Evolution of Semiconductor Demand Forecasts
by
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1 Introduction

In this paper, we will attempt to quantify the differences between forecasts and actual demands for semiconductors. The primary concern will be the evolution of such differences as time passes. Generation of forecasts is outside the scope of this study. In practice, more information becomes available with time. Thus, forecasts for a particular month's demand improve in terms of accuracy as that month is approached. This process has not been systematically studied. In addition, dependencies among product demands will be captured.

We will now make the forecast evolution concept lucid via an example. Presume that we want to forecast domestic PC sales in December; suppose we start forecasting in January 1988 and update the forecast every month. At the beginning of December 1998, we will have produced 11 forecasts for December sales. Actual sales figure for December will be available at the end of December. This figure can be thought as a forecast with zero error. Putting these 12 numbers, all being forecasts for December sales, into chronological order, a series is obtained. This series shows how forecasts for December sales evolve over time from a highly uncertain forecast in January to a sharply accurate one in December.

The reason for forecast updates is the availability of additional information. In a month information regarding sales, customer behaviors, promotion campaigns and the like become available. Thus, it is expected that the forecasts of the de-
mand in a given month become more reliable (less variable) as time advances. Following up on the PC sales example above, December sales forecasted in January usually will be more variable than those forecasted in February.

The relevant literature for this study can be divided into three groups. The first group contains demand modeling work. Chang and Fyffe [5] visualize monthly demands as a fixed fraction of total demand. Total demands are modified as monthly sales reveal. Murray and Silver [11] represent demand as a binomial random variable where the number of potential buyers is constant but the probability of a customer purchase is updated using past sales. Azoury [3] assumes that demand has a particular density and updates parameters of that density in a Bayesian fashion. Angelus et al. [2] ties the current period’s demand to the last period’s demand through a random multiplier whose expected value is greater than one. None of these papers’ principal aim is demand modeling; demand models are treated as a side issue.


The third group studies forecast revisions. The main driving idea in Guerrero and Elizondo [6], and Kekre et al. [8] is the specific way they compute forecasts from previous demands. For example, in one of the models it is assumed that the ratio of future demands to past demands is constant. After this constant is estimated, forecasts are readily generated from the past demands.

It should be noted that with one exception previous work always attempted to generate forecasts in lieu of understanding their covariance structure. The exception is due to Heath and Jackson [7] where covariance matrix of unbiased and stationary demand forecasts is estimated. That matrix is also used to simulate demand forecasts. This paper will extend Heath and Jackson approach to nonstationary semiconductor demand forecasts.
2 Evolution Model

The model has product families at one echelon and products at a finer echelon. Chips having the same function are put into the same product family. For example memory chips constitute a product family. Chips of the same product family are further grouped according to line width (e.g. Cmos 12). The product family and the line width define a product uniquely for our purposes. An example of a product defined as such is (memory, Cmos12). Each of such doublets will represent a product.

In semiconductor industry, there is a trend towards shorter line widths; within each product family, demand for a given line width dies out and is replaced by demand for a finer line width. This process is called migration of product families. In Figure 1, each S-curve represents forecasted demand for memory chips for the line width in question plus demands for all shorter line widths. Due to migration of product families, one expects those curves to be nondecreasing. The vertical distance between S-curves is the demand for memory chip with a single line-width.

![Figure 1: Technology migration in a product family](image)

In addition to working with absolute quantities on the vertical axis of Figure 1, the ratio of those absolute figures to the total product family demand will be of interest. Let \( D_{s,t}^{p,lw} \) be the demand of product family \( p \) and line width \( lw \), forecasted from period \( s \) for period \( t \). Let \( CD_{s,t}^{p,lw} \) be the demand for all line widths shorter than or equal to \( lw \) of family \( p \), forecasted from period \( s \) for period \( t \). When \( s = t \), the quantity in question is no longer a forecast but an observation. Furthermore,
define $D_{s,t}^p$, the forecast for family $p$, and $F_{s,t}^{p,lw}$, the fractional forecast, as

$$D_{s,t}^p = \sum_{lw} D_{s,t}^{p, lw}$$

$$CD_{s,t}^{p, lw} = \sum_{lw \leq lw_0} D_{s,t}^{p, lw}$$

$$F_{s,t}^{p, lw} = \frac{CD_{s,t}^{p, lw}}{D_{s,t}^p}$$

Note that $F_{s,t}^{p, lw}$ can be easily calculated from the past demand forecasts.

For the rest of this section, consider a single product family and line width so that $lw$ and $p$ indices on $F$ can be dropped. Without loss of generality, we impose a parameterization on $F_{s,t}$ of the following form,

$$F_{s,t} = F_{t-H,t} + g_t \delta_{s,t}$$ (1)

where $g_t$ is a deterministic but unknown parameter. This parameter is a function of product family and line width. $H$ is the forecast horizon, so that $F_{t-H,t}$ is the first forecast made for the ramp up demand fraction of time $t$. We call $\delta_{s,t}$ a parameterized forecast, because it is a measure of forecast shifts between the first forecast $F_{t-H,t}$ and $(s - t + H + 1)$ st forecast $F_{s,t}$. Equation (1) represents forecast update as a product of two components. The first component, $\delta_{s,t}$, is the shift due to the time lag between periods $s$ and $t$. The second component, $g_t$, is due to the rapidness of the migration. We assume that lag update $\delta_{s,t}$ depends only on $t - s$. Thus, $g_t$ is used to equalize the volatility of forecast updates.

Given fractional forecasts $F_{s,t}$ and parameters $g_t$, $\delta_{s,t}$ can be calculated via equation (1). The Heath-Jackson forecast evolution model considers updates in parameterized forecasts, which would be calculated as

$$u_{s,t} = \delta_{s,t} - \delta_{s-1,t}$$ (2)

Note that the fractional forecast $F_{s,t}$, the parameterized forecast $\delta_{s,t}$ and the update $u_{s,t}$ are all observed at time $s$ and they are data in periods $r$, $r \geq s$. In periods $r$, $r \leq s$, they are random variables because more information will be obtained between period $r$ and period $s$ when they are produced by the forecaster. Let $\mathcal{I}_s$ be the information available at time $s$ ($\mathcal{I}_s$,
stands for the \( \sigma \)-field at time \( s \)). We will use notation inspired from conditioning to distinguish between the versions of the forecasts as seen from different time periods. Specifically, \( F_{s,t} | \mathcal{S}_r \), \( \delta_{s,t} | \mathcal{S}_r \), and \( u_{s,t} | \mathcal{S}_r \) refer to the random variables \( F_{s,t} \), \( \delta_{s,t} \) and \( u_{s,t} \) as seen from period \( r \). With this notation equation (2) becomes

\[
u_{s,t} | \mathcal{S}_r = \delta_{s,t} | \mathcal{S}_r - \delta_{s-1,t} | \mathcal{S}_r\] (3)

Let \( U_{s,t} \) be the generic update random variable defined on \( (\Omega, P, \mathcal{S}_s) \). We make the following assumptions on the update random variable:

(A1) No learning: \( U_{s,t} | \mathcal{S}_{r_1} = U_{s,t} | \mathcal{S}_{r_2} \) in distribution for \( r_1 < s, r_2 < s \). That is, nothing is learned about \( U_{s,t} \) before period \( s \).

(A2) Stationarity: \( U_{s,t} = U_{s+h,t+h} \) in distribution for any increment \( h \).

(A3) Zero expected value: \( E[U_{s,t}] = 0 \).

(A4) Independence: \( U_{s_1,t_1} \) and \( U_{s_2,t_2} \) are independent if \( s_1 \neq s_2 \).

The main interest here is in calculating the variance of the forecasts. Forecasts made for periods too far into the future are not useful, so we have a finite forecast horizon \( H \). Thus, parameterized forecasts \( \delta_{s,t} | \mathcal{S}_r \) for \( s < t - H \) will not be defined. Useful information regarding period \( t \) demand comes in after period \( t - H \), so for \( r < t - H \), \( \delta_{s,t} | \mathcal{S}_r = \delta_{t-H,t} | \mathcal{S}_{t-H} \). In the more interesting case of \( s \geq t - H \) and \( r \geq t - H \), from Equation (3), the parameterized forecast is

\[
\delta_{s,t} | \mathcal{S}_r = \sum_{j=t-H+1}^{s} u_{j,t} | \mathcal{S}_r = \sum_{j=r-H+1}^{r} u_{j,t} + \sum_{j=r+1}^{s} U_{j,t} \quad s \geq t - H, r \geq t - H \] (4)

The second equality follows from the assumption of no learning about updates \( U_{j,t} \) before they are observed. Note that values \( u_{j,t} \) are data observed before time \( r \) whereas \( U_{j,t} \) is random. Thus, as \( r \) increases, the deterministic component of parameterized forecasts (for fixed \( s \) and \( t \)) grows, the forecast eventually becomes deterministic at \( r = s \). It follows from equation (4) that the stochastic parts of \( \delta_{s,t} | \mathcal{S}_r \) and \( \delta_{s+h,t+h} | \mathcal{S}_{r+h} \) have the same distribution for any increment \( h \).
Although $\delta_{s,t}|\mathcal{Z}_r$, $\delta_{s,t}|\mathcal{Z}_{r+1}$ have different means, parameterized forecasts as given by equation (4) satisfy

$$E(\delta_{s,t}|\mathcal{Z}_r) = E(\delta_{s,t}|\mathcal{Z}_r)$$  \hfill (5)

Thus, all (parameterized and fractional) forecasts are unbiased, a consequence of no learning and zero expected value assumptions, and an important assumption in [7]. It is possible to deduce (A1), (A3) and (A4) as a consequence of unbiasedness of forecasts. For $r \leq t - H$, the expected values in 5 become zero. Also note that the parameterized forecasts constitute a martingale.

Obtaining forecasts via equation (4) has a nice feature: The mean square error of the parameterized forecasts is non-increasing and goes to zero as $s$ approaches $t$ for any $r$, i.e.,

$$E[(\delta_{s,t} - \delta_{s,t})^2|\mathcal{Z}_r] = \sum_{j=s+1}^{t} \text{var}(U_{j,t})$$

The last equality follows from the independence assumption. Thus, as time advances towards the period the forecast is made for, the accuracy of the forecast increases.

3 Normalized Updates

In the previous section, we illustrated the evolution of parameterized forecasts by focusing on a fixed product $p_{lw}$. This allowed us to illustrate the stochastic mechanisms with notational simplicity. On the other hand, we will now obtain some statistics on these mechanisms. Thus, unless otherwise stated, all the notation refers to the observed data.

We introduce a specific form for the multipliers $g_{t^{p,\text{lw}}}$. Our assumption relies on the observation that updates tend to be smaller when $E_{s,t}$ is close to either 0 or 1. Specifically, if $g_{t^{p,\text{lw}}}$ is proportional to the derivative of the ramp up curve, then $\delta_{s,t}$ becomes stationary for fixed $t - s$. Thus, we select $g_{t^{p,\text{lw}}}$ to be the derivative of the ramp up curve.
The ramp up curve is forecasted as well. For a product \((p, lw)\), \(F_{t-H,t}^{p,lw}\) is the earliest forecast of the fractional demand at time \(t\). A generic ramp up function \(F(t)\) will be fit to historical data \(F_{t-H,t}^{p,lw}\), say once every 48-60 periods. For all line widths, \(F(t)\) will be 0 at \(t = 0\) and increase up to 1. Then, for all \(p, lw\), we will set

\[
g_t^{p,lw} = \frac{dF(t - \Delta)}{dt}
\]

where \(\Delta\) is the period the product \((p, lw)\) is introduced.

We will now divert our attention to obtaining parameterized forecast updates. Towards that end, it is assumed that for different line widths \(lw_1 \neq lw_2\), fractional cumulative demands \(\delta_{s_1,t_1}^{p_1,lw_1}\) and \(\delta_{s_2,t_2}^{p_2,lw_2}\) are stochastically pairwise independent. Thus, the ramp up curves of figure 1 are independent. An example will clarify and motivate the assumption. Let \(A, B\) denote the fractional demands for line widths 8 and 10 of the same product family at a particular time, respectively. Then the fractional cumulative demands are \(A+B\), and \(B\) for 8 and 10, respectively. If the forecast for \(B\) increases (corresponding to faster migration than estimated), then the forecast for \(A\) will probably decrease, because demand for \(B\) replaces demand for \(A\). However, it is not clear how \(A+B\) will behave. The sign of the updates in \(A\) or \(B\) may not agree with the sign of the updates in \(A+B\). Thus, assuming independence of \(A+B\) and \(B\) is justifiable.

The updates, being a function of the time lag and the product, can be calculated as

\[
u_{s,t}^{p,lw} = \frac{F_{s, t}^{p,lw} - F_{s-1, t}^{p,lw}}{g_t^{p,lw}} \tag{6}
\]

These figures are called normalized updates because their distribution depends on the difference between \(t\) and \(s\), but not on \(t\) or \(s\). The update vectors will be constructed as in [7]. For example, assume that memory and X86 are the only two product families. At time \(s\), for line widths 10 and 8 and product families memory and X86 chips, we would have vectors as

\[
U_s^{10} = [u_{s,8}^{X86,10}, \ldots, u_{s,s+H-1}^{X86,10}, u_{s,8}^{Mem,10}, \ldots, u_{s,s+H-1}^{Mem,10}]
\]

\[
U_s^8 = [u_{s,8}^{X86,8}, \ldots, u_{s,s+H-1}^{X86,8}, u_{s,8}^{Mem,8}, \ldots, u_{s,s+H-1}^{Mem,8}]
\]
This specific construction lets us observe several update vectors (each corresponding to an active line width) at a single period. Note that different line widths do not need to be put into the same update vector. This is because they are assumed to be independent.

In general, $U^{lw}_{s}$ has entries for each product $p$ and each lag $t - s$, $0 \leq t - s < H$. Note that $U^{lw_{1}}_{s_{1}}$ and $U^{lw_{2}}_{s_{2}}$ are independently and identically distributed. If $s_{1} \neq s_{2}$, this is a consequence of (A2) and (A4). If $lw_{1} \neq lw_{2}$, it is a consequence of the independence of fractional cumulative demands for different line-widths. Though updates in forecasts for different technologies are independent, the components of the vector $U_{s}$ will be dependent among themselves. Thus, we still estimate demand correlations among different product families as well as among time periods.

However, there is one problem with update vectors as written above. That is, not all components of the update vectors will be observed at all periods. $H$ time periods before the first introduction of a certain line width, there will not be any forecasts for that line width. For example, if $lw = 10$ is introduced into family X86 at time $t$, then no parameterized forecast $\delta^{X86,10}_{s,\ast}$ will be available in period $s = t - H - 1$. Only two periods later in period $t - H + 1$, it is possible to observe the first update, $u^{X86,10}_{t - H + 1, t}$. In the next period, two updates $u^{X86,10}_{t - H + 2, t}$ and $u^{X86,10}_{t - H + 2, t + 1}$ will be observed, and so forth. At some point in time, a particular line width may be used for some product families, but not for the others. In that case, only forecast updates of product families using that line width will be observed. Basically, some data will be missing in many of the update vectors. Conceptually, this is not a big issue; nevertheless it poses numerical obstacles for estimation of the covariance matrix.

4 Covariance Matrix with EM

Updates in parameterized forecasts have an expected value of zero, because forecasts are assumed to be unbiased. Thus, we will focus on the estimation of the covariance matrix. Denote the length of an update vector $U_{s}$ with $p$, where $p$ is the number of product families times the length of the forecast horizon, $H$. Suppose $U_{s}$ is a normal random vector with the $p \times p$
covariance matrix $\Sigma$. Note that the components of $U^*_s$ are not indexed by line widths. The entries of $\Sigma$ will be indexed by (product family)x(time lag).

Estimation will be based on a maximum likelihood framework. Suppose that the vector $U^*_s$ are numbered from 1 to $N$, to obtain the sample \{$U_i : i = 1..N$\}. $N$ is approximately the number of time periods times the average number of active line widths being produced at a given time. Then the maximum likelihood function would be

$$L(\Sigma) = (2\pi)^{pN/2} |\Sigma|^{-N/2} \exp\left(\frac{-1}{2} \sum_{i=1}^{N} U_i \Sigma^{-1} U_i^T \right)$$

After taking the logarithm and dropping constant terms, the MLE estimator of $\Sigma$ would solve the following minimization problem.

$$\min_{\Sigma} \frac{N}{2} \log|\Sigma| + \frac{1}{2} \sum_{i=1}^{N} \frac{U_i \Sigma^{-1} U_i^T}{g_i}$$

The solution to this minimization problem is easily found when no data is missing (see [1]). When some data is missing, an iterative procedure, called the EM algorithm, is available (see [13]). The details of the EM algorithm are outside the scope of this paper, but it is worth mentioning that the EM maximizes the likelihood function given the observed data. In a sense it does the best we can hope for.

5 Demand Covariances

After running the EM algorithm, the resulting covariance matrix $\Sigma$ is the covariance of updates in parameterized forecasts. We now discuss recovering the covariance of line width demands. First, suppose that a classical Heath-Jackson [7] algorithm is run on product family demand forecasts to obtain the covariance matrix $\Lambda$ of $D^p_s,t$ updates. These updates are calculated by $v^p_{s,t} = D^p_{s,t} - D^p_{s-1,t}$ and the update vector $V_s$, whose values are observed in period $s$, is a concatenation of the product
family update vectors $V_s^P = [v_{s,t}, v_{s,t+1}, ..., v_{s,t+H-1}]$. Since product family demands are rather stable, they are likely to satisfy the stationarity assumption in the Heath-Jackson framework. By the definition of fractional demands, we obtain:

$$D_{s,t}^P|\mathcal{F}_r = \{D_{s,t}^P|\mathcal{F}_r\} \{F_{s,t}^{p,lw}|\mathcal{F}_r - F_{s,t}^{p,lw+}|\mathcal{F}_r\}$$

where $lw$ stands for line width and $lw+$ denotes the next line width introduced after $lw$. We assume that the product family demands and fractional demands are independent because family demands are primarily driven by the market (exterior forces). Contrary to that, fractional demands are driven mainly by the company’s technology (interior forces). Next, let us look at the calculation of covariances of fractional forecasts:

$$F_{s,t}^{p,lw}|\mathcal{F}_r = F_{s,t-H,t}^{p,lw} + g_{s,t}^{p,lw} \{\delta_{s,t}^{p,lw}(lw)|\mathcal{F}_r\}$$

$$F_{s,t}^{p,lw+}|\mathcal{F}_r = F_{s,t-H,t}^{p,lw+} + g_{s,t}^{p,lw+} \{\delta_{s,t}^{p,lw+}(lw+)|\mathcal{F}_r\}$$

The parameterized forecasts $\delta_{s,t}^{p,lw}(lw)|\mathcal{F}_r$ and $\delta_{s,t}^{p,lw+}(lw+)|\mathcal{F}_r$ are i.i.d. random variables having the same distribution as $\delta_{s,t}^{p,lw}|\mathcal{F}_r$ (see Equation (4)). Thus,

$$\text{Cov}(D_{s_1,t_1}^{p_1,lw_1}|\mathcal{F}_r, D_{s_2,t_2}^{p_2,lw_2}|\mathcal{F}_r) = E\{(D_{s_1,t_1}^{p_1}|\mathcal{F}_r)(D_{s_2,t_2}^{p_2}|\mathcal{F}_r)\} \text{Cov}(\{g_{s_1,t_1}^{p_1,lw_1}(lw_1)|\mathcal{F}_r, g_{s_2,t_2}^{p_2,lw_2}(lw_2)|\mathcal{F}_r\}$$

$$+ E\{F_{s_1,t_1}^{p_1,lw_1}|\mathcal{F}_r - F_{s_1,t_1}^{p_1,lw_1+}|\mathcal{F}_r\} E\{F_{s_2,t_2}^{p_2,lw_2}|\mathcal{F}_r - F_{s_2,t_2}^{p_2,lw_2+}|\mathcal{F}_r\} \text{Cov}(D_{s_1,t_1}^{p_1}|\mathcal{F}_r, D_{s_2,t_2}^{p_2}|\mathcal{F}_r)$$

where

$$g_{s,t}^{p,lw}(lw)|\mathcal{F}_r = g_{s,t}^{p,lw}|\mathcal{F}_r - g_{s,t}^{p,lw+}\delta_{s,t}^{p,lw+}(lw+)|\mathcal{F}_r$$

and

$$\text{Cov}(g_{s_1,t_1}^{p_1,lw_1}(lw_1)|\mathcal{F}_r, g_{s_2,t_2}^{p_2,lw_2}(lw_2)|\mathcal{F}_r) = \begin{cases} 
\{g_{s_1,t_1}^{p_1,lw_1}g_{s_2,t_2}^{p_2,lw_2} + g_{s_1,t_1}^{p_1,lw_1+}g_{s_2,t_2}^{p_2,lw_2+}\} \text{Cov}(\delta_{s_1,t_1}^{p_1}|\mathcal{F}_r, \delta_{s_2,t_2}^{p_2}|\mathcal{F}_r) & \text{if } lw_2 = lw_1 \\
-g_{s_1,t_1}^{p_1,lw_1+}g_{s_2,t_2}^{p_2,lw_2+} \text{Cov}(\delta_{s_1,t_1}^{p_1}|\mathcal{F}_r, \delta_{s_2,t_2}^{p_2}|\mathcal{F}_r) & \text{if } lw_2 = lw_1+ \\
0 & \text{otherwise}
\end{cases}$$

See the appendix for the details of these derivations. What remains is connecting $\text{Cov}(D_{s_1,t_1}^{p_1}|\mathcal{F}_r, D_{s_2,t_2}^{p_2}|\mathcal{F}_r)$ to $\Lambda$ and $\text{Cov}(\delta_{s_1,t_1}^{p_1}|\mathcal{F}_r, \delta_{s_2,t_2}^{p_2}|\mathcal{F}_r)$ to $\Sigma$, for $r \geq t_1 - H$ and $r \geq t_2 - H$.
\[
\text{Cov}(D_{s_1,t_1}^{p_1}, D_{s_2,t_3}^{p_2} | \mathcal{Z}_r) = \sum_{i=t_1 - H+1 \vee t_2 - H+1}^{s_1 \wedge s_2} \Lambda(p_1, s_1 - i; p_2, s_2 - i)
\]

\[
\text{Cov}(\delta_{s_1,t_1}^{p_1}, \delta_{s_2,t_2}^{p_2} | \mathcal{Z}_r) = \sum_{i=t_1 - H+1 \vee t_2 - H+1}^{s_1 \wedge s_2} \Sigma(p_1, s_1 - i; p_2, s_2 - i)
\]

Both equalities are due to the fact that updates revealed at different times are uncorrelated. Using our estimates of \(\text{Cov}(D_{s_1,t_1}^{p_1}, D_{s_2,t_3}^{p_2} | \mathcal{Z}_r)\) and \(\text{Cov}(\delta_{s_1,t_1}^{p_1}, \delta_{s_2,t_2}^{p_2} | \mathcal{Z}_r)\) in equation (7) will finally yield the covariance between forecasted demands of two ramp ups of two product families.

6 Conclusion

Importance of accurate forecasts for decision making is indisputable in the semiconductor industry. Thus, quantifying the accuracy of forecasts can have significant implications. First, a measure of forecast accuracy would be a helpful tool to single out deteriorating forecasts. Second, decision makers are likely to adapt more flexible strategies, avoiding or postponing commitments, if they understand the degree of uncertainty in their forecasts. As actions are delayed, more accurate forecasts will become available. On the other hand, delaying actions brings certain risks, e.g. delaying tool orders may lead to insufficient capacity. In that respect, forecast accuracy is an important component in postponement vs. commitment trade offs.

Our method can also be used in simulating forecasts realistically, because it recognizes dependences among product demands.

As for the future research, relaxing our assumption that ramps are independent will make the model more defendable. However, this ought to be done without increasing the computational burden, e.g. without making update vectors very long. Given this computational handicap and lack of evidence for strong correlations among ramps, our model strikes a good balance between complexity and utility. Herein, we have used a specific parameterization to stationarize forecasts. Comparing the quality of estimation under different parameterizations will also be valuable.
References


7 Appendix: Details of equation (7)

For two independent stochastic processes $X_t$ and $Y_t$,

$$
\text{Cov}(X_t, Y_t) = \text{Cov}(X_{t_1}, X_{t_2}) E(Y_{t_1} Y_{t_2}) + \text{Cov}(Y_{t_1}, Y_{t_2}) E(Y_{t_1}) E(Y_{t_2})
$$

Noting that all the random variables are conditioned on the information at time $\mathfrak{S}_r$, we remove $\mathfrak{S}_r$ for convenience. Let

$$
X_{t_1} = (F^{p_1,lw_1}_{s_1,t_1} - F^{p_1,lw_1+}_{s_1,t_1}) \quad X_{t_2} = (F^{p_2,lw_2}_{s_2,t_2} - F^{p_2,lw_2+}_{s_2,t_2})
$$

$$
Y_{t_1} = D^{p_1}_{s_1,t_1} \quad Y_{t_2} = D^{p_2}_{s_2,t_2}
$$

Then

$$
\text{Cov}(D^{p_1,lw_1}_{s_1,t_1}, D^{p_2,lw_2}_{s_2,t_2}) = \text{Cov}(F^{p_1,lw_1}_{s_1,t_1} - F^{p_1,lw_1+}_{s_1,t_1}, F^{p_2,lw_2}_{s_2,t_2} - F^{p_2,lw_2+}_{s_2,t_2}) E(D^{p_1}_{s_1,t_1} D^{p_2}_{s_2,t_2})
$$

$$
+ \text{Cov}(D^{p_1}_{s_1,t_1}, D^{p_2}_{s_2,t_2}) E(F^{p_1,lw_1}_{s_1,t_1} - F^{p_1,lw_1+}_{s_1,t_1}) E(F^{p_2,lw_2}_{s_2,t_2} - F^{p_2,lw_2+}_{s_2,t_2})
$$

where

$$
\text{Cov}(F^{p_1,lw_1}_{s_1,t_1} - F^{p_1,lw_1+}_{s_1,t_1}, F^{p_2,lw_2}_{s_2,t_2} - F^{p_2,lw_2+}_{s_2,t_2}) = \text{Cov}(g_1^{p_1}, g_2^{p_2})(lw)
$$

$$
E(F^{p_1,lw_1}_{s_1,t_1} - F^{p_1,lw_1+}_{s_1,t_1}) = (F^{p_1,lw_1}_{t_1-H,t_1} - F^{p_1,lw_1+}_{t_1-H,t_1}) E(g_1^{p_1})
$$

$$
E(F^{p_2,lw_2}_{s_2,t_2} - F^{p_2,lw_2+}_{s_2,t_2}) = (F^{p_2,lw_2}_{t_2-H,t_2} - F^{p_2,lw_2+}_{t_2-H,t_2}) E(g_2^{p_2})(lw)
$$

$$
E(g_1^{p_1} g_2^{p_2})(lw) = g_1^{p_1} \sum_{j=t-H+1}^{r} u_j^{p_1}(lw) - g_1^{p_1+} \sum_{j=t-H+1}^{r} u_j^{p_1+}(lw+)
$$

13
Besides covariances, one may be interested in the evolution of demand forecast from period $s - 1$ to period $s$.

\[
D_{s,t}^{p, lw} = D_{s, t}^{p, lw} | \mathcal{Z}_{s-1} = (D_{s, t}^{p} | \mathcal{Z}_{s-1}) \{ F_{t-H, t}^{p, lw} + g_{t}^{p, lw}(\delta_{s, t}(lw)) | \mathcal{Z}_{s-1} - F_{t-H, t}^{p, lw+} - g_{t}^{p, lw+}(\delta_{s, t}(lw+)) | \mathcal{Z}_{s-1} \} \\
- (D_{s-1, t}^{p} | \mathcal{Z}_{s-1}) \{ F_{t-H, t}^{p, lw} + g_{t}^{p, lw}(\delta_{s-1, t}(lw)) | \mathcal{Z}_{s-1} - F_{t-H, t}^{p, lw+} - g_{t}^{p, lw+}(\delta_{s-1, t}(lw+)) | \mathcal{Z}_{s-1} \}
\]

\[
\text{Var}(D_{s,t}^{p, lw} - D_{s-1,t}^{p, lw} | \mathcal{Z}_{s-1}) = \text{Var}(D_{s,t}^{p, lw} | \mathcal{Z}_{s-1}) = E((D_{s,t}^{p})^{2} | \mathcal{Z}_{s-1})(g_{t}^{p, lw})^{2}(g_{t}^{p, lw+})^{2} \text{Cov}(\delta_{s,t}^{p} | \mathcal{Z}_{s-1}, \delta_{s,t}^{p} | \mathcal{Z}_{s-1}) \\
+ \{E(F_{s,t}^{p, lw} | \mathcal{Z}_{s-1} - F_{s,t}^{p, lw+} | \mathcal{Z}_{s-1})\}^{2} \text{Cov}(D_{s,t}^{p} | \mathcal{Z}_{s-1}, D_{s,t}^{p} | \mathcal{Z}_{s-1}) \\
= E((D_{s,t}^{p})^{2} | \mathcal{Z}_{s-1})(g_{t}^{p, lw})^{2}(g_{t}^{p, lw+})^{2}\sum(p, 1; p, 1) + \{E(F_{s,t}^{p, lw} | \mathcal{Z}_{s-1} - F_{s,t}^{p, lw+} | \mathcal{Z}_{s-1})\}^{2} \Lambda(p, 1; p, 1)
\]