Domains of attraction for exponential families

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Abstract

With the df F of the rv X we associate the natural exponential family of df's F_λ where

$$dF_\lambda(x) = e^{\lambda x} dF(x)/Ee^{\lambda X}$$

for $\lambda \in \Lambda := \{\lambda \in \mathbb{R} \mid Ee^{\lambda X} < \infty\}$. Assume $\lambda_\infty = \sup \Lambda \leq \infty$ does not lie in Λ. Let $\lambda \uparrow \lambda_\infty$, then non-degenerate limit laws for the normalised distributions $F_\lambda(a_\lambda x + b_\lambda)$ are the normal and gamma distributions. Their domains of attractions are determined. Applications to saddlepoint and gamma approximations are considered.

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1 Introduction

This paper is a continuation of Balkema, Klüppelberg and Resnick [1998]. In that paper, which we shall refer to as BKR [1998] it was shown that a one-dimensional exponential family may sometimes be normalised (by translation and scaling) to yield a non-degenerate limit law. The only possible limit laws are the normal distribution and the gamma distributions on $(0, \infty)$ and on $(-\infty, 0)$. These are exactly the distributions which generate exponential families which are stable under a continuous one-parameter group of positive affine transformations; see also Bar-Lev and Casalis [1998].

In the present paper we start with a new characterisation of a stable exponential family, but the main aim of this paper is to describe the domains of attraction of these limit distributions. We shall investigate the same question in the multivariate setting in a subsequent paper.

Exponential families are of fundamental importance in probability theory and statistics. The theory developed here has applications to the asymptotic theory of Laplace transforms and their inversion, and for saddlepoint approximations. In the latter case, distributions in the domain of attraction of the normal law have the property that the saddlepoint approximation of the density becomes exact in the limit; see Barndorff-Nielsen and Klüppelberg [1992, 1998]. The theory of this paper sheds new light on the subject of regular variation and on a class of distribution functions with very thin tails. A good understanding of the asymptotics of univariate exponential families is essential if one wants to investigate the boundary behaviour of multivariate Laplace transforms.

Domains of attraction for exponential families bear a strong similarity with the domains of attraction for extreme value theory. For exponential families the limit distributions also form a continuous three parameter family, the extended gamma family, and here too the domain of attraction of the central normal distribution and the domains of attraction of the gamma distributions are very different. For a gamma limit, the domain of attraction can be described in terms of regular variation, for the description of the domain of attraction of the normal law we need a higher order theory and have to impose conditions on the second derivative of certain functions.

The normalising constants can be chosen to have a simple form. If the limit is normal then we can normalise by subtracting the expectation and dividing by the standard deviation. For the positive gamma limit the upper endpoint λ_∞ of Λ is finite and we can normalise by the factor $\lambda_\infty - \lambda$. For the negative gamma limit the upper endpoint x_∞ of F is finite. A translation ensures that $x_\infty = 0$. We can then normalise by multiplication with λ.

Weak convergence of the normalised random variables in the exponential family to a non-degenerate limit implies convergence of the mgf's. It is well-known that this implies convergence of all moments. It will be shown that with the normalisation introduced above it suffices that the third moment vanishes for $\lambda \to \lambda_\infty$ to obtain a normal limit and convergence of the first moment to a constant $c \neq 0$ yields a gamma limit with parameter $\alpha = |c|$.

Although the domains of attraction for the normal and the gamma limits are so different both reveal a strong symmetry between the behaviour of the probability distribution and the mgf. In the case of a gamma limit this is Karamata's celebrated Tauberian theorem on the Laplace transform of measures whose distribution varies regularly. In the case of a normal limit this symmetry holds for the density and is visible if the density is strongly unimodal. The standard normal density and its mgf are each other's inverse up to the factor $\sqrt{2\pi}$. In the general case the density and the mgf are linked by conjugate convex functions.

This paper is organised as follows. Section 2 provides the necessary background, Section 3 describes the domain of attraction of the normal distribution and Section 4 describes the domains of attraction of the gamma distributions on $(0, \infty)$ and on $(-\infty, 0)$. In Section 5 we apply our results to saddlepoint and gamma approximations.

2 Setting the stage

With the rv X with df F we associate the *natural exponential family* consisting of rv's X^λ with df's $dF_\lambda(x) = e^{\lambda x} dF(x)/K(\lambda)$. The norming constant $K(\lambda) = \int e^{\lambda x} dF(x)$ is the *moment generating function (mgf)* of X evaluated at λ. The domain of K is

$$\Lambda = \{\lambda \in \mathbb{R} \mid K(\lambda) = Ee^{\lambda X} < \infty\}.$$

This is a connected subset of $\mathbb{R}$ which contains the origin. The mgf $\lambda \mapsto K(\lambda)$ is continuous and strictly positive on Λ. The *cumulant generating function (cgf)* $\kappa = \log K$ is a convex analytic function on Λ. It will play an important role in our investigations. This paper studies the asymptotic behaviour of the df's F_λ for $\lambda \to \lambda_\infty := \sup \Lambda$, where $\lambda \to \lambda_\infty$ means that λ lies in

Λ and converges to λ_∞ from below. Note that $\lambda_\infty \geq 0$. We are interested in the case that the *upper endpoint* λ_∞ of the mgf K does not lie in the set Λ. Hence $\lambda_\infty > 0$.

If $\lambda_\infty \in \Lambda$, then $F_\lambda \downarrow F_{\lambda_\infty}$ pointwise while if $\lambda_\infty \notin \Lambda$, then $F_\lambda(x) \downarrow 0$ for every $x < x_\infty$ where $x_\infty = \sup\{F < 1\}$ is the *upper endpoint* of the df F. Then $X^\lambda \to x_\infty$ in probability. In the latter case the exponential family $\{X^\lambda,\, \lambda \in \Lambda\}$, may have a *limit law* for $\lambda \to \lambda_\infty$. By this we mean that it may sometimes be possible to normalise the variables X^λ (by translation and scaling) so that

$$A_\lambda X^\lambda := \frac{X^\lambda - b_\lambda}{a_\lambda} \Rightarrow V \qquad \lambda \to \lambda_\infty. \tag{2.1}$$

Here $\Rightarrow$ denotes convergence in law to a rv with a *non-degenerate* df. The rv's X^λ are introduced for convenience of exposition. The exponential family X^λ, $\lambda \in \Lambda$, is not a stochastic process. There are no multivariate distributions.

We shall say that X or F or the exponential family X^λ *lies in the domain of attraction of* the rv V or of its df G and write $X \in \mathcal{D}(V)$ or $F \in \mathcal{D}(G)$ as shorthand for (2.1).

The main result, Theorem 3.3, of BKR [1998] states that if there is a non-degenerate limit variable V in (2.1) then we can choose the centering $b_\lambda \in \mathbb{R}$ and scaling $a_\lambda > 0$ so that V is a standard normal variable, or so that V or $-V$ has a gamma distribution on $(0, \infty)$. This yields a continuous three parameter family of possible non-degenerate limit distributions, the extended gamma family, see BKR [1998, Example 2.9].

We start with a simple characterisation of the limit distributions. To this end we introduce the notation:

$$\varphi_\beta(\xi) = \begin{cases} -\beta^{-2}\log(1-\xi\beta) - \xi/\beta & \text{if } \beta \neq 0,\ \xi\beta < 1 \\ \xi^2/2 & \text{if } \beta = 0. \end{cases}$$

The functions φ_β depend continuously on the parameter β as one sees either by using L'Hospital's rule or by noting that $\varphi_\beta(0) = \varphi'_\beta(0) = 0$ and

$$\varphi''_\beta(\xi) = \frac{1}{(1-\beta\xi)^2} \qquad \beta\xi < 1.$$

A gamma distributed rv Z with parameter γ has density $z^{\gamma-1}e^{-z}/\Gamma(\gamma)$ for $z > 0$ and cgf $\lambda \mapsto -\gamma\log(1-\lambda)$ on $(-\infty, 1)$. Set $\gamma = 1/\beta^2$. Then $Y_\beta = \beta Z - 1/\beta$ has cgf φ_β for $\beta \neq 0$, both positive and negative. For $\beta \to 0$ the variable Y_β converges in distribution to the standard normal rv Y_0 with density $e^{-y^2/2}/\sqrt{2\pi}$. The *extended gamma family* is the set of the probability distributions of the variables $aY_\beta + b$ with $a > 0$ and $\beta, b \in \mathbb{R}$. This is a continuous three parameter family.

Theorem 2.1 *Let Y^λ, $\lambda \in \Lambda$, be the exponential family generated by Y. The exponential family is stable in the sense that the standardised variables*

$$V_\lambda = \frac{Y^\lambda - \mu}{\sigma}, \qquad \mu = \kappa'(\lambda) = EY^\lambda,\ \sigma^2 = \kappa''(\lambda) = Var(Y^\lambda), \qquad \lambda \in \Lambda$$

all have the same distribution if and only if there is a non-empty open interval $J \subset \Lambda$ so that EV_λ^3 does not depend on λ for $\lambda \in J$.

Corollary 2.2 *The only stable exponential families are those generated by the random variables in the extended gamma family.*

Proof The condition is obviously necessary. We may assume that J contains the origin and that $EY = 0$ and $EY^2 = 1$. Then $V_0 = Y^0 = Y$. Let κ_λ be the cgf of Y^λ. The cgf of V_λ is

$$\log Ee^{\xi(Y^\lambda - \mu)/\sigma} = \kappa_\lambda(\xi/\sigma) - \xi\mu/\sigma = \kappa(\lambda + \xi/\sigma) - \kappa(\lambda) - \xi\mu/\sigma.$$

If V_λ is distributed like Y it has cgf κ and hence

$$\kappa(\lambda + \xi/\sigma) = \kappa(\lambda) + \kappa(\xi) + \xi\mu/\sigma. \tag{2.2}$$

Taking the second derivative with respect to ξ we obtain

$$\sigma^2(\lambda + \xi/\sigma)/\sigma^2 = \sigma^2(\xi), \qquad \sigma = \sigma(\lambda). \tag{2.3}$$

Write $\tau = 1/\sigma$ to get $\tau(\lambda + \tau\xi)/\tau = \tau(\xi)$ and differentiate this expression to find

$$\tau'(\lambda + \tau\xi) = \tau'(\xi). \tag{2.4}$$

Setting $\xi = 0$ we see that $\tau'(\lambda)$ is constant in λ. Note that

$$\tau'(\lambda) = (1/\sigma(\lambda))' = -\frac{\kappa'''(\lambda)}{2\sigma^3} = -\frac{1}{2}E(\frac{Y^\lambda - \mu}{\sigma})^3. \tag{2.5}$$

If EV_λ is constant, then (2.5) (2.4),(2.3),(2.2) all hold and $V_\lambda \stackrel{d}{=} Y$.

If $\tau'(\lambda) = EV_\lambda^3 = -\beta$ does not depend on λ then from (2.5) $\tau(\lambda) = 1 - \beta\lambda$, (since $\sigma(0) = 1$) and $\kappa''(\lambda) = 1/(1-\beta\lambda)^2$ on J. Hence, by analytic continuation $\kappa(\lambda) = \varphi_\beta(\lambda)$ for $\beta\lambda < 1$. □

In the present paper we describe the domains of attraction of these limit laws. This description relates the behaviour of the df F in the neighbourhood of its upper endpoint x_∞ with the behaviour of the mgf K in the neighbourhood of $\lambda_\infty = \sup\Lambda$ and the behaviour of the exponential family F_λ for $\lambda \to \lambda_\infty$. Indications of the relations between asymptotic behaviour are as follows. Recall that two distribution functions F and G are *tail equivalent* if they have the same upper endpoint, $x_\infty = \sup\{F < 1\} = \sup\{G < 1\}$, if they are continuous in the upper endpoint, $F(x_\infty - 0) = G(x_\infty - 0) = 1$, and if

$$1 - G(x) \sim 1 - F(x) \qquad x \to x_\infty - 0 \tag{2.6}$$

in the sense that the quotient tends to 1.

Theorem 2.3 *Let X have a df F which is continuous in its upper endpoint $x_\infty = \sup\{F < 1\}$, and mgf K with domain Λ. Assume that $\lambda_\infty = \sup \Lambda$ does not lie in Λ. Let Y have df G and mgf M. Assume that F and G are tail equivalent in the sense of (2.6). Then*

1) K and M have the same upper endpoint λ_∞ and $M(\lambda) \sim K(\lambda)$ for $\lambda \to \lambda_\infty$;

2) $G_\lambda(x) - F_\lambda(x) \to 0$ uniformly in $x \in \mathbb{R}$ for $\lambda \to \lambda_\infty$;

3) $(1 - G_\lambda(x))/(1 - F_\lambda(x)) \to 1$ uniformly in $x < x_\infty$ for $\lambda \to \lambda_\infty$.

Proof Note that $\lambda_\infty(G) = \lambda_\infty(F)$ since for $\lambda > 0$ the integral $\int e^{\lambda x}(1 - G(x))dx = M(\lambda)/\lambda$ converges if and only if this holds for the corresponding integral for the df F. Since 3) implies 2) it remains to prove the asymptotic equivalences in 1) and 3).

Now first assume that $G \equiv F$ on an interval $[x_0, x_\infty)$. Then

$$K(\lambda)(1 - F_\lambda(x)) = \int_{(x,\infty)} e^{\lambda t} dF(t) = M(\lambda)(1 - G_\lambda(x)) \qquad x_0 \leq x < x_\infty. \tag{2.7}$$

For fixed $x < x_\infty$ the central term grows without bound for $\lambda \uparrow \lambda_\infty$ since $K(\lambda) \uparrow \infty$ if $\lambda_\infty \notin \Lambda$. Hence $M(\lambda) \uparrow \infty$ and $G_\lambda(x) \to 0$ for $x < x_\infty$ by the first sentence of the proof. Since this also holds for $F_\lambda(x)$ we see that $M(\lambda) \sim K(\lambda)$ for $\lambda \to \lambda_\infty$. This in turn implies 3) by (2.7) for $x_0 \leq x < x_\infty$.

Now let $\epsilon > 0$. There exists a constant $x_0 < x_\infty$ so that

$$|\log(1 - G(x)) - \log(1 - F(x))| < \epsilon \qquad x \in [x_0, x_\infty).$$

Let $F^* = F1_{[x_0,\infty)}$ and define G^* similarly. The inequality

$$\int h(t) dG^*(t) < e^\epsilon \int h(t) dF^*(t)$$

holds for all indicator functions $h = 1_{[x,\infty)}$. Hence it holds for all non-negative increasing functions. Take $h(t) = e^{\lambda t}$ with $\lambda \in (0, \lambda_\infty)$ to conclude that $M^*(\lambda) < e^\epsilon K^*(\lambda)$ and take $h(t) = e^{\lambda t} 1_{[x,\infty)}(t)$ to conclude that $M^*(\lambda)(1 - G^*_\lambda(x)) < e^\epsilon K^*(\lambda)(1 - F^*_\lambda(x))$ for $\lambda \in (0, \lambda_\infty)$ and all x. By symmetry the inequalities hold if we interchange F and G (and K and M). Hence we conclude that $|\log M^*(\lambda) - \log K^*(\lambda)| < \epsilon$ for $\lambda \in (0, \lambda_\infty)$ and $|\log(1-G^*_\lambda(x)) - \log(1-F^*_\lambda(x))| < 2\epsilon$ for $\lambda \in (0, \lambda_\infty)$ and all $x < x_\infty$.

Since $F^* \equiv F$ and $G^* \equiv G$ on $[x_0, x_\infty)$ we know that $K^*(\lambda) \sim K(\lambda)$ and $M^*(\lambda) \sim M(\lambda)$. Thus $|\log M(\lambda) - \log K(\lambda)| < 2\epsilon$ for $\lambda \in (\lambda_1, \lambda_\infty)$. Similarly $|\log(1-F_\lambda(x)) - \log(1-G_\lambda(x))| < 3\epsilon$ for $\lambda \in (\lambda_2, \lambda_\infty)$ and all $x < x_\infty$. Since ϵ is arbitrary this proves the asymptotics in 1) and 3).
□

Corollary 2.4 *Let X and Y be as in Theorem 2.3 and suppose X^λ and Y^λ are the associated exponential families. For any sequence $\lambda_n \to \lambda_\infty$ and any rv V*

$$\frac{X^{\lambda_n} - b_n}{a_n} \Rightarrow V \iff \frac{Y^{\lambda_n} - b_n}{a_n} \Rightarrow V.$$

Proof By 2) weak convergence $G_{\lambda_n}(a_n x + b_n) \to H(x)$ holds if and only if $F_{\lambda_n}(a_n x + b_n)$ converges weakly to $H(x)$. □

Now assume (2.1). Introduce the *Esscher transform*

$$E^\lambda X = X^\lambda \qquad \lambda \in \Lambda. \tag{2.8}$$

The operators E^λ form a semigroup: $E^\alpha E^\beta = E^{\alpha+\beta}$ if α and $\alpha + \beta$ lie in Λ.

Theorem 2.5 *Convergence $V_\lambda := A_\lambda X^\lambda \Rightarrow V$ to a non-constant rv V implies convergence of the exponential families $E^\gamma V_\lambda \Rightarrow E^\gamma V$ and of the mgf's $Ee^{\gamma V_\lambda} \to Ee^{\gamma V}$ as $\lambda \to \lambda_\infty$, for all $\gamma \in \mathbb{R}$ for which $Ee^{\gamma V}$ is finite.*

Proof Convergence of the mgf's is proved in BKR [1998, Theorem 3.4]. Observe that the set $\Gamma = \{\gamma \in \mathbb{R} \mid Ee^{\gamma V} < \infty\}$ is open if V is normal or if V or $-V$ has a gamma distribution. Hence convergence of the mgf's implies for any $\gamma \in \Gamma$

$$\int \varphi(x) e^{\gamma x} d\pi_\lambda(x) \to \int \varphi(x) e^{\gamma x} d\pi(x) \qquad \lambda \to \lambda_\infty$$

for all continuous bounded functions φ on $\mathbb{R}$. Here π is the distribution of V and π_λ the distribution of V_λ. This gives the asserted weak convergence of the exponential families. □

Convergence of densities, when they exist, is of particular interest in applications. Suppose F has density f and set $f_\lambda(x) = e^{\lambda x} f(x)/K(\lambda)$ for the density of X^λ. Assume

$$g_\lambda(c) = a_\lambda f_\lambda(a_\lambda c + b_\lambda) \to g(c) > 0 \qquad \lambda \to \lambda_\infty \tag{2.9}$$

in some point c. Write $c_\lambda = b_\lambda + a_\lambda c$. This yields an asymptotic expression for the mgf:

$$K(\lambda) \sim a_\lambda f(c_\lambda) e^{\lambda c_\lambda}/g(c) \qquad \lambda \to \lambda_\infty. \tag{2.10}$$

Feigin and Yashchin [1983] discuss this asymptotic relation. Their paper forms the basis of our work on exponential families and Gaussian densities. Their point of view is slightly different. They take a more analytical approach and consider the exponential family of rv's Y^λ, $0 < \lambda < \lambda_\infty$, generated by the Radon measure with density $f^*(y) = 1_F(y)$. The density f^*_λ of Y_λ is $e^{\lambda y}(1 - F(y))/K^*(\lambda)$ where $K^*(\lambda) = \int e^{\lambda y}(1 - F(y)dy = K(\lambda)/\lambda$ by partial integration. Their Theorem 1 gives the Tauberian relation

$$1 - F(c^*_\lambda) \sim K(\lambda) e^{-\lambda c^*_\lambda} g^*(c)/\lambda a^*_\lambda \tag{2.11}$$

provided that $a^*_\lambda f^*_\lambda(a^*_\lambda c + b^*_\lambda) \to g^*(c)$.

Since weak convergence of variables Y^λ, properly normalised, is assumed in their results, the theory developed in BKR [1998] shows that only the normal and the gamma densities can occur as limit in these asymptotic relations.

Relation (2.9) and the results developed in BKR [1998] have been applied in Barndorff-Nielsen and Klüppelberg [1992] to show that the saddlepoint approximation becomes exact in the tail for all densities in the domain of attraction of the normal law. The next section characterises this domain.

3 The domain of attraction of the normal law

Let $U = N_{01}$ denote a standard normal random variable with probability distribution γ_{01} and density $g_{01}(u) = (1/\sqrt{2\pi})\exp(-u^2/2)$. A sequence of random variables $\{X_n\}$ is *asymptotically normal* if there exist $a_n > 0$ and $b_n \in \mathbb{R}$ so that the *normalised* variables converge in law to a standard normal variable: $(X_n - b_n)/a_n \Rightarrow U$. Asymptotic normality of a sequence $\{X_n\}$ does not imply that the second moments exist, and if these exist this does not imply that $U_n := (X_n - E(X_n))/\sqrt{\mathrm{Var}(X_n)} \Rightarrow U$.

Thus one can distinguish a number of variations on asymptotic normality; see e.g. Embrechts, Klüppelberg and Mikosch [1997, Section 2.2.]. Set $U_n = (X_n - \mu_n)/\sigma_n$, where $\mu_n = EX_n$ and $\sigma_n^2 = \mathrm{Var}(X_n)$, assuming these moments exist. Let $\kappa = \log K$ denote the cgf of the standard normal variable and let $\kappa_n = \log K_n$ be the cgf of the standardised variable U_n. With the notation introduced here one has the following possibilities:

AN1) $(X_n - b_n)/a_n \Rightarrow U$;

AN2) $U_n \Rightarrow U$;

AN3) $EU_n^3 \to 0$;

AN4) $EU_n^k \to EU^k$ for $k = 1, 2, \ldots$;

AN5) $\kappa_n(\lambda) \to \lambda^2/2$ uniformly on bounded intervals.

The limit relation AN5) implies that $Eh(U_n) \to Eh(U)$ for all continuous functions of exponential growth. It will be called *strong asymptotic normality.* One aim of this paper is to show that for exponential families these five limit relations are equivalent. In particular asymptotic normality of an exponential family implies strong asymptotic normality and a necessary and sufficient condition for asymptotic normality is that EU_λ^3 vanish for $\lambda \to \lambda_\infty$.

If the exponential family is generated by a random variable X with a continuous density f then each variable X^λ of the exponential family has a continuous density f_λ. Asymptotic normality of the exponential family does not imply convergence of the densities. Let g_λ denote the density of the standardised random variable

$$U_\lambda = (X^\lambda - \mu(\lambda))/\sigma(\lambda) \tag{3.1}$$

with μ and σ the mean and variance of X^λ; see (3.4). One can distinguish the following forms of convergence of the densities for $\lambda \to \lambda_\infty$:

D1) $g_\lambda \to g_{01}$ in $\mathcal{L}^1$;

D2) $g_\lambda \to g_{01}$ uniformly on $\mathbb{R}$;

D3) for all $M > 1$

$$\sup_{u \in \mathbb{R}} M e^{M|u|} |g_\lambda(u) - g_{01}(u)| \to 0. \tag{3.2}$$

If the density f of the variable X exists and is *strongly unimodal*, i.e. $f = e^{-\varphi}$ with φ

convex, then asymptotic normality of the exponential family is equivalent with each of the three convergence relations for the densities g_λ above.

This section contains proofs of these statements. We shall first prove the equivalence of the five statements above about asymptotic normality for exponential families. Since convergence of mgf's implies weak convergence and convergence of moments, it suffices to prove AN1)⇒AN5) and AN3)⇒AN1). We shall then formulate conditions on the cgf, the density and the left tail of the df F which imply asymptotic normality of the natural exponential family generated by F.

Theorem 3.1 *If the exponential family $\{X^\lambda,\ \lambda \in \Lambda\}$ is asymptotically normal for $\lambda \to \lambda_\infty$ it is strongly asymptotically normal and hence AN1) implies AN5).*

Proof This is Theorem 2.5 above. It is well known that pointwise convergence of the mgfs implies uniform convergence on bounded sets. □

If X has cgf κ then the cgf of X_λ satisfies

$$\kappa_\lambda(\xi) = \kappa(\lambda + \xi) - \kappa(\lambda). \tag{3.3}$$

For each $\lambda \in (0, \lambda_\infty)$ the cgf κ_λ exists on a neighbourhood of the origin. We may compute the moments of X^λ by differentiating the cgf:

$$\mu(\lambda) = E\, X^\lambda = \kappa'(\lambda) \qquad \sigma^2(\lambda) = \mathrm{Var}(X_\lambda) = \kappa''(\lambda) \qquad 0 < \lambda < \lambda_\infty. \tag{3.4}$$

In particular, if F is non-degenerate then so is F_λ, and we see that the function κ then has a strictly positive second derivative. The following result is implicit in the proof of Corollary 1 in Feigin and Yashchin [1983]. See also Balkema, Klüppelberg and Resnick [1993].

Theorem 3.2 *Let X have cgf κ with domain Λ. Suppose $\lambda_\infty = \sup \Lambda$ does not lie in Λ. If the function σ in (3.4) satisfies the relation*

$$\sigma(\lambda + x/\sigma(\lambda))/\sigma(\lambda) \to 1 \qquad \lambda \to \lambda_\infty \tag{3.5}$$

for each $x \in \mathbb{R}$ then the family X^λ is strongly asymptotically normal.

Proof Since the cgf κ of X is a convex analytic function so is the cgf γ_λ of the normalized variable U_λ given by (3.1). Relation (3.3) gives

$$\begin{aligned} \gamma_\lambda(\xi) &= \kappa_\lambda(\xi/\sigma(\lambda)) - \xi\mu(\lambda)/\sigma(\lambda) \\ &= \kappa(\lambda + \xi/\sigma(\lambda)) - \kappa(\lambda) - \xi\mu(\lambda)/\sigma(\lambda). \end{aligned} \tag{3.6}$$

Note that we have normalized the convex function κ to make $\gamma_\lambda(0) = 0,\ \gamma'_\lambda(0) = 0,\ \gamma''\lambda(0) = 1$. The condition (3.5) is assumed to hold pointwise. By continuity of the function σ it will hold uniformly on bounded sets by Bloom's theorem (see Bingham, Goldie and Teugels (henceforth BGT) [1987, Section 2.11]. It thus implies that the second derivative of γ_λ will be close to 1 uniformly on any bounded interval around the origin, and hence $\gamma_\lambda(\xi) \to \xi^2/2$ uniformly on bounded intervals, which implies strong asymptotic normality.

If $\Lambda \neq \mathbb{R}$ one has to check that $\gamma_\lambda(\xi)$ is well-defined in the sense that for any ξ the point $\lambda + \xi/\sigma(\lambda)$ lies in Λ eventually. Note that $\sigma^2(\lambda) \to \infty$ if λ_∞ is finite since $\lambda_\infty \notin \Lambda$ then implies $\kappa(\lambda) \to \infty$. Below we shall see that (3.5) implies that $1/\sigma(\lambda) = o(\lambda)$ if $\lambda_\infty = \infty$ and $1/\sigma(\lambda) = o(\lambda_\infty - \lambda)$ if λ_∞ is finite. This ensures that $\lambda + \xi/\sigma(\lambda) \in \Lambda$ eventually. □

We shall prove a converse result in this section. First we introduce some terminology:

A positive function s is *self-neglecting* (or *Beurling slowly varying*) in $t_\infty \leq \infty$ if it is defined on a left neighbourhood of t_∞ and if

$$s(t + xs(t))/s(t) \to 1 \qquad t \to t_\infty \tag{3.7}$$

holds uniformly on bounded x-intervals. Again $t \to t_\infty$ means convergence from the left. If the endpoint t_∞ is finite we also assume that $s(t) \to 0$ for $t \to t_\infty$.

If t_∞ is infinite and the first derivative of s exists and vanishes at ∞ then the function s is self-neglecting. If t_∞ is finite then s is self-neglecting if both s and s' vanish at t_∞. Any self-neglecting function is asymptotic to such a function with a vanishing derivative. Hence if s is self-neglecting then $s(t) = o(t)$ if $t_\infty = \infty$ and $s(t) = o(t_\infty - t)$ if t_∞ is finite. A function which is asymptotic to a self-neglecting function is self-neglecting. For a continuous function s it suffices to assume pointwise convergence in (3.7) by Bloom's theorem. See BGT [1987, Section 2.11] for a very readable account of self-neglecting functions. The condition in Theorem 3.2 above may now be formulated as: The function $s(\lambda) = 1/\sigma(\lambda)$ should be self-neglecting for $\lambda \to \lambda_\infty$.

A function ψ is *asymptotically parabolic in* $t_\infty \leq \infty$ if it is defined, convex and C^2 on a left neighbourhood of t_∞ with $\psi'' > 0$ and if $s = 1/\sqrt{\psi''}$ is self-neglecting in t_∞. (Cf. Balkema, Klüppelberg and Resnick [1993].)

Lemma 3.3 *If ψ is asymptotically parabolic in t_∞ then $|\psi(t)| \to \infty$ as $t \uparrow t_\infty$.*

Proof Let $s(t) = 1/\sqrt{\psi''(t)}$. If t_∞ is finite then $\psi''(t) >> 1/(t_\infty - t)^2$ since $s(t) = o(t_\infty - t)$.

Hence $\psi(t) >> |\log(t_\infty - t)|$. Similarly $\psi''(t) >> 1/t^2$ if $t_\infty = \infty$. For a convex function ψ the limit $\psi(\infty)$ exists in $[-\infty, \infty]$. If $\psi(\infty)$ is finite then $\psi'(\infty) = 0$ and $-\psi'(t) >> 1/t$ implies $\psi(t) - \psi(t_\infty) \to \infty$. This is a contradiction. □

By the arguments following (3.6) any asymptotically parabolic function ψ satisfies

$$\psi(t + xs(t)) = \psi(t) + xs(t)\psi'(t) + x^2/2 + o(1) \qquad t \to t_\infty \tag{3.8}$$

uniformly on bounded x-intervals. For asymptotic normality of the exponential family X^λ it thus suffices that the cgf κ be asymptotically parabolic at λ_∞. Condition (3.8) on the cgf implies that the cgf's $\gamma_\lambda(\xi)$ of the standardised variables U_λ converge to the standard normal cgf $\xi^2/2$ as $\lambda \to \lambda_\infty$.

Consider the following list of statements for $\lambda \to \lambda_\infty$:

AP1) κ is asymptotically parabolic in λ_∞;

AP2) $s = 1/\sqrt{\kappa''}$ is self-neglecting in λ_∞;

AP3) the derivative of $s(\lambda) = 1/\sigma(\lambda)$ vanishes in λ_∞, and so does $1/\sigma$ if λ_∞ is finite.

The implications AP3) $\Rightarrow$ AP2) $\Rightarrow$ AP1) hold from the discussion prior to Lemma 3.3. Relation AP1) implies strong asymptotic normality of the exponential family (Theorem 3.2) and relation AP3) is equivalent to the condition that $\kappa(\lambda_\infty) = \infty$ if λ_∞ is finite (by Lemma 3.3), and

$$EU_\lambda^3 \to 0 \qquad \lambda \to \lambda_\infty \tag{3.9}$$

since $\kappa'''(\lambda) = E(X^\lambda - \mu(\lambda))^3$ and $(1/\sigma(\lambda))' = -\kappa'''(\lambda)/2\sigma^3(\lambda)$. Hence in the context of exponential families, AN3) $\Rightarrow$ AN1).

We have now arrived at the central result of this section.

Theorem 3.4 *Let X be a rv with cgf κ having upper endpoint λ_∞. The exponential family generated by X is asymptotically normal if and only if κ is asymptotically parabolic in λ_∞.*

Proof Sufficiency has been shown above. So assume that X^λ is asymptotically normal. Then the family is strongly asymptotically normal (Theorem 3.1) and hence all moments converge and (3.9) holds. Also λ_∞ does not lie in Λ. (Else $X^\lambda \Rightarrow X^{\lambda_\infty}$. However X^{λ_∞} can not be normal since then $\Lambda = \mathbb{R}$ which is inconsistent with $\lambda_\infty \in \Lambda$.) So $\kappa(\lambda) \to \infty$ and $1/\sigma(\lambda)$ vanishes for $\lambda \to \lambda_\infty$ if λ_∞ is finite. Thus AP3) holds, and this implies AP1). □

Corollary 3.5 *The conditions AP1) – AP3) are equivalent.*

A second corollary is the following simple criterion for asymptotic normality.

Theorem 3.6 *Suppose* $\lambda_\infty = \sup \Lambda \notin \Lambda$. *The exponential family* X^λ *is asymptotically normal for* $\lambda \to \lambda_\infty$ *if the third moment of the standardised random variable* U_λ *in (3.1) vanishes for* $\lambda \to \lambda_\infty$.

The condition that a function is asymptotically parabolic is a condition on the second derivative. For cgf's this conditions also determines the asymptotic behaviour of the higher derivatives:

Proposition 3.7 *Suppose the cgf* κ *is asymptotically parabolic in the upper endpoint* λ_∞ *of its domain. Define* $\sigma(\lambda) = \sqrt{\kappa''(\lambda)}$ *as in (3.4). Then for all integers* $n > 2$:

$$\kappa^{(n)}(\lambda)/\sigma^n(\lambda) \to 0 \qquad \lambda \to \lambda_\infty.$$

Proof Strong asymptotic normality of the associated exponential family implies $EU_\lambda^n \to EU^n$ for all $n \geq 1$. Hence the cgf γ_λ of U_λ has the property $\gamma_\lambda^{(n)}(0) \to \gamma^{(n)}(0)$ where $\gamma(\xi) = \xi^2/2$. (The relation also follows directly from the normal convergence of analytic functions.) □

Cumulant generating functions are C^∞ and convex. Given an asymptotically parabolic function ψ it is not hard to construct a convex C^∞ function which is asymptotic to ψ but which itself is not asymptotically parabolic. For cgf's this is not possible.

Proposition 3.8 *Let the rv* X *have mgf* $K = e^\kappa$ *with domain* Λ. *Let* $\lambda_\infty = \sup \Lambda$. *Suppose* $K(\lambda) \sim e^{\psi(\lambda)}$ *for* $\lambda \to \lambda_\infty$ *where* ψ *is asymptotically parabolic in* λ_∞. *Then*

1) $\kappa''(\lambda) \sim \psi''(\lambda)$ *for* $\lambda \to \lambda_\infty$;

2) κ *is asymptotically parabolic in* λ_∞.

Proof Set $b(\lambda) = \psi'(\lambda)$ and $a(\lambda) = \sqrt{\psi''(\lambda)}$. Then $1/a(\lambda)$ is self-neglecting and

$$\kappa(\lambda + \xi/a(\lambda)) - \kappa(\lambda) - b(\lambda)\xi/a(\lambda) \to \xi^2/2 \qquad x \in \mathbb{R}$$

since this holds for ψ, and the difference $\kappa(\lambda) - \psi(\lambda) = o(1)$.

It follows that $(X^\lambda - b(\lambda))/a(\lambda) \Rightarrow U$. So 2) holds by Theorem 3.2. One even has strong asymptotic normality, which implies convergence in law of the standardised variables $U_\lambda = (X^\lambda - \mu(\lambda))/\sigma(\lambda)$. Khinchine's convergence of types theorem, see Feller [1966, II, Lemma VIII.2.1], then gives $\sigma(\lambda) \sim a(\lambda)$ which is 1). □

Specifying the domain of attraction of the normal by means of the density or distribution.

In the preceding we characterised the domain of attraction of the normal law for exponential families in terms of transforms. A natural question to ask is: can one decide from inspection of the upper tail of a df F or the density f whether the associated exponential family is asymptotically normal? Here we give a partial answer. The full answer will be given in the analysis of the multivariate situation.

Theorem 3.9 *Let the rv X have a bounded density f with upper endpoint x_∞. Suppose that $f(x) \sim e^{-\psi(x)}$ for $x \to x_\infty$ where ψ is asymptotically parabolic. Then the associated exponential family is asymptotically normal and the densities g_λ of the standardized variables U_λ in (3.1) satisfy (3.2) for any $M > 1$.*

Proof First assume $f = e^{-\psi}$. Let $x_0 < x_\infty$. Then $\lambda = \psi'(x_0)$ is the slope of the convex function ψ in x_0. Set $a_0 = 1/\sqrt{\psi''(x_0)}$. Then (3.8) gives

$$\psi_\lambda(u) := \psi(x_0 + a_0 u) - \psi(x_0) - \lambda a_0 u \to u^2/2 \qquad x_0 \to x_\infty. \tag{3.10}$$

The density h_λ of $(X^\lambda - x_0)/a_0$ is $c_0 e^{-\psi_\lambda(u)}$ for some normalising constant $c_0 = c(\lambda) > 0$. Thus $h_\lambda \to g_{01}$ by (3.10) and convexity of ψ_λ. The convexity also gives (3.2). See BKR [1993, Theorem 6.4] for further details. □

Corollary 3.10 *Suppose the density f is bounded, and f is positive and C^3 on a left neighbourhood of the upper endpoint x_∞ and vanishes for $x \to x_\infty$. Set $\psi = -\log f$. If $\psi''(x)$ is positive and $\psi'''(x)/(\psi''(x))^{3/2} \to 0$ for $x \to x_\infty$ then (3.2) holds for all $M > 1$.*

Proof The conditions imply that ψ is asymptotically parabolic. □

In Balkema, Klüppelberg and Stadtmüller [1995] a number of Tauberian conditions were formulated which ensure that the rv X with asymptotically parabolic cgf κ has a density f with Gaussian tail. This means that the upper endpoint x_∞ is infinite and $f \sim e^{-\psi}$ for some asymptotically parabolic function ψ. The results of that paper were formulated in the framework of densities with upper endpoint $x_\infty = \infty$ but the theorem below remains valid in the case where the upper endpoint is finite. For the proof of this theorem we refer to Section 2 in that paper.

Theorem 3.11 *Suppose the exponential family $\{X^\lambda,\ \lambda \in \Lambda\}$, is asymptotically normal. If X has a strongly unimodal density f then f is bounded and $f \sim e^{-\psi}$ where ψ is asymptotically parabolic.*

Corollary 3.12 *The conditions D1) – D3) are equivalent if the density is strongly unimodal.*

Proof of Corollary 3.12 The weakest condition D1) implies asymptotic normality. Hence by Theorem 3.11 $f \sim e^{-\psi}$ with ψ asymptotically parabolic and D3) holds by Theorem 3.9. □

The conditions above concern the upper tail of the density. Similar results are valid if these conditions hold for the upper tail of the distribution function. We first need a lemma.

Lemma 3.13 *If ψ is asymptotically parabolic in $t_\infty > 0$ then so is $\varphi(t) = \psi(t) + \log t$.*

Proof It suffices to observe that $\varphi''(t) \sim \psi''(t)$ for $t \to t_\infty$ since $\psi''(t) >> 1/t^2$ in t_∞. □

Theorem 3.14 *Suppose the df F has upper endpoint x_∞ and tail $1 - F(x) \sim e^{-\psi(x)}$ for $x \to x_\infty$ where ψ is asymptotically parabolic in x_∞. Then the associated exponential family is asymptotically normal.*

Proof Define the bounded density $f^*(x) = e^x(1 - F(x))/c$. Then $f^*(x) \sim e^{-\phi(x)}$ where $\phi(x) = \psi(x) - x + \log c$ is asymptotically parabolic since $\phi'' = \psi''$. So from Theorem 3.9, the exponential family generated by f^* is asymptotically normal. Let K be the mgf of F. We then have from Proposition 3.4 that

$$\log \int e^{\lambda x} f^*(x)dx = \log \int e^{(\lambda+1)x}(1 - F(x))dx/c = \log(\frac{K(1+\lambda)}{(1+\lambda)c})$$

is asymptotically parabolic. This implies that $\log K(1+\lambda)$ is asymptotically parabolic by Lemma 3.13 and hence so is $\log K$. □

The converse of this implication does not hold. Asymptotic normality of an exponential family does not imply that the underlying df has a tail $1 - F(x) \sim e^{-\psi(x)}$ with ψ asymptotically parabolic. The tail need not even be asymptotically continuous.

Example 3.15 The Poisson distributions form an exponential family which is well known to be asymptotically normal. The tail of a Poisson distribution with expectation 1 is very irregular: $(1 - F(n-0))/(1 - F(n)) \sim n \not\to 1$ for $n \to \infty$. □

One can introduce measures with increasingly smooth densities by setting $f_1 = 1 - F$ and $f_{n+1}(x) = \int_x^\infty f_n(t)dt$. The cgf's corresponding to f_n are $\kappa(\lambda) - n\log\lambda$, and these are asymptotically parabolical if and only if κ is. If F is the Poisson distribution with expectation 1 then none of the densities f_n has a Gaussian tail, even though they all generate exponential families which are asymptotically normal.

Remark 3.16 Let Y be a perturbation of a Poisson variable with expectation 1. The variable Y has mass $e^{-1}/n!$ at $y_n = n + \delta_n$ for $n \geq 0$ with $\delta_n \to 0$. One can choose the sequence (δ_n) so that the exponential family $\{Y^\lambda,\ \lambda \in \mathbb{R}\}$, is not asymptotically normal. □

Asymptotically parabolical functions abound. Let us give some examples. We are looking for functions which are convex and unbounded at their upper endpoint.

Example 3.17 The function x^2 is asymptotically parabolic in infinity, and so is any monic polynomial of degree ≥ 2. So too are the functions x^α for $\alpha > 1$, $x - x^\alpha$ for $\alpha \in (0,1)$ and e^{x^α} for $\alpha > 0$. Positive linear combinations of such functions are again asymptotically parabolic. The functions $1/(c-x)^\alpha$ with $\alpha > 0$ and $|\log(c-x)|^\beta$ for $\beta > 1$ are asymptotically parabolic in the point c. □

Not every asymptotically parabolic function is the cgf of a probability measure. Cumulant generating functions are very special convex functions. A moment generating function is totally positive, its derivatives are all strictly positive on Λ, and it extends to an analytic function on the vertical strip $\{\Re z \in \Lambda\}$. So one may ask which of the functions in the example above is asymptotic to a cumulant generating function. The final result of the section addresses this question.

We shall make use of a beautiful result which links the asymptotic behaviour of a density and its mgf. This result is based on the conjugate Legendre transform ψ^* of a convex function ψ with domain D

$$\psi^*(t) = \sup\{xt - \psi(x) \mid x \in D\}. \tag{3.11}$$

If $f = e^{-\psi}$ is a strongly unimodal density and ψ is asymptotically parabolic, then (2.10) with $c = 0$ gives

$$K(\lambda) \sim \sqrt{2\pi}a_\lambda f(b_\lambda)e^{\lambda b_\lambda} \sim \sqrt{2\pi}\sigma(\lambda)e^{\psi^*(\lambda)} \qquad \lambda \to \lambda_\infty \tag{3.12}$$

if we choose b_λ so that $\psi'(b_\lambda) = \lambda$, thus maximising $\lambda x - \psi(x)$ in (3.11). In that case $a_\lambda \sim \sigma(\lambda)$. One can get rid of the factor $\sqrt{2\pi}\sigma(\lambda)$ in (3.12) since this function is practically constant (flat) on intervals of length $\sigma(\lambda)$.

Theorem 3.18 *Let φ be asymptotically parabolic in λ_∞. There exists a rv X with mgf K so that $K(\lambda) \sim e^{\varphi(\lambda)}$ for $\lambda \to \lambda_\infty$. We may choose X to have a strongly unimodal density.*

Proof We may assume that φ is convex and that φ'' is continuous and strictly positive. Let $t_\infty = \sup\{\varphi'(\lambda) \mid \lambda < \lambda_\infty\}$ and let $\psi(t) = \varphi^*(t)$ be the conjugate (Legendre) transform of $\varphi(\lambda)$. The function ψ is defined on a left neighbourhood of t_∞ and is asymptotically parabolic in t_∞ by Theorem 5.3 in BKR [1993] with scale function $a(t) = 1/\sqrt{\psi''(t)}$. Now apply Theorem 6.6 in BKR [1993] with a bounded density $f \sim \gamma e^{-\psi}$ where $\gamma(t) = 1/(\sqrt{2\pi}a(t))$. The function γ is flat (see BKR [1993, p.580]) for a since a is self-neglecting. This implies that we may choose f strongly unimodal. Note that $\psi^* = \varphi^{**} = \varphi$. Hence the mgf K of f satifies $K(\lambda) \sim e^{\varphi(\lambda)}$ by relation (6.6) in BKR [1993]. □

4 Domains of attraction for the gamma limits

For the domains of attraction of the gamma limits there is a simple and complete description in terms of regular variation. In fact the limit theory for exponential families with a gamma limit leads to a novel approach to regular variation. We shall obtain a new derivation of Karamata's Tauberian theorem. It will also be seen that smoothly varying functions occur naturally in the limit theory of exponential families.

For the definition and properties of regular variation we refer to BGT [1987], Geluk and De Haan [1987], Resnick [1987]; Embrechts, Klüppelberg, and Mikosch [1997]. A simple introduction is given in Feller [1966].

Let γ_α for $\alpha > 0$ denote the probability distribution on $(0, \infty)$ with density

$$g_\alpha(y) = y^{\alpha-1}e^{-y}/\Gamma(\alpha) \qquad y > 0. \tag{4.1}$$

The mgf $1/(1-\lambda)^\alpha$ of the distribution γ_α is finite on $(-\infty, 1)$. The gamma variable V with density (4.1) satisfies a stability relation. For a normal rv the Esscher transform has the effect of a translation, for a gamma rv the Esscher transform has the effect of a multiplication:

$$E^\xi V = V^\xi \stackrel{\mathrm{d}}{=} Q_\xi V \qquad Q_\xi v = v/(1-\xi) \qquad \xi < 1. \tag{4.2}$$

We are interested in random variables in the domain of attraction of V and of the rv $W = -V$ with probability distribution $\bar{\gamma}_\alpha$, mgf $1/(1+\lambda)^\alpha$, $\lambda > -1$, and density

$$\bar{g}_\alpha(y) = (-y)^{\alpha-1}e^y/\Gamma(\alpha) \qquad y < 0.$$

Our first aim is to prove that $\lambda_\infty = \sup\Lambda$ is finite if F lies in the domain of attraction $\mathcal{D}(\gamma_\alpha)$ of the positive gamma distribution with parameter α and that the upper endpoint x_∞ of the df F is finite for $F \in \mathcal{D}(\bar{\gamma}_\alpha)$.

For the following proof we make use of the following result (see (2.2) in BKR [1998]): If $Ax = ax + b$ and $\lambda \in \Lambda,\ a > 0, b \in \mathbb{R}$, then

$$AE^\lambda X = E_{\lambda/a}AX. \tag{4.3}$$

Proposition 4.1 *Suppose $X \in \mathcal{D}(\gamma_\alpha)$. Then $\lambda_\infty = \sup\Lambda$ is finite and*

$$(\lambda_\infty - \lambda)X^\lambda \Rightarrow V \qquad \lambda \to \lambda_\infty\,. \tag{4.4}$$

Proof There exist positive affine transformations A_λ depending continuously on the parameter λ, see BKR [1998, Lemma 2.8], so that as $\lambda \to \lambda_\infty$

$$U_\lambda := A_\lambda X^\lambda \Rightarrow V.$$

Let $\xi = 1/2$. For some $\lambda_0 < \lambda_\infty$, $Ee^{\xi U_\lambda}$ is finite for $\lambda \in [\lambda_0, \lambda_\infty)$. Use (4.3) to see that it is possible to choose $\lambda_0 < \lambda_1 < \cdots$ and positive affine transformations $B_n x = (x - b_n)/a_n$ so that the variables $Z_n = U_{\lambda_n}$ satisfy $B_{n+1}E^\xi Z_n = Z_{n+1}$ and $Z_0 = B_0 E^{\lambda_0} X$. Then $B_n x \to Qx = x/2$ by (4.2). This means that $a_n \to 2$ and $b_n \to 0$. Observe that from repeated use of (4.3),

$$Z_{n+1} = B_{n+1}E^\xi B_n E^\xi \cdots B_1 E^\xi A_{\lambda_0} E^{\lambda_0} X =: D_n E^{\xi_n} X$$

with

$$\xi_n = \lambda_0 + \xi/a_0 + \cdots + \xi/(a_0 \cdots a_n) \uparrow \xi_\infty < \infty$$

since $a_n \to 2$, and $D_n = B_n \circ \cdots \circ B_0$. Set $D_n x = c_n x + d_n$. Then $c_n = 1/(a_0 \cdots a_n) \to 0$ and hence $\|D_n\| := \sqrt{(\log c_n)^2 + d_n^2} \to \infty$ and therefore, by BKR [1998, Proposition 2.10], $\xi_\infty = \lambda_\infty \notin \Lambda$. Since $a_n \to 2$, we have

$$\lambda_\infty - \xi_n \sim \xi/(a_0 \cdots a_n) \sim c_n/2.$$

The relation $D_n x = c_n x + d_n$ gives $d_n = D_n(0) = B_n(D_{n-1}(0)) = B_n(d_{n-1}) = (d_{n-1} - b_n)/a_n$. Due to $b_n \to 0$ and $a_n \to 2$, we get $d_n \to 0$ and

$$(\lambda_\infty - \xi_n)E^{\xi_n} X \Rightarrow V.$$

Finally write $\lambda = \xi_n + \theta_n(\lambda_\infty - \xi_n)$ for $\lambda \in [\xi_n, \xi_{n+1})$. Then $\theta_n = \theta_n(\lambda) \in [0, 2/3]$ eventually, $V_\lambda \overset{\mathrm{d}}{=} B_n^{\theta_n} Z_n$ with $Q_{\theta_n} B_n^{\theta_n} \to \mathrm{id}$ uniformly in $\theta_n \in [0, 2/3]$. This implies

$$(\lambda_\infty - \lambda)X^\lambda = (1 - \theta_n)(\lambda_\infty - \xi_n)E^{\theta_n} U_{\lambda_n} = (1 - \theta_n)E^{\theta_n} Z_n \Rightarrow V$$

which is the desired relation (4.4). □

Proposition 4.2 *Let $W = -V$ have distribution $\bar{\gamma}_\alpha$ on $(-\infty, 0)$. Suppose $X \in \mathcal{D}(\bar{\gamma}_\alpha)$ has df F. Then $x_\infty = \sup\{F < 1\}$ is finite and*

$$\lambda(X^\lambda - x_\infty) \Rightarrow W \qquad \lambda \to \infty\,. \tag{4.5}$$

Proof The proof of this Proposition is similar. Take $\xi = 1$. Then $B_n x \to 2x$ and $D_n x = c_n(x + \delta_n)$ with c_n as above and

$$\delta_n = \frac{d_n}{c_n} = \frac{d_{n-1} - b_n}{a_n c_n} = \delta_{n-1} - \frac{b_n}{c_{n-1}} \to \delta_\infty < \infty.$$

We thus find $\xi_n \sim 2\xi/(a_0 \cdots a_{n-1}) \sim c_n \sim \lambda_n$ and $D_n E^{\xi_n} X \Rightarrow W$ gives

$$\lambda_n(X_{\lambda_n} - x_\infty) \Rightarrow W.$$

Assume $x_\infty = 0$ for simplicity. Set $Z_n = \lambda_n X_{\lambda_n}$. Then $Z_n \Rightarrow W$ implies $\theta E^\theta Z_n \Rightarrow W$ uniformly in $\theta \in [1, 3]$. Hence writing $\lambda = \theta_n \lambda_n$ for $\lambda_n \le \lambda < \lambda_{n+1}$ we find

$$\lambda X^\lambda = \theta_n E^{\theta_n} Z_n \Rightarrow W.$$

For general x_∞ one obtains (4.5). □

Remark 4.3 The basic idea of these proofs is well known from applications of regular variation in extreme value theory. If a distribution lies in the domain of attraction of a Fréchet or a Weibull law then the norming transformations may be chosen to have the simple form $a_n(x - b)$ or $a_n x$. Only for the Gumbel limit law $\Lambda(x) = \exp(-e^{-x})$ both coefficients a_n and b_n play a role. See Resnick [1987] or Embrechts, Klüppelberg and Mikosch [1997]. See also Balkema [1973] for a more algebraic approach. □

Suppose $\lambda \to \lambda_\infty < \infty$. Let the limit variable $V > 0$ have distribution γ_a. As in the case of a normal limit distribution a number of limit relations turn out to be equivalent for a gamma limit:

G1) $V_\lambda = (\lambda_\infty - \lambda)X^\lambda \Rightarrow V$;

G2) $EV_\lambda \to EV = \alpha$;

G3) $EV_\lambda^n \to EV^n$ for $n \in \mathbb{N}$;

G4) $K_\lambda(\xi) = Ee^{\xi V_\lambda} \to 1/(1-\xi)^\alpha$ for $\xi < 1$.

Proposition 4.4 *The limit relations G4) and G1) are equivalent.*

Proof The implication G4) $\Rightarrow$ G1) is well-known. (Convergence of the mgf's implies convergence of the distributions.) The implication G1)$\Rightarrow$G4) follows from Theorem 2.5. □

Infinite Radon measures occur naturally in the theory of exponential families in the domain of attraction of a gamma distribution on $[0, \infty)$. If F lies in the domain of attraction of γ_α then λ_∞ is finite. The measure $d\mu(y) = e^{\lambda_\infty y} dF(y)$ has infinite mass since $\lambda_\infty \notin \Lambda$ implies $K(\lambda) \to \infty$ for $\lambda \to \lambda_\infty$. Note however that

$$\widehat{M}(\tau) = \int e^{\tau y} d\mu(y) = K(\lambda_\infty + \tau) < \infty \qquad \tau_0 < \tau < 0$$

for some $\tau_0 < 0$.

The natural exponential family generated by the Radon measure μ consists of rv's Y^τ with distribution

$$dG_\tau(y) = e^{\tau y} d\mu(y)/\widehat{M}(\tau) \qquad \tau_0 < \tau < 0.$$

This is also the exponential family generated by the df F up to a shift in the parametrization: $G_\tau = F_\lambda$ for $\lambda = \lambda_\infty + \tau$.

The df $M(y) = \mu((-\infty, y])$ of the measure μ plays a key role in the description of the domain of attraction of the positive gamma distributions. We first give an example.

Example 4.5 (i) Let μ be a Radon measure on $\mathbb{R}$ with density m which vanishes off $[0, \infty)$. Suppose $m(x) \to 1$ for $x \to \infty$. Let Y^τ, $\tau < \tau_\infty = 0$, be the exponential family generated by μ. The rv Y^τ has density $e^{\tau y} m(y)/\widehat{M}(\tau)$. Set $\xi = -\tau$. The normalized rv $V_\tau = \xi Y^\tau$ has density $e^{-y}m(y/\xi)/(\xi\widehat{M}(\tau))$ which converges to the standard exponential density for $\xi \downarrow 0$ since $m(y/\xi) \to 1$. Note that $\widehat{M}(\tau) \sim 1/\xi$ for $\tau \uparrow 0$ and $M(y) = \mu((-\infty, y]) \sim y$ for $y \to \infty$.

(ii) More generally start with a measure μ on $\mathbb{R}$ with distribution function $M(y) = \mu((-\infty, y])$ which varies regularly at ∞ with exponent $\alpha > 0$. Assume that $\int e^{\lambda_0 y} d\mu(y)$ is finite for some $\lambda_0 < 0$. The corresponding exponential family Y^λ, $\lambda_0 \le \lambda < 0$, with distribution

$$d\pi_\lambda(y) = e^{\lambda y} d\mu(y)/\widehat{M}(\lambda) \qquad \widehat{M}(\lambda) = \int e^{\lambda y} d\mu(y) \qquad \lambda_0 \le \lambda < 0$$

satisfies G1) in the list above with $\lambda_\infty = 0$.

Proof Regular variation with exponent α implies for $\beta > \alpha$ that $M(y) = o(y^\beta)$ for $y \to \infty$. Hence $e^{\lambda y} d\mu(y)$ is a finite measure for $\lambda_0 \le \lambda < 0$. For $\xi > 0$, let S^ξ be multiplication by ξ, so $S^\xi y = \xi y$. The measure $S^\xi \mu$ is the image of μ under the map S^ξ. It has df $M(y/\xi)$. Let $A(\xi) = M(1/\xi)\Gamma(\alpha + 1)$. Then

$$\frac{M(y/\xi)}{A(\xi)} \to \frac{y_+^\alpha}{\Gamma(\alpha+1)} \qquad \text{weakly on } \mathbb{R} \text{ for } \xi \downarrow 0.$$

Note that for $y \le 0$,

$$\frac{M(y/\xi)}{M(1/\xi)} \le \frac{M(0)}{M(1/\xi)} \to 0,$$

as $\xi \downarrow 0$, since $M(1/\xi) \to \infty$ as a consequence of regular variation. The finite measures $d\nu_\xi(y) = e^{-y} d(S^\xi \mu)/A(\xi)$, $\xi > 0$, satisfy

$$d\nu_\xi(y) \to e^{-y} y_+^{\alpha-1} dy/\Gamma(\alpha) \qquad \xi \downarrow 0$$

vaguely on $[-\infty, \infty)$ and even weakly since $\nu_\xi(\mathbb{R}) \to \int e^{-y} dy_+^\alpha/\Gamma(\alpha+1) = 1$ because of the relation $M(y) = o(y^\beta)$ for $\beta > \alpha$ mentioned above. We conclude that $A(\xi) \sim \widehat{M}(-\xi) = \int e^{-\xi y} d\mu(y)$ for $\xi \downarrow 0$ and hence for $\lambda = -\xi$ the probability measure $e^{-y} d(S^\xi \mu)(y)/\widehat{M}(\lambda)$ of ξY^λ tends to γ_α weakly for $\lambda \uparrow 0$. □

The ideas of this example suggest the following general result.

Theorem 4.6 *Let V have a gamma distribution with parameter $\alpha > 0$. Let the rv X with distribution F have mgf K with finite upper endpoint λ_∞. Let $M(y) = \mu((-\infty, y])$ be the df of the Radon measure $d\mu(x) = e^{\lambda_\infty x} dF(x)$. Then the following five statements are equivalent:*

1) $V_\xi = \xi X^\lambda \Rightarrow V$, $\qquad \xi = \lambda_\infty - \lambda \downarrow 0$;

2) M varies regularly at ∞ with exponent α;

3) K varies regularly at $\lambda_\infty - 0$ with exponent $-\alpha$; that is

$$\lim_{t\downarrow 0} \frac{K(\lambda_\infty - tx)}{K(\lambda_\infty - t)} = x^{-\alpha}, \quad x > 0.$$

4) the mgf of V_ξ converges to $1/(1-\tau)^\alpha$ for $\tau < 1$ when $\xi \downarrow 0$;

5) the df M and the mgf K are asymptotically related:

$$\frac{M(y/\xi)}{K(\lambda_\infty - \xi)} \to \frac{y_+^\alpha}{\Gamma(\alpha+1)} \quad \text{weakly on } \mathbb{R} \text{ when } \xi \downarrow 0. \tag{4.6}$$

Proof We proceed in six steps.

3) $\Longleftrightarrow$ 4) since the mgf of V_ξ is $\tau \to K(\lambda_\infty - \xi + \tau\xi)/K(\lambda_\infty - \xi)$.

4) $\Rightarrow$ 1) Convergence of mgf's implies weak convergence.

1) $\Rightarrow$ 5) The rv V_ξ has distribution $d\pi_\xi^*(y) = e^{-y} d(S^\xi\mu)(y)/K(\lambda)$ where S^ξ as in the proof of the example above is multiplication by ξ. Then

$$d\pi_\xi^*(y) \to e^{-y} y^{\alpha-1} dy/\Gamma(\alpha) \qquad \xi \downarrow 0. \tag{4.7}$$

Multiply by e^y and integrate over $(-\infty, y]$. Since $S^\xi\mu$ has df $M(y/\xi)$ we obtain (4.6).

5) $\Rightarrow$ 2) is obvious.

5) $\Rightarrow$ 3) by symmetry: $K(\lambda_\infty - \eta\xi)/M(1/\xi) \to \Gamma(\alpha+1)/\eta^\alpha$ on $(0, \infty)$.

2) $\Rightarrow$ 1) is proved in the Example 4.5 (ii) above.

So we have established 1) $\Rightarrow$ 5) $\Rightarrow$ 2) $\Rightarrow$ 1) and 1) $\Rightarrow$ 5) $\Rightarrow$ 3) $\Rightarrow$ 4) $\Rightarrow$ 1). □

Note incidentally that we have proven Karamata's celebrated Tauberian theorem 3) $\Rightarrow$ 2). We have also proven that weak convergence implies convergence of the mgf's for exponential families with the limit distribution γ_α, see Theorem 2.5.

Let $\kappa = \log K$ denote the cgf in the theorem above and set

$$\varphi(t) = \kappa(\lambda_\infty - e^{-t}).$$

Then regular variation of K with exponent $-\alpha$ in $\lambda_\infty - 0$ just means that φ increases roughly with rate α: for any $x \in \mathbb{R}$

$$\varphi(t+x) - \varphi(t) \to \alpha x \qquad t \to \infty \tag{4.8}$$

uniformly on bounded intervals in $\mathbb{R}$. If $\varphi \in C^1$, then it satisfies (4.8) if

$$\varphi'(t) \to \alpha \qquad t \to \infty. \tag{4.9}$$

In our case the function φ is analytic and

$$\varphi_t(x) = \varphi(t+x) - \varphi(t) = \log\left(\frac{K(\lambda_\infty - e^{-(t+x)})}{K(\lambda_\infty - e^{-t})}\right) \to \log(e^{\alpha x}) = \alpha x \qquad t \to \infty.$$

Hence $\varphi'(t) \to \alpha$ and $\varphi^{(n)}(t) = \varphi_t^{(n)}(0) \to 0$ for $n \geq 2$. This means that the mgf K varies smoothly in $\lambda_\infty - 0$. See BGT [1987, Section 1.8]. We have shown:

Proposition 4.7 *If the df F lies in $\mathcal{D}(\gamma_\alpha)$ then the mgf K has a finite upper endpoint λ_∞ and varies smoothly in $\lambda_\infty - 0$ with exponent $-\alpha$.*

In the case of asymptotic normality there exists a necessary and sufficient condition in terms of the third moment of the standardised variables: $EU_\lambda^3 \to 0$. For exponential families in the domain of attraction of the gamma distribution γ_α we can formulate a condition in terms of the first moment of the normalised variables $V_\lambda = (\lambda_\infty - \lambda)X^\lambda$.

Theorem 4.8 *Suppose X is a rv with exponential family X^λ, $\lambda \in \Lambda$, and the upper endpoint $\lambda_\infty = \sup \Lambda$ is finite. Then $X \in \mathcal{D}(\gamma_\alpha)$ if and only if $(\lambda_\infty - \lambda)EX^\lambda \to \alpha$ for $\lambda \to \lambda_\infty$.*

Proof Necessity of the condition has been proved above: G4) $\Rightarrow$ G2. For sufficiency note that the condition can be formulated in terms of the function $\widehat{M}$ as

$$|\tau|\widehat{M}'(\tau)/\widehat{M}(\tau) \to -\alpha \qquad \tau \uparrow 0.$$

This is the well-known von Mises sufficient condition for regular variation with exponent $-\alpha$. See (4.9) or BGT [1987]. □

We now turn our attention to the gamma distribution on $(-\infty, 0]$. The theory for the domain of attraction of $\bar{\gamma}_\alpha$ is even simpler.

Let X have df $F \in \mathcal{D}(\bar{\gamma}_\alpha)$ and mgf K. Since the upper endpoint x_∞ of F is finite we may assume $x_\infty = 0$. Since F is continuous in its upper endpoint the mgf $K(\lambda)$ vanishes for $\lambda \to \infty$. The probability measure η of $-X$ has df $H(y) = 1 - F(y - 0)$. The positive rv $-\lambda X^\lambda$ has probability distribution

$$e^{-y} d(S^\lambda \eta)(y)/K(\lambda)$$

and $S^\lambda\eta$ has df $H(y/\lambda)$. The following two weak limit relations for $\lambda\to\infty$ are equivalent:

$$\begin{aligned} e^{-y}d(S^\lambda\eta)(y)/K(\lambda) &\to d\gamma_\alpha(y)=e^{-y}y^{\alpha-1}dy/\Gamma(\alpha)\\ H(y/\lambda)/K(\lambda) &\to y^\alpha/\Gamma(\alpha+1).\end{aligned}$$

Theorem 4.9 *Let W have probability distribution $\bar\gamma_\alpha$ on $(-\infty,0]$ for some $\alpha>0$. Let X have df F with upper endpoint x_∞ and mgf K. Then the upper endpoint of K is ∞. The following conditions are equivalent:*

1) $V_\lambda=\lambda(X^\lambda-x_\infty)\Rightarrow W$ for $\lambda\to\infty$;

2) $1-F$ varies regularly with exponent α in $x_\infty-0$;

3) $e^{-x_\infty\lambda}K(\lambda)$ varies regularly in ∞ with exponent $-\alpha$;

4) the mgf of V_λ converges to the mgf of W on $(1,\infty)$ for $\lambda\to\infty$;

5) the tail $1-F$ and the mgf K are asymptotically related: For $x>0$

$$\frac{1-F(x_\infty-x/\lambda)}{e^{-\lambda x_\infty}K(\lambda)}\to\frac{x^\alpha}{\Gamma(\alpha+1)}\qquad\lambda\to\infty.$$

Proof The proof is similar to that of Theorem 4.6 and omitted. □

Similarly setting $\varphi(t)=\kappa(e^t)$ we find $\varphi_t(x)=\varphi(t+x)-\varphi(t)\to-\alpha x$ which proves

Proposition 4.10 *For a df in $\mathcal{D}(\bar\gamma_\alpha)$ the mgf varies smoothly with exponent $-\alpha$.*

Theorem 4.11 *Let X^λ, $\lambda\in\Lambda$, be the exponential family generated by the rv X with df F. Suppose* $\sup\{F<1\}=x_\infty$ *is finite. Then* $\sup\Lambda=\infty$ *and* $F\in\mathcal{D}(\bar\gamma_\alpha)$ *if and only if*

$$\lambda(x_\infty-EX^\lambda)\to\alpha\in(0,\infty)\qquad\lambda\to\infty. \tag{4.10}$$

Proof If x_∞ is finite then $Ee^{\lambda x}$ is finite for all $\lambda>0$, so $\lambda_\infty=\infty$. Relation (4.10) follows from convergence of the mgf's since that implies convergence of the moments. Now assume $x_\infty=0$. Then (4.10) implies $\lambda EX^\lambda=\lambda K'(\lambda)/K(\lambda)\to-\alpha$. This implies regular variation for the decreasing function K, see (4.8). □

Our final result is a companion to Theorem 3.18. It is not clear whether the function H defined below will vary regularly in some sector of the complex plane. Compare BGT [1987, Theorem 7.4.3].

Theorem 4.12 *Let $M : [0,\infty) \to [0,\infty)$ be an increasing function which varies regularly in ∞ with exponent $\alpha \neq 0$. There exists an analytic function H defined on a half space $\{\Re z > 0\}$ which is real valued and increasing on the positive half line such that $H(t) \sim M(t)$ for $t \to \infty$. Moreover the function H varies smoothly.*

Proof Define $H(z) = \widehat{M}(-1/z)/\Gamma(\alpha+1)$ where $\widehat{M}$ is the mgf of the measure μ on $[0,\infty)$ with df $M(y+0)$, and apply (4.6). □

5 On the tail accuracy of the saddlepoint and gamma approximation

Approximations for densities of sums of rv's is an important problem in statistics. Saddlepoint approximations have been suggested by Daniels [1954] and since then the accuracy of the approximation in the tail has been considered by various authors. The methods traditionally applied involve complex integration (cf. Jensen [1988]).

We show that limit laws for exponential families can be applied to prove tail-accuracy of certain densities. These results enable us to approximate densities of sums of the form

$$X = \sum_{i=1}^{n} X_i$$

where $n \in \mathbb{N}$ and X_i, $i = 1, \ldots, n$, are independent but not necessarily identically distributed.

The key results concern the convolution of distribution functions and densities from the domains of attraction. The parameter to determine the domain of attraction of a gamma distribution is the parameter α in (4.1), and we denote the corresponding domain of attraction by $\mathcal{D}(\alpha)$ for $\alpha > 0$. The normal limit is a member of the extended gamma family corresponds to $\alpha = \infty$, hence we denote its domain of attraction by $\mathcal{D}(\infty)$.

For densities in $\mathcal{D}(\infty)$ results of this kind and some statistical examples can be found in Barndorff-Nielsen and Klüppelberg [1992]. The following summarises these results.

Proposition 5.1 *Suppose F, $G \in \mathcal{D}(\infty)$, then $F * G \in \mathcal{D}(\infty)$.*

Proof Notice that the cgf of $F * G$ is the sum $\kappa = \kappa_F + \kappa_G$ of the factors, and the variances add. Hence $\sigma \geq \sigma_F, \sigma_G$. So if $1/\sigma_F$ and $1/\sigma_G$ are self-neglecting, then also $1/\sigma$. The result follows then by Theorem 3.2. □

Proposition 5.2 (BKR [1993])

The class of densities with Gaussian tails are closed w.r.t convolutions.

Let f be a density, defined and positive on an interval I that is unbounded above. The *(unnormalised) saddlepoint approximation* to $f(x)$ may be expressed as

$$f^\dagger(x) = \frac{1}{\sqrt{2\pi\kappa''(\lambda)}}\, e^{-(\lambda x - \kappa(\lambda))} \tag{5.1}$$

where $\kappa(\lambda) = \log K(\lambda)$ denotes the cgf and λ is the saddlepoint, i.e. it satisfies $\kappa'(\lambda) = x$. The ratio $f^\dagger(x)/f(x)$ expresses the relative accuracy of the saddlepoint approximation and we obtain immediately from (3.12) that $f^\dagger(x) \sim f(x)$ as $x \to \infty$ and hence for the relative error

$$RE^\dagger(x) = \left|\log\left(f^\dagger(x)/f(x)\right)\right| \to 0 \qquad x \to \infty\,.$$

We now turn to $\mathcal{D}(\alpha)$ for finite α and start with a convolution result, which can be found in Cline [1986], Theorem 3.4.

Proposition 5.3 *Suppose $F \in \mathcal{D}(\alpha_1)$, $G \in \mathcal{D}(\alpha_2)$ for $\alpha_1, \alpha_2 < \infty$, then*

$$\overline{F * G}(x) \sim \frac{\Gamma(\alpha_1)\ \Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}\, x\, e^x\, \overline{F}(x)\, \overline{G}(x) \qquad x \to \infty\,. \tag{5.2}$$

In particular, $F * G \in \mathcal{D}(\alpha_1 + \alpha_2)$.

Corollary 5.4 *If F and G as in Proposition 5.3 have densities f and g, then*

$$f * g(x) \sim \frac{\Gamma(\alpha_1)\ \Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}\, x\, e^x\, f(x)\, g(x) \qquad x \to \infty\,,$$

where $f * g$ is the density of $F * G$, where $\overline{F * G}$ satisfies relation (5.2).

Proof This is an immediate consequences of the fact that for $F \in \mathcal{D}(\alpha)$ with density f the hazard rate $f(x)/\overline{F}(x) \to 1$ as $x \to \infty$. □

Now assume that $f(x) \sim e^{-x}\, x^{\alpha-1}\, \ell(x)$, $x \to \infty$, for $\alpha > 0$ and $\ell \in SV$ (i.e. $\lim_{x\to\infty} \ell(xt)/\ell(x) = 1$ for all $t > 0$). Then $\sup \Lambda = \lambda_\infty = 1$ and $F \in \mathcal{D}(\alpha)$ by Theorem 4.6. Indeed, it has been shown already in Theorem 7.1 of Daniels [1954] that the associated exponential family is asymptotically gamma. By an immediate consequence of regular variation for the derivatives of the mgf and the cgf we obtain

$$K^{(j)}(\lambda) \quad \sim \quad \frac{\Gamma(b+j)}{(1-\lambda)^{\alpha+j}}\ell\left(\frac{1}{1-\lambda}\right) \qquad j \in \mathbb{N}_0\,, \tag{5.3}$$

$$\kappa^{(j)}(\lambda) \quad \sim \quad \frac{\alpha}{(1-\lambda)^j} \qquad j \in \mathbb{N}. \tag{5.4}$$

Furthermore, since $f(x)/\overline{F} \to 1$ as $x \to \infty$, (5.3) implies that

$$f^{\dagger}(x) \sim \frac{(1-\lambda)}{\sqrt{2\pi\,\alpha}}\;\frac{\Gamma(\alpha)}{(1-\lambda)^{\alpha}}\;\ell\left(\frac{1}{1-\lambda}\right)\;e^{-\lambda x}$$

and for λ satisfying $\kappa'(\lambda) = \frac{\alpha}{1-\lambda}(1+o(1)) = x$ as $\lambda \to 1$ (see Theorem 4.8), we obtain

$$\begin{aligned} f^{\dagger}(x) &\sim \frac{\Gamma(\alpha)}{\sqrt{2\pi\,\alpha}}\;\left(\frac{x}{\alpha}\right)^{\alpha-1}\;\ell\left(\frac{x}{\alpha}\right)\;e^{-(x-\alpha)(1+o(1))} \\ &\sim \frac{\Gamma(\alpha)}{\sqrt{2\pi\,\alpha}}\;\alpha^{-(\alpha-1)}\;x^{\alpha-1}\;\ell(x)\;e^{-x}\;e^{\alpha} \\ &= \frac{\Gamma(\alpha)\,e^{\alpha}}{\sqrt{2\pi\,\alpha}\;\alpha^{\alpha-1}}\;f(x) \qquad x \to \infty\,. \end{aligned}$$

Hence $RE^{\dagger}(x)$ is bounded and independent of x.

On the other hand, for densities in the domain of attraction of a gamma distribution, a gamma approximation as e.g. suggested by Bower is more appropriate [cf. Beard, Pentikäinen and Pesonen [1984], see also Jensen [1988], equation (3.7)]. The gamma approximation is defined as follows.

$$f^{\dagger\dagger}(x) = \frac{\kappa'(\lambda)}{\kappa''(\lambda)}\;\gamma\left(\frac{(\kappa'(\lambda))^{2}}{\kappa''(\lambda)}\right)\;e^{-(\lambda x-\kappa(\lambda))} \tag{5.5}$$

where $\gamma(u) = u^{u-1}\,e^{-u}/\Gamma(u)$ and λ is such that $\kappa'(\lambda) = x$. We use Theorem 4.8 which gives $\kappa'(\lambda) = x \sim \alpha/(1-\lambda)$ and hence $\lambda = 1 - \frac{\alpha}{x}\,(1+o(1))$, which implies that

$$\begin{aligned} f^{\dagger\dagger}(x) &\sim (1-\lambda)\;\gamma(\alpha)\;e^{-x}\;e^{\alpha(1+o(1))}\;\frac{\Gamma(\alpha)}{(1-\lambda)^{\alpha}}\;\ell\left(\frac{1}{1-\lambda}\right) \\ &\sim e^{-x}\;\left(\frac{\alpha}{1-\lambda}\right)^{\alpha-1}\;\ell\left(\frac{\alpha}{1-\lambda}\right) \\ &\sim e^{-x}\;x^{\alpha-1}\;\ell(x) \\ &= f(x)\,, \qquad x \to \infty\,. \end{aligned}$$

Hence

$$RE^{\dagger\dagger}(x) = \left|\log\left(f^{\dagger\dagger}(x)/f(x)\right)\right| \to 0 \quad \text{as } x \to \infty\,,$$

i.e. the gamma approximation becomes exact in the tail.

By Proposition 5.3 the gamma family is closed w.r.t finite convolutions and hence the saddle-point approximation $f^{\dagger}$ and the gamma approximation $f^{\dagger\dagger}$ are asymptotically exact for densities of finite sums of independent rv's in the gamma family, which are not necessarily identically distributed.

Example 5.5 (i) Suppose $X_1, \ldots, X_n$ are independent rv's with densities

$$f_i(x) = e^{-x}\, x^{\alpha_i - 1}\, \ell_i(x)$$

for $\ell_i \in SV$, $\alpha_i > 0$, for $i = 1, \ldots, n$. For $n \in \mathbb{N}$ we are interested in the density f of

$$X = \sum_{i=1}^{n} X_i \,.$$

By Corollary 5.4 the density f satisfies for $x \to \infty$

$$f(x) \sim \frac{\Gamma(\alpha_1) \cdot \ldots \cdot \Gamma(\alpha_n)}{\Gamma(\alpha_1 + \ldots + \alpha_n)}\, e^{-x}\, x^{\alpha_1 + \ldots + \alpha_n - 1}\, \ell_1(x) \cdot \ldots \cdot \ell_n(x)\,.$$

Hence $f \in \mathcal{D}(\alpha_1 + \ldots + \alpha_n)$ and the gamma approximation $f^{\dagger\dagger}$ as defined in (5.5) is asymptotically exact in the tail.

(ii) The situation changes if we consider a weighted sum with possibly different weights, i.e. we are now interested in the approximation of the density f_0 of the weighted sum

$$X = \sum_{i=1}^{n} w_i\, X_i$$

for certain weights w_i. For simplicity we assume that $X_1, \ldots, X_n$ are iid with density

$$g(x) = e^{-x}\, x^{b-1}\, \ell(x)\,.$$

Then for $i = 1, \ldots, n$ the rv's X_i has density $f_i(x) = f\left(x/w_i\right)/w_i$ for

$$f_i(x) = \exp\left\{-\frac{x}{w_i}\right\}\, x^{b-1}\, \ell(x)/w_i^b\,.$$

W.l.o.g. assume that $w_1 = \max\left(w_1, \ldots, w_n\right)$, and that $w_i < w_1$ for $i = 2, \ldots, n$. Then the mgf $K_i\left(\frac{a}{w_1}\right) < \infty$ for $i = 2, \ldots, n$ and a classical result by Breimann [1965, Prop. 3] yields

$$\begin{aligned} f(x) \quad &\sim \quad \prod_{i=2}^{n} K_i\left(\frac{1}{w_1}\right)\, f_1(x) \\ &= \quad \exp\left\{-\frac{1}{w_1}\, x\right\}\, x^{\alpha-1}\, \widetilde{\ell}(x)\,, \quad \widetilde{\ell} \in SV\,. \end{aligned}$$

Hence $f_1^{\dagger\dagger}(x)$ approximates $f(x)$ asymptotically exact in the tail.

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