Analytic Results For A Periodic Review Inventory Control Problem With Lost Sales *

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Abstract

We study a single item, single location, periodic review inventory procurement problem in this paper. Demand is modeled as a continuous time stochastic process. An order for procurement of additional inventory is placed at the beginning of every period and is received before the beginning of the next period. In other words, the lead time is between zero and the length of a period. Demand that can not be satisfied with inventory on hand is lost. Properties of the optimal ordering policy are the main focus of this study. We also consider the class of order-up-to-S policies and prove the convexity of a discounted cost function with respect to the order-up-to quantity.

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1 Introduction

We study a single item, single location, periodic review inventory procurement system which operates as follows. At the beginning of each period, a procurement order is placed on a supplier. The lead time for an order is a random variable, which is strictly positive but always assumes a value less than the length of a period. Demand occurs throughout the period. Demand is assumed to be described by a continuous random variable. All demand that arises before the receipt of this order in excess of the inventory on hand at the time the order is placed is assumed to be lost. Next, the inventory that was ordered at the beginning of that period arrives, which increases the quantity of stock on hand. Following the receipt of this order random demand continues to occur. Demand occurring from the time just after the receipt of this period's order until the placement of the next order in excess of supply is also lost. We also assume demand independent and identically distributed from period-to-period. However, non-stationarity of demand throughout a period and correlation of demand within a period are permitted. Slightly more restrictive assumptions about the demand process and the lead times will be specified in the statement of two propositions we make in section 4. We assume linear purchase costs and linear holding costs. Holding costs are charged at the beginning of every period based on inventory on hand. At the end of a period, linear lost sales costs are charged proportional to the number of units of sales lost during that period. The discounted sum of these expected period costs over a finite horizon represents the problem's objective function.

Our goals in this paper are to (1) show the objective function is a convex function, (2) establish properties of the optimal ordering policy, (3) determines bounds on the optimal order quantities, (4) derive monotonicity properties for the probability of not stocking out and on-hand stock at the end of a period as a function of lead time length, and (5) prove a convexity property of the objective function when the commonly used base stock policy is followed.

There are very few papers that contain analytic results for periodic review problems with lost sales and positive lead times. The seminal paper by Karlin and Scarf (1958) is the first paper that analyzed this problem. They proved some basic properties about the optimal ordering function (the function relating the optimal order quantity to the on-hand inventory) with the assumption that the lead time of all orders is exactly one period. However, they did not provide an efficient method for computing the optimal order quantities. Later, Morton (1969) generalized these basic results to the periodic review lost sales problems with arbitrary, but fixed lead times that are integer multiples of the period's length. In addition, he developed easily computable bounds on the optimal order quantity for a given inventory vector (the vector of the quantities of inventory in different stages of the pipeline). Subsequently Morton (1971) proposed and evaluated myopic policies as effective heuristics for these problems. Later, Nahmias (1979) considered more general periodic review lost sales problems that include fixed ordering costs or partial backordering or random lead times. He developed myopic policies for these problems and investigated their effectiveness. In addition, he also proposed the use of (s,S) policies or order-up-to-S policies, when the fixed cost is zero, for problems with lead times greater than two periods. Given that the analytic determination of the best policy, among the class of such (s,S) or order up to S policies, may not be possible, he used simulation to determine the best policies. More recently, the convexity of the cost function has been established for lost sales problems with positive lead times when an order-up-to policy is followed (Downs et al. (2001) and Roundy and Janakiraman (2000)). Downs et al. (2001) assume arbitrary but fixed lead times that are an integer multiple of the period's length. They prove this convexity result and use it to develop a linear program to determine optimal order-up-to stocking levels for multiple products in the presence of budget constraints. Roundy and Janakiraman (2000) assume random integer lead times such that orders do not cross and show the convexity result for the order-up-to policy case.

It is important to note that almost all papers that deal with periodic review inventory procurement problems assume that lead times are integral multiples of the period lengths. Our problem differs from the existing literature on this count. Our assumption that lead times are between zero and the length of a period may appear to be a minor alteration of the ones made by others; doing so, however, surprisingly complicates the analysis significantly. Also, by permitting non-stationarity of demand throughout a period and correlation of demand within a period, we also extend previous results.

The remainder of this paper is organized as follows. In section 2, we describe the notation and assumptions used throughout the paper. In section 3, we develop key analytic results about the optimal ordering quantities. In section 4, we consider the case when the lead times are deterministic and present two results on the effect of increasing the lead times. In section 5, we restrict attention to the class of order-up-to policies and prove a useful convexity result. In sections 3.1 and 5.1, we briefly discuss extensions of the finite horizon results of sections 3 and 5 respectively to the infinite horizon problem. Finally, we conclude the paper in section 6.

The results developed in section 3 are extensions of those found in Karlin and Scarf (1958) and Morton (1969). Using methods similar to theirs, we are able to prove the convexity of the cost function, monotonicity of order quantities as a function of on-hand stock at the beginning of a period, and some bounds on the order quantity. In section 5 we consider the class of order-up-to policies. While these policies are known to be suboptimal for lost sales problems, they are commonly used in practice. See Karlin and Scarf (1958) and Nahmias (1979), for example.

2 Notation and Assumptions

2.1 Notation

Assume that there are N periods in the planning horizon which we index periods in a backward fashion, (i.e.) where period $N-1$ occurs after period N, and period 1 is the last period in the planning horizon. As we mentioned earlier, a linear purchase cost is allowed; but, we adopt a technique used recently in Roundy and Janakiraman (2000) to show that this problem is equivalent to a problem where the purchase cost is zero. This transformation is useful in simplifying the analysis. To make this simplification possible, we assume that the on-hand inventory at the end of period 1 can be salvaged at the purchase price.

Let us now state the notation that is used in this paper.

α = discount factor .

x_n = on-hand inventory at the beginning of period n .

x_0 = on hand inventory at the end of period 1.

q_n = quantity ordered in period n .

D_{n1} = random variable representing the demand that occurs between the start of period n and the time the order of size q_n is received.

D_n = random variable representing the total demand that occurs in period n .

D_{n2} = $D_n - D_{n1}$

= the random variable representing the demand that occurs between the time that we receive the order of size q_n and the end of period n .

$\overline{D}_n$ = the random vector (D_{n1}, D_{n2}) .

y_n = on-hand inventory just after receiving q_n .

= $(x_n - D_{n1})^+ + q_n$.

h = holding cost incurred per unit of inventory (charged at the beginning of a period) .

b = lost sales cost incurred for every unit of sales lost during a period (charged at the end of a period) .

c = purchase cost per unit

(charged at the beginning of a period) .

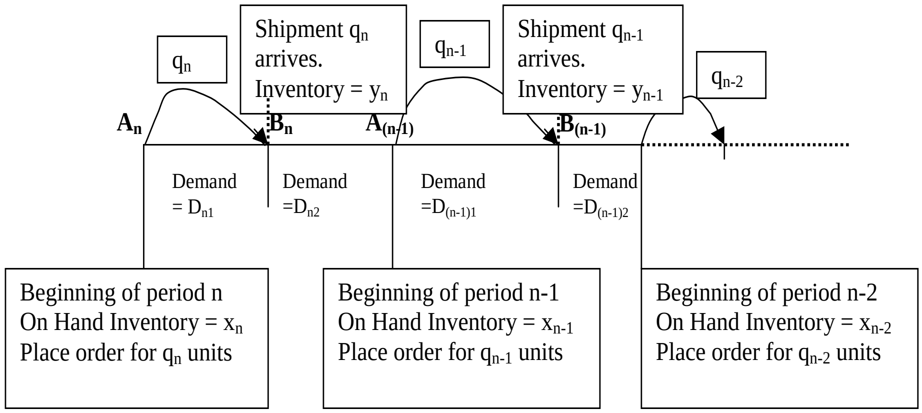


Figure 1: Dynamics of the order/demand process

The dynamics of the system's operation as discussed in section 1 are depicted in Figure 1. The timing of events and the manner in which costs are incurred are critical to our analysis. Each period is linked to the following one through the evolution of the on-hand inventory at the beginning of a period. The equation that describes this relationship from period n to $n-1$ is given by

$$x_{n-1} \quad = \quad ((x_n - D_{n1})^+ + q_n - D_{n2})^+. \tag{2.1}$$

Next, we define the one-period cost and the expected discounted cost. Note that all the expectation operators in this paper will be subscripted with the set of random vectors or variables over which the expectation is taken.

$$\begin{aligned} V_n(x_n, q_n) \quad &= \quad \text{expected period } n \text{ purchase, holding and lost sales costs} \\ &\quad\quad \text{if we start period } n \text{ with } x_n \text{ units of inventory on hand} \\ &\quad\quad \text{and we order } q_n \text{ units} \\ &= \quad E_{\overline{D}_n}\left[c \cdot q_n + h \cdot x_n + b \cdot (D_{n1} - x_n)^+ + b \cdot (D_{n2} - y_n)^+\right]. \end{aligned} \tag{2.2}$$

$$\begin{aligned} f_n(x_n) \quad &= \quad \text{minimum expected sum of all discounted future costs over the} \\ &\quad\quad \text{planning horizon if we start period } n \text{ with } x_n \text{ units of inventory on hand} \end{aligned}$$

$$= \min_{q_n \geq 0} \left\{ V_n(x_n, q_n) + \alpha \cdot E_{\overline{D}_n}[f_{n-1}(x_{n-1})] \right\}. \quad (2.3)$$

$$\begin{aligned} f_n(x_n, q_n) &= \text{minimum expected sum of all discounted future costs if we start period } n \\ &\quad \text{with } x_n \text{ units of inventory on hand and if we order } q_n \text{ units in period } n \\ &= V_n(x_n, q_n) + \alpha \cdot E_{\overline{D}_n}[f_{n-1}(x_{n-1})]. \quad (2.4) \\ q_n^*(x) &= \arg\min_{q \geq 0} (f_n(x, q)), \text{ and therefore} \\ f_n(x) &= f_n(x, q_n^*(x)). \end{aligned}$$

In addition to these costs incurred in periods N, $N-1$, ..., 1 there is an end of horizon cost $f_0(x_0)$ which depends on x_0, the inventory on hand at the end of period 1.

$$\begin{aligned} f_0(x_0) &= \text{cost incurred at the end of the horizon, (i.e.) the end of period 1} \\ &= h \cdot x_0 - c \cdot x_0. \quad (2.5) \end{aligned}$$

Observe that at the end of the horizon, we are charged a holding cost for the inventory on hand less the salvage value associated with that inventory.

The finite horizon problem is to determine the optimal ordering policy, that is, the optimal order function $q_n^*(x)$, for all $n \in \{N, N-1, \ldots, 1\}$ and for all $x \geq 0$.

2.2 Assumptions

The following conditions are assumed throughout the paper unless specifically stated to be otherwise.

1. $0 \leq \alpha \leq 1$.

2. The cost parameters satisfy the relation $\alpha(b-h) \geq c$ (such a set of costs will be called "valid"). This implies that the present value of losing a sale in the period is greater than the purchase price of the unit plus the present value of holding that unit in inventory throughout the period. Thus if the sale of a unit of stock is certain to occur some time during the period, it will be best to purchase the unit of stock.

3. $\{\overline{D}_n\} = \{(D_{n1}, D_{n2})\}$ is a sequence of i.i.d. random vectors.

4. All the random variables representing demand possess continuous density functions. This assumption implies the existence of all the derivatives that appear in section 3. Let us now define the necessary probability density functions and cumulative distribution functions.

$$
\begin{aligned}
\phi_1(u_1) &= \text{ the probability density of } D_{n1} \text{ at } u_1. \\
\Phi_1(u_1) &= \text{ the cumulative distribution function of } D_{n1} \text{ at } u_1 \\
&= P(\, D_{n1} \, \leq \, u_1 \,). \\
\phi_2(u_2) &= \text{ the probability density of } D_{n2} \text{ at } u_2. \\
\Phi_2(u_2) &= \text{ the cumulative distribution function of } D_{n2} \text{ at } u_2 \\
&= P(\, D_{n2} \, \leq \, u_2 \,). \\
\phi(u) &= \text{ the probability density of } D_n \text{ at } u. \\
\Phi(u) &= \text{ the cumulative distribution function of } D_n \text{ at } u \\
&= P(\, D_n \, \leq \, u \,). \\
\Phi * \Phi_1(u) &= P(\, D_n + D_{n-1,1} \, \leq u \,). \\
\overline{\Phi}_1(u_1) &= \text{ the complementary cumulative distribution function of } D_{n1} \text{ at } u_1 \\
&= 1 \, - \, \Phi_1(u_1). \\
\overline{\Phi}_2(u_2) &= \text{ the complementary cumulative distribution function of } D_{n2} \text{ at } u_2 \\
&= 1 \, - \, \Phi_2(u_2). \\
\tilde{\Phi}(u) &= P(D_{n2} + D_{(n-1)1} \leq u). \\
\phi_{12}(u_1, u_2) &= \text{ probability density function of } \overline{D}_n \text{ at } (u_1, u_2).
\end{aligned}
$$

The randomness in the lead time is captured in the probability density function of $\overline{D}_n = (D_{n1}, D_{n2})$.

3 Properties of the Optimal Ordering Function

Our first lemma states that the finite horizon problem with a positive unit purchase cost of c can be transformed into a finite horizon problem with zero purchase cost per unit. This transformation is useful in simplifying the analysis.

In the following lemma the superscript denotes the purchase, holding and lost sales cost parameters in that order.

Lemma 1 For all valid sets of cost parameters $(\tilde{c}, \tilde{h}, \tilde{b})$, $\exists$ another set of cost parameters $(0, h, b)$ such that

$$f_n^{(\tilde{c},\tilde{h},\tilde{b})}(x_n, q_n) \;=\; f_n^{(0,h,b)}(x_n, q_n) \;+\; \textit{a term independent of the policy.} \tag{3.6}$$

$$\text{and } b \;\geq\; h. \tag{3.7}$$

Proof: Appendix.

As a consequence of this lemma, we know that the optimal ordering policy for a problem with parameters $(\tilde{c}, \tilde{h}, \tilde{b})$ is the same as the optimal policy for an equivalent transformed problem with parameters $(0, h, b)$. From now on, we will be working only with the transformed problem. In other words, we will assume that the purchase cost $c = 0$ in the remainder of the paper.

We now derive basic properties of the function f_n and the optimal ordering function, q_n^*. The key facts that we present in the following theorem are (i) the function $f_n(x)$ is convex, (ii) there exists a limiting quantity $\bar{x}_n$ on the on-hand inventory beyond which the optimal order quantity is zero, and (iii) the optimal ordering function $q_n^*(x)$ is a non-increasing function with slope greater than -1, that is $q_n^*(x+1) - q_n^*(x) \leq 1 \; \forall \; x \geq 0$.

Theorem 1 For any period $1 \leq n \leq N$,

$$\text{(a)} \qquad f_n^{''}(x) \;\geq 0 \;, \tag{3.8}$$

$$\text{(b)} \qquad f_n^{'}(\infty) \ \geq \ 0 \ , \tag{3.9}$$

$$\text{(c)} \qquad F_n(x,q) \ = \ \frac{\partial f_n(x,q)}{\partial q} \ \text{ increases with } q \, \forall \, x \ \geq \ 0 \ , \tag{3.10}$$

$$\text{(d)} \qquad f_n^{'}(x) \ \geq \ h - b \ , \tag{3.11}$$

$$\text{(e)} \qquad \exists \, \bar{x}_n \text{ such that } \ q_n^*(x) > 0 \text{ if and only if } x \ < \ \bar{x}_n \ , \tag{3.12}$$

$$\text{(f)} \qquad -1 \ \leq \ \frac{dq_n^*(x)}{dx} \ \leq 0 \ \forall \ x \ \geq \ 0. \tag{3.13}$$

Proof: The proof is by induction. Recall that $f_0(x) = h \cdot x$. Thus statements (a),(b) and (d) are trivially true for f_0. Now, we assume that the statements (a), (b) and (d) in the theorem are true for functions $f_0, f_1, \ldots, f_{n-1}$. Now, let us show that all the statements (a) - (e) are true for period n as well.

Next, to simplify notation, let

$$\begin{aligned} \theta &= ((x - u_1)^+ + q_n^*(x) - u_2), \\ \theta_1 &= x - u_1 + q_n^*(x) - u_2, \\ \theta_2 &= q_n^*(x) - u_2 \end{aligned}$$

where u_1 represents a realization of D_{n1} and u_2 represents a realization of D_{n2}.

From (2.2) and (2.4), we get

$$\begin{aligned} f_n(x,q) &= h \cdot x + b \cdot E_{\overline{D}_n}[(D_{n1} - x)^+] + b \cdot E_{\overline{D}_n}[(D_{n2} - ((x - D_{n1})^+ + q)^+] \\ &\quad + \alpha \cdot E_{\overline{D}_n}[f_{n-1}(((x - D_{n1})^+ + q - D_{n2})^+)]. \end{aligned}$$

Differentiating this expression with respect to q, we get

$$\begin{aligned} F_n(x,q) &= -b \cdot P((x - D_{n1})^+ + q < D_{n2}) \\ &\quad + \alpha \cdot E_{\overline{D}_n}[f_{n-1}^{'}((x - D_{n1})^+ + q - D_{n2}) 1((x - D_{n1})^+ + q - D_{n2} > 0)] \qquad (3.14) \\ &= -b \int_{u_1=0}^{x} \int_{u_2=x-u_1+q}^{\infty} \phi_{12}(u_1,u_2)\, du_2 du_1 - b \int_{u_1=x}^{\infty} \int_{u_2=q}^{\infty} \phi_{12}(u_1,u_2)\, du_2 du_1 \end{aligned}$$

$$+\alpha \int_{u_1=0}^{x} \int_{u_2=0}^{x-u_1+q} f_{n-1}^{'}(x-u_1+q-u_2)\phi_{12}(u_1,u_2)\, du_2 du_1$$
$$+\alpha \int_{u_1=x}^{\infty} \int_{u_2=0}^{q} f_{n-1}^{'}(q-u_2)\, \phi_{12}(u_1,u_2)\, du_2 du_1. \tag{3.15}$$

But $F_n(x,q)$ is increasing in q and is non-negative for sufficiently high values of q. Also, $F_n(x,q)$ increases in x. Note that $q_n^*(x)$ is the value at which $\frac{\partial F_n(x,q)}{\partial q}$ equals zero.
This implies that

$$\exists \bar{x}_n \text{ such that } q_n^*(x) > 0 \text{ if and only if } x < \bar{x}_n.$$

Thus we have shown statements (c) and (e).

$$\begin{aligned}
f_n^{'}(x) &= \frac{df_n(x, q_n^*(x))}{dx} \\
&= h - b\cdot P(x < D_{n1}) - b\cdot P(x > D_{n1}, \theta_1 < 0) \\
&\quad +\alpha \cdot E_{\overline{D}_n}[f_{n-1}^{'}(\theta^+)1(x > D_{n1}, \theta > 0)] \qquad (3.16)\\
&= h - b\cdot \overline{\Phi}_1(x) - b\cdot \int_{u_1=0}^{x} \int_{u_2=x-u_1+q_n^*(x)}^{\infty} \phi_{12}(u_1,u_2)\, du_2\, du_1 \\
&\quad + \alpha \cdot \int_{u_1=0}^{x} \int_{u_2=0}^{x-u_1+q_n^*(x)} f_{n-1}^{'}(x + q_n^*(x) - u_1 - u_2)\, \phi_{12}(u_1,u_2)\, du_2\, du_1\ . \qquad (3.17)
\end{aligned}$$

But, $f_n^{'}(0) = h$ - b and, by assumption statement (d) holds for n-1. Thus, $f_n^{'}(x) \geq h$ - b. Also $f_n^{'}(\infty) \geq 0$, using (3.16), and statement (b) which holds for n-1. Thus we have shown statements (b) and (d).

Differentiating both sides of equation (3.17) we get

$$\begin{aligned}
f_n^{''}(x) &= b\cdot\phi_1(x) + b\cdot \int_{u_1=0}^{x} \phi_{12}(u_1, x-u_1+q_n^*(x))\, du_1 \cdot (1 + \frac{dq_n^*(x)}{dx}) \\
&\quad -b\cdot \int_{u_2=q_n^*(x)}^{\infty} \phi_{12}(x,u_2)\, du_2 \\
&\quad +\alpha\cdot \int_{u_1=0}^{x} \int_{u_2=0}^{x-u_1+q_n^*(x)} \left[f_{n-1}^{''}(x + q_n^*(x) - u_1 - u_2)\cdot (1 + \frac{dq_n^*(x)}{dx}) \right] \phi_{12}(u_1,u_2)\, du_2\, du_1 \\
&\quad + \alpha\cdot \int_{u_1=0}^{x} \left[f_{n-1}^{'}(0)\phi_{12}(u_1, x-u_1+q_n^*(x))\cdot (1 + \frac{dq_n^*(x)}{dx}) \right] du_1 \\
&\quad +\alpha\cdot \int_{u_2=0}^{q_n^*(x)} f_{n-1}^{'}(q_n^*(x) - u_2)\, \phi_{12}(x,u_2)\, du_2. \qquad (3.18)
\end{aligned}$$

Along the curve $y = q_n^*(x)$, $F_n(x, q_n^*(x)) = 0 \; \forall \; x \in (0, \bar{x}_n)$.

Let $H(x) = F_n(x, q_n^*(x))$. Thus

$$\begin{aligned} \frac{dH}{dx} &= 0,\; 0 < x < \bar{x}_n \;\; \text{and} \\ \frac{dH}{dx} &= \left(\frac{\partial F_n(x,q)}{\partial x}\right)_{q=q_n^*(x)} + \left(\frac{\partial F_n(x,q)}{\partial q}\right)_{q=q_n^*(x)} \left(\frac{\partial q_n^*(x)}{\partial x}\right), \;\; \text{when } 0 < x < \bar{x}_n. \end{aligned}$$

Observe that

$$\begin{aligned} \frac{dH}{dx} &= (b + \alpha \cdot f_{n-1}'(0)) \left(1 + \frac{\partial q_n^*(x)}{\partial x}\right) \int_{u_1=0}^{x} \phi_{12}(u_1, x - u_1 + q_n^*(x))\, du_1 \\ &\quad + (b + \alpha \cdot f_{n-1}'(0)) \left(\frac{\partial q_n^*(x)}{\partial x}\right) \int_{u_1=x}^{\infty} \phi_{12}(u_1, q_n^*(x))\, du_1 \\ &\quad + \alpha \int_{u_1=0}^{x} \int_{u_2=0}^{x-u_1+q_n^*(x)} f_{n-1}''(x + q_n^*(x) - u_1 - u_2) \phi_{12}(u_1, u_2)\, du_2 du_1 \cdot \left(1 + \frac{\partial q_n^*(x)}{\partial x}\right) \\ &\quad + \alpha \int_{u_1=x}^{\infty} \int_{u_2=0}^{q_n^*(x)} f_{n-1}''(q_n^*(x) - u_2) \left(\frac{\partial q_n^*(x)}{\partial x}\right) \phi_{12}(u_1, u_2)\, du_2 du_1 \text{ in the region } 0 < x < \bar{x}_n. \end{aligned}$$

In this region $\frac{dH}{dx}$ equals zero, which implies that

$$-1 \;\le\; \frac{dq_n^*(x)}{dx} \;\le 0\,,\; \forall\; x \;\ge\; 0$$

because $(b + \alpha \cdot f_{n-1}'(0)) \ge 0$ (by statement (d) for n-1) and $f_{n-1}''(\cdot) \ge 0$. Thus we have proved statement (f). Observe from equation (3.18) and statements (d) and (f) that $f_n''(x)$ is strictly positive for $n \ge 1$. In other words, $f_n(x)$ is strictly convex, $n \ge 1$. We have now proved all the statements listed in the theorem. □

We next derive upper and lower bounds on the derivative of the cost function f_n, which are used to construct bounds on the optimal order quantities.

Theorem 2

$-(b + \alpha h - h) + \Phi_1(x)(b + \alpha h) \;\le\; f_n'(x) \;\le (h - b) + (b + \alpha h)\Phi_1(x),\; \forall\, x \;\in\; (0, \bar{x}_n),\; 1 \le n \le N.$

Proof: Define $\bar{q}_n$ to be $q_n^*(0)$. As a consequence of (3.8) and (3.13) we have the following four inequalities.

$$0 \;\le\; q_n^*(x) \;\le\; \bar{q}_n,\; 0 \le x \le \bar{x}_n. \tag{3.19}$$

$$
\begin{aligned}
x &\leq x + q_n^*(x) \leq \bar{x}_n, \; 0 \leq x \leq \bar{x}_n. && (3.20)\\
\bar{q}_n &\leq \bar{x}_n. && (3.21)\\
x_1 &< x_2 \text{ if and only if } f_n^{'}(x_1) < f_n^{'}(x_2). && (3.22)
\end{aligned}
$$

To prove this theorem, we first establish some additional relationships.

Using (3.14) at $(x, q) = (0, \bar{q}_n)$, we get

$$
F_n(0, \bar{q}_n) = 0 = -bP(D_{n2} > \bar{q}_n) + \alpha E_{\overline{D}_n}\left[f_{n-1}^{'}(\bar{q}_n - D_{n2})1(D_{n2} \leq \bar{q}_n)\right].
$$

Since $f_{n-1}(\cdot)$ is convex, and $\bar{q}_n < \bar{x}_n$, we have

$$
f_{n-1}^{'}(q_n^*(x) - u) \leq f_{n-1}^{'}(q_n^*(x)) \leq f_{n-1}^{'}(\bar{q}_n) \leq f_{n-1}^{'}(\bar{x}_n) = h \;, \text{ where } u \geq 0.
$$

Hence
$$
\begin{aligned}
F_n(0, \bar{q}_n) &\leq -b\overline{\Phi}_2(\bar{q}_n) + \alpha f_{n-1}^{'}(\bar{q}_n)\Phi_2(\bar{q}_n) && (3.23)\\
&= -b + b\Phi_2(\bar{q}_n) + \alpha f_{n-1}^{'}(\bar{q}_n)\Phi_2(\bar{q}_n).
\end{aligned}
$$

Therefore,
$$
f_{n-1}^{'}(\bar{q}_n) \geq \frac{b\,\overline{\Phi}_2(\bar{q}_n)}{\alpha\,\Phi_2(\bar{q}_n)} \geq 0. \qquad (3.24)
$$

$$
\begin{aligned}
f_n^{'}(x) &= h - b\,P(x < D_{n1}) - b\,P(x > D_{n1}, \theta_1 < 0)\\
&\quad +\alpha\,E_{\overline{D}_n}[f_{n-1}^{'}(\theta^+)1(\theta > 0)] - \alpha\,E_{\overline{D}_n}[f_{n-1}^{'}(\theta^+)1(x < D_{n1}, \theta > 0)]\\
&\quad (\text{ from (3.16)})\\
&= h - b\,P(x < D_{n1}) - b\,P(x > D_{n1}, \theta_1 < 0)\\
&\quad +[F_n(x, q_n^*(x)) + b\,P(\theta < 0)] - \alpha\,E_{\overline{D}_n}[f_{n-1}^{'}(\theta^+)1(x < D_{n1}, \theta > 0)]\\
&\quad (\text{from equation (3.14)})\\
&= h - b\,P(x < D_{n1}) - b\,P(x > D_{n1}, x + q_n^*(x) < D_{n1} + D_{n2})\\
&\quad +b\,P((x - D_{n1})^+ + q_n^*(x) < D_{n2}) - \alpha\,E_{\overline{D}_n}[f_{n-1}^{'}(\theta^+)1(x < D_{n1}, \theta > 0)]\\
&\quad (\text{since } F_n(x, q_n^*(x)) = 0)\\
&= h - b\,P(x < D_{n1}) + b\,P(x < D_{n1}, q_n^*(x) < D_{n2})\\
&\quad -\alpha\,E_{\overline{D}_n}[f_{n-1}^{'}(q_n^*(x) - D_{n2})1(x < D_{n1}, q_n^*(x) > D_{n2})]\\
&= h - b\,P(x < D_{n1}, q_n^*(x) > D_{n2})\\
&\quad -\alpha\,E_{\overline{D}_n}[f_{n-1}^{'}(q_n^*(x) - D_{n2})1(x < D_{n1}, q_n^*(x) > D_{n2})]. && (3.25)
\end{aligned}
$$

Evaluating the above expression at $x = \bar{x}_n$, we see that $f_n^{'}(\bar{x}_n) = h$ by using the fact that $q_n^*(\bar{x}_n) = 0 \ \forall \ n$.

Since $f_{n-1}^{'}(\bar{x}_n) = h$ and $f_n^{'}(\cdot)$ is convex,

$$
\begin{aligned}
f_n^{'}(x) &\geq h - (b + \alpha h) P(x < D_{n1}, q_n^*(x) > D_{n2}) \\
&\geq h - \overline{\Phi}_1(x)(b + \alpha\, h) \\
&= -(b + \alpha\, h - h) + \Phi_1(x)(b + \alpha\, h)
\end{aligned}
$$

which establishes one side of the inequality stated in the theorem. Now, we prove the other side of the inequality.

Rewriting (3.16), we get

$$
\begin{aligned}
f_n^{'}(x) &= h - bP(x < D_{n1}) - bP(x > D_{n1}, x + q_n^*(x) < D_{n1} + D_{n2}) \\
&\quad + \alpha\, E_{\overline{D}_n}[f_{n-1}^{'}((x - D_{n1})^+ + q_n^*(x) - D_{n2}) 1(x > D_{n1}, x + q_n^*(x) > D_{n1} + D_{n2})] \\
&= h - bP(x < D_{n1}) - bP(x > D_{n1}, x + q_n^*(x) < D_{n1} + D_{n2}) \\
&\quad + \alpha\, E_{\overline{D}_n}[f_{n-1}^{'}(x - D_{n1} + q_n^*(x) - D_{n2}) 1(x > D_{n1}, x + q_n^*(x) > D_{n1} + D_{n2})] \\
&\leq h - b\overline{\Phi}_1(x) + \alpha E[h 1(x > D_{n1}, x + q_n^*(x) > D_{n1} + D_{n2})] \\
&\quad \text{(since } f_{n-1}^{'}(x - D_{n1} + q_n^*(x) - D_{n2}) \ \leq \ f_{n-1}(\bar{x}_n) \ = \ h) \\
&\leq h - b\overline{\Phi}_1(x) + \alpha h \Phi_1(x) \\
&\leq (h - b) + (b + \alpha h)\Phi_1(x)\ ,
\end{aligned}
$$

which proves the other side of the inequality stated in the theorem. □

Next, we construct bounds on the probability of not running out of stock in the interval of time between the receipt of two successive orders.

$$
\begin{aligned}
\text{Let } \pi_n(x, q) &= P((x - D_{n1})^+ + q \geq D_{n2} + D_{(n-1)1}) \\
&= P(x > D_{n1}, x + q - D_{n1} \geq D_{n2} + D_{(n-1)1}) \\
&\quad + P(x < D_{n1}, q \geq D_{n2} + D_{(n-1)1}), \text{ and} \\
\pi_n^*(x) &= \pi_n(x, q_n^*(x)).
\end{aligned}
$$

$\pi_n(x,q)$ is the probability that we will not run out of stock in the time between the receipt of the order for q units, given x are on-hand when the order is placed, and the receipt of the subsequent order.

$$\text{But} \qquad \pi_n(x,q) = E_{\overline{D}_n} E_{D_{(n-1)1}} \left[1((x - D_{n1})^+ + q - D_{n2} \geq D_{(n-1)1}) \right] \tag{3.26}$$

$$= E_{\overline{D}_n} \left(\Phi_1((x - D_{n1})^+ + q - D_{n2}) \right). \tag{3.27}$$

The following theorem gives bounds on the probability of not stocking out while using the optimal policy. Let $L = \frac{b-h}{b+\alpha h}$ and $U = \frac{b}{b+\alpha h}$.

Theorem 3

$$L \leq \pi_n^*(x) \leq U, \quad \forall\, x \in (0, \bar{x}_n).$$

Proof :

$$
\begin{aligned}
0 &= F_n(x, q_n^*(x)) \\
&= -bP((x - D_{n1})^+ + q_n^*(x) < D_{n2}) + \alpha E_{\overline{D}_n}[f_{n-1}^{'}(\theta^+)1(\theta > 0)] \\
&= -bP(\theta < 0) + \alpha E_{\overline{D}_n}[f_{n-1}^{'}(\theta)1(\theta > 0)] \\
&= -b \; + \; E_{\overline{D}_n} \left[(b + \alpha \cdot f_{n-1}^{'}(\theta))1(\theta > 0) \right] \\
&\geq -b + E_{\overline{D}_n} \left[(b + \alpha \cdot [(h - b - \alpha \cdot h) + \Phi_1(\theta)(b + \alpha \cdot h)])1(\theta > 0) \right] \\
&\quad \text{(using Theorem 2)} \\
&= -b + E_{\overline{D}_n} E_{\overline{D}_{n-1}} \left[(b + \alpha \cdot (h - b - \alpha \cdot h) + \alpha(b + \alpha \cdot h)1(D_{(n-1)1} < \theta))1(\theta > 0) \right] \\
&\quad \text{(because } \Phi_1(\theta) \; = \; P(D_{(n-1)1} < \theta) \; = \; E_{\overline{D}_{n-1}}[1(D_{(n-1)1} < \theta)]) \\
&\geq -b + E_{\overline{D}_n} E_{\overline{D}_{n-1}} \left[(b + \alpha \cdot (h - b - \alpha \cdot h) + \alpha(b + \alpha \cdot h)1(D_{(n-1)1} < \theta))1(\theta > D_{(n-1)1}) \right] \\
&\quad \text{(because } 1(\theta > 0) \; \geq \; 1(\theta > D_{(n-1)1})) \\
&= -b \; + \; (b + \alpha \cdot h)P(\theta > D_{(n-1)1}) \\
&= -b \; + \; (b + \alpha \cdot h)P((x - D_{n1})^+ + q_n^*(x) - D_{n2} > D_{(n-1)1}) \\
&= -b \; + \; (b + \alpha \cdot h)\pi_n^*(x) \\
&\quad \text{(from the definition of } \pi_n^*(x)).
\end{aligned}
$$

Therefore, we have $\pi_n^*(x) \leq \frac{b}{b+\alpha h}$.

To prove the other inequality, we again start with the expression $F_n(x, q_n^*(x)) = 0$.

$$
\begin{aligned}
0 &= F_n(x, q_n^*(x)) \\
&= -b + bP((x - D_{n1})^+ + q_n^*(x) > D_{n2}) + \alpha E_{\overline{D}_n}[f'_{n-1}(\theta^+)1(\theta > 0)] \\
&= -b + E_{\overline{D}_n}[(b + \alpha f'_{n-1}((x - D_{n1})^+ + q_n^*(x) - D_{n2}))1((x - D_{n1})^+ + q_n^*(x) > D_{n2})] \\
&\leq -b + E_{\overline{D}_n}[(b + \alpha(h - b) + \alpha(b + \alpha h)\Phi_1((x - D_{n1})^+ + q_n^*(x) - D_{n2})) \cdot \\
&\qquad\qquad 1((x - D_{n1})^+ + q_n^*(x) > D_{n2})] \\
&\quad \text{(using Theorem 2)} \\
&\leq -b + E_{\overline{D}_n}[b + \alpha(h - b) + \alpha(b + \alpha h) \cdot \Phi_1((x - D_{n1})^+ + q_n^*(x) - D_{n2})] \\
&\quad \text{(since } 1((x - D_{n1})^+ + q_n^*(x) > D_{n2}) \ \leq\ 1 \\
&\quad\ \text{and } (b + \alpha(h - b) + \alpha(b + \alpha h) \cdot \Phi_1((x - D_{n1})^+ + q_n^*(x) - D_{n2})) \ \geq\ 0) \\
&= \alpha(h - b) + \alpha(b + \alpha h)\pi_n^*(x) \\
&\quad \text{(using equation (3.27)).}
\end{aligned}
$$

Therefore,

$$\pi_n^*(x) \quad \geq \quad \frac{b - h}{b + \alpha h} \ . \ \Box$$

Before stating easily computable bounds on the optimal order quantity $q_n^*(x)$, we first prove

Lemma 2

$$q_n^*(x) \quad \leq \quad \tilde{\Phi}^{-1}(U).$$

Proof:

$$
\begin{aligned}
\tilde{\Phi}(q_n^*(x)) &= P(D_{(n-1)1} + D_{n2} \leq q_n^*(x)) \\
&\leq P(D_{(n-1)1} + D_{n2} \leq (x - D_{n1})^+ + q_n^*(x)) \\
&= \pi_n^*(x) \ \leq\ U. \ \Box
\end{aligned}
$$

Another upper bound on $q_n^*(x)$ is established in

Lemma 3

$$\forall\, x \ \text{ for which } 0 \ \le \ x \ \le \ \bar{x}_n, \ q_n^*(x) \ \le \ (\Phi * \Phi_1)^{-1}(U) - x,$$
$$\text{or } x + q_n^*(x) \ \le \ (\Phi * \Phi_1)^{-1}(U).$$

Proof:

$$\begin{aligned}
\Phi * \Phi_1(x + q_n^*(x)) &= P(D_{n1} + D_{n2} + D_{(n-1)1} \le x + q_n^*(x)) \\
&= P(D_{n2} + D_{(n-1)1} \le x - D_{n1} + q_n^*(x)) \\
&\le P(D_{n2} + D_{(n-1)1} \le (x - D_{n1})^+ + q_n^*(x)) \\
&= \pi_n^*(x) \ \le \ U. \ \square
\end{aligned}$$

We now establish a lower bound on $q_n^*(x)$, as shown in

Lemma 4

$$q_n^*(x) \ \ge \ \tilde{\Phi}^{-1}(L) \ - \ x.$$

Proof :

$$\begin{aligned}
L \ \le \ \pi_n^*(x) &= P((x - D_{n1})^+ + q_n^*(x) \ge D_{n2} + D_{(n-1)1}) \\
&\le P(x + q_n^*(x) \ge D_{n2} + D_{(n-1)1}) \ = \ \tilde{\Phi}(x + q_n^*(x)). \ \square
\end{aligned}$$

When $x = 0$, $q_n^*(0) = \bar{q}_n$; consequently,

$$\bar{q}_n \ \ge \ \tilde{\Phi}^{-1}(L).$$

A second lower bound on $q_n^*(x)$ is stated in

Lemma 5

$$x + q_n^*(x) \ \ge \ (\Phi * \Phi_1)^{-1}\left[L - \Phi_1(\tilde{\Phi}^{-1}(U))\overline{\Phi}_1(x)\right].$$

Proof :

$$L \ \le \ \pi_n^* \ = \ P(x \ge D_{n1}, x + q_n^*(x) \ge D_{n1} + D_{n2} + D_{(n-1)1}) + P(x \le D_{n1}, q_n^*(x) \ge D_{n2} + D_{(n-1)1})$$

$$\begin{aligned}
&\le \Phi * \Phi_1(x + q_n^*(x)) + P(x \le D_{n1}, q_n^*(x) \ge D_{(n-1)1}) \\
&= \Phi * \Phi_1(x + q_n^*(x)) + \overline{\Phi}_1(x)\Phi_1(q_n^*(x)) \\
&\le \Phi * \Phi_1(x + q_n^*(x)) + \overline{\Phi}_1(x)\Phi_1(\tilde{\Phi}^{-1}(U)), \\
&\quad \text{since } \tilde{\Phi}(q_n^*(x)) \le U \text{ from Lemma 2.} \quad \square
\end{aligned}$$

A third lower bound on $q_n^*(x)$ is given by

Lemma 6

$$\forall\, x \ \text{ for which } 0 \le x \le \bar{x}_n, \ \ q_n^*(x) \ \ge \ \Phi_1^{-1}\left[\frac{L - U\Phi_1(x)}{\overline{\Phi}_1(x)}\right].$$

Proof : Observe from Theorem 2 that

$$f_n'(q_n^*(x)) \ \le \ (h-b) \ + \ (b+\alpha h)\Phi_1(q_n^*(x)). \tag{3.28}$$

Note also that equation (3.25), along with the fact that $f_{n-1}'(\cdot)$ is an increasing function, implies that

$$\begin{aligned}
f_n'(x) \ &\ge \ h - b\, P(x < D_{n1}, q_n^*(x) > D_{n2}) \\
&\quad -\alpha\, E_{\overline{D}_n}[f_{n-1}'(q_n^*(x))1(x < D_{n1}, q_n^*(x) > D_{n2})] \qquad (3.29) \\
&\ge \ h - b\, P(x < D_{n1}) - \alpha\, E_{\overline{D}_n}[f_{n-1}'(q_n^*(x))1(x < D_{n1})]. \qquad (3.30)
\end{aligned}$$

Combining (3.30) and (3.28), we get

$$f_n'(x) \ \ge \ h \ - \ [b + \alpha(h-b) + \alpha(b + \alpha\cdot h)\Phi_1(q_n^*(x))]\, \overline{\Phi}_1(x)\ . \tag{3.31}$$

Finally, using (3.31) and Theorem 2, we see that

$$(h-b) \ + \ (b+\alpha\cdot h)\Phi_1(x) \ \ge \ h \ - \ [b + \alpha(h-b) + \alpha(b+\alpha\cdot h)\Phi_1(q_n^*(x))]\, \overline{\Phi}_1(x)\ .$$

Simplifying this equation and using the definitions of L and U yields the result. $\square$

We can also state a lower bound on $\bar{x}_n$ as follows.

Lemma 7

$$\bar{x}_n \ \ge \ (\Phi * \Phi_1)^{-1}(L).$$

Proof : Recall that by definition $q_n^*(\bar{x}_n) = 0$ and $\pi_n^*(\bar{x}_n) = \pi_n(\bar{x}_n, 0)$. But

$$\begin{aligned} \pi_n^*(\bar{x}_n) &= P(\bar{x}_n \geq D_{n1} + D_{n2} + D_{(n-1)1}) \\ &= (\Phi * \Phi_1)(\bar{x}_n) \ \geq \ L. \ \square \end{aligned}$$

Thus we have developed a series of bounds on the optimal order quantity $q_n^*(x)$. These bounds can potentially be used to establish heuristics for computing order quantities.

3.1 Infinite Horizon Extensions

In this sub-section we consider a sequence of finite horizon problems and state convergence results as the length of the horizon approaches infinity. For this purpose, we will index all functions and quantities corresponding to the finite horizon problem of length N periods with the superscript N. We assume that $x_N^N = I \; \forall \; N$. That is, in each of the finite horizon problems the starting inventory level is the same. It is also assumed that $I \leq B = (\Phi * \Phi_1)^{-1}(U)$. An obvious consequence of Lemma 3 and this assumption is that $x_n^N \leq B$. Thus, the inventory level in any period is bounded and $(x_n^N, q_n^{N*}) \in \{(x, q) : 0 \leq x \leq B, 0 \leq q \leq B\}$. Since the probability density functions of the demand random variables are continuous, they are also bounded over compact intervals. These facts are useful for showing that $(f_n^N)''(\cdot)$ is bounded over intervals of interest and this result is necessary for proving the following theorem.

The following theorem summarizes the important infinite horizon results and is stated without proof. The proof is a very long exercise in analysis and is identical to the one in Morton (1968) when lead times are integer multiples of period lengths.

Theorem 4 Under the assumption that $x_N^N = I \; \forall \; N$, $I \leq B = (\Phi * \Phi_1)^{-1}(U)$, and, $\overline{D} = (D_1, D_2) \sim_d \overline{D}_n$,

(a) $f_N^N(x, q)$ converges uniformly to a function $f(x, q)$.

(b) $f_N^N(x)$ converges uniformly to $f(x) \equiv$ minimum of $f(x,q)$ over all $q \geq 0$.

(c) The functions $f(x,q)$ and $f(x)$ satisfy the integral equation

$$\begin{aligned} f(x,q) &= E_{\overline{D}}\left[h \cdot x + b \cdot (D_1 - x)^+ + b \cdot (D_2 - ((x - D_1)^+ + q))^+\right] \\ &\quad + \alpha \cdot E_{\overline{D}}\left[f(((x - D_1)^+ + q - D_2)^+)\right]. \end{aligned}$$

(d) $q_N^{N*}(x)$ converges uniformly to a function $q^*(x)$.

(e) $q^*(x)$ uniquely minimizes $f(x,q)$ in the interval $\{q : q \geq 0\}$; $f(x,q)$ is convex in q; $f(x)$ is convex.

(f) $\exists\, \overline{x}$ such that $q^*(x) > 0$ if and only if $x < \overline{x}$.

A direct consequence of these convergence results is that all the bounds on the optimal ordering quantities for the finite horizon problems proved in Lemmas 2 through 7 hold for the infinite horizon problem too.

4 Monotonicity properties with respect to the lead times

In this section we assume that the lead time for all the orders is a known constant between 0 and the length of one period. For simplicity, we assume that time is scaled so that the length of a period is one time unit. Thus, the order lead time, L, is less than 1. We examine the effect of increasing the lead time on the probability of not stocking out and on the on hand inventory at the end of a period and at the time of receipt of the next period's order. We will show these results by examining two situations, one when the lead time is L and the other, when the lead time is $L + \epsilon$, where $0 < L + \epsilon \leq 1$. Also, to prove these lemmas, we make the following two assumptions about the demand process.

1. For any (t_1, t_2) such that $0 \leq t_1 \leq t_2 \leq 1$, $D_n(t_1, t_2)$ is i.i.d., from period to period where $D_n(t_1, t_2)$ is the random variable for the demand that occurs in the interval (t_1, t_2) of period n.

2. The stochastic process $D_n(0,t)$ has independent increments. That is, $D_n(t_1,t_2)$ and $D_n(t_3,t_4)$ are probabilistically independent if the intervals (t_1,t_2) and (t_3,t_4) are disjoint subintervals of the interval $(0,1)$.

The first lemma in this section says that the probability of not stocking out is a decreasing function of the lead time for a given pair (x,q), that is, when we start period n with x units on-hand and we place an order for q units. Before stating the lemma, let us define $\pi_n^L(x,q)$ and $\pi_n^{(L+\epsilon)}(x,q)$ to be the same as $\pi_n(x,q)$ as defined in section 3 when the lead time is L and $L+\epsilon$, respectively.

Lemma 8

$$\pi_n^L(x,q) \;\; \geq \;\; \pi_n^{(L+\epsilon)}(x,q).$$

Proof : Let the superscript L (and $L+\epsilon$) represent the order lead time in the situation considered. As a result of our assumptions, $D_{n2}^L + D_{(n-1)1}^L$ has the same distribution as $D_{n2}^{L+\epsilon} + D_{(n-1)1}^{L+\epsilon}$. Let us define D as a random variable with this distribution. Since $D_{n1}^{L+\epsilon} \geq D_{n1}^L$ for every sample path,

$$\pi_n^{(L+\epsilon)}(x,q) \;=\; P((x - D_{n1}^{L+\epsilon})^+ + q \geq D) \;\leq\; P((x - D_{n1}^L)^+ + q \geq D) \;=\; \pi_n^L(x,q). \;\square$$

In other words, Lemma 8 says that for the same on-hand inventory x, the order size q that is required to maintain a certain probability of not stocking out increases with the lead time.

Next, we compare the on-hand inventory random variable at the end of the current period and at the time of receiving an order in the next period in the two situations. Denote the on-hand inventory at the end of period n by I_A^L and at the time of receiving the order q_{n-1} (in period $n-1$) by I_B^L when the lead time is L. The corresponding quantities when the lead time is $L+\epsilon$ are $I_A^{L+\epsilon}$ and $I_B^{L+\epsilon}$.(The subscripts A and B refer to the time points A_{n-1} and B_{n-1} of figure 1 respectively.)

Lemma 9 For a given pair $(x_n^L, q_n^L) = (x_n^{L+\epsilon}, q_n^{L+\epsilon}) = (x,q)$,

$$I_A^{L+\epsilon} \;\; \geq \;\; I_A^L,$$

$$\forall\, t \;\geq\; 0,\; P(I_B^{L+\epsilon} \;\geq\; t) \;\;\leq\;\; P(I_B^L \;\geq\; t).$$

Proof : Define the random variable Δ to be $D_{n1}^{L+\epsilon} - D_{n1}^{L}$. Clearly $\Delta \geq 0$ for any sample path of the stochastic process $D_n(0,t)$. In addition, for every sample path of $D_n(0,t)$, $D_{n1}^{L} + D_{n2}^{L}$ $= D_{n1}^{L+\epsilon} + D_{n2}^{L+\epsilon}$ by definition. Then, we have $D_{n2}^{L+\epsilon} = D_{n2}^{L} - \Delta$.

Let us compare I_A^L and $I_A^{L+\epsilon}$ for any such sample path of demands.

$$\begin{aligned}
I_A^{L+\epsilon} &= ((x - D_{n1}^{L+\epsilon})^+ + q - D_{n2}^{L+\epsilon})^+ \\
&= ((x - D_{n1}^{L} - \Delta)^+ + q - D_{n2}^{L} + \Delta)^+ \\
&\geq ((x - D_{n1}^{L})^+ - \Delta + q - D_{n2}^{L} + \Delta)^+ \\
&= ((x - D_{n1}^{L})^+ + q - D_{n2}^{L})^+ \ = \ I_A^L
\end{aligned}$$

Now let us prove the second part. Let D^L denote $(D_{n2}^{L} + D_{(n-1)1}^{L})$ and $D^{L+\epsilon}$ denote $(D_{n2}^{L+\epsilon} + D_{(n-1)1}^{L+\epsilon})$. Note that D_{n2}^{L} represents the demand $D_n(L,1]$, the demand in the time interval $(L,1]$ of period n, and $D_{(n-1)1}^{L}$ represents the demand $D_{n-1}(0,L]$, the demand that happens in the time interval $(0,L]$ of period $n-1$. Then, our assumptions about the demand process implies that $(D_{n2}^{L} + D_{(n-1)1}^{L})$ has the same distribution as D, the random variable representing total demand in any period. In the same way, $(D_{n2}^{L+\epsilon} + D_{(n-1)1}^{L+\epsilon})$ has the same distribution as D. We say $X \sim_d Y$ if the two random variables X and Y have the same distribution function.

$$\begin{aligned}
\text{Since } I_B^L &= ((x - D_{n1}^{L})^+ + q - D_{n2}^{L} - D_{(n-1)1}^{L})^+, \\
I_B^L &\sim_d ((x - D_{n1}^{L})^+ + q - D_L)^+. \\
\text{Since } I_B^{L+\epsilon} &= ((x - D_{n1}^{L+\epsilon})^+ + q - D_{n2}^{L+\epsilon} - D_{(n-1)1}^{L+\epsilon})^+, \\
I_B^{L+\epsilon} &\sim_d ((x - D_{n1}^{L} - \Delta)^+ + q - D_L)^+.
\end{aligned}$$

Comparing the expressions for the distributions of $I_B^{L+\epsilon}$ and I_B^L, we can see that the second inequality stated in the lemma is just a consequence of the fact that $\Delta \geq 0$ for every sample path of $D_n(0,t)$. □

5 Base Stock Policies : A Convexity Result

We now consider a particular class of policies, the "base-stock" or "order-up-to S" policies. Clearly, such policies are not necessarily optimal, but are often used in practice. When following such policies, an order is placed in each period to raise the inventory position to S. One interesting research question is "how well do such policies perform?" Another equally important and interesting question is "how do we determine the value of S that minimizes the cost among this class of policies?". The latter is the issue that we address in this section, and the former in a subsequent paper. In the absence of simple analytic methods to determine the optimal value of S, it is common to use simple search techniques in combination with simulation to decide on a value of S. Recently, "infinitesimal perturbation analysis" has been advocated as an effective and efficient technique for computing stock levels. However, these techniques can be guaranteed to yield optimal solutions only if the cost function is known to be convex. With this as the motivation, we next prove the convexity of the discounted cost function for our finite horizon problem in the parameter S. To obtain the result, we show that it is true for almost every sample path of the demand and lead time processes. It is important to note that the result is valid under much weaker assumptions about the demand distributions than we have been making. Convexity results for other lost-sales inventory problems using base-stock policies are available in Downs et al. (2001) and Roundy and Janakiraman (2000).

First, we develop the notation required in the analysis. Let x_n = on hand inventory at the beginning of period n, that is, the point in time at which the order is placed in period n, q_n = order placed in period n which equals $S - x_n$, and l_n = the amount of lost sales in period n.

Assume $x_N = S$, that is, we start the system with S units of inventory on-hand. Now we write recursive equations describing the evolution of the on-hand inventory through time. Subsequently, we show that the on-hand inventory at the beginning of any period can be written as a piecewise linear function of S. We also characterize the set of values of S at which this function could potentially have a derivative discontinuity. Then, we show that the one-period cost function is linear in S and the on-hand inventory at the end of a period or, equivalently, at the beginning of the next period. This, along with the piecewise linearity of $x_n(S)$, implies that the discounted cost function is a piecewise linear function in S and therefore trivially convex at all these values of S. However, to prove the convexity of the discounted cost function over the entire domain of S, we need to verify that the slope, or derivative, of this cost function increases at every derivative discontinuity, that is, the left hand derivative is smaller than the right hand derivative.

Suppose we have a given sample path of demands throughout the horizon, $\{(D_{N1}, D_{N2}), \ldots \ldots, (D_{11}, D_{12})\}$. We know that

$$x_N = S, \text{ and} \tag{5.32}$$

$$x_n + q_n = S. \tag{5.33}$$

Furthermore,

$$x_{n-1} = \Big[(x_n - D_{n1})^+ + q_n - D_{n2}\Big]^+ \tag{5.34}$$

$$= [x_n - min(x_n, D_{n1}) + q_n - D_{n2}]^+ \tag{5.35}$$

$$= [S - D_{n2} - min(x_n, D_{n1})]^+ \text{ , and} \tag{5.36}$$

$$l_n = (D_{n1} - x_n)^+ + [D_{n2} - (x_n - D_{n1})^+ - q_n]^+. \tag{5.37}$$

At the beginning of a period, just prior to the time an order is placed, all the inventory in the system is on hand. As soon as the order is placed, the inventory position takes the value S. Hence, x_{n-1} can be computed as the inventory position at the beginning of the previous period(S) less the total depletion in inventory in the previous period(D_n-l_n). Thus

$$x_{n-1} = S - (D_n - l_n), \text{ or} \tag{5.38}$$

$$L_n = D_n + x_{n-1} - S. \tag{5.39}$$

Define the one period cost function in period n to be

$$v_n(S) = h \cdot x_{n-1} + b \cdot l_n \text{ , or} \tag{5.40}$$

$$v_n(S) = (h+b)x_{n-1} - b \cdot S + b \cdot D_n \text{ .} \tag{5.41}$$

Compare this definition with definition (2.2) in section 2. The definition of the one-period cost function here (5.40) is the same as (2.2) without the expected value operator except for the following difference. The holding cost for the on-hand inventory at the end of period n (beginning of period $n-1$) is charged in period n in (5.40) whereas it was charged proportional to the inventory on hand at the beginning of period n in (2.2). We define the cost function v_n this way purely for algebraic simplicity. We are interested in proving the convexity of the function $\sum_{m=1}^{N} \alpha^{N-m} v_n(S)$.

For the given sample path, define

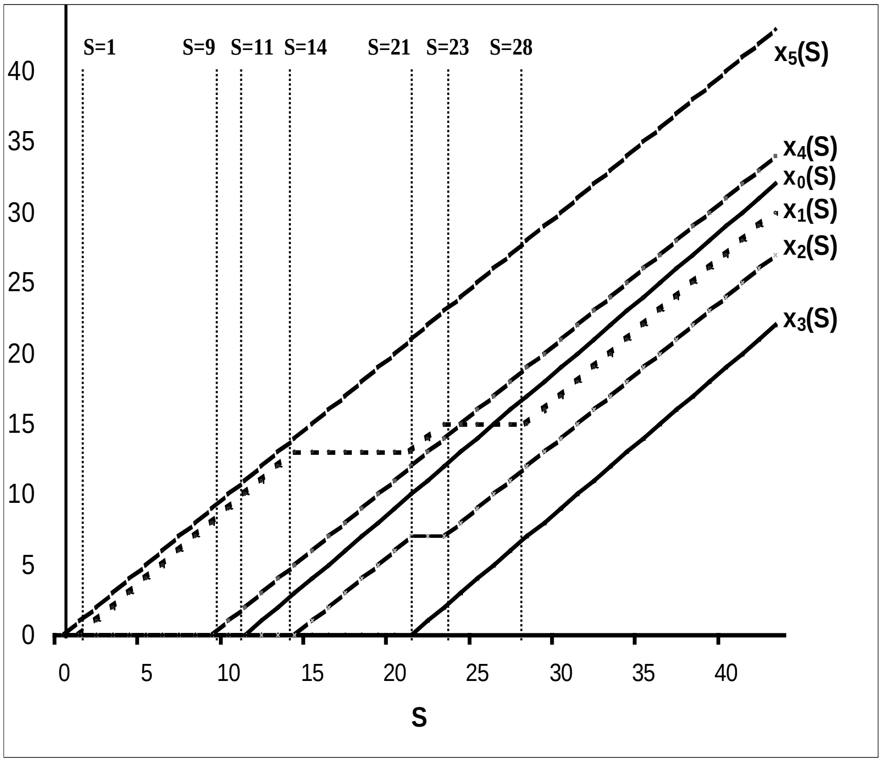
$$\delta_{n-1} = inf\{S : x_{n-1} > 0\}, \text{ and}$$

$$\gamma_{n-1} = inf\{S : x_n > D_{n1}\}.$$

By examining equation (5.36), it is apparent that δ_{n-1} and γ_{n-1} are two obvious values of S where x_{n-1} could be potentially discontinuous. We illustrate the dynamics of the sequence $\{x_n\}$ and also record the values of δ_n and γ_n in the following example.

Period n	5	4	3	2	1
D_{n1}	7	12	2	12	1
D_{n2}	2	9	14	1	10
δ_{n-1}	9	21	14	1	11
γ_{n-1}	7	21	23	28	2

In the following graph, equations (5.32) and (5.36) are used to generate the plots for $x_n(S)$ for this example. These equations also determine the values of δ_n and γ_n which are given in the table. It is important to note from the graph that all the values of S where some $x_n(S)$ has a derivative discontinuity belong to the set $\{\delta_n\} \cup \{\gamma_n\}$.



Since we assume that the random variables D_{n1} and D_{n2} have continuous density functions, we can see from equation (5.36) that for almost every sample path, if $j \neq k$, then $\delta_k \neq \delta_j$, $\gamma_k \neq \gamma_j$ and $\delta_k \neq \gamma_j$. Though we do not state it explicitly, we consider only such sample paths in all the results in this section.

In the following lemma we show that $x_n(S)$ is piecewise linear in S with derivative discontinuities only at values of S that are equal to δ_k or γ_k for some k. In addition, we also show that $x_n(S)$ has a slope of 0 or 1 in all its linear segments and also that it always lies between 0 and S. We use $x_n^R(S)$ and $x_n^L(S)$ to denote the left hand and right hand derivatives of $x_n(S)$ with respect to S. $x_n^{'}(S)$ will denote the derivative of $x_n(S)$ when it exists.

Lemma 10

(a) $x_n(S)$ is continuous in S.

(b) $\forall\ S \neq \delta_k$ or γ_k, $k \geq n$, $x_n(S)$ is linear at S and $x_n^{'}(S) \in \{0, 1\}$.

(c) $0 \leq x_n \leq S$.

Proof : The proof is by induction. Observe that the three statements are trivially true when $n = N$ since $x_N(S) = S$. Now, assume that the statement is true for all $j \geq n$ ($n \geq 1$). We will now verify that the statements are true for $n - 1$. Recall that

$$x_{n-1}(S) \quad = \quad [S - (D_{n2} + min(x_n(S), D_{n1}))]^+ \ .$$

We use this fact throughout the proof. First, observe that by assumption, $x_n(S)$ is a continuous function, and therefore $x_{n-1}(S)$ is a continuous function of S as well. Second, since $0 \leq x_n(S) \leq S$, $0 \leq x_{n-1}(S) \leq S$, too. To prove statement (b) for $n - 1$, assume that $S \neq \delta_k$ or γ_k, $k \geq n - 1$. Suppose S is in the region $\{S > \delta_{n-1}, S > \gamma_{n-1}\}$. Then we see that $x_{n-1}(S)$ $= S - D_{n2} - D_{n1}$, and therefore x_{n-1} is linear at S with slope 1 in this region. Next, suppose that S is in the set $\{S > \delta_{n-1}, S < \gamma_{n-1}\}$. For all S in this region, $x_{n-1}(S) = S - x_n - D_{n1}$. Since $x_n(S)$ is linear and $x_n^{'}(S) \in \{0, 1\}$ at all these values of S, $x_{n-1}(S)$ is linear in S and $x_n^{'}(S) \in \{0, 1\}$. Finally, suppose that S is in the set $\{S < \delta_{n-1}\}$. This condition implies that $x_{n-1}(S) = 0$ in this region. Consequently, x_{n-1} is linear with slope 0 at these values of S. Thus we have shown statement (b) to be true for $x_{n-1}(S)$. □

This lemma implies that $x_n(S)$ is a piecewise linear function of S. Consequently, equation (5.41) implies that $\sum_{m=1}^{N} \alpha^{N-m} v_n(S)$ is a piecewise linear function of S. Therefore, to prove the convexity of $\sum_{m=1}^{N} \alpha^{N-m} v_n(S)$, we need to make sure that the slopes of the linear segments of $\sum_{m=1}^{N} \alpha^{N-m} x_{n-1}(S)$ increase as S increases. This is proved as a corollary to the following theorem.

Theorem 5 $W_n(S) = \sum_{m=n}^{N} a_m \cdot (x_{m-1}^R(S) - x_{m-1}^L(S)) \geq 0\ \forall\ n,\ S$, any non-negative and non-decreasing sequence $\{a_m\}$.

Proof : As was the case for the preceding theorem, we prove this one by induction. First observe that $(x^R_{m-1}(S) - x^L_{m-1}(S)) = 0$ if $x_{m-1}(S)$ is differentiable, that is, the left hand derivative equals the right hand derivative at S. When $n = N$, $W_N(S) = a_N \cdot (x^R_{N-1}(S) - x^L_{N-1}(S))$. Also $x_{N-1} = [(S - D_{N1})^+ - D_{N2}]^+$, which is the same as $(S - D_N)^+$. This is clearly differentiable $\forall\ S \neq D_N$ and the slope increases from 0 to 1 at $S = D_N$. Hence the statement of the theorem is true for $n = N$.

Assume the conjecture is true for all $j \geq n$ for some $n > 1$ and let us now try to prove the statement for $n - 1$.

Recall that $$W_n = \sum_{m=n+1}^{N} a_m \cdot (x^R_{m-1}(S) - x^L_{m-1}(S)) + a_n \cdot (x^R_{n-1}(S) - x^L_{n-1}(S)).$$

Case I: $S < \delta_{n-1}$ and $x_{n-1} = 0$. In this region $x_{n-1}(S)$ is differentiable everywhere and hence $(x^R_{n-1}(S) - x^L_{n-1}(S)) = 0$ everywhere.

Case IIa: $S \geq \delta_{n-1},\ S < \gamma_{n-1}\ ,\ x_{n-1} = S - x_n - D_{n2}$.

Case IIb: $S \geq \delta_{n-1},\ S \geq \gamma_{n-1}\ ,\ x_{n-1} = S - D_{n1} - D_{n2}$. In this region, $x_{n-1}(S)$ is differentiable everywhere and hence $(x^R_{n-1}(S) - x^L_{n-1}(S)) = 0$ everywhere.

Since we have just seen that $(x^R_{n-1}(S) - x^L_{n-1}(S)) = 0$ in the regions represented by Case I and Case IIb, the theorem is clearly true for $n-1$ if S lies in one of these regions. To complete the proof, we need to consider Case IIa and the points of intersection between the three cases. In region I, $x'_{n-1}(S)$ is zero and in region IIa, $x'_{n-1}(S) \in \{0, 1\}$. So at the boundary of these two regions, the change in the derivative of $x_{n-1}(S)$ is non-negative. Using almost identical reasoning, we see that the change in the derivative of $x_{n-1}(S)$ is non-negative at the point of intersection between regions IIa and IIb and between I and IIb.

Now, let us examine Case IIa. In this region, $x_{n-1} = S - x_n - D_{n2}$. Therefore $(x^R_{n-1}(S) - x^L_{n-1}(S)) = -(x^R_n(S) - x^L_n(S))$. Substituting this in the expression for $W_n(S)$, we get $W_n(S) = \sum_{m=n+2}^{N} a_m \cdot (x^R_{m-1}(S) - x^L_{m-1}(S)) + (a_{n+1} - a_n) \cdot (x^R_{n-1}(S) - x^L_{n-1}(S))$. This is nothing but $W_{n+1}(S)$ for the sequence of coefficients $\{0, \ldots, 0, a_{n+1} - a_n, a_{n+2}, \ldots, a_{N-1}, a_N\}$ which is clearly non-negative and non-decreasing. But we have assumed the statement of the theorem to be true for $n + 1$. Therefore, $W_n(S) \geq 0$ in region IIa. We have thus shown the theorem to be true for all values of S. $\Box$

Corollary 1 $\sum_{m=1}^{N} \alpha^{N-m} v_m(S)$ is convex in S.

Proof : Application of the theorem above with $a_m = \alpha^{N-m}$, which is non-negative and non-decreasing, shows that $\sum_{m=1}^{N} \alpha^{N-m}(x_{m-1}^R(S) - x_{m-1}^L(S)) \geq 0$. Consequently, equation (5.41) implies that $\sum_{m=1}^{N} \alpha^{N-m}(v_m^R(S) - v_m^L(S)) \geq 0 \; \forall \; S$. Therefore, the slope of $\sum_{m=1}^{N} \alpha^{N-m} v_m(S)$ increases at every value of S where the derivative is not defined. At all the other values of S, this function is linear. Therefore, $\sum_{m=1}^{N} \alpha^{N-m} v_m(S)$ is convex in S. $\Box$

Thus we have shown that the discounted cost function is convex in S for almost every realization of the sequence of random variables $\{(D_{n1}, D_{n2})\}$.

5.1 Infinite Horizon Extension

As in section 3.1 we state the infinite horizon version of the main result developed in this section without proof. The proof is a straightforward extension of the finite horizon convexity result. It is based on bounding the expected one period cost which can then be used to bound the finite horizon cost using a geometric series with the multiplicative factor α. We use the superscript N to denote the finite horizon problem with N periods. Let us define $g^N(S) = \mathrm{E}[\sum_{m=1}^{N} \alpha^{N-m} v_m(S)]$ where the expectation is taken over the sequence of random vectors $\{(D_{n1}, D_{n2}) : 1 \leq n \leq N\}$.

Theorem 6 If $E(D_{n1}) < \infty$ and $E(D_{n2}) < \infty \; \forall \; n$, $g^N(S)$ converges uniformly to a convex function $g(S)$ as N approaches infinity.

6 Conclusions

We have considered a periodic review, lost sales inventory procurement problem which differs from existing problems in the literature. Significant differences are the incorporation of lead times that are random and between zero and the length of a period, and non-stationary and correlated demand within a period. An expected discounted cost function was considered as the objective function to be minimized. We proved that the cost that results from the use of the optimal ordering policy is a convex function of the inventory on hand at the beginning of a period. The optimal ordering quantity was shown to be a monotonic function of the beginning inventory. Easily computable bounds were derived for the optimal order size. These bounds can easily be used to develop a variety of heuristics to decide the order quantity. We also considered the class of order-up-to policies and proved a convexity result that justifies the use of simple search methods to determine the optimal value of the order-up-to parameter.

7 Appendix

Proof of Lemma 0: In this proof, it is assumed that $1 \leq n \leq N$ unless specified otherwise. First, we fix a sequence $\{(x_n, q_n, D_{n1}, D_{n2})\}$ that satisfies the relation $x_{(n-1)} = ((x_n - D_{n1})^+ + q_n - D_{n2})^+$. Then, for this sequence we write down the expression for the actual costs incurred in period n through the function G_2 with the original set of cost parameters $(\tilde{c}, \tilde{h}, \tilde{b})$. Next we derive the expression for the actual discounted value of all these one period costs as seen from period N through the function H_2 with the original cost parameters. Then we show that the function H_2 with a different set of cost parameters $(0, h, b)$ differs from the original H_2 function by a term that depends only on $(\tilde{c}, \tilde{h}, \tilde{b})$ and the demand sequence $\{(D_{n1}, D_{n2})\}$. Taking the expected value over the relation explained in the previous sentence proves the lemma.

Let us first make some useful definitions. IC_n, PC_n are the purchase cost and the inventory holding cost in period n respectively. IC_0 is the inventory holding cost paid at the end of the horizon for any inventory on hand. LC_{n1} and LC_{n2} are the lost sales costs incurred in the first (before the shipment q_n arrives) and second (after q_n arrives) part of period n

respectively. LC_n is the total lost sales cost incurred in period n. SV is the salvage value obtained at the end of the horizon.

Note: In the following, the relationship $\simeq$ is defined between two functions A and B if the function A-B is a constant that depends only on $(\tilde{c}, \tilde{h}, \tilde{b})$ and the demand sequence $\{(D_{n1}, D_{n2})\}$.

$$\begin{aligned}
G_2^{(\tilde{c},\tilde{h},\tilde{b})}(n) &\stackrel{\text{def}}{=} \tilde{c} \cdot q_n + \tilde{h} \cdot x_n + \tilde{b} \cdot (D_{n1} - x_n)^+ + \tilde{b} \cdot (D_{n2} - ((x_n - D_{n1})^+ + q_n))^+ \\
G_2^{(\tilde{c},\tilde{h},\tilde{b})}(0) &\stackrel{\text{def}}{=} - \ \tilde{c} \cdot x_0 \ + \ \tilde{h} \cdot x_0 \\
H_2^{(\tilde{c},\tilde{h},\tilde{b})}(N) &\stackrel{\text{def}}{=} \sum_{n=0}^{N} \alpha^{N-n} \cdot G_2^{(\tilde{c},\tilde{h},\tilde{b})}(n) \\
IC_n &\stackrel{\text{def}}{=} \alpha^{N-n} \cdot \tilde{h} \cdot x_n, \qquad 0 \leq \ n \leq N \\
LC_{n1} &\stackrel{\text{def}}{=} \alpha^{N-n} \cdot \tilde{b} \cdot (D_{n1} - x_n)^+ \\
LC_{n2} &\stackrel{\text{def}}{=} \alpha^{N-n} \cdot \tilde{b} \cdot (D_{n2} - ((x_n - D_{n1})^+ + q_n))^+ \\
LC_n &\stackrel{\text{def}}{=} LC_{n1} \ + \ LC_{n2}. \\
PC_n &\stackrel{\text{def}}{=} \alpha^{N-n} \cdot c \cdot q_n \\
SV &\stackrel{\text{def}}{=} \textit{salvage value at epoch } 0 \\
&= \alpha^N \cdot \tilde{c} \cdot x_0 \ = \ \frac{\tilde{c}}{\tilde{h}} IC_0 \\
\text{Thus, } H_2^{(\tilde{c},\tilde{h},\tilde{b})}(N) &= \sum_{n=1}^{N} [PC_n + IC_n + LC_{n1} + LC_{n2}] \ + \ IC_0 \ - \ SV; \\
x_{n-1} &= ((x_n - D_{n1})^+ + q_n - D_{n2})^+ \\
&= ((x_n - D_{n1})^+ + q_n - D_{n2}) + (D_{n2} - ((x_n - D_{n1})^+ + q_n))^+ \\
&= x_n - D_{n1} + (D_{n1} - x_n)^+ + q_n - D_{n2} + (\frac{LC_{n2}}{\alpha^{N-n} \cdot \tilde{b}}) \\
x_{n-1} &= x_n - D_n + \frac{PC_n}{\alpha^{N-n} \cdot \tilde{c}} + \frac{LC_n}{\alpha^{N-n} \cdot \tilde{b}} \\
\frac{IC_{n-1}}{\alpha^{N-n+1} \cdot \tilde{h}} &= \frac{IC_n}{\alpha^{N-n} \cdot \tilde{h}} - D_n + \frac{PC_n}{\alpha^{N-n} \cdot \tilde{c}} + \frac{LC_n}{\alpha^{N-n} \cdot \tilde{b}} \\
PC_n &\simeq \tilde{c} \cdot \frac{(IC_{n-1} - \alpha \cdot IC_n)}{\alpha \cdot \tilde{h}} - \tilde{c} \cdot \frac{LC_n}{\tilde{b}}
\end{aligned}$$

$$
\begin{aligned}
\sum_{n=1}^{N} PC_n &\simeq \frac{\tilde{c}}{\alpha \cdot \tilde{h}} \left[IC_0 - \alpha \cdot IC_N + \sum_{n=1}^{N-1} (1-\alpha) \cdot IC_n \right] - \frac{\tilde{c} \cdot \sum_{n=1}^{N} LC_n}{\tilde{b}} \\
\sum_{n=1}^{N} PC_n &\simeq \frac{\tilde{c}}{\alpha \cdot \tilde{h}} \left[\alpha \cdot IC_0 + (1-\alpha) \cdot IC_0 + \sum_{n=1}^{N-1} (1-\alpha) \cdot IC_n \right] - \frac{\tilde{c}}{\tilde{b}} \sum_{n=1}^{N} LC_n \\
& \quad \text{(taking } IC_N \text{ to be a constant)} \\
&\simeq \frac{\tilde{c}}{\tilde{h}} IC_0 + \frac{(1-\alpha)}{\alpha} \frac{\tilde{c}}{\tilde{h}} \sum_{n=0}^{N} IC_n - \frac{c}{b} \sum_{n=1}^{N} LC_n \\
\sum_{n=1}^{N} PC_n - SV &\simeq \frac{(1-\alpha)}{\alpha} \frac{\tilde{c}}{\tilde{h}} \sum_{n=0}^{N} IC_n - \frac{\tilde{c}}{\tilde{b}} \sum_{n=1}^{N} LC_n
\end{aligned}
$$

The last relationship proved above shows that the present value of all the purchase costs and the salvage value can be captured by a linear combination of the present value of holding costs and the present value of lost sales costs. This implies the following relationship.

$$H_2^{(\tilde{c},\tilde{h},\tilde{b})}(N) \simeq H_2^{(0,h,b)}$$

where $h = \tilde{h} + \frac{(1-\alpha)}{\alpha}\tilde{c}$ and $b = \tilde{b} - \tilde{c}$.
Also, $b \geq h$ because $\alpha(\tilde{b} - \tilde{h}) \geq \tilde{c}$. Taking the expected value of both sides of this relationship yields the statement of the lemma. □.

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