponding discrete optimal stopping problem reduces to an effective implementation of the dynamic programming principle. The conditional expectations involved in the iterations of dynamic programming cause the main difficulty for the development of Monte-carlo techniques. One way of treating this problem is to use least squares regression on a finite set of functions as a proxy for conditional expectation. This is the method used by Longstaff and Schwartz [3]. Another type of least squares regression is proposed by Tsitsiklis and Van Roy [5].

The purpose of this paper is to analyze the Longstaff-Schwartz algorithm, which seems to have become popular among practitioners. More precisely, we will prove the convergence of the algorithm and establish a type of central limit theorem for the rate of convergence, thus providing the asymptotic normalized error. We note that partial convergence results are stated in [3], together with excellent empirical results, but with no study of the rate of convergence. On the other hand, convergence (but not the rate nor the error distribution) is provided in [5] for a somewhat different algorithm.

The paper is organized as follows. In Section 2, a precise description of the Longstaff-Schwartz algorithm and the notation is established. In Section 3, we prove the convergence of the algorithm. In Section 4, we study the rate of convergence.

## 2 The Longstaff-Schwartz algorithm and notations

### 2.1 Description of the algorithm

As mentioned in the introduction, the first step in all probabilistic approximation methods is to replace the original optimal stopping problem in continuous time by an optimal stopping problem in discrete time. Therefore, we will present the Longstaff-Schwartz algorithm in the context of discrete optimal stopping.

We will consider a probability space \((\Omega, \mathcal{A}, P)\), equipped with a discrete filtration \((\mathcal{F}_j)_{j=0,\ldots,L}\). Here, the positive integer \(L\) denotes the (discrete) time horizon. Given an adapted payoff process \((Z_j)_{j=0,\ldots,L}\), where \(Z_0, Z_1, \ldots, Z_L\) are square integrable random variables, we are interested in computing

\[
\sup_{\tau \in \mathcal{T}_{j,L}} \mathbb{E} Z_{\tau},
\]

where \(\mathcal{T}_{j,L}\) denotes the set of all stopping times with values in \(\{j, \ldots, L\}\).
Following classical optimal stopping theory (for which we refer to [4], chapter 6), we introduce the Snell envelope \((U_j)_{j=0,...,L}\) of the payoff process \((Z_j)_{j=0,...,L}\), defined by
\[
U_j = \text{ess- sup}_{\tau \in T_{j,L}} \mathbb{E} (Z_\tau \mid \mathcal{F}_j), \quad j = 0, \ldots, L.
\]
The dynamic programming principle can be written as follows:
\[
\begin{cases}
U_L = Z_L \\
U_j = \max (Z_j, \mathbb{E} (U_{j+1} \mid \mathcal{F}_j)), \quad 0 \leq j \leq L - 1.
\end{cases}
\]
We also have \(U_j = \mathbb{E} \left( Z_{\tau_j} \mid \mathcal{F}_j \right) \), with
\[
\tau_j = \min \{ k \geq j \mid U_k = Z_k \}.
\]
In particular \(\mathbb{E} U_0 = \sup_{\tau \in T_{0,L}} \mathbb{E} Z_\tau = \mathbb{E} Z_{\tau_0}\).

The dynamic programming principle can be rewritten in terms of the optimal stopping times \(\tau_j\), as follows:
\[
\begin{cases}
\tau_L = L \\
\tau_j = j1_{\{Z_j \geq \mathbb{E}(Z_{\tau_{j+1}} \mid \mathcal{F}_j)\}} + (j+1)1_{\{Z_j < \mathbb{E}(Z_{\tau_{j+1}} \mid \mathcal{F}_j)\}}, \quad j \leq L - 1,
\end{cases}
\]
This formulation in terms of stopping rules (rather than in terms of value functions) plays an essential role in the Longstaff-Schwartz method.

The method also requires that the underlying model be a Markov chain. Therefore, we will assume that there is an \((\mathcal{F}_j)\)-Markov chain \((X_j)_{j=0,...,L}\) with state space \((E, \mathcal{E})\) such that, for \(j = 0, \ldots, L\),
\[
Z_j = f(j, X_j),
\]
for some Borel function \(f(j, \cdot)\). We then have \(U_j = V(j, X_j)\) for some function \(V(j, \cdot)\) and \(\mathbb{E} \left( Z_{\tau_{j+1}} \mid \mathcal{F}_j \right) = \mathbb{E} \left( Z_{\tau_{j+1}} \mid X_j \right)\). We will also assume that the initial state \(X_0 = x\) is deterministic, so that \(U_0\) is also deterministic.

The first step of the Longstaff-Schwartz algorithm is to approximate the conditional expectation with respect to \(X_j\) by the orthogonal projection on the space generated by a finite number of functions of \(X_j\). Let us consider a sequence \((e_k(x))_{k \geq 1}\) of measurable real valued functions defined on \(E\) and satisfying the following conditions:

\textbf{A}1: For \(j = 1\) to \(L - 1\), the sequence \((e_k(X_j))_{k \geq 1}\) is total in \(L^2(\sigma(X_j))\).

\textbf{A}2: For \(j = 0\) to \(L - 1\) and \(m \geq 1\), if \(\sum_{k=1}^{m} \lambda_k e_k(X_j) = 0\) a.s. then \(\lambda_k = 0\) for \(k = 1\) to \(m\).
For \( j = 1 \) to \( L - 1 \), we denote by \( P_j^m \) the orthogonal projection from \( L^2(\Omega) \) onto the vector space generated by \( \{e_1(X_j), \ldots, e_m(X_j)\} \) and we introduce the stopping times \( \tau_j^{[m]} \):
\[
\begin{cases}
\tau_L^{[m]} = L \\
\tau_j^{[m]} = j1 \{ Z_j \geq P_j^m (Z_{j+1}^{[m]}) \} + \tau_{j+1}^{[m]} 1 \{ Z_j < P_j^m (Z_{j+1}^{[m]}) \}, \ j \leq L - 1,
\end{cases}
\]
From these stopping times, we obtain an approximation of the value function:
\[
U_0^m = \max \left( Z_0, \mathbb{E} Z_{\tau_1^{[m]}} \right).
\]
Recall that \( Z_0 = f(0,x) \) is deterministic. The second step of the algorithm is then to evaluate numerically \( \mathbb{E} Z_{\tau_1^{[m]}} \) by a Monte-Carlo procedure. We assume that we can simulate \( N \) independent paths \( (X_j^{(1)}), \ldots, (X_j^{(n)}), \ldots (X_j^{(N)}) \) of the Markov chain \( (X_j) \) and we denote by \( Z_j^{(n)} \) the associated payoff for \( j = 0 \) to \( L \) and \( n = 1 \) to \( N \) \( (Z_j^{(n)} = f(j, X_j^{(n)})) \). For each path \( n \), we then estimate recursively the stopping times \( \tau_j^{[m]} \) by:
\[
\begin{cases}
\tau_L^{n,m,N} = L \\
\tau_j^{n,m,N} = j1 \{ Z_j^{(n)} \geq \alpha_j^{(m,N)}, e^m(X_j^{(n)}) \} + \tau_{j+1}^{n,m,N} 1 \{ Z_j^{(n)} < \alpha_j^{(m,N)}, e^m(X_j^{(n)}) \}, \ j \leq L - 1,
\end{cases}
\]
Here, \( x \cdot y \) denotes the usual inner product in \( \mathbb{R}^m \), \( e^m \) is the vector valued function \( (e_1, \ldots, e_m) \) and \( \alpha_j^{(m,N)} \) is the least square estimator:
\[
\alpha_j^{(m,N)} = \arg \min_{a \in \mathbb{R}^m} \sum_{n=1}^{N} \left( Z_j^{(n)} - a \cdot e^m(X_j^{(n)}) \right)^2,
\]
Remark that for \( j = 1 \) to \( L - 1 \), \( \alpha_j^{(m,N)} \in \mathbb{R}^m \). Finally, from the variables \( \tau_j^{n,m,N} \), we derive the following approximation for \( U_0^m \):
\[
U_0^{n,m,N} = \max \left( Z_0, \frac{1}{N} \sum_{n=1}^{N} Z_j^{(n)} \right).
\]
In the next section, we prove that, for any fixed \( m \), \( U_0^{n,m,N} \) converges almost surely to \( U_0^m \) as \( N \) goes to infinity, and that \( U_0^m \) converges to \( U_0 \) as \( m \) goes to infinity. Before stating these results, we devote a short section to notation.

2.2 Notation

For \( m \geq 1 \) we denote by \( e^m(x) \) the vector \( (e_1(x), \ldots, e_m(x)) \) and for \( j = 1 \) to \( L - 1 \) we note:
\[
P_j^m \left( Z_{j+1}^{[m]} \right) = \alpha_j^m \cdot e^m(X_j)
\]
We remark that the $m$ dimensional parameter $\alpha_j^m$ has the explicit expression:

$$\alpha_j^m = (A_j^m)^{-1} \mathbb{E}(Z_{j+1}^{m}e^m(X_j)),$$

(2.4)

for $j = 1$ to $L - 1$, where $A_j^m$ is an $m \times m$ matrix, with coefficients given by

$$(A_j^m)_{1 \leq k, l \leq m} = \mathbb{E}(e_k(X_j)e_l(X_j)).$$

(2.5)

Similarly, the estimators $\alpha_j^{(m,N)}$ are equal to

$$\alpha_j^{(m,N)} = (A_j^{(m,N)})^{-1} \frac{1}{N} \sum_{n=1}^{N} Z_{j+1,n}^{(n)} e^m(X_j^{(n)}),$$

(2.6)

for $j = 1$ to $L - 1$, where $A_j^{(m,N)}$ is an $m \times m$ matrix, with coefficients given by

$$(A_j^{(m,N)})_{1 \leq k, l \leq m} = \frac{1}{N} \sum_{n=1}^{N} e_k(X_j^{(n)})e_l(X_j^{(n)}).$$

(2.7)

We note $\alpha^m = (\alpha_1^m, \ldots, \alpha_{L-1}^m)$ and $\alpha^{(m,N)} = (\alpha_1^{(m,N)}, \ldots, \alpha_{L-1}^{(m,N)})$.

Given a parameter $a^m = (a_1^m, \ldots, a_{L-1}^m)$ in $\mathbb{R}^m \times \cdots \times \mathbb{R}^m$ and deterministic vectors $z = (z_1, \ldots, z_L) \in \mathbb{R}^L$ and $x = (x_1, \ldots, x_L) \in E^L$, we define a vector field $F = (F_1, \ldots, F_L)$ by:

$$F_L(a^m, z, x) = z_L$$
$$F_j(a^m, z, x) = z_j 1_{\{z_j \geq a_j^m \cdot e^m(x_j)\}} + F_{j+1}(a^m, z, x) 1_{\{z_j < a_j^m \cdot e^m(x_j)\}}, \text{ for } j = 1, \ldots, L - 1.$$

We have

$$F_j(a^m, z, x) = z_j 1_{B_j} + \sum_{i=j+1}^{L-1} z_i 1_{B_j \ldots B_{i-1} B_i} + z_L 1_{B_j \ldots B_{L-1}},$$

with

$$B_j = \{z_j < a_j^m \cdot e^m(x_j)\}.$$  

We remark that $F_j(a^m, Z, X)$ does not depend on $(a_1^m, \ldots, a_{j-1}^m)$ and that we have

$$F_j(a^m, Z, X) = Z_{j}^{[m]}$$
$$F_j(\alpha^{(m,N)}, Z^{(n)}, X^{(n)}) = Z_{j}^{(n)}_{n,m,n}.$$  

For $j = 2$ to $L$, we denote by $G_j$ the vector valued function

$$G_j(a^m, z, x) = F_j(a^m, z, x)e^m(x_{j-1}),$$
and we define the functions $\phi_j$ and $\psi_j$ by

$$
\phi_j(a^m) = E F_j(a^m, Z, X) 
$$

$$
\psi_j(a^m) = E G_j(a^m, Z, X). 
$$

Observe that with this notation, we have

$$
\alpha_j^m = (A_j^m)^{-1} \psi_{j+1}(\alpha^m), 
$$

and similarly, for $j = 1$ to $L - 1$,

$$
\alpha_j^{(m,N)} = (A_j^{(m,N)})^{-1} \sum_{n=1}^N F_{j+1}(\alpha^{(m,N)}, Z^{(n)}, X^{(n)}) e^m(X^{(n)}). 
$$

3 Convergence

The convergence of $U_0^m$ to $U_0$ is a direct consequence of the following result.

**Theorem 3.1** Assume that $A_1$ is satisfied, then for $j = 0$ to $L$ we have

$$
\lim_{m \to +\infty} E(Z_{\tau_j} | F_j) = E(Z_{\tau_j} | F_j), 
$$

in $L^2$.

**Proof:** We proceed by induction on $j$. The result is true for $j = L$. Let us prove that if it holds for $j + 1$, it is true for $j$ ($j \leq L - 1$). Since

$$
Z_{\tau_j} = Z_j 1_{\{Z_j \geq \alpha_j^m \cdot e^m(X_j)\}} + Z_{\tau_{j+1}} 1_{\{Z_j < \alpha_j^m \cdot e^m(X_j)\}}, 
$$

for $j \leq L - 1$, we have

$$
E(Z_{\tau_j} - Z_{\tau_j} | F_j) = (Z_j - E(Z_{\tau_{j+1}} | F_j)) \left(1_{\{Z_j \geq \alpha_j^m \cdot e^m(X_j)\}} - 1_{\{Z_j < E(Z_{\tau_{j+1}} | F_j)\}}\right)
$$

$$
+ E(Z_{\tau_{j+1}} - Z_{\tau_{j+1}} | F_j) 1_{\{Z_j < \alpha_j^m \cdot e^m(X_j)\}}. 
$$

By assumption, the second term of the right side of the equality converges to 0 and we just have to prove that $B_j^m$ defined by

$$
B_j^m = (Z_j - E(Z_{\tau_{j+1}} | F_j)) \left(1_{\{Z_j \geq \alpha_j^m \cdot e^m(X_j)\}} - 1_{\{Z_j < E(Z_{\tau_{j+1}} | F_j)\}}\right), 
$$

converges to 0 in $L^2$. Observe that

$$
|B_j^m| = |Z_j - E(Z_{\tau_{j+1}} | F_j)| 1_{\{E(Z_{\tau_{j+1}} | F_j) > Z_j \geq \alpha_j^m \cdot e^m(X_j)\}} - 1_{\{\alpha_j^m \cdot e^m(X_j) > Z_j \geq E(Z_{\tau_{j+1}} | F_j)\}}|
$$

$$
\leq |Z_j - E(Z_{\tau_{j+1}} | F_j)| 1_{\{Z_j - E(Z_{\tau_{j+1}} | F_j) | \leq |\alpha_j^m \cdot e^m(X_j) - E(Z_{\tau_{j+1}} | F_j)|\}}
$$

$$
\leq |\alpha_j^m \cdot e^m(X_j) - E(Z_{\tau_{j+1}} | F_j)| 
$$

$$
\leq |\alpha_j^m \cdot e^m(X_j) - P_j^m(E(Z_{\tau_{j+1}} | F_j))| + |P_j^m(E(Z_{\tau_{j+1}} | F_j)) - E(Z_{\tau_{j+1}} | F_j)|. 
$$
But
\[ \alpha_j^m \cdot e^m(X_j) = P_j^m(Z_{j+1}) = P_j^m(\mathbb{E}(Z_{j+1} | \mathcal{F}_j)), \]
and consequently
\[ ||B_j^m||_2 \leq ||\mathbb{E}(Z_{j+1} | \mathcal{F}_j) - \mathbb{E}(Z_{j+1} | \mathcal{F}_j)||_2 + ||P_j^m(\mathbb{E}(Z_{j+1} | \mathcal{F}_j)) - \mathbb{E}(Z_{j+1} | \mathcal{F}_j)||_2. \]

The first term of the right side of this inequality tends to 0 by the induction hypothesis and the second one by \( A_1 \).

In what follows, we fix the value \( m \) and we study the properties of \( U_0^{m,N} \) as \( N \) the number of Monte-Carlo simulations, goes to infinity. For notational simplicity, we drop the superscript \( m \) throughout the rest of the paper.

**Theorem 3.2** We assume \( A_1, A_2 \) and that for \( j = 1 \) to \( L - 1 \), \( \mathbb{P}(\alpha_j \cdot e(X_j) = Z_j) = 0 \). Then \( U_0^{m,N} \) converges almost surely to \( U_0^m \) as \( N \) goes to infinity.

Note that with the notation of the preceding section, we have to prove that
\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} F_1(\alpha^{(N)}, Z^{(n)}, X^{(n)}) = \phi_1(\alpha). \] (3.1)

The proof is based on the following lemmas.

**Lemma 3.1** For \( j = 1 \) to \( L \), we have:
\[ |F_j(a, Z, X) - F_j(b, Z, X)| \leq \sum_{i=j}^{L} |Z_i| \sum_{i=j}^{L-1} 1_{|Z_i - b_i \cdot e(X_i)| \leq |a_i - b_i||e(X_i)|}. \]

**Proof:** Let \( B_j = \{ Z_j \geq a_j \cdot e(X_j) \} \) and \( \bar{B}_j = \{ Z_j \geq b_j \cdot e(X_j) \} \). We have:
\[ F_j(a, Z, X) - F_j(b, Z, X) = Z_j (1_{B_j} - 1_{\bar{B}_j}) \]
\[ + \sum_{i=j+1}^{L-1} Z_i (1_{B_j \ldots B_i \bar{B}_i} - 1_{B_j \ldots \bar{B}_i}) \]
\[ + Z_L (1_{B_j \ldots B_{L-1}} - 1_{B_j \ldots \bar{B}_{L-1}}). \]

But
\[ |1_{B_j} - 1_{\bar{B}_j}| = 1_{\{a_j \cdot e(X_j) \leq Z_j < b_j \cdot e(X_j)\}} - 1_{\{b_j \cdot e(X_j) \leq Z_j \leq a_j \cdot e(X_j)\}} \]
\[ = |1_{\{b_j \cdot e(X_j) < Z_j \leq a_j \cdot e(X_j)\}} - 1_{\{a_j \cdot e(X_j) < Z_j \leq b_j \cdot e(X_j)\}}| \]
\[ = 1_{\{b_j \cdot e(X_j) < Z_j \leq a_j \cdot e(X_j)\}} + 1_{\{a_j \cdot e(X_j) < Z_j \leq b_j \cdot e(X_j)\}} \]
\[ \leq 1_{\{Z_j - b_j \cdot e(X_j) \leq |a_j - b_j||e(X_j)|\}} \]

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Moreover
\[ |1_{B_{j...B_{i-1}B_i}} - 1_{B_{j...B_{i-1}}|B_i} | \leq \sum_{k=j}^{i-1} |1_{B_k} - 1_{B_k}| + |1_{B_i} - 1_{B_i}| \]
\[ = \sum_{k=j}^{i-1} |1_{B_k} - 1_{B_k}|, \]
this gives
\[ |F_j(a, Z, X) - F_j(b, Z, X)| \leq \sum_{i=j}^{L} |Z_i| \sum_{i=j}^{L-1} |1_{B_i} - 1_{B_i}|. \]

Combining these inequalities, we obtain the result of Lemma 3.1.

**Lemma 3.2** Assume that for \( j = 1 \) to \( L - 1 \), \( \mathbb{P}(\alpha_j \cdot e(X_j) = Z_j) = 0 \) then \( \alpha^{(N)}_j \) converges almost surely to \( \alpha_j \).

**Proof:** we proceed by induction on \( j \). For \( j = L - 1 \), the result is a direct consequence of the law of large numbers. Now, assume that the result is true for \( i = j \) to \( L - 1 \). We want to prove that it is true for \( j - 1 \). We have
\[ \alpha^{(N)}_{j-1} = (A^{(N)}_{j-1})^{-1} \frac{1}{N} \sum_{n=1}^{N} G_j(\alpha^{(N)}_n, Z^{(N)}, X^{(n)}). \]

By the law of large numbers, we know that \( A^{(N)}_{j-1} \) converges almost surely to \( A_{j-1} \) and it remains to prove that \( \frac{1}{N} \sum_{n=1}^{N} G_j(\alpha^{(N)}_n, Z^{(n)}, X^{(n)}) \) converges to \( \psi_j(\alpha) \). From the law of large numbers, we have the convergence of \( \frac{1}{N} \sum_{n=1}^{N} G_j(\alpha, Z^{(n)}, X^{(n)}) \) to \( \psi_j(\alpha) \) and it suffices to prove that :
\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left( G_j(\alpha^{(N)}_n, Z^{(n)}, X^{(n)}) - G_j(\alpha, Z^{(n)}, X^{(n)}) \right) = 0. \]

We note \( G_N = \frac{1}{N} \sum_{n=1}^{N} \left( G_j(\alpha^{(N)}_n, Z^{(n)}, X^{(n)}) - G_j(\alpha, Z^{(n)}, X^{(n)}) \right) \). We have :
\[ |G_N| \leq \frac{1}{N} \sum_{n=1}^{N} |e(X^{(n)}_{j-1})| |F_j(\alpha^{(N)}_n, Z^{(n)}, X^{(n)}) - F_j(\alpha, Z^{(n)}, X^{(n)})| \]
\[ \leq \frac{1}{N} \sum_{n=1}^{N} |e(X^{(n)}_{j-1})| \sum_{i=j}^{L} |Z_i| \sum_{i=j}^{L-1} 1_{|Z_i - \alpha \cdot e(X^{(n)}_i)| \leq |e(X^{(n)}_i)|}. \]

Since, for \( i = j \) to \( L - 1 \), \( \alpha^{(N)}_i \) converges almost surely to \( \alpha^{(N)}_i \), we have for each \( \epsilon > 0 \):
\[ \limsup |G_N| \leq \limsup \frac{1}{N} \sum_{n=1}^{N} |e(X^{(n)}_{j-1})| \sum_{i=j}^{L} |Z_i| \sum_{i=j}^{L-1} 1_{|Z_i - \alpha \cdot e(X^{(n)}_i)| \leq \epsilon |e(X^{(n)}_i)|}. \]
\[ = E|e(X_{j-1})| \sum_{i=j}^{L} |Z_i| \sum_{i=j}^{L-1} 1_{|Z_i - \alpha \cdot e(X_i)| \leq \epsilon |e(X_i)|}, \]

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where the last equality follows from the law of large numbers. Letting \( \epsilon \) go to 0, we obtain the convergence to 0, since for \( j = 1 \) to \( L - 1 \), \( \mathbb{P}(\alpha_j \cdot e(X_j) = Z_j) = 0. \)

The proof of Theorem 3.2 is similar to the proof of Lemma 3.2.

4 Rate of convergence of the Longstaff-Schwartz algorithm

4.1 Tightness

In this section we are interested in the rate of convergence of \( \frac{1}{N} \sum_{n=1}^{N} Z_{j,n}^{(n)} \), for \( j = 1 \) to \( L \).

Recall that \( m \) is fixed.

We assume that:

\( \textbf{H}_1 \): The random variable \( Z \) and the functions \( e_1, \ldots, e_m \) are bounded.

\( \textbf{H}_2 \): \( \forall j = 1, \ldots, L-1 \), \( \limsup_{\epsilon \to 0} \mathbb{P}(|Z_j - \alpha_j \cdot e(X_j)| \leq \epsilon) = 0. \)

Note that \( H_2 \) implies that \( \mathbb{P}(Z_j = \alpha_j \cdot e(X_j)) = 0 \) and, consequently, under \( H_2 \) we know from Section 3 that \( \frac{1}{N} \sum_{n=1}^{N} F_j(\alpha^{(N)}, Z^{(n)}, X^{(n)}) \) converges almost surely to \( \phi_j(\alpha) \). Remark too that \( H_2 \) is satisfied if the random variable \( (Z_j - \alpha_j \cdot e(X_j)) \) has a bounded density near zero.

Theorem 4.1 Under \( \textbf{H}_1 \) and \( \textbf{H}_2 \), the sequences \( \left( \frac{1}{\sqrt{N}} \sum_{n=1}^{N} (F_j(\alpha^{(N)}, Z^{(n)}, X^{(n)}) - \phi_j(\alpha)) \right)_{N \geq 1} \) \( (j = 1, \ldots, L) \) and \( (\sqrt{N}(\alpha_j^{(N)} - \alpha_j))_{N \geq 1} \) \( (j = 1, \ldots, L-1) \) are tight.

The proof of Theorem 4.1 is based on the following Lemma.

Lemma 4.1 Let \( (U^{(n)}) \) be a sequence of identically distributed random variables such that

\[ \limsup_{\epsilon \to 0} \frac{\mathbb{P}(|U^{(1)}| \leq \epsilon)}{\epsilon} < +\infty, \]

and \( (\theta_N) \) a sequence of positive random variables such that \( (\sqrt{N}\theta_N) \) is tight, then the sequence

\[ \left( \frac{1}{\sqrt{N}} \sum_{n=1}^{N} 1_{|U^{(n)}| \leq \theta_N} \right)_{N \geq 1} \]

is tight.

Proof: We note \( \sigma_N(\theta) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} 1_{|U^{(n)}| \leq \theta_N} \). Observe that \( \sigma_N \) is a non decreasing function
of $\theta$. Let $A > 0$, we have
\[
\mathbb{P}(\sigma_N(\theta_N) \geq A) \leq \mathbb{P}(\sigma_N(\theta_N) \geq A, \sqrt{N}\theta_N \leq B) + \mathbb{P}(\sqrt{N}\theta_N > B)
\leq \mathbb{P}(\sigma_N(\frac{B}{\sqrt{N}}) \geq A) + \mathbb{P}(\sqrt{N}\theta_N > B)
\leq \frac{1}{N} \mathbb{E}_{\sigma_N(\frac{B}{\sqrt{N}})} + \mathbb{P}(\sqrt{N}\theta_N > B)
= \frac{\sqrt{N}}{A} \mathbb{P}(|U^{(1)}| \leq \frac{B}{\sqrt{N}}) + \mathbb{P}(\sqrt{N}\theta_N > B).
\]
From the assumption on $(U^{(n)})$ and the tightness of $(\sqrt{N}\theta_N)$, we deduce easily the tightness of $\sigma_N(\theta_N)$.

**Proof of Theorem 4.1:** We know from the classical Central Limit Theorem that the sequence $(1/\sqrt{N}) \sum_{n=1}^{N} (F_j(\alpha, Z^{(n)}, X^{(n)} - \phi_j(\alpha))$ is tight and it remains to prove the tightness of $(1/\sqrt{N}) \sum_{n=1}^{N} (F_j(\alpha^{(N)}, Z^{(n)}, X^{(n)} - F_j(\alpha, Z^{(n)}, X^{(n)}))$, for $j = 1$ to $L$. Similarly, to prove the tightness of $(\sqrt{N}(\alpha_j^{(N)} - \alpha_j))_{N \geq 1}$, for $j = 1$ to $L - 1$, we just have to prove the tightness of $(1/\sqrt{N}) \sum_{n=1}^{N} (G_j(\alpha^{(N)}, Z^{(n)}, X^{(n)} - G_j(\alpha, Z^{(n)}, X^{(n)}))$ (see Section 2 for the notation). We proceed by induction on $j$. The tightness of $(1/\sqrt{N}) \sum_{n=1}^{N} (F_L(\alpha^{(N)}, Z^{(n)}, X^{(n)} - F_L(\alpha, Z^{(n)}, X^{(n)}))$ and $(\sqrt{N}(\alpha_L^{(N)} - \alpha_L - 1))$ is straightforward.

Assume that $(1/\sqrt{N}) \sum_{n=1}^{N} (F_i(\alpha^{(N)}, Z^{(n)}, X^{(n)} - F_i(\alpha, Z^{(n)}, X^{(n)}))$ and $(\sqrt{N}(\alpha_i^{(N)} - \alpha_i - 1))$ are tight for $i = j$ to $L$. We set
\[
F_N = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} (F_j(\alpha^{(N)}, Z^{(n)}, X^{(n)} - F_j(\alpha, Z^{(n)}, X^{(n)})).
\]
Now from Lemma 3.1, we have:
\[
|F_N| \leq C \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \sum_{i=1}^{L-1} \sum_{j=1}^{L-1} 1(|z_i^{(n)} - e_j e(\alpha_i^{(N)})| \leq |\alpha_i - \alpha_i^{(N)}||e||e|)
\]
from Lemma 4.1 and by the induction hypothesis, we deduce that $F_N$ is tight. In the same way, we prove that $(\sqrt{N}(\alpha_{j-2}^{(N)} - \alpha_{j-2}))$ is tight.

**4.2 A central limit theorem**

We prove in this section that under some stronger assumptions than in section 4.1, the vector $\left(\frac{1}{\sqrt{N}} \sum_{n=1}^{N} (Z_j^{(n)} - \mathbb{E}Z_j^{[n]})\right)_{j=1, \ldots, L}$ converges weakly to a Gaussian vector. With the
preceding notation, we have

\[
\frac{1}{\sqrt{N}} \sum_{n=1}^{N} (Z_{i_j, N}^{(n)} - E Z_{i_j}^{[m]}) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} (F_j(\alpha^{(N)}, Z^{(n)}, X^{(n)}) - \phi_j(\alpha)).
\]  

(4.1)

In the following, we will denote by \( Y \) the couple \( (Z, X) \) and by \( Y^{(n)} \) the couple \( (Z^{(n)}, X^{(n)}) \).

We make some more hypothesis:

\( H_2 \): For \( j = 1 \) to \( L-1 \), there exists a neighborhood \( V_j \) of \( \alpha_j \) such that for \( a_j \in V_j \),

\( Z_j - a_j \cdot e(X_j) \) admits a density \( f_{a_j} \) near 0, such that, for some \( \eta > 0 \),

\[
\sup_{a_j \in V_j, |z| \leq \eta} f_{a_j}(z) < +\infty.
\]

\( H_3 \): For \( j = 1 \) to \( L-1 \), \( \phi_j \) and \( \psi_j \) are \( C^1 \) in a neighborhood of \( \alpha \).

Observe that \( H_2^* \) is stronger than \( H_2 \).

**Theorem 4.2** Under \( H_1, H_2^*, H_3 \), the vector

\[
\left( \frac{1}{\sqrt{N}} \sum_{n=1}^{N} (Z_{i_j, N}^{(n)} - E Z_{i_j}^{[m]}), \sqrt{N}(\alpha_j^{(N)} - \alpha_j) \right)_{j=1, \ldots, L-1}
\]

converges in law to a Gaussian vector as \( N \) goes to infinity.

To prove Theorem 4.2, we use the following decomposition:

\[
\frac{1}{\sqrt{N}} \sum_{n=1}^{N} (F_j(\alpha^{(N)}, Y^{(n)}) - \phi_j(\alpha)) \equiv \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \left( F_j(\alpha^{(N)}, Y^{(n)}) - F_j(\alpha, Y^{(n)}) - (\phi_j(\alpha^{(N)}) - \phi_j(\alpha)) \right)
\]

\[
+ \frac{1}{\sqrt{N}} \sum_{n=1}^{N} (F_j(\alpha, Y^{(n)}) - \phi_j(\alpha)) + \sqrt{N}(\phi_j(\alpha^{(N)}) - \phi_j(\alpha)).
\]

From the classical Central Limit Theorem, we know that

\[
\left( \frac{1}{\sqrt{N}} \sum_{n=1}^{N} (F_j(\alpha, Y^{(n)}) - \phi_j(\alpha)) \right)_{j=1, \ldots, L}
\]

converges in law to a Gaussian vector. Moreover, we have:

\[
\sqrt{N}(\alpha_j^{(N)} - \alpha_j) = (A_j^{(N)})^{-1} \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \left( G_j+1(\alpha^{(N)}, Y^{(n)}) - \psi_{j+1}(\alpha) \right)
\]

\[
-(A_j)^{-1} \sqrt{N}(A_j^{(N)} - A_j)(A_j^{(N)})^{-1} \psi_{j+1}(\alpha),
\]

where \( A_j^{(N)} \) converges almost surely to \( A_j \) and \( \sqrt{N}(A_j^{(N)} - A_j) \) converges in law. From these decompositions, it is straightforward to check that Theorem 4.2 is a consequence of the differentiability of the functions \( \phi \) and \( \psi \) and of the following theorem.

**Theorem 4.3** Under \( H_1, H_2^*, H_3 \), the variables

\[
\frac{1}{\sqrt{N}} \sum_{n=1}^{N} \left( F_j(\alpha^{(N)}, Y^{(n)}) - F_j(\alpha, Y^{(n)}) - (\phi_j(\alpha^{(N)}) - \phi_j(\alpha)) \right)
\]

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\[
\frac{1}{\sqrt{N}} \sum_{n=1}^{N} \left( G_{j+1}(\alpha^{(N)}, Y^{(n)}) - G_{j+1}(\alpha, Y^{(n)}) - (\psi_{j+1}(\alpha^{(N)}) - \psi_{j+1}(\alpha)) \right)
\]
converge to 0 in \( L^2 \), for \( j = 1 \) to \( L - 1 \).

The proof of Theorem 4.3 requires some Lemmas. In the following, we denote by \( I(Y_i, a_i, \epsilon) \) the function

\[
I(Y_i, a_i, \epsilon) = 1_{\{ |Z_i - a_i, \epsilon(x_i)| \leq \|\epsilon\|_{\infty} \}}. \quad (4.2)
\]

Note that \( I(Y_i, a_i, \epsilon) \leq I(Y_i, b_i, \epsilon + |b_i - a_i|) \).

**Lemma 4.2** For \( j = 1 \) to \( L - 1 \), and \( a, b \) in \( (\mathbb{R}^m)^{L-1} \), we have under \( H_1 \)

\[
|F_j(a, Y) - F_j(b, Y)| \leq \|Z\|_{\infty} \sum_{i=j}^{L-1} I(Y_i, a_i, |a_i - b_i|)
\]

\[
|G_j(a, Y) - G_j(b, Y)| \leq \|Z\|_{\infty} \|\epsilon\|_{\infty} \sum_{i=j}^{L-1} I(Y_i, a_i, |a_i - b_i|)
\]

This result follows easily from Lemma 3.1.

**Lemma 4.3** Assume \( H_1 \) and \( H_2^* \), then for \( j = 1 \) to \( L - 1 \), we have, for all \( \delta > 0 \),

\[
\lim_{N \to +\infty} N \mathbb{P}(|\alpha_j^{(N)} - \alpha_j| \geq \delta) = 0.
\]

**Proof:** Let us recall that if \((U_n)_n\) is a sequence of i.i.d. bounded variables, we have

\[
\forall \delta > 0, \quad \lim_{N \to +\infty} N \mathbb{P} \left( \left| \frac{1}{N} \sum_{i=1}^{N} U_i - E U_1 \right| \geq \delta \right) = 0 \quad (4.3)
\]

Observe that

\[
\alpha_j^{(N)} - \alpha_j = (A_j^{(N)})^{-1} \frac{1}{N} \sum_{n=1}^{N} (G_{j+1}(\alpha^{(N)}, Y^{(n)}) - \psi_{j+1}(\alpha))
\]

\[-(A_j)^{-1}(A_j^{(N)} - A_j)(A_j^{(N)})^{-1}\psi_{j+1}(\alpha),
\]

We note \( \Omega_j^c = \{||A_j^{(N)} - A_j|| \leq \epsilon\} \) and we choose \( \epsilon \) such that \( ||(A_j^{(N)})^{-1}|| \leq 2||(A_j)^{-1}|| \) on \( \Omega_j^c \). From (4.3), we know that \( N \mathbb{P}((\Omega_j^c)^c) \) tends to 0, for \( j = 1 \) to \( L - 1 \) and that, on \( \Omega_j^c \),

\[
|\alpha_j^{(N)} - \alpha_j| \leq K' \left| \frac{1}{N} \sum_{n=1}^{N} (G_{j+1}(\alpha^{(N)}, Y^{(n)}) - \psi_{j+1}(\alpha)) \right| + K \epsilon.
\]

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Now, since $G_L(\alpha^{(N)}, Y^{(n)}) = Z_L^{(n)} e(X_L^{(n)})$, we deduce that

$$\lim_{N \to +\infty} \mathbb{P}(|\alpha_L^{(N)} - \alpha_{L-1}| \geq \delta) = 0.$$ 

Assume now that the result of Lemma 4.3 is true for $j + 1, \ldots, L - 1$. We will prove that

$$\lim_{N \to +\infty} \mathbb{P}(|\alpha_j^{(N)} - \alpha_j| \geq \delta) = 0.$$ 

We have

$$\frac{1}{N} \sum_{n=1}^{N} (G_{j+1}(\alpha^{(N)}, Y(n)) - \psi_{j+1}(\alpha)) = \frac{1}{N} \sum_{n=1}^{N} (G_{j+1}(\alpha^{(N)}, Y(n)) - G_{j+1}(\alpha, Y(n)))$$

$$+ \frac{1}{N} \sum_{n=1}^{N} (G_{j+1}(\alpha, Y(n)) - \psi_{j+1}(\alpha)).$$

From Lemma 4.2, we obtain on $\Omega_j^*$,

$$|\alpha_j - \alpha_j^{(N)}| \leq K\epsilon + \frac{K'}{N} \sum_{n=1}^{N} \sum_{i=j+1}^{L-1} I(Y_i^{(n)}, \alpha_i, |\alpha_i - \alpha_i^{(N)}|)$$

$$+ \frac{K''}{N} \sum_{n=1}^{N} (G_{j+1}(\alpha, Y(n)) - \psi_{j+1}(\alpha)).$$

The last term can be treated using (4.3). Therefore, it suffices to prove that

$$\forall \delta > 0 \lim_{N \to +\infty} \mathbb{P}(S_N \geq \delta) = 0,$$

where $S_N = \frac{1}{N} \sum_{n=1}^{N} \sum_{i=j+1}^{L-1} I(Y_i^{(n)}, \alpha_i, |\alpha_i - \alpha_i^{(N)}|)$. But

$$\mathbb{P}(S_N \geq \delta) \leq \mathbb{P} \left( \frac{1}{N} \sum_{n=1}^{N} \sum_{i=j+1}^{L-1} I(Y_i^{(n)}, \alpha_i, \epsilon) \geq \delta \right) + \sum_{i=j+1}^{L-1} \mathbb{P} \left( |\alpha_i^{(N)} - \alpha_i| \geq \epsilon \right).$$

By assumption, for $i = j + 1$ to $L - 1$, we have $\lim_{N \to +\infty} \mathbb{P}(|\alpha_i^{(N)} - \alpha_i| \geq \epsilon) = 0$. Moreover, we know from $H_2^*$ that $\sum_{i=j+1}^{L-1} \mathbb{E} I(Y_i^{(n)}, \alpha_i, \epsilon)$ goes to zero as $\epsilon$ tends to 0, so for $\epsilon$ small enough we have $\delta - \sum_{i=j+1}^{L-1} \mathbb{E} I(Y_i^{(n)}, \alpha_i, \epsilon) > 0$ and from (4.3), we see that $\mathbb{P} \left( \frac{1}{N} \sum_{n=1}^{N} \sum_{i=j+1}^{L-1} I(Y_i^{(n)}, \alpha_i, \epsilon) \geq \delta \right)$ tends to 0 as $N$ goes to infinity. This completes the proof of Lemma 4.3.

We now state a technical Lemma which requires some more notation. Given $C$ and $C'$ two constant vectors in $\mathbb{R}_{+}^{L-1}$ and $a \in (\mathbb{R}^m)^{L-1}$, we define a sequence of random vectors
\[ \mathcal{U}^{(N-2-k)}_{L-1}(C, C', a) = \frac{C_{L-1}}{N} \]
\[ \mathcal{U}^{(N-2-k)}_{j}(C, C', a) = \frac{C_j}{N} + \frac{C'_j}{N} \sum_{n=1}^{N-2-k} \sum_{i=j+1}^{L-1} I(Y^{(n)}_i, a_i, \mathcal{U}^{(N-2-k)}_{(C, C', a)}) \]

for \( j = 1, \ldots, L - 2 \). Note that \( \mathcal{U}^{(N-2-k)} \) is \( \sigma(Y^{(1)}, \ldots, Y^{(N-2-k)}) \) measurable.

**Lemma 4.4** Assume \( H_1 \) and \( H_2^* \), then for all \( C, C' \in R_{L-1} \) and \( j = 1 \) to \( L - 2 \), there exist some constant \( K \) such that

\[ \mathbb{E}\mathcal{U}^{(N-2)}_j(C, C', \alpha^{(N-2)}) \leq \frac{K}{N}. \]

**Proof**: For \( j = 1 \) to \( L - 2 \), we will prove by induction on \( k, 1 \leq k \leq L - 1 - j \) that there exist \( C_j(k) \) and \( C'_j(k) \) in \( R_{L-1} \) such that

\[ \mathbb{E}\mathcal{U}^{(N-2)}_j(C, C', \alpha^{(N-2)}) \leq \frac{K}{N} + K' \sum_{i=j+k}^{L-1} \mathbb{E}\mathcal{U}^{(N-2-k)}(C_j(k), C'_j(k), \alpha^{(N-2-k)}). \]

We have

\[ \mathbb{E}\mathcal{U}^{(N-2)}_j(C, C', \alpha^{(N-2)}) = \frac{C_j}{N} + \frac{C'_j}{N} \sum_{n=1}^{N-2} \sum_{i=j+1}^{L-1} \mathbb{E}I(Y^{(n)}_i, \alpha^{(N-2)}, \mathcal{U}^{(N-2)}_i) \]
\[ = \frac{C_j}{N} + \frac{C'_j}{N} (N - 2) \sum_{i=j+1}^{L-1} \mathbb{E}I(Y^{(N-2)}_i, \alpha^{(N-2)}, \mathcal{U}^{(N-2)}_i). \]

But \( I(Y^{(N-2)}_i, \alpha^{(N-2)}, \mathcal{U}^{(N-2)}_i) \leq I(Y^{(N-2)}_i, \alpha^{(N-3)}, \mathcal{U}^{(N-2)}_i + |\alpha^{(N-2)}_i - \alpha^{(N-3)}_i|) \). As in the proof of Lemma 4.3, we can choose \( \Omega_N \) such that \( NP(\Omega_N^c) \) goes to 0 as \( N \) goes to infinity and such that on \( \Omega_N \)

\[ |\alpha^{(N-2)}_{L-1} - \alpha^{(N-3)}_{L-1}| \leq \frac{C_{L-1}}{N} \]
\[ |\alpha^{(N-2)}_j - \alpha^{(N-3)}_j| \leq \frac{C_j}{N} + \frac{C'_j}{N} \sum_{n=1}^{N-2} \sum_{i=j+1}^{L-1} I(Y^{(n)}_i, \alpha^{(N-3)}_i, |\alpha^{(N-2)}_i - \alpha^{(N-3)}_i|). \]

So, on \( \Omega_N \), there exist \( C(1) \) and \( C'(1) \) in \( R_{L-1} \) such that, for \( i = j + 1 \) to \( L - 1 \),

\[ \mathcal{U}^{(N-2)}_i(C, C', \alpha^{(N-2)}) + |\alpha^{(N-2)}_i - \alpha^{(N-3)}_i| \leq \mathcal{U}^{(N-3)}_i(C(1), C'(1), \alpha^{(N-3)}). \]

Consequently, we obtain

\[ \mathbb{E}\mathcal{U}^{(N-2)}_j \leq \frac{C_j}{N} + \frac{C'_j}{N} (N - 2) \sum_{i=j+1}^{L-1} \mathbb{E}I(Y^{(N-2)}_i, \alpha^{(N-3)}_i, \mathcal{U}^{(N-3)}(C(1), C'(1), \alpha^{(N-3)})). \]
Conditioning on $\sigma(Y^{(1)}, \ldots, Y^{(N-3)})$ and using $H^*_2$, we finally get

$$\mathbb{E}U^{(N-2)}_j(C, C', \alpha^{(N-2)}) \leq \frac{K}{N} + K' \sum_{i=j+1}^{L-1} \mathbb{E}U^{(N-3)}_i(C(1), C'(1), \alpha^{(N-3)}).$$

This proves the step $k = 1$ of the induction. We prove the implementation from $k$ to $k + 1$ in a similar way. Applying this result to $k = L - 1 - j$ gives Lemma 4.4.

**Lemma 4.5** For $i, j = 1$ to $L - 1$, we have, under $H_1, H^*_2$,

$$\lim_N N\mathbb{E} \left[ I \left( Y^{(N-1)}_i, \alpha_i^{(N-2)}, |\alpha_i^{(N)} - \alpha_i^{(N-2)}| \right) I \left( Y^{(N)}_j, \alpha_j, |\alpha_j^{(N)} - \alpha_j| \right) \right] = 0$$

$$\lim_N N\mathbb{E} \left[ I \left( Y^{(N-1)}_i, \alpha_i^{(N-2)}, |\alpha_i^{(N)} - \alpha_i^{(N-2)}| \right) I \left( Y^{(N-1)}_j, \alpha_j, |\alpha_j^{(N)} - \alpha_j| \right) \right] = 0$$

**Proof:** We choose $\tilde{\Omega}_N$ such that $NP(\tilde{\Omega}_N)$ tends to 0 as $N$ goes to infinity and such that, on $\tilde{\Omega}_N$,

$$|\alpha_{L-1}^{(N)} - \alpha_{L-1}^{(N-2)}| \leq \frac{C_{L-1}}{N}$$

$$|\alpha_j^{(N)} - \alpha_j^{(N-2)}| \leq \frac{C_j}{N} + \frac{C'}{N} \sum_{n=1}^{N-2} \sum_{i=j+1}^{L-1} I(Y^{(n)}_i, \alpha_i^{(N-2)}, |\alpha_i^{(N)} - \alpha_i^{(N-2)}|),$$

for $j = 1$ to $L - 2$. Now, with the notation $U^{(N-2)} = U^{(N-2)}(C, C', \alpha^{(N-2)})$, we remark that for $j = 1$ to $L - 1$, $|\alpha_j^{(N)} - \alpha_j^{(N-2)}| \leq U_j^{(N-2)}$ on $\tilde{\Omega}_N$. Define

$$\Omega_N = \tilde{\Omega}_N \cap \{|\alpha^{(N)} - \alpha| \leq \delta\} \cap \{|\alpha^{(N-2)} - \alpha| \leq \eta\}.$$

From Lemma 4.3, $N\mathbb{P}^{e}(\Omega_N^c)$ tends to 0. Now we have

$$N\mathbb{E} \left[ I \left( Y^{(N-1)}_i, \alpha_i^{(N-2)}, |\alpha_i^{(N)} - \alpha_i^{(N-2)}| \right) I \left( Y^{(N)}_j, \alpha_j, |\alpha_j^{(N)} - \alpha_j| \right) 1_{\Omega_N} \right]$$

$$\leq N\mathbb{E} \left[ I \left( Y^{(N-1)}_i, \alpha_i^{(N-2)}, U_i^{(N-2)} \right) I \left( Y^{(N)}_j, \alpha_j, \delta \right) 1_{\{|\alpha^{(N-2)} - \alpha| \leq \eta\}} \right].$$

But

$$\mathbb{E} \left[ I \left( Y^{(N)}_j, \alpha_j, \delta \right) |Y^{(1)}, \ldots, Y^{(N-1)}| \right] \leq C\delta,$$

and, from $H^*_2$,

$$\mathbb{E} \left[ I \left( Y^{(N-1)}_i, \alpha_i^{(N-2)}, U_i^{(N-2)} \right) 1_{\{|\alpha^{(N-2)} - \alpha| \leq \eta\}} |Y^{(1)}, \ldots, Y^{(N-2)}| \right] \leq C U_i^{(N-2)},$$
this gives

\[ N \mathbb{E} \left[ I \left( Y^{(N-1)}_i, \alpha^{(N-2)}_i, |\alpha^{(N)}_i - \alpha^{(N-2)}_i| \right) I \left( Y^{(N)}_{j_1}, \alpha_{j_1}, |\alpha^{(N)}_{j_1} - \alpha_{j_1}| \right) \mathbf{1}_{\alpha_{n_1}} \right] \leq CN \mathbb{E} U^{(N-2)}_i \delta. \]

Applying Lemma 4.4, we know that \( \mathbb{E} U^{(N-2)}_i \leq K / N \) and Lemma 4.5 is proved.

**Proof of Theorem 4.3:** We prove that

\[ \lim_{N \to \infty} \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \left( F_j(\alpha^{(N)}_n, y^{(n)}) - F_j(\alpha, y^{(n)}) - (\phi_j(\alpha^{(N)}_n) - \phi_j(\alpha)) \right) = 0 \]

in \( L^2 \). The proof is similar for the second term of the Theorem. We introduce the notation

\[ \Delta_j(a, b, Y) = F_j(a, Y) - F_j(b, Y) - (\phi_j(a) - \phi_j(b)). \]

We have to prove that

\[ \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \left( \sum_{n=1}^{N} \Delta_j(\alpha^{(N)}_n, \alpha, y^{(n)}) \right)^2 = 0. \]

Remark that for \( n = 1 \) to \( N \), the couples \( (\alpha^{(N)}_n, y^{(n)}) \) and \( (\alpha^{(N)}, y^{(1)}) \) have the same law, and for \( n \neq m \), \( (\alpha^{(N)}_n, y^{(n)}, y^{(m)}) \) and \( (\alpha^{(N)}, y^{(N)}, y^{(N-1)}) \) have the same distribution. So we obtain

\[ \frac{1}{N} \mathbb{E} \left( \sum_{n=1}^{N} \Delta_j(\alpha^{(N)}_n, \alpha, y^{(n)}) \right)^2 = \mathbb{E} \Delta_j^2(\alpha^{(N)}, \alpha, y^{(1)}) + (N - 1) \mathbb{E} \Delta_j(\alpha^{(N)}_n, \alpha, y^{(N-1)}) \Delta_j(\alpha^{(N)}_n, \alpha, y^{(N)}). \]

But \( |\Delta_j(\alpha^{(N)}, \alpha, y^{(1)})| \leq 4||Z||_{\infty} \). Since the sequence \( (\alpha^{(N)}) \) goes to \( \alpha \) almost surely and \( \mathbb{P}(Z_j = \alpha \cdot \tilde{e}(X_j)) = 0 \) for \( j = 1 \) to \( L \) by assumption, we deduce that \( \Delta_j(\alpha^{(N)}, \alpha, y^{(1)}) \) goes to 0 almost surely. Consequently, we obtain that \( \mathbb{E} \Delta_j^2(\alpha^{(N)}, \alpha, y^{(1)}) \) tends to 0. It remains to prove that

\[ \lim_{N \to \infty} N \mathbb{E} \Delta_j(\alpha^{(N)}, \alpha, y^{(N-1)}) \Delta_j(\alpha^{(N)}, \alpha, y^{(N)}) = 0. \]  

(4.4)

We observe that

\[ \mathbb{E} \left( \Delta_j(\alpha^{(N-2)}, \alpha, y^{(N)})|y^{(1)}, \ldots, y^{(N-1)} \right) = 0, \]

since \( \mathbb{E} \left( F_j(\alpha^{(N-2)}, y^{(N)})|y^{(1)}, \ldots, y^{(N-1)} \right) = \phi_j(\alpha^{(N-2)}) \) almost surely. This gives

\[ \mathbb{E} \Delta_j(\alpha^{(N-2)}, \alpha, y^{(N-1)}) \Delta_j(\alpha^{(N-2)}, \alpha, y^{(N)}) = 0, \]

and we just have to prove that

\[ \lim_{N \to \infty} N \mathbb{E} \left( \Delta_j(\alpha^{(N)}, \alpha, y^{(N-1)}) \Delta_j(\alpha^{(N)}, \alpha, y^{(N)}) - \Delta_j(\alpha^{(N-2)}, \alpha, y^{(N-1)}) \Delta_j(\alpha^{(N-2)}, \alpha, y^{(N)}) \right) = 0. \]
We have the equality

\[ \Delta_j(\alpha^{(N)}, \alpha, Y^{(N-1)}) \Delta_j(\alpha^{(N)}, \alpha, Y^{(N)}) - \Delta_j(\alpha^{(N-2)}, \alpha, Y^{(N-1)}) \Delta_j(\alpha^{(N-2)}, \alpha, Y^{(N)}) = \Delta_j(\alpha^{(N)}, \alpha^{(N-2)}, Y^{(N-1)}) \Delta_j(\alpha^{(N)}, \alpha, Y^{(N)}) + \Delta_j(\alpha^{(N-2)}, \alpha, Y^{(N-1)}) \Delta_j(\alpha^{(N)}, \alpha^{(N-2)}, Y^{(N)}) , \]

We want to prove that \( \lim_{N \to \infty} NE \Delta_j(\alpha^{(N)}, \alpha^{(N-2)}, Y^{(N-1)}) \Delta_j(\alpha^{(N)}, \alpha, Y^{(N)}) = 0 \) and \( \lim_{N \to \infty} NE \Delta_j(\alpha^{(N-2)}, \alpha, Y^{(N-1)}) \Delta_j(\alpha^{(N)}, \alpha^{(N-2)}, Y^{(N)}) = 0 \). Both equalities can be proved in a similar manner. We give some details for the first one. Using Lemma 4.2 and Lemma 4.4, we see that there exists a subset \( \Omega_N \) such that \( \lim_{N \to \infty} NP(\Omega_N) = 0 \), on which

\[ \left| F_j(\alpha^{(N)}, Y^{(N-1)}) - F_j(\alpha^{(N-2)}, Y^{(N-1)}) \right| \leq C \sum_{i=j}^{L-1} I(Y_{i}^{(N-1)}, \alpha_i^{(N-2)}, U_i^{(N-2)}) , \]

with \( U_i^{(N-2)} = U_i^{(N-2)}(C_i, C_i', \alpha^{(N-2)}) \) for some constant vectors \( C_i, C_i' \). We may also assume that, for a given \( \delta > 0 \), with a proper choice of \( \Omega_N \), we have, on \( \Omega_N \),

\[ \left| \Delta_j(\alpha^{(N)}, \alpha, Y^{(N)}) \right| \leq C \sum_{i=j}^{L-1} I(Y_{i}^{(N)}, \alpha_i, \delta) + \sup_{|\beta - \alpha| \leq \delta} |\phi_j(\beta) - \phi_j(\alpha)| . \]

We now condition with respect to \( \sigma(Y^{(1)}, \ldots, Y^{(N-2)}) \) to obtain

\[ E \left| F_j(\alpha^{(N)}, Y^{(N-1)}) - F_j(\alpha^{(N-2)}, Y^{(N-1)}) \right| \left| \Delta_j(\alpha^{(N)}, \alpha, Y^{(N)}) \right| \leq \rho(\delta) EU_i^{(N-2)} + o(1/N) , \]

where \( \lim_{\delta \to 0} \rho(\delta) = 0 \). It follows that

\[ \lim_{N \to \infty} N E \left| F_j(\alpha^{(N)}, Y^{(N-1)}) - F_j(\alpha^{(N-2)}, Y^{(N-1)}) \right| \left| \Delta_j(\alpha^{(N)}, \alpha, Y^{(N)}) \right| = 0. \]

It remains to show that

\[ \lim_{N \to \infty} N E \left| \phi_j(\alpha^{(N)}) - \phi_j(\alpha^{(N-2)}) \right| \left| \Delta_j(\alpha^{(N)}, \alpha, Y^{(N)}) \right| = 0. \]

We can find \( \Omega_N \) such that \( \lim_{N \to \infty} NP(\Omega_N) = O \), on which

\[ \left| \phi_j(\alpha^{(N)}) - \phi_j(\alpha^{(N-2)}) \right| \leq C \sum_{k=j}^{L-1} \left| \alpha_k^{(N)} - \alpha_k^{(N-2)} \right| . \]

Here, we use the fact that \( \phi_j \) is \( C^1 \) near \( \alpha \). If \( j = L - 1 \), the right handside can be estimated by \( C/N \) (on a suitable \( \Omega_N \)). For \( j < L - 1 \), it can be controlled by

\[ \frac{C}{N} \sum_{n=1}^{N-2} \sum_{k=j+1}^{L-1} I(Y_{k}^{(n)}, \alpha_k^{(N-2)}, \alpha_k^{(N)} - \alpha_k^{(N-2)}) , \]

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and the expectation can be estimated in the same way as above.

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**References**


