HOW MISLEADING CAN SAMPLE ACF'S OF STABLE MA'S BE? (VERY!)

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ABSTRACT. For the stable moving average process

$$X_t = \int_{-\infty}^{\infty} f(t+x)M(dx), t = 1, 2, ...$$

we find the weak limit of its sample autocorrelation function as the sample size n increases to ∞. It turns out that, as a rule, this limit is random! This shows how dangerous it is to rely on sample correlation as a model fitting tool in the heavy tailed case. We discuss for what functions f this limit is non-random for all (or only some – this can be the case, too!) lags.

1. INTRODUCTION

The sample autocorrelation function (acf) of a stationary process $\{X_t\}_{1\le t<\infty}$ has played a central statistical role in traditional time series analysis, where the assumption is made that the marginal distribution has a second moment (see e.g. Brockwell and Davis, 1991). However, more and more data sets from fields like telecommunications, economics, insurance and finance exhibit infinite variance (see Duffy et al., 1993, 1994; Meier-Hellstern et al., 1991; Resnick, 1997a; Willinger et al., 1997). It is therefore natural to question whether the classical methods based on acfs are still applicable in heavy tailed modeling, where the corresponding version of the acf is often defined by

$$\hat\rho_n(h) := \hat\gamma_n(h)/\hat\gamma_n(0),\ h = 0, 1, 2, \dots, \tag{1.1}$$

and

$$\hat\gamma_n(h) := \frac{1}{n}\sum_{t=1}^{n} X_t X_{t+h},\ h = 0, 1, 2, \dots \tag{1.2}$$

are the sample covariance functions.

Continuing interest in the sample acf for the heavy tailed case seems to be based on the relative success of the acf for analyzing data from an infinite order moving average process (MA(∞)). Consider the process

$$X_t = \sum_{j=-\infty}^{\infty} c_j Z_{t-j},\ t = 1, 2, \dots,$$

where $\{Z_k\}$ are iid random variables in the domain of attraction of an α-stable law, $0 < \alpha < 2$. Davis and Resnick (1985) have shown under appropriate summability conditions on the coefficients $\{c_j\}$ that for all $h > 0$,

$$\hat\rho_n(h) \xrightarrow{P} \rho(h)$$

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where "$\xrightarrow{P}$" denotes convergence in probability and

$$\rho(h) = \frac{\sum_{j=-\infty}^{\infty} c_j c_{j+h}}{\sum_{j=-\infty}^{\infty} c_j^2}$$

is a constant.

However, if for some heavy tailed processes, the sample acf loses this desirable feature of converging to a constant, the usual model fitting and diagnostic tools such as the Akaike Information Criterion or Yule-Walker estimators will be of questionable applicability. In this case, the mischief potential for misspecifying a model is great, and more care must be taken in using the sample correlations for model fitting and estimation (see e.g. Resnick, 1997b).

Recent studies seem to indicate that processes with asymptotically degenerate sample acf (like MA(∞)) form a very limited class in the heavy tailed world. For bilinear time series and some variations of MA(∞) (sum of two MA(∞)'s, coefficient permutation with reset), it is shown that the sample acf converges in finite dimensional distribution to a random limit (Davis and Resnick, 1996; Resnick and Van Den Berg, 1998; Cohen et al., 1997).

In order to understand what happens to sample correlations in heavy tailed cases, it is natural to look at stationary α-stable processes, $0 < \alpha < 2$. This class of processes can be viewed as a heavy tailed analog of Gaussian processes, and its structure is relatively well understood. Cohen et al. (1997) conducted empirical studies on two examples of ergodic symmetric α-stable (SαS) processes of the form

$$X_t = \int_E f_t(x) M(dx),\ t = 1, 2, \ldots,$$

where M is a SαS random measure on E with σ-finite control measure m, $f_t \in L^\alpha(E, m)$ for all t, and $0 < \alpha < 2$ (see Samorodnitsky and Taqqu, 1994). Simulation evidence was found in both cases that the limit of the sample acf as the sample size n goes to ∞ is random .

This article focuses on the class of α-stable moving average processes

$$X_t = \int_{-\infty}^{\infty} f(t + x) M(dx),\ t = 1, 2, \ldots, \tag{1.3}$$

where $f \in L^\alpha(\mathbb{R}^1)$, M is a SαS random measure on $\mathbb{R}^1$ with Lebesgue control measure, and $0 < \alpha < 2$. Although one might think this class is quite close to the MA(∞) class, that is not the case. We will evaluate for these processes the weak limits of the sample acfs, using the series representation of $\{X_t\}$ and certain results on tetrahedral multi-linear forms provided by Samorodnitsky and Szulga (1989). Despite $\{X_t\}$'s kinship with MA(∞), these limits are usually (with notable exceptions) random, thus confirming the empirical results. The limits, of course, depend on the lag h and the function f.

In §2, we give the series representation of the sample covariance $\hat{\gamma}_n(h)$ and write it as sum of "diagonal" and "off-diagonal" parts; §3 finds the weak limit of the diagonal part under suitable normalization; §4 shows that the off-diagonal part, when compared with the diagonal part, can be neglected. In §5, we summarize our findings and discuss when the weak limit of $\hat{\rho}_n(h)$ is degenerate. Examples are used to demonstrate the arbitrary limit behavior of acfs when different lags are studied. In particular we construct examples which show that the sample acf may be asymptotically constant for some lags, but asymptotically random for other lags.

A simulation result is presented in Figure 1 for one particular stable moving average process which can be written as sum of two MA(∞) processes. The sample acfs of eight independent copies are drawn in the first eight plots and overlaid in the last. For this process, the sample acfs appear to have a degenerate limit for lags no bigger then 10, but randomness takes over afterwards, as indicated by the fuzziness in the last plot. Evidence continues to accumulate which casts doubt on the appropriateness of the acf as a tool for model fitting and parameter estimation in heavy tailed models.

2. Decomposition of the Series Representation of Covariance Functions

Let $q(x)$ be any density function that is strictly positive on $\mathbb{R}^1$. A change of variable (Samorodnitsky and Taqqu, 1994) in (1.3) gives

$$\{X_t, t=1,2,\dots\} \stackrel{d}{=} \left\{\int_{-\infty}^{\infty} f(t+x)q(x)^{-1/\alpha} M_1(dx), t=1,2,\dots\right\} \tag{2.1}$$

(the equality is in the sense of finite dimensional distributions), where M_1 is a symmetric α-stable random measure on $\mathbb{R}^1$ whose control measure has density $q(x)$ with respect to the Lebesgue measure. Unlike M, M_1 has a finite control measure, hence has the following series representation (Samorodnitsky and Taqqu, 1994):

$$\{M_1(A), A\in\mathcal{B}\} \stackrel{d}{=} \left\{C_\alpha^{1/\alpha}\sum_{i=1}^{\infty}\epsilon_i\Gamma_i^{-1/\alpha}\mathbf{1}(V_i\in A), A\in\mathcal{B}\right\},$$

where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}^1$,

$$C_\alpha := \left(\int_{-\infty}^{\infty} x^{-\alpha}\sin x\,dx\right)^{-1} = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}, & \text{if } \alpha\neq 1\\ 2/\pi, & \text{if } \alpha=1\end{cases} \tag{2.2}$$

is a constant, and

(2.3) $\{\epsilon_j\}$ are iid Rademacher random variables with $P[\epsilon_i=1]=P[\epsilon_i=-1]=1/2$;

(2.4) $\{\Gamma_j\}$ are arrival times of a Poisson process with unit rate on $[0,\infty)$;

(2.5) $\{V_j\}$ are iid random variables with the density $q(x)$.

All of the above three sequences are independent.

We now write down the series representation of X_t. Define

$$S_t := C_\alpha^{1/\alpha}\sum_{i=1}^{\infty}\epsilon_i\Gamma_i^{-1/\alpha}f(V_i+t)q(V_i)^{-1/\alpha}, t=1,2,\dots. \tag{2.6}$$

Then the series in (2.6) converges almost surely (Samorodnitsky and Taqqu, 1994), and

$$\{X_t, t\geq 1\} \stackrel{d}{=} \{S_t, t\geq 1\}.$$

With $\hat{\gamma}_n(h)$ defined by (1.2), we have for all $H\geq 0$,

$$\{n\hat{\gamma}_n(h), h=0,1,\dots,H\} \stackrel{d}{=} \left\{\sum_{t=1}^{n} S_tS_{t+h}, h=0,1,\dots,H\right\}. \tag{2.7}$$

From (2.6), the following holds almost surely.

$$\begin{aligned}\sum_{t=1}^{n} S_tS_{t+h} &= \sum_{t=1}^{n}\left(C_\alpha^{1/\alpha}\sum_{i=1}^{\infty}\left(\epsilon_i\Gamma_i^{-1/\alpha}f(V_i+t)q(V_i)^{-1/\alpha}S_{t+h}\right)\right)\\ &= \sum_{t=1}^{n}\left(C_\alpha^{1/\alpha}\sum_{i=1}^{\infty}\left(\epsilon_i\Gamma_i^{-1/\alpha}f(V_i+t)q(V_i)^{-1/\alpha}C_\alpha^{1/\alpha}\right.\right.\end{aligned} \tag{2.8}$$

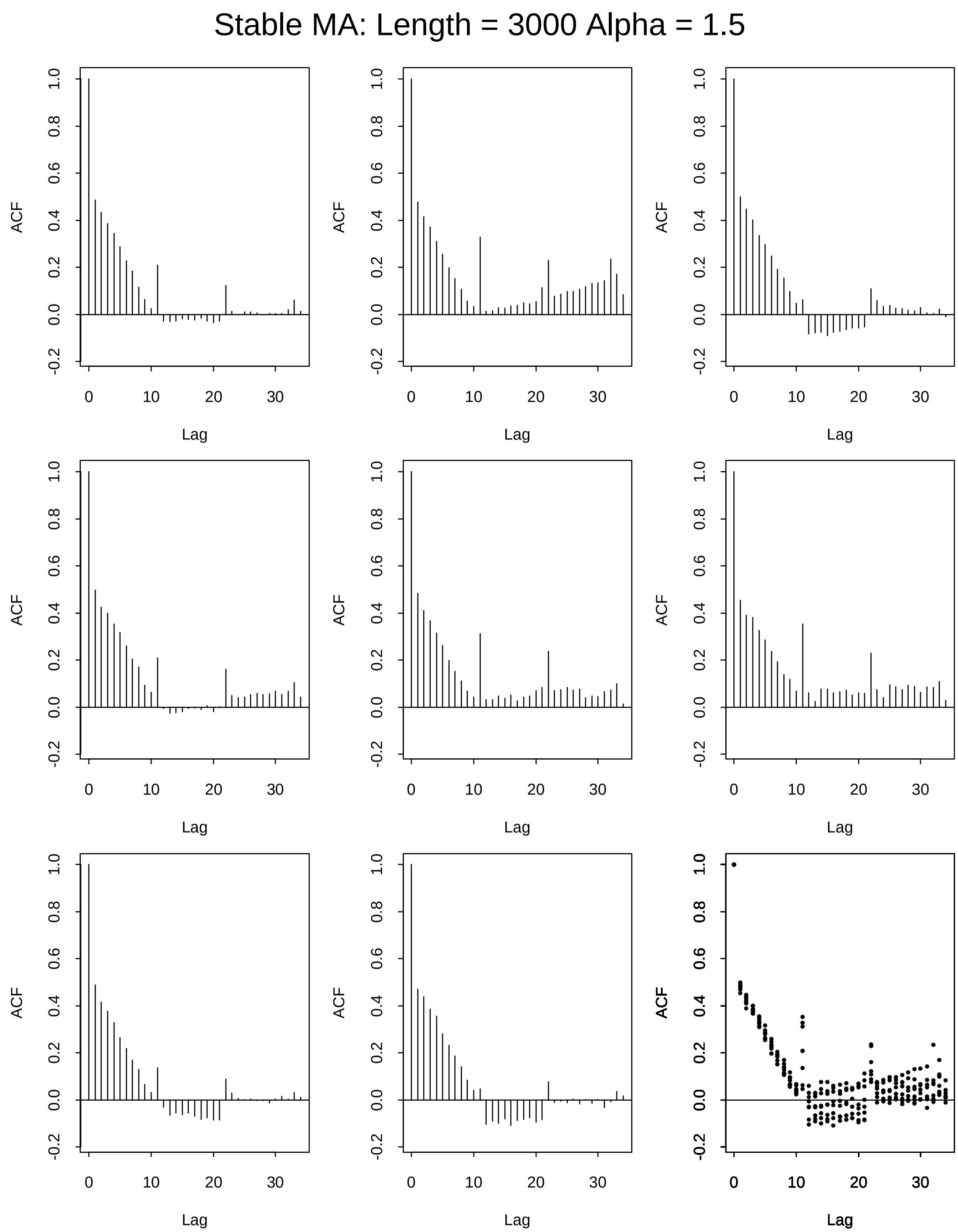

FIGURE 1. Stable Moving Average: Sample Correlation Functions

$$
\begin{aligned}
&\left.\left(\epsilon_i \Gamma_i^{-1/\alpha} f(V_i+t+h) q(V_i)^{-1/\alpha} + \sum_{j\neq i} \epsilon_j \Gamma_j^{-1/\alpha} f(V_j+t+h) q(V_j)^{-1/\alpha}\right)\right)\Bigg) \\
= \;& C_\alpha^{2/\alpha} \sum_{i=1}^{\infty} \Gamma_i^{-2/\alpha} \sum_{t=1}^{n} f(V_i+t) f(V_i+t+h) q(V_i)^{-2/\alpha} \\
&+ C_\alpha^{2/\alpha} \sum_{i=1}^{\infty} \sum_{j\neq i} \epsilon_i \epsilon_j \Gamma_i^{-1/\alpha} \Gamma_j^{-1/\alpha} \sum_{t=1}^{n} f(V_i+t) f(V_j+t+h) q(V_i)^{-1/\alpha} q(V_j)^{-1/\alpha} \\
=: \;& Y_n'(h) + Y_n''(h),
\end{aligned}
$$

where

$$
Y_n'(h) = \left(\frac{C_\alpha}{C_{\alpha/2}}\right)^{2/\alpha} C_{\alpha/2}^{2/\alpha} \sum_{i=1}^{\infty} \Gamma_i^{-2/\alpha} \sum_{t=1}^{n} f(V_i+t) f(V_i+t+h) q(V_i)^{-2/\alpha} \tag{2.9}
$$

is the sum of the "diagonal" terms where $i = j$ in the double sum $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}$ and $Y_n''(h)$ is the sum of the "off-diagonal" terms. We will see that both series converge almost surely. As a matter of fact, for all $H \geq 0$,

$$
\begin{aligned}
&\left\{Y_n'(h), h = 0, 1, \ldots, H\right\} \stackrel{d}{=} \\
&\quad \left\{ \left(\frac{C_\alpha}{C_{\alpha/2}}\right)^{2/\alpha} \int_{-\infty}^{\infty} \sum_{t=1}^{n} f(t+x) f(t+h+x) q(x)^{-2/\alpha} \tilde{M}_1(dx), h = 0, 1, \ldots, H \right\},
\end{aligned} \tag{2.10}
$$

where $\tilde{M}_1$ is a positive strictly $\alpha/2$-stable random measure on $\mathbb{R}^1$, whose control measure has density $q(x)$ with respect to Lebesgue measure (Samorodnitsky and Taqqu, 1994). Being the series representation of the stable integrals in (2.10), the series of the diagonal terms (2.9) converges almost surely to $Y_n'(h)$. So the series of the off-diagonal terms also converges almost surely. But a Rademacher series converges unconditionally whenever it converges almost surely (Samorodnitsky and Szulga, 1989). Hence the convergence to $Y_n''(h)$ is unconditional. This will enable us to rewrite this sum with an arbitrary deterministic change of order.

With (2.7), (2.8) and a change of variable in (2.10), we have the following.

Proposition 2.1. *For any $H \geq 0$ and any $n > 0$,*

$$
(n\hat{\gamma}_n(h), h = 0, 1, \ldots, H) \stackrel{d}{=} \left(Y_n'(h) + Y_n''(h), h = 0, 1, \ldots, H\right),
$$

with

$$
\begin{aligned}
&\left\{Y_n'(h), h = 0, 1, \ldots, H\right\} \stackrel{d}{=} \\
&\quad \left\{ \left(\frac{C_\alpha}{C_{\alpha/2}}\right)^{2/\alpha} \int_{-\infty}^{\infty} \sum_{t=1}^{n} f(t+x) f(t+h+x) \tilde{M}(dx), h = 0, 1, \ldots, H \right\},
\end{aligned} \tag{2.11}
$$

and

$$
Y_n''(h) = C_\alpha^{\alpha/2} \sum_{\substack{1 < i,j < \infty \\ i \neq j}} \epsilon_i \epsilon_j {\Gamma_i}^{-1/\alpha} {\Gamma_j}^{-1/\alpha} \sum_{t=1}^{n} f\left(t+V_i\right) f\left(t+h+V_j\right) q(V_i)^{-1/\alpha} q(V_j)^{-1/\alpha}, \tag{2.12}
$$

and where $\tilde{M}$ is a positive strictly stable random measure on $\mathbb{R}^1$ with index $\alpha/2$ and Lebesgue control measure, $q(x)$ is any density function that is strictly positive on $\mathbb{R}^1$, $\{\epsilon_j\}$, $\{\Gamma_j\}$ and $\{V_j\}$

are independent sequences of random variables defined by (2.3), (2.4) and (2.5), and the constant C_α *is defined by (2.2).*

Remark 2.1. Here we are only interested in the distributions of $Y_n'(h)$ and $Y_n''(h)$ and will not care about the dependence structure between them. As we will see later, $Y_n''(h)$ is dominated asymptotically by $Y_n'(h)$, and Slutsky's Theorem (see e.g. Durrett, 1996) will be used to deduce the limit behavior of $\hat{\gamma}_n(h)$ based on the limit behavior of $Y_n'(h)$.

Remark 2.2. Although the density $q(x)$ appears in (2.12), it is not involved in (2.11). Thus the distribution of $Y_n'(h)$ does not depend on $q(x)$, and it turns out that neither does the distribution of $Y_n''(h)$. This is because $Y_n''(h)$ has the same distribution as the stable integral of $\tilde{f}(x,y)$ on $\mathbb{R}^2$ with respect to the product measure $M \times M$, if we let

$$\tilde{f}(x,y) = \begin{cases} \sum_{t=1}^n f(t+x)f(t+h+y), & \text{if } x \neq y; \\ 0, & \text{if } x = y. \end{cases}$$

Note that, if desired, q could be chosen to depend on n.

3. The Diagonal Part

We begin with several lemmas used in the derivation of the weak limit of Y_n' when normalized by $n^{-2/\alpha}$.

First a notation: $a^{<p>} := |a|^p \operatorname{sign}(a)$.

Lemma 3.1. *If* $0 < \beta < 1$, *then for any real number* a, b *and* c,

$$\left| (a+b)^{<\beta>} - (a+c)^{<\beta>} \right| \leq 2\left(|b|^\beta + |c|^\beta \right)$$

Proof. If $(a+b)(a+c) \geq 0$, then the triangle inequality gives

$$\left| (a+b)^{<\beta>} - (a+c)^{<\beta>} \right| \leq |(a+b)-(a+c)|^\beta \leq \left(|b|^\beta + |c|^\beta \right)$$

If $(a+b)(a+c) < 0$, then either $a(a+b) \leq 0$ or $a(a+c) \leq 0$. Without loss of generality, assume $a(a+b) \leq 0$, which means $ab \leq 0$ and $|a| \leq |b|$, thus

$$\left| (a+b)^{<\beta>} - (a+c)^{<\beta>} \right| = |a+b|^\beta + |a+c|^\beta \leq |b|^\beta + \left(|a|^\beta + |c|^\beta \right) \leq 2|b|^\beta + |c|^\beta.$$

□

Lemma 3.2. *If* $0 < \beta < 1$ *and* $\phi(x) \in L^\beta(-\infty, \infty)$, *then*

$$\frac{1}{n}\int_{-\infty}^{\infty} \left| \sum_{t=1}^{n} \phi(t+x) \right|^\beta dx \longrightarrow \int_0^1 \left| \sum_{t=-\infty}^{\infty} \phi(t+x) \right|^\beta dx, \tag{3.1}$$

and

$$\frac{1}{n}\int_{-\infty}^{\infty} \left(\sum_{t=1}^{n} \phi(t+x) \right)^{<\beta>} dx \longrightarrow \int_0^1 \left(\sum_{t=-\infty}^{\infty} \phi(t+x) \right)^{<\beta>} dx. \tag{3.2}$$

Proof. First note that $\phi(x) \in L^\beta$ guarantees that all the above integrals are finite. We will only prove (3.2) when n takes just even values. The odd case can be treated exactly the same, and the proof of (3.1) is similar and actually easier.

Let

$$\begin{aligned} A_n &= \frac{1}{2n}\int_{-\infty}^{\infty}\left(\sum_{t=1}^{2n}\phi(t+x)\right)^{<\beta>}dx, \\ B_n &= \frac{1}{2n}\int_{-n}^{n}\left(\sum_{t=-n}^{n-1}\phi(t+x)\right)^{<\beta>}dx, \\ C_n &= \int_0^1\left(\sum_{t=-n}^{n-1}\phi(t+x)\right)^{<\beta>}dx, \\ D &= \int_0^1\left(\sum_{t=-\infty}^{\infty}\phi(t+x)\right)^{<\beta>}dx. \end{aligned}$$

Since

$$\left|\left(\sum_{t=-n}^{n-1}\phi(t+x)\right)^{<\beta>}\right| \le \sum_{t=-\infty}^{\infty}|\phi(t+x)|^\beta$$

and

$$\int_0^1\sum_{t=-\infty}^{\infty}|\phi(t+x)|^\beta dx = \int_{-\infty}^{\infty}|\phi(x)|^\beta dx < \infty,$$

from the dominated convergence theorem,

$$\lim_{n\to\infty} C_n = D. \tag{3.3}$$

Moreover,

$$A_n = \frac{1}{2n}\int_{-\infty}^{\infty}\left(\sum_{t=-n}^{n-1}\phi(t+x)\right)^{<\beta>}dx,$$

and

$$\begin{aligned} |A_n - B_n| &= \frac{1}{2n}\left|\int_{|x|>n}\left(\sum_{t=-n}^{n-1}\phi(t+x)\right)^{<\beta>}dx\right| \\ &\le \frac{1}{2n}\int_{|x|>n}\sum_{t=-n}^{n-1}|\phi(t+x)|^\beta\, dx \\ &= \frac{1}{2n}\left(\sum_{t=-n}^{n-1}\int_{n+t}^{\infty}|\phi(x)|^\beta dx + \sum_{t=-n}^{n-1}\int_{-\infty}^{t-n}|\phi(x)|^\beta dx\right) \\ &= \frac{1}{2n}\left(\sum_{t=1}^{2n}\int_{t-1}^{\infty}|\phi(x)|^\beta dx + \sum_{t=1}^{2n}\int_{-\infty}^{-t}|\phi(x)|^\beta dx\right) \\ &\le \frac{1}{2n}\sum_{t=1}^{2n}\int_{|x|>t-1}|\phi(x)|^\beta dx. \end{aligned}$$

But this is the Cesaro mean of the sequence $\int_{|x|>n} |\phi(x)|^\beta dx$, which goes to zero as n goes to ∞, so

$$\lim_{n\to\infty} |A_n - B_n| = 0. \tag{3.4}$$

Next we estimate the distance between B_n and C_n.

$$\begin{aligned}
|B_n - C_n| &= \left| \frac{1}{2n} \sum_{j=-n}^{n-1} \int_j^{j+1} \left(\sum_{t=-n}^{n-1} \phi(t+x) \right)^{<\beta>} dx - C_n \right| \\
&= \frac{1}{2n} \left| \sum_{j=-n}^{n-1} \int_0^1 \left(\sum_{t=-n}^{n-1} \phi(t+j+x) \right)^{<\beta>} dx - 2nC_n \right| \\
&= \frac{1}{2n} \left| \sum_{j=-n}^{n-1} \int_0^1 \left(\sum_{t=-n+j}^{n+j-1} \phi(t+x) \right)^{<\beta>} dx - \sum_{j=-n}^{n-1} \int_0^1 \left(\sum_{t=-n}^{n-1} \phi(t+x) \right)^{<\beta>} dx \right| \\
&\le \frac{1}{2n} \sum_{j=-n}^{n} \int_0^1 \left| \left(\sum_{t=-n+j}^{n+j-1} \phi(t+x) \right)^{<\beta>} - \left(\sum_{t=-n}^{n-1} \phi(t+x) \right)^{<\beta>} \right| dx.
\end{aligned}$$

Applying Lemma 3.1, the above can be bounded by

$$\begin{aligned}
& \frac{1}{n} \sum_{j=1}^{n} \int_0^1 \left(\left| \sum_{t=n}^{n+j-1} \phi(t+x) \right|^\beta + \left| \sum_{t=-n}^{-n+j-1} \phi(t+x) \right|^\beta \right) dx + \\
& \quad \frac{1}{n} \sum_{j=-n}^{-1} \int_0^1 \left(\left| \sum_{t=-n+j}^{-n-1} \phi(t+x) \right|^\beta + \left| \sum_{t=n+j}^{n-1} \phi(t+x) \right|^\beta \right) dx \\
\le \; & \frac{1}{n} \sum_{j=1}^{n} \int_0^1 \left(\sum_{t=n}^{n+j-1} |\phi(t+x)|^\beta + \sum_{t=-n}^{-n+j-1} |\phi(t+x)|^\beta \right) dx + \\
& \quad \frac{1}{n} \sum_{j=-n}^{-1} \int_0^1 \left(\sum_{t=-n+j}^{-n-1} |\phi(t+x)|^\beta + \sum_{t=n+j}^{n-1} |\phi(t+x)|^\beta \right) dx \\
\le \; & \frac{1}{n} \sum_{j=1}^{n} \int_0^1 \left(\sum_{t=n}^{\infty} |\phi(t+x)|^\beta + \sum_{t=-\infty}^{-n+j-1} |\phi(t+x)|^\beta \right) dx + \\
& \quad \frac{1}{n} \sum_{j=-n}^{-1} \int_0^1 \left(\sum_{t=-\infty}^{-n-1} |\phi(t+x)|^\beta + \sum_{t=n+j}^{\infty} |\phi(t+x)|^\beta \right) dx \\
= \; & \int_n^\infty |\phi(x)|^\beta dx + \frac{1}{n} \sum_{j=1}^{n} \int_{-\infty}^{-n+j} |\phi(x)|^\beta dx + \int_{-\infty}^{-n} |\phi(x)|^\beta dx + \frac{1}{n} \sum_{j=-n}^{-1} \int_{n+j}^{\infty} |\phi(x)|^\beta dx \\
= \; & \int_{|x|>n} |\phi(x)|^\beta dx + \frac{1}{n} \sum_{j=0}^{n-1} \int_{|x|>j} |\phi(x)|^\beta dx.
\end{aligned}$$

With the same reasoning as applied to (3.4),

$$\lim_{n\to\infty} |B_n - C_n| = 0. \tag{3.5}$$

From (3.3),(3.4) and (3.5), $A_n \to D$ as $n \to \infty$. □

Proposition 3.3. *Suppose $\tilde{M}$ is a positive strictly stable random measure on $\mathbb{R}^1$ with index $\alpha/2$ and Lebesgue control measure, and*

$$\hat{\gamma}(h) := \left(\frac{C_\alpha}{C_{\alpha/2}}\right)^{2/\alpha} \int_0^1 \sum_{t=-\infty}^{\infty} f(t+x)f(t+x+h)\tilde{M}(dx). \tag{3.6}$$

Then for all $H \geq 0$,

$$\left\{n^{-2/\alpha}Y_n'(h), h=0,1,\dots,H\right\} \Longrightarrow \{\hat{\gamma}(h), h=0,1,\dots,H\},\ \textit{as } n \longrightarrow \infty, \tag{3.7}$$

where "$\Longrightarrow$" denotes weak convergence.

Proof. For any real $\theta_0, \theta_1, \dots, \theta_H$, if we take $\phi(x) = f(x)\sum_{h=0}^H f(x+h)\theta_h$ in Lemma 3.2, then (2.11) shows that both the scale parameter and skewness parameter of the strictly $\alpha/2$-stable random variable $n^{-2/\alpha}\sum_{h=0}^H \theta_h Y_n'(h)$ converge to the corresponding parameters of $\sum_{h=0}^H \theta_h \hat{\gamma}(h)$, which is also a strictly $\alpha/2$-stable random variable. So

$$n^{-2/\alpha}\sum_{h=0}^H \theta_h Y_n'(h) \Longrightarrow \sum_{h=0}^H \theta_h \hat{\gamma}(h), \text{ as } n \longrightarrow \infty. \tag{3.8}$$

Thus (3.7) follows from the Cramér-Wold device (see e.g. Billingsley, 1995). □

4. The Off-Diagonal Part

In this section, we need the following notation:

$$\ln_+ x := \begin{cases} \ln x, & \text{if } x > 1; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 4.1. *(Samorodnitsky and Szulga, 1989, Proposition 5.1) If $\{\epsilon_j\}_{1\leq j<\infty}$ and $\{\Gamma_j\}_{1\leq j<\infty}$ are independent sequences that are defined by (2.3) and (2.4), then*

(a) *there exist constants m_2, C, and $\beta < \alpha$, such that for any $m \geq m_2$ and any identically distributed random variables W_{ij} that are independent of $\{\epsilon_j\}$ and $\{\Gamma_j\}$,*

$$\begin{aligned}
\mathbf{E}\left|\sum_{m<i<j<\infty} \epsilon_i\epsilon_j\Gamma_i^{-1/\alpha}\Gamma_j^{-1/\alpha}W_{ij}\mathbf{1}_{\{|W_{ij}|^\alpha\leq ij\}}\right|^\alpha &\leq C(\mathbf{E}(|W_{ij}|^\alpha(1+\ln_+|W_{ij}|)))^\beta, \\
\mathbf{E}\left|\sum_{m<i<j<\infty} \epsilon_i\epsilon_j\Gamma_i^{-1/\alpha}\Gamma_j^{-1/\alpha}W_{ij}\mathbf{1}_{\{|W_{ij}|^\alpha> ij\}}\right|^\alpha &\leq C\mathbf{E}(|W_{ij}|^\alpha(1+\ln_+^2|W_{ij}|));
\end{aligned}$$

(b) *there exist constants m_1, C, and $\beta < \alpha$, such that for any $m \geq m_1$ and any identically distributed random variables W_j that are independent of $\{\epsilon_j\}$ and $\{\Gamma_j\}$,*

$$\mathbf{E}\left|\sum_{j=m+1}^{\infty} \epsilon_j\Gamma_j^{-1/\alpha}W_j\mathbf{1}_{\{|W_j|^\alpha\leq j\}}\right|^\alpha \leq C(\mathbf{E}|W_j|^\alpha)^\beta,$$

$$\mathbf{E}\left|\sum_{j=m+1}^{\infty}\epsilon_j\Gamma_j^{-1/\alpha}W_i\mathbf{1}_{\{|W_j|^\alpha>j\}}\right|^\alpha \leq C\mathbf{E}(|W_j|^\alpha(1+\ln_+|W_j|)).$$

Lemma 4.2. *Using the notation of §2, define*

$$U_{ij}^{(n)} := n^{-2/\alpha}\sum_{t=1}^{n} f\left(t+V_i\right) f\left(t+h+V_j\right) q\left(V_i\right)^{-1/\alpha} q\left(V_j\right)^{-1/\alpha}. \tag{4.1}$$

Then for all $i \neq j$, $\mathbf{E}\left|U_{ij}^{(n)}\right|^\alpha \to 0$, *as* $n\to\infty$.

Proof.

$$\mathbf{E}\left|U_{ij}^{(n)}\right|^\alpha = n^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\sum_{t=1}^{n} f(t+x)f(t+h+y)\right|^\alpha dxdy.$$

If $\alpha\leq 1$, then from the triangle inequality,

$$\begin{aligned}\mathbf{E}\left|U_{ij}^{(n)}\right|^\alpha &\leq n^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{t=1}^{n}|f(t+x)f(t+h+y)|^\alpha dxdy\\ &= \frac{1}{n}\left(\int_{-\infty}^{\infty}|f(x)|^\alpha dx\right)^2 \longrightarrow 0.\end{aligned}$$

If $\alpha>1$, then from the convexity of $|x|^\alpha$,

$$\begin{aligned}\mathbf{E}\left|U_{ij}^{(n)}\right|^\alpha &= n^{\alpha-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{1}{n}\sum_{t=1}^{n} f(t+x)f(t+h+y)\right|^\alpha dxdy\\ &\leq n^{\alpha-2}\frac{1}{n}\sum_{t=1}^{n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(t+x)f(t+h+y)|^\alpha dxdy\\ &\leq n^{\alpha-2}\left(\int_{-\infty}^{\infty}|f(x)|^\alpha dx\right)^2 \longrightarrow 0.\end{aligned}$$

□

We are now ready to prove that the off-diagonal part $Y_n''(h)$ does not grow as fast as the diagonal part $Y_n'(h)$.

Proposition 4.3. *For all* $h\geq 0$, $n^{-2/\alpha}Y_n''(h)\xrightarrow{P} 0$.

Proof. From (4.1), we write

$$n^{-2/\alpha}Y_n''(h) = C_\alpha^{2/\alpha}\sum_{\substack{1<i,j<\infty\\ i\neq j}}\epsilon_i\epsilon_j\Gamma_i^{-1/\alpha}\Gamma_j^{-1/\alpha}U_{ij}^{(n)} = C_\alpha^{2/\alpha}\sum_{\substack{1<i,j<\infty\\ i\neq j}}\tilde{U}_{ij}^{(n)}, \tag{4.2}$$

where

$$\tilde{U}_{ij}^{(n)} := \epsilon_i\epsilon_j\Gamma_i^{-1/\alpha}\Gamma_j^{-1/\alpha}U_{ij}^{(n)}. \tag{4.3}$$

Due to symmetry and the unconditional convergence of the series in (4.2) (cf. comments before Proposition 2.1), it is enough to show $\sum_{i<j}\tilde{U}_{ij}^{(n)}\to 0$. For m_1 and m_2 specified by Lemma 4.1, we

can always assume $m_1 > m_2$. Since

$$\begin{aligned}\sum_{i<j} \tilde{U}_{ij}^{(n)} &= \sum_{i=1}^{m_2}\sum_{j=i+1}^{\infty} \tilde{U}_{ij}^{(n)} + \sum_{m_2<i<j<\infty} \tilde{U}_{ij}^{(n)} \\ &= \sum_{i=1}^{m_2}\sum_{j=i+1}^{m_1} \tilde{U}_{ij}^{(n)} + \sum_{i=1}^{m_2}\sum_{j=m_1+1}^{\infty} \tilde{U}_{ij}^{(n)} + \sum_{m_2<i<j<\infty} \tilde{U}_{ij}^{(n)},\end{aligned}$$

we need only prove

(i) $\tilde{U}_{ij}^{(n)} \xrightarrow{P} 0$ for all i, j;

(ii) $\sum_{j=m_1+1}^{\infty} \tilde{U}_{ij}^{(n)} \to 0$ in L^α for all i;

(iii) $\sum_{m_2<i<j<\infty} \tilde{U}_{ij}^{(n)} \to 0$ in L^α.

From Lemma 4.2, $U_{ij}^{(n)} \to 0$ in L^α, thus in probability, so $\tilde{U}_{ij}^{(n)} \to 0$ in probability, yielding (i).

To prove (ii), we observe that $\sum_{j=m_1+1}^{\infty} \tilde{U}_{ij}^{(n)} = \epsilon_i \Gamma_i^{-1/\alpha} \sum_{j=m_1+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} U_{ij}^{(n)}$. Because of Lemma 4.1, it will be enough to prove $\mathbf{E}\left|U_{ij}^{(n)}\right|^\alpha (1 + \ln_+ \left|U_{ij}^{(n)}\right|) \to 0$.

For (iii), Lemma 4.1 says that $\mathbf{E}\left|U_{ij}^{(n)}\right|^\alpha (1 + \ln_+^k |U_{ij}^{(n)}|) \to 0, k = 1, 2$ will suffice.

With the help of Lemma 4.2, though, all of (i), (ii) and (iii) will follow if $\ln_+ \left|U_{ij}^{(n)}\right|$ are uniformly bounded, that is, for any fixed $h \geq 0$, the functions

$$B_h^{(n)}(x, y) := n^{-2/\alpha} \sum_{t=1}^{n} f(t+x) f(t+h+y) q(x)^{-1/\alpha} q(y)^{-1/\alpha}$$

are bounded uniformly in $(x, y) \in \mathbb{R}^2$ and $n \geq 1$.

Now recall that for each n the density $q(x)$ can be chosen arbitrarily without affecting the distribution of Y_n'' (cf. Remark 2.2). To suit our need, let $q(x) = Q(x) / \int_{-\infty}^{\infty} Q(u) du$ with

$$Q(x) := \max\left\{ q_0(x), \left(\sum_{t=1}^{n+h} f(x+t)^2 \right)^{\alpha/2} \right\},$$

where $q_0(x)$ is any density that is strictly positive on $(-\infty, \infty)$. With this choice, by the Cauchy-Schwartz Inequality,

$$\begin{aligned}\left|B_h^{(n)}(x,y)\right| &= n^{-2/\alpha} \left| \sum_{t=1}^{n} f(t+x) f(t+h+y) \right| (Q(x)Q(y))^{-1/\alpha} \left(\int_{-\infty}^{\infty} Q(u) du \right)^{2/\alpha} \\ &\leq n^{-2/\alpha} \left(\sum_{t=1}^{n} f(t+x)^2 \sum_{t=1}^{n} f(t+h+y)^2 \right)^{1/2} \\ &\qquad (Q(x)Q(y))^{-1/\alpha} \left(\int_{-\infty}^{\infty} \left(q_0(u) + \left(\sum_{t=1}^{n+h} f(u+t)^2 \right)^{\alpha/2} \right) du \right)^{2/\alpha} \\ &\leq n^{-2/\alpha} \left(Q(x)^{2/\alpha} Q(y)^{2/\alpha} \right)^{1/2} (Q(x)Q(y))^{-1/\alpha} \left(1 + \int_{-\infty}^{\infty} \sum_{t=1}^{n+h} |f(u+t)|^\alpha du \right)^{2/\alpha}\end{aligned}$$

$$= \quad n^{-2/\alpha}\left(1+(n+h)\int_{-\infty}^{\infty}|f(u)|^{\alpha}du\right)^{2/\alpha},$$

which only depends on n and has a finite limit, thus uniformly bounded. □

5. Sample Correlation Functions

Proposition 4.3 and Proposition 3.3 have established the asymptotic dominance of the diagonal part over the off-diagonal part. Together with Proposition 2.1, they yield the following theorem.

Theorem 5.1. *Let X_t, $\hat{\gamma}_n(h)$ and $\hat{\rho}_n(h)$ be defined by (1.3),(1.2) and (1.1). For all $H \geq 0$,*

$$\left(n^{1-2/\alpha}\hat{\gamma}_n(h), h=0,1,\dots,H\right) \Longrightarrow (\hat{\gamma}(h), h=0,1,\dots,H)$$

and $\hat{\rho}_n(h) \Rightarrow \hat{\rho}(h)$, as $n \to \infty$, where $\hat{\gamma}(h)$ is defined by (3.6) and $\hat{\rho}(h) = \hat{\gamma}(h)/\hat{\gamma}(0)$.

What Theorem 5.1 indicates is that $\hat{\rho}_n(h)$ usually has a random weak limit. The following corollary specifies when this limit is nonrandom.

Corollary 5.2. *For $\hat{\rho}_n(h)$ to have a constant limit, it is necessary and sufficient that there exists a constant ρ, such that $\sum_{t=-\infty}^{\infty} f(x+t)f(x+t+h) = \rho\sum_{t=-\infty}^{\infty} f(x+t)^2$ almost everywhere in $[0,1)$. In this case, $\hat{\rho}_n(h) \xrightarrow{P} \hat{\rho}(h) = \rho$.*

Proof. Sufficiency follows from Theorem 5.1, using the definition (3.6).

Conversely, suppose the distribution of $\hat{\rho}(h)$ concentrates on one point ρ. Then $\hat{\gamma}(h)/\hat{\gamma}(0) = \rho$ and

$$\begin{aligned} 0 &= \hat{\gamma}(h) - \rho\hat{\gamma}(0) \\ &= \left(\frac{C_\alpha}{C_{\alpha/2}}\right)^{2/\alpha}\int_0^1\left(\sum_{t=-\infty}^{\infty} f(x+t)f(x+t+h) - \rho\sum_{t=-\infty}^{\infty} f(x+t)^2\right)\tilde{M}(dx). \end{aligned}$$

But the right hand side is a stable random variable, and it is zero only if its scale parameter is zero (Samorodnitsky and Taqqu, 1994, page 5). Hence

$$0 = \int_0^1\left|\sum_{t=-\infty}^{\infty} f(x+t)f(x+t+h) - \rho\sum_{t=-\infty}^{\infty} f(x+t)^2\right|^{\alpha/2} dx,$$

and

$$\sum_{t=-\infty}^{\infty} f(x+t)f(x+t+h) = \rho\sum_{t=-\infty}^{\infty} f(x+t)^2 \text{ almost everywhere.}$$

□

Before we present some examples that illustrate Corollary 5.2, we define for all $f \in L^\alpha(-\infty,\infty)$ the following periodic function

$$g_h(f,x) = \sum_{t=-\infty}^{\infty} f(x+t)f(x+t+h), \tag{5.1}$$

usually abbreviated as $g_h(x)$ when there is no ambiguity. With this notation, what Corollary 5.2 says is that $\hat{\rho}_n(h)$ has a nonrandom limit ρ if and only if $g_h(x) = \rho g_0(x)$ almost everywhere.

Example 5.1. Suppose $f(x) = \sum_{k=-\infty}^{\infty} c_k \mathbf{1}_{[0,1)}(x-k)$, with $\{c_k\} \in L^\alpha(\mathbb{Z}) =: l^\alpha$. In this case, $g_h(x) = \sum_{t=-\infty}^{\infty} c_t c_{t+h}$ are constants, therefore $\hat{\rho}_n(h)$ have degenerate limits $g_h(x)/g_0(x)$. Actually, if we let $Z_{-k} = M([k, k+1))$, then $\{Z_k\}$ are iid stable (thus with regularly varying tails) with index α and $X_t = \sum_{k=-\infty}^{\infty} c_k Z_{t-k}$ is a traditional moving average process MA(∞) (see Davis and Resnick, 1985).

Example 5.2. Set $f(x) = \sum_{i=1}^{m} \sum_{k=-\infty}^{\infty} c_k^{(i)} \mathbf{1}_{A_i}(x-k)$, where $\left\{c_k^{(i)}, k \in \mathbb{Z}\right\} \in l^\alpha, i = 1, 2, \ldots, m$, and $A_1, A_2, \ldots, A_m$ are Borel sets with $\cup_{i=1}^m A_i = [0,1)$, $A_{i'} \cap A_{i''} = \emptyset$ if $i' \neq i''$. This time $g_h(x) = \sum_{i=1}^{m} \sum_{k=-\infty}^{\infty} c_k^{(i)} c_{k+h}^{(i)} \mathbf{1}_{A_i}(x)$, and

$$\hat{\rho}_n(h) \Longrightarrow \frac{\int_{-\infty}^{\infty} g_h(x) \tilde{M}(dx)}{\int_{-\infty}^{\infty} g_0(x) \tilde{M}(dx)} = \frac{\sum_{i=1}^{m} \xi_i \sum_{k=-\infty}^{\infty} c_k^{(i)} c_{k+h}^{(i)}}{\sum_{i=1}^{m} \xi_i \sum_{k=-\infty}^{\infty} \left(c_k^{(i)}\right)^2},$$

where $\xi^{(i)} = \tilde{M}(A_i)$ are positive strictly stable random variables with index $\alpha/2$. This limit is usually random unless $\sum c_k^{(i)} c_{k+h}^{(i)} / \sum \left(c_k^{(i)}\right)^2$ does not depend on i. If we let $Z_{-k}^{(i)} = M(A_i + k)$, then $\left\{Z_k^{(i)}\right\}$ are independent sequences of iid stable random variables with index α, and $X_t = \sum_{i=1}^{m} \sum_{k=-\infty}^{\infty} c_k^{(i)} Z_{t-k}^{(i)}$ is a sum of m independent moving average processes (see Cohen et al., 1997).

Besides the MA(∞) process in Example 5.1, are there any other stable moving average processes with the same property that the limits $\hat{\rho}(h)$ are degenerate for all lags h? We will see from examples later that the answer is yes. However, we have the following conditions which guarantee that the process must be a finite order classical moving average.

Corollary 5.3. *Suppose $g_h(f,x)$ is defined by (5.1), and*

(i) *there exists $q > 0$ such that $f(x) = 0$ whenever $x < 0$ or $x > q+1$;*
(ii) *$f(x)$ is continuous on $(k, k+1)$ for all $k \in \mathbb{Z}$;*
(iii) *$g_0(f,x) > 0$ for all $x \in (0,1)$;*
(iv) *there exist constants $\rho_h, h \geq 0$, such that $\hat{\rho}(h) = \rho_h$ almost surely.*

Then there exist constants $c_0, c_1, \ldots, c_q$ and a sequence of iid $S\alpha S$ random variables $\{Z_k\}_{-\infty<k<\infty}$, such that $X_t = \sum_{k=0}^{q} c_k Z_{t-k}, t = 1, 2, \ldots$.

Proof. Let $g(x) = (g_0(f,x))^{1/2}$. Then for any $k \in \mathbb{Z}$, we have on $(k, k+1)$ that $g(x) > 0$ and is continuous, thanks to assumptions (i), (ii) and (iii). So we can define on $\mathbb{R}\backslash\mathbb{Z}$ a function $\tilde{f}(x) = f(x)/g(x)$, which is also continuous on $(k, k+1)$ for all $k \in \mathbb{Z}$, and satisfies for all $x \in (0,1)$ and $h \geq 0$

$$\sum_{t=0}^{q-h} \tilde{f}(x+t)\tilde{f}(x+t+h) = \sum_{t=-\infty}^{\infty} \frac{f(x+t)f(x+t+h)}{g_0(f,x)} = \frac{g_h(f,x)}{g_0(f,x)} = \rho_h, \tag{5.2}$$

where the infinite sum has actually only $q+1-h$ nonzero terms, since f has a compact support.

We now proceed by introducing the following polynomials. Let

$$F_x(z) := \sum_{k=0}^{q} \tilde{f}(x+k) z^k$$

and

$$H(z) := z^q + \sum_{k=1}^{q} \rho_k \left(z^{q-k} + z^{q+k} \right)$$

$$= \rho_q + \rho_{q-1} z + \cdots + \rho_1 z^{q-1} + z^q + \rho_1 z^{q+1} + \rho_2 z^{q+2} + \cdots + \rho_q z^{2q}. \tag{5.3}$$

By (5.2)

$$H(z) = \left(\sum_{k=0}^{q} \tilde{f}(x+k) z^{q-k} \right) \left(\sum_{k=0}^{q} \tilde{f}(x+k) z^k \right)$$

$$= z^q F_x\left(z^{-1}\right) F_x(z). \tag{5.4}$$

For each $x \in (0,1)$, there exist $K \in \mathbb{C}$, $r \in \{0,1,\ldots,q\}$, and $b_i \in \mathbb{C}\backslash\{0\}$, $i = 1,2,\ldots,r$, such that

$$F_x(z) = K z^{q-r} \prod_{i=1}^{r} (z - b_i). \tag{5.5}$$

Substitute (5.5) into (5.4), we have

$$H(z) = z^q K^2 \prod_{i=1}^{r} \left(z^{-1} - b_i \right) (z - b_i)$$

$$= z^{q-r} \left(K^2 \prod_{i=1}^{r} (-b_i) \right) \left(\prod_{i=1}^{r} \left(z - b_i^{-1} \right) (z - b_i) \right). \tag{5.6}$$

Comparing (5.3) with (5.6), we observe

(a) $r = \max\{h : \rho_h \neq 0, 0 \leq h \leq q\}$ is completely determined by the ρ's and doesn't depend on x.
(b) $b_1, b_1^{-1}, b_2, b_2^{-1}, \ldots, b_r, b_r^{-1}$ are all the non-zero roots of $H(z)$. So given $H(z)$, there are at most 2^r possible choices for the set $\{b_1, b_2, \ldots, b_r\}$ (the number of choices can be less than 2^r if $H(z)$ has repeated non-zero roots).
(c) $K^2 \prod_{i=1}^{r}(-b_i) = \rho_r$. So given $\{b_1, b_2, \ldots, b_r\}$, there are at most 2 choices for K.

To sum up, given $H(z)$, there are at most $2 \cdot 2^r$ polynomials $F_x(z)$ that satisfy (5.4). Consequently, for each k, $\tilde{f}(x+k)$ can take at most $2 \cdot 2^r$ possible values. Therefore from the continuity assumption, for every fixed k, $\tilde{f}(x+k)$ has to be a constant for all $x \in (0,1)$. Call this constant c_k, and we have $f(x+k) = c_k g(x)$ for all $x \in (0,1)$. If $Z_{-k} = \int_k^{k+1} g(x) M(dx)$, then $\{Z_k\}$ are iid and

$$X_t = \int_{-\infty}^{\infty} f(x+t) M(dx) = \sum_{k=-\infty}^{\infty} c_{k+t} \int_k^{k+1} g(x) M(dx) = \sum_{k=0}^{q} c_k Z_{t-k},$$

since $c_k = 0$ when $k < 0$ or $k > q$. □

The following examples indicate that assumptions (ii) and (iii) are necessary in Corollary 5.3.

Example 5.3. In the setting of Example 5.2, let $m = 2$, $A_1 = [0, 1/2)$, $A_2 = [1/2, 1)$, and

$c'_k = c''_k = 0$, if $k < 0$ or $k > 2$,
$c'_0 = 2, c'_1 = 9, c'_2 = 4$,
$c''_0 = 1, c''_1 = 6, c''_2 = 8$,

where c'_k denotes $c_k^{(1)}$ and c''_k denotes $c_k^{(2)}$ for every $k \in \mathbb{Z}$. In this case, (ii) of Corollary 5.3 fails. But since $g_0(x) = 101$, $g_1(x) = 54$, $g_2(x) = 8$ are all constants and $g_k(x) = 0$ if $k < 0$ or $k > 2$, $\hat{\rho}(h)$ is nonrandom for all h. However, this process is not a classical finite moving average.

Example 5.4. The process of Example 5.3 has, up to a multiplicative constant, another representation. In the notation of that example, let

$$f(x) = \begin{cases} c'_{[x]}\{x\}, & \text{if } \{x\} \le 0 \\ c''_{[x]}\{x\}, & \text{if } \{x\} > 0 \end{cases}$$

where $[x] := \max(\mathbb{Z} \cap (0, x])$, $\{x\} := x - [x] - 1/2$, and c'_k and c''_k are defined in Example 5.3. Here f is continuous on $(k, k+1)$ for all $k \in \mathbb{Z}$, but (iii) of Corollary 5.3 fails as $g_h(f, 1/2) = 0$.

The next example shows that without assumption (i) in Corollary 5.3, assumptions (ii), (iii) and (iv) are not enough to guarantee that the process is a classical moving average of finite or infinite order.

Example 5.5. Let $\phi : (0,1) \mapsto (0,1)$ be any continuous function. For all $x \in (0,1)$, the function $F_x(z) := \exp\left(\phi(x)\left(z - z^{-1}\right)\right)$ is analytic on $\{z : 0 < |z| < \infty\}$, thus has Laurent expansion (see e.g. Ahlfors, 1979)

$$F_x(z) = \sum_{k=-\infty}^{\infty} a_k(x) z^k, \tag{5.7}$$

where

$$a_k(x) = \frac{1}{2\pi i} \int_{|z|=1} \frac{F_x(z)}{z^k} dz = \begin{cases} \sum_{j=0}^{\infty} \frac{(-1)^j \phi(x)^{2j+k}}{j!(j+k)!}, & \text{if } k \ge 0; \\ (-1)^k a_{|k|}(x), & \text{if } k < 0. \end{cases}$$

Let $f(x+k) = a_k(x)$ for all $x \in (0,1)$ and $k \in \mathbb{Z}$. Then $f(x)$ satisfies (ii) and (iii) of Corollary 5.3, and is in $L^\alpha(\mathbb{R})$, since

$$\begin{aligned} \int_{-\infty}^{\infty} |f(x)|^\alpha dx &= \sum_{k=-\infty}^{\infty} \int_0^1 |a_k(x)|^\alpha \, dx \\ &\le \sum_{k=-\infty}^{\infty} \int_0^1 \left(\sum_{j=0}^{\infty} \frac{\phi(x)^{2j+|k|}}{j!(j+|k|)!} \right)^\alpha dx \\ &\le \sum_{k=-\infty}^{\infty} \left(\sum_{j=0}^{\infty} \frac{1}{j!(j+|k|)!} \right)^\alpha \quad < \quad \infty. \end{aligned}$$

Moreover, for all $x \in (0,1)$, the Laurent series $F_x(z^{-1}) = \sum_{k=-\infty}^{\infty} a_k(x) z^{-k}$ and (5.7) both converge absolutely on $\{z : 0 < |z| < \infty\}$, so

$$\begin{aligned} 1 &= F_x(z^{-1}) F_x(z) \\ &= \left(\sum_{k=-\infty}^{\infty} a_k(x) z^{-k} \right) \left(\sum_{k=-\infty}^{\infty} a_k(x) z^k \right) \\ &= \sum_{h=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_k(x) a_{k+h}(x) z^h \\ &= \sum_{h=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} f(x+k) f(x+k+h) z^h \end{aligned}$$

$$= \sum_{h=-\infty}^{\infty} g_h(f,x) z^h.$$

From the uniqueness of the Laurent series of the constant function 1, we have $g_0(f,x) = 1$ and $g_h(f,x) = 0$ for all $h > 0$. So $\hat{\rho}(0) = 1$ almost surely, and for all $h > 0$, $\hat{\rho}(h) = 0$ almost surely. However, $\{X_t\}$ is rarely a classical moving average process (not, e.g., if $\phi(x) = x$, since the spectral measure of (X_1, X_2) is not discrete).

Remark 5.1. This example shows that one classical method of testing whether data comes from an iid model, namely testing if $\hat{\rho}(h) \approx 0$, $h > 0$, is extremely unreliable. The process in Example 5.5 is far from iid.

Our final result considers special cases of Example 5.2. It is significant because it shows the variety of the asymptotic behavior of acfs, which seriously questions the viability of the sample correlation function as an appropriate tool for statistical estimation or model fitting of heavy tailed time series models.

Proposition 5.4. *Under the setting of Example 5.2 with $m = 2$, let $\mathbb{N}$ be the set of positive integers and A,B subsets of $\mathbb{N}$. If A and B satisfy any one of the following three conditions, then we can choose c'_k and c''_k, such that $\hat{\rho}(h)$ is degenerate when $h \in A$ and random when $h \in B$.*

(i) $A = \{H, H+1, \dots\}, B = \mathbb{N} \backslash A, H \in \mathbb{N}$.
(ii) $B = \{H, H+1, \dots\}, A = \mathbb{N} \backslash B, H \in \mathbb{N}$.
(iii) *A,B are finite and* $A \cap B = \emptyset$.

Proof. As we have seen in Example 5.2, $\hat{\rho}(h)$ is degenerate if and only if

$$\frac{\sum_{k=-\infty}^{\infty} c'_k c'_{k+h}}{\sum_{k=-\infty}^{\infty} {c'_k}^2} = \frac{\sum_{k=-\infty}^{\infty} c''_k c''_{k+h}}{\sum_{k=-\infty}^{\infty} {c''_k}^2}, \tag{5.8}$$

where c'_k denotes $c_k^{(1)}$ and c''_k denotes $c_k^{(2)}$ for every $k \in \mathbb{Z}$. There are many ways to choose c'_k and c''_k to make (5.8) hold when $h \in A$ and fail when $h \in B$. We will show just one example.

(i) If $c'_k = c''_k = 0$ whenever $k > H$ or $k \le 0$, then (5.8) holds for all $h \in A$. Most choices of c'_k and c''_k will fail (5.8) when $h \in B$.
(ii) Let $c'_k = c''_k \neq 0$ if $k = 0, -1, -2, \dots, -H+1$, and $c'_0 \neq -5/3$, $c'_0 \neq 5$;
$c'_H = 2, c''_H = 3$;
$c'_{kH} = 2^{3-k}, c''_{kH} = 2^{1-k}$, if $k = 2, 3, \dots$;
$c'_k = c''_k = 0$, otherwise.
With these c'_k and c''_k,

$$\sum_{k=-\infty}^{\infty} {c'_k}^2 = \sum_{k=1-H}^{0} {c'_k}^2 + 4 + \sum_{k=2}^{\infty} 2^{6-2k} = \sum_{k=1-H}^{0} {c''_k}^2 + 9 + \sum_{k=2}^{\infty} 2^{2-2k} = \sum_{k=-\infty}^{\infty} {c''_k}^2.$$

If $h < H$,

$$\sum_{k=-\infty}^{\infty} c'_k c'_{k+h} = \sum_{k=1-H}^{-h} c'_k c'_{k+h} = \sum_{k=1-H}^{-h} c''_k c''_{k+h} = \sum_{k=-\infty}^{\infty} c''_k c''_{k+h}.$$

If $h = k_1 H + k_2$, $k_1 \ge 1$, $k_2 = 1, 2, \dots, H-1$,

$$\sum_{k=-\infty}^{\infty} c'_k c'_{k+h} = c'_{-k_2} c'_{k_1 H} \neq c''_{-k_2} c''_{k_1 H} = \sum_{k=-\infty}^{\infty} c''_k c''_{k+h},$$

since $c'_{-k_2} = c''_{-k_2} \neq 0$ and $c'_{k_1 H} \neq c''_{k_1 H}$.

If $h = k_1 H$, $k_1 \geq 1$, it can be similarly checked that $\sum_{k=-\infty}^{\infty} c'_k c'_{k+h} \neq \sum_{k=-\infty}^{\infty} c''_k c''_{k+h}$.

(iii) Suppose $l = \max(A \cup B)$. Pick c'_k, such that $c'_k = 0$ if $k < 0$ or $k > l$ and ${c'_0}^2 + {c'_l}^2 > 0$. Define $a'_h = \sum_{t=0}^{l-h} c'_t c'_{t+h}$ and

$$a''_h = \begin{cases} a'_h + \epsilon, & \text{if } h \in B \\ a'_h, & \text{otherwise} \end{cases} \tag{5.9}$$

where ϵ awaits to be decided. Let $A' = \left[a'_{|j-k|}\right]_{j,k=0}^{l}$ and $A'' = \left[a''_{|j-k|}\right]_{j,k=0}^{l}$ be two $(l+1) \times (l+1)$ matrices. Linear algebra shows that A' is positive definite. Since all the main subdeterminants of A'' are continuous functions of ϵ, we can find an $\epsilon \neq 0$ to keep A'' positive definite. This achieved, there must exist $c''_0, c''_1, \ldots, c''_l$ such that $a''_h = \sum_{t=0}^{l-H} c''_t c''_{t+h}, h = 0, 1, \ldots, l$. The last assertion can be proved via linear algebra or through a probability approach (see Brockwell and Davis, 1991, Theorem 1.5.1, Proposition 3.2.1, Theorem 3.2.1).

With c'_k and c''_k chosen this way, (5.8) becomes

$$\frac{a'_h}{a'_0} = \frac{a''_h}{a''_0}. \tag{5.10}$$

From (5.9), we have that (5.10) fails if and only if $h \in B$.

□

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