The Maximum of the Periodogram for a Heavy-Tailed Sequence

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October 20, 1997

ABSTRACT We consider the maximum of the periodogram based on an infinite variance heavy-tailed sequence. For $\alpha < 1$ we show that the maxima constitute a weakly convergent sequence, and find its limiting distribution. For $1 \leq \alpha < 2$ we show that the sequence of the maxima is not tight, and find a normalization that makes it tight.

*This research was supported in part by NSF Grant DMS-97-04982 at Cornell University.

AMS 1991 Subject Classification: Primary: 62M15 Secondary: 60F05 60G10 60G55

Key Words and Phrases. Periodogram, discrete Fourier transform, stable random variable, stable process, stochastic integral, infinite variance, linear process, point process convergence
1 Introduction

For a sequence $X = (X_t, t \geq 1)$ of random variables we consider its periodogram defined by

$$I_{n,X}(x) = |J_{n,X}(x)|^2 = \left| b_n^{-1} \sum_{t=1}^n X_te^{-i2\pi xt} \right|^2$$

$$= \left( b_n^{-1} \sum_{t=1}^n X_t \cos(2\pi xt) \right)^2 + \left( b_n^{-1} \sum_{t=1}^n X_t \sin(2\pi xt) \right)^2, \quad x \in [0,0.5],$$

where $(b_n)$ is an appropriate norming sequence to be specified later. We are interested in the limit behavior of the sequences

$$M_{n,X} = \max_{x \in [0,0.5]} I_{n,X}(x) \quad \text{and} \quad \widetilde{M}_{n,X} = \max_{j=1,\ldots,q} I_{n,X}(\omega_j),$$

for a stationary infinite variance sequence $X$. Here

$$2\pi \omega_j = 2\pi j/n, \quad j = 1,\ldots,q, \quad q = q_n = \max\{j : 1 \leq j < n/2\},$$

denote the Fourier frequencies in $(0, \pi)$.

First we recall some results for finite and infinite variance sequences on the limit behaviour of $I_{n,X}$, $M_{n,X}$ and $\widetilde{M}_{n,X}$. We commence with an iid sequence which we always denote by $Z = (Z_t)$. Assume for the moment that $EZ_1 = 0$ and $EZ_1^2 = 1$, and choose $b_n = n^{1/2}$. If the $Z_t$’s are iid $N(0,1)$, then the vector of the periodogram ordinates

$$I_{n,Z}(\omega_j), \quad j = 1,\ldots,q,$$

constitutes an iid standard exponential sequence. Hence the extreme value theory for these vectors is the well-known theory for the extremes of an iid exponential sequence. In particular,

$$\widetilde{M}_{n,Z} - \ln q \Rightarrow Y,$$

where \( \Rightarrow \) denotes weak convergence and $Y$ has the standard Gumbel distribution $P[Y \leq x] = \exp\{-\exp\{-x\}\}, \ x \in \mathbb{R}$. For iid $N(0,1)$ random variables $Z_t$, Turkman and Walker (1984) showed that an analogue of (1.4) holds for $M_{n,Z}$ but the centering constants have to be slightly modified.

The quantity $\widetilde{M}_{n,Z}$ has been used in time series analysis for a long time; see Section 6.1.4 in Priestley (1981) who gives a historical account. The normalized statistic $\widetilde{M}_{n,Z}$, or its studentised version

$$g = \frac{\widetilde{M}_{n,Z}}{\sum_{j=1}^q I_{n,Z}(\omega_j)},$$

was used to construct asymptotic or exact tests based on the maximum of the periodogram for the null hypothesis that the $Z_t$’s are iid $N(0,1)$. The perhaps best known result in this context was
proved by Fisher (1929). He calculated the exact distribution of \( g \) for iid \( N(0,1) \) \( Z_t \)'s. Therefore the expression \( g \) is known as \textit{Fisher's} \( g \)-\textit{statistic}. If the \( Z_t \)'s are not Gaussian, an exact result for the distribution of \( g \) is difficult to obtain and therefore an asymptotic result such as (1.4) seems more appropriate. Davis and Mikosch (1997) showed that relation (1.4) remains valid if the \( Z_t \)'s are iid non-Gaussian with \( EZ_1 = 0, EZ_1^2 = 1 \) and \( E|Z_1|^{2+\delta} < \infty \) for some \( \delta > 0 \). Thus the asymptotic relation (1.4) may serve as the theoretical basis for a large sample test based on the maximum of the periodogram for the null hypothesis that the \( Z_t \)'s are iid with finite variance.

We note that the periodogram ordinates (1.3) have many properties in common with an iid standard exponential sequence, provided the second moment of the \( Z_t \)'s is finite. This claim is supported by relation (1.4), but also by various other results. For example, Freedman and Lane (1980) proved that the empirical distribution function, constructed from (1.3), converges uniformly in probability to the exponential distribution function. Chen and Hannan (1980) extended this result from convergence in probability to convergence a.s. See Davis and Mikosch (1997) for more references to this phenomenon.

It is the aim of this paper to study the maximum of the periodogram for sequences of heavy-tailed random variables. In Section 2 we assume that \((Z_t, t \in \mathbb{Z})\) is a sequence of iid random variables whose common distribution \( F \) is in the domain of attraction of an \( \alpha \)-stable random variable where \( 0 < \alpha < 1 \). (The definition of an \( \alpha \)-stable random variable is given below.) This means there exist \( p, q \geq 0 \) with \( p + q = 1 \) and a slowly varying function \( L(x) \), such that

\[
\lim_{x \to \infty} \frac{P[Z_1 > x]}{P[|Z_1| > x]} = p, \quad \lim_{x \to \infty} \frac{P[Z_1 \leq -x]}{P[|Z_1| > x]} = q
\]

(1.6)

and

\[
P[|Z_1| > x] \sim x^{-\alpha} L(x), \quad x \to \infty.
\]

(1.7)

The normalizing sequence \( (b_n) \) is defined via the quantile function of the distribution of \( |Z_1| \), namely

\[
b_n = \left( \frac{1}{P[|Z_1| > \cdot]} \right)^{\uparrow} (n).
\]

In Section 3 we will assume \((Z_t)\) is an iid sequence of infinite variance symmetric \( \alpha \)-stable \((S\alpha S)\) random variables, \( 1 \leq \alpha < 2 \). Recall that a random variable \( Y_\alpha \) is said to have a stable distribution \((Y_\alpha \sim S_\alpha(\sigma, \beta, \mu))\) if there are parameters \( 0 < \alpha \leq 2, \sigma \geq 0, -1 \leq \beta \leq 1, \) and \( \mu \) real such that its characteristic function has the form

\[
Ee^{itY_\alpha} = \begin{cases} 
\exp \{i\mu - \sigma^\alpha |t|^\alpha (1 - i\beta \text{sign}(t) \tan(\pi\alpha/2))\} & \text{if } \alpha \neq 1, \\
\exp \{i\mu - \sigma |t| (1 + (2i\beta/\pi) \text{sign}(t) \ln |t|)\} & \text{if } \alpha = 1.
\end{cases}
\]

If \( \beta = \mu = 0 \), then \( Y_\alpha \) is \( S\alpha S \). In this case, we set \( b_n = n^{1/\alpha} \) and for convenience we also assume in Section 3 that \( Z_1 \sim S_\alpha(1,0,0) \).
Recall that the heavy-tailed random variables considered in Sections 2 and 3 have infinite variance and, therefore, the limit theory for $I_{nZ}$ and $M_{nZ}$ is totally different from the finite variance case. In particular, there is no asymptotic independence of the periodogram ordinates. An indication of this fact is provided by a result of Freedman and Lane (1981) who showed that the empirical distribution function of the sequence (1.3) converges weakly to a random non-degenerate limit. This is in contrast to the aforementioned result in the finite variance case.

In the SoS case, Klüppelberg and Mikosch (1993) studied convergence of the finite-dimensional distributions of the process $(I_{nZ}(x))_{x \in [0,0.5]}$. In particular, they showed that

$$(I_{nZ}(x_j))_{j=1,...,m} \Rightarrow \left(\alpha^2(x_j) + \beta^2(x_j)\right)_{j=1,...,m}, \quad (1.8)$$

where the $x_j$'s are distinct frequencies in (0,0.5) such that 1, $x_1$, ..., $x_m$ are 
rationally independent.

This means that $r_0 + r_1x_1 + \cdots + r_mx_m = 0$ for rational $r_i$'s implies that the $r_i$'s must be zero. Clearly, the $x_i$'s are then all irrational numbers. The limit vector

$$A_m = (\alpha(x_1), \beta(x_1), \ldots, \alpha(x_m), \beta(x_m))$$

has characteristic function

$$E e^{i(t, A_m)} = \exp \left\{ - \int_{(0,1)^m} \left[ \sum_{j=1}^m (t_{2j-1} \cos(2\pi y_j) + t_{2j} \sin(2\pi y_j)) \right] dy \right\}, \quad t \in \mathbb{R}^{2m}. \quad (1.9)$$

Hence $A_m$ is an SoS random vector in $\mathbb{R}^{2m}$; see Samorodnitsky and Taqqu (1994), Section 2.1.

The limit result (1.8) turns out to be helpful for understanding the asymptotic behaviour of $M_{nZ}$ and $\tilde{M}_{nZ}$. When $\alpha \in (0,1)$ and the $Z_t$'s are SoS, it will be shown that both $(M_{nZ})$ and $(\tilde{M}_{nZ})$ have the distributional limit $\max_{x \in I_0} (\alpha^2(x) + \beta^2(x))$, where $I_0 \subset (0,0.5)$ is a countable set of rationally independent numbers which is dense in [0,0.5]. This limit will also be identified as the weak limit of the sequence $((b_n^{-1} \sum_{i=1}^n |Z_t|^2)^2$, i.e. the square of a positive $\alpha$-stable random variable. The case $\alpha \in [1,2)$ is more complicated. The sequence $(M_{nZ})$ is not tight, and so additional normalization is required. This is not surprising since the case $\alpha \in (1,2]$ may be considered as intermediate between the finite variance case (see (1.4)) and the infinite first moment case.

The asymptotic results for an iid heavy-tailed sequence $(Z_t, t \in \mathbb{Z})$ are the theoretical basis for results on the weak limit behaviour of $(M_{nX})$, where $X$ is a linear process given by

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \quad t \in \mathbb{Z}. \quad (1.10)$$
To guarantee the a.s. convergence of the latter series we always assume that the real coefficients $\psi_j$ obey the condition
\[ \sum_{j=-\infty}^{\infty} |\psi_j|^a L(1/|\psi_j|) < \infty. \] (1.11)

We use the slowly varying function $L$ from (1.7) in the setup of Section 2, where $0 < \alpha < 1$, and $L \equiv 1$ in the SoS setup of Section 3. If $\psi_j = 0$ we interpret $|\psi_j|^a L(1/|\psi_j|)$ as zero. Moreover, even in the case $0 < \alpha < 1$ we will assume that
\[ \sum_{j=-\infty}^{\infty} |\psi_j|^a < \infty. \] (1.12)

Note that for certain slowly varying functions $L$ the condition (1.12) is superfluous in light of (1.11).

In what follows, we present the results on the asymptotic behaviour of the maximum of the periodogram both for an iid heavy-tailed sequence $Z$ and the corresponding linear process $X$ defined by (1.10). In Section 2 we give the complete solution to the limit problem in the case $\alpha \in (0, 1)$. In Section 3 we deal with the tightness of the sequence of maxima of the periodogram in the case $\alpha \in [1, 2)$ and the $Z_t$'s are SoS.

## 2 The case $\alpha \in (0, 1)$

In this section, $Z = (Z_t)$ is a sequence of iid heavy-tailed random variables satisfying (1.6) and (1.7) for some $\alpha \in (0, 1)$. In what follows, we consider three independent sequence $(\Gamma_j)$, $(U_j)$ and $(B_j)$, defined on the same probability space. The first one, $(\Gamma_j)$, is the arrival sequence of a unit rate Poisson process on $\mathbb{R}_+$. The random variables $U_j$ are iid $U(0, 1)$, and the $B_j$'s are iid satisfying
\[ P[B_1 = 1] = p, \quad \text{and} \quad P[B_1 = -1] = q, \]
where $p$ and $q$ are defined in (1.6).

Recall the definitions of $M_{n,Z}$ and $\widetilde{M}_{n,Z}$ from (1.2). The following result completely characterises the distributional limit of the maximum of the periodogram $I_{n,Z}$.

**Theorem 2.1** For $\alpha \in (0, 1)$, the limit relations
\[ M_{n,Z} \Rightarrow Y_\alpha^2 \quad \text{and} \quad \widetilde{M}_{n,Z} = \max_{j=1,\ldots,q} I_{n,Z}(\omega_j) \Rightarrow Y_\alpha^2 \] (2.1)
hold, where $Y_\alpha = \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \sim S_\alpha(C_\alpha^{-1/\alpha}, 1, 0)$ with
\[ C_\alpha = \left( \int_{0}^{\infty} x^{-\alpha} \sin x \, dx \right)^{-1}. \] (2.2)

The proof rests on the following proposition. First recall the definition of $J_{n,Z}$ from (1.1).
**Proposition 2.2** For $0 < \alpha < 1$, the limit relation
\[
J_{n,z}(x/n) \Rightarrow J_\infty(x) := \sum_{j=1}^{\infty} B_j \Gamma_j^{-1/\alpha} e^{-2\pi i \alpha u_j},
\]  
(2.3)

as $n \to \infty$, holds in $C[0, \infty)$.

**Proof of Proposition 2.2.** Let $M_p(\mathbb{E})$ be the set of point measures on
\[
\mathbb{E} := [0, 1] \times ([-\infty, \infty] \setminus \{0\}),
\]
topologized by vague convergence; cf. Resnick (1987). A typical element of $M_p(\mathbb{E})$ is represented by
\[
\sum_k \epsilon_{x_k}(\cdot)
\]
where $x_k \in \mathbb{E}$. The basic result we need is from Proposition 3.21 of Resnick (1987)
\[
N_n := \sum_{k=1}^{n} \epsilon_{(k/n, z_k/b_n)} \Rightarrow N := \sum_{j=1}^{\infty} \epsilon_{(u_j, B_j \Gamma_j^{-1/\alpha})}
\]
in $M_p(\mathbb{E})$.

Now pick $\eta > 0$ and define
\[
T_\eta : M_p(\mathbb{E}) \mapsto C[0, \infty)
\]
in the following way. If
\[
m = \sum_j \epsilon_{(t_j, v_j)} \in M_p(\mathbb{E})
\]
and all $v_j$'s are finite, then
\[
(T_\eta m)(x) = \sum_j v_j 1_{|v_j| \geq \eta} e^{-2\pi i \alpha t_j}.
\]
Otherwise, set $(T_\eta m)(x) \equiv 0$.

**Lemma 2.3** The map
\[
T_\eta : M_p(\mathbb{E}) \mapsto C[0, \infty)
\]
is continuous a.s. with respect to the distribution of $N$.

**Proof of Lemma 2.3.** It suffices to show that $x_n \to x \geq 0$ and $m_n \overset{\nu}{\to} m$ in $M_p(\mathbb{E})$, where
\[
m\left(\partial([0, 1] \times \{|v| \geq \eta\}) \cup [0, 1] \times \{-\infty, \infty\}\right) = 0,
\]
($\partial(A)$ is the boundary of any set $A$) implies
\[
(T_\eta m_n)(x_n) \to (T_\eta m)(x).
\]
To do this denote

\[ m_n = \sum_j \varepsilon_{(t_j^n, v_j^n)} \quad \text{and} \quad m = \sum_j \varepsilon_{(t_j, v_j)} . \]

The set

\[ K_\eta := [0, 1] \times \{ v : |v| \geq \eta \} \]

is compact in \( \mathbb{E} \) with \( m(\partial K_\eta) = 0 \). For \( n \geq n_0 \),

\[ m_n(K_\eta) = m(K_\eta) =: l \]

say and there is an enumeration of the points in \( K_\eta \) such that

\[ \left( (t_k^{(n)}, v_k^{(n)}), 1 \leq k \leq l \right) \to ((t_k, v_k), 1 \leq k \leq l) \]

and in fact, without loss of generality we may assume for given \( \xi > 0 \) that

\[ \sup_{n \geq n_0} |x_n| \lor \sup_{k=1, \ldots, l} |v_k^{(n)}| \leq \xi . \]

Therefore

\[
|\langle T_\eta m_n \rangle(x_n) - \langle T_\eta m \rangle(x) | = \left| \sum_{k=1}^l v_k^{(n)} e^{-2\pi ix_k t_k^n} - \sum_{k=1}^l v_k e^{-2\pi ix_k t_k} \right| \\
\leq \sum_{k=1}^l \left| v_k^{(n)} e^{-2\pi ix_k t_k^n} - v_k e^{-2\pi ix_k t_k} \right| .
\]

Now

\[
\left| v_k^{(n)} e^{-2\pi ix_k t_k^n} - v_k e^{-2\pi ix_k t_k} \right| = \left| v_k^{(n)} e^{-2\pi ix_k t_k^n} - v_k e^{-2\pi ix_k t_k^n} + v_k e^{-2\pi ix_k t_k^n} - v_k e^{-2\pi ix_k t_k} \right| \\
\leq \left| v_k^{(n)} - v_k \right| + \left| v_k \right| \left| e^{-2\pi ix_k t_k^n} - e^{-2\pi ix_k t_k} \right| .
\]

Thus

\[
|\langle T_\eta m_n \rangle(x_n) - \langle T_\eta m \rangle(x) | \leq \sum_{k=1}^l \left| v_k^{(n)} - v_k \right| + \sum_{k=1}^l \left| v_k \right| \left| e^{-2\pi ix_k t_k^n} - e^{-2\pi ix_k t_k} \right| ,
\]

so

\[
\lim_{n \to \infty} |\langle T_\eta m_n \rangle(x_n) - \langle T_\eta m \rangle(x) | = 0 .
\]

This completes the proof of Lemma 2.3. \( \Box \)

We now continue with the proof of Proposition 2.2. We apply the functional \( T_\eta \) to both sides in (2.4) to obtain

\[
\sum_{j=1}^n Z_j \frac{Z_j}{B_n} e^{-2\pi iz_j/n} 1_{[Z_j > \eta B_n]} \Rightarrow \sum_{j=1}^\infty B_j \Gamma_j^{1/\alpha} e^{-2\pi iz_j} 1_{[\Gamma_j^{1/\alpha} > \eta]} := J_\infty(x) \tag{2.5}
\]
in $C[0, \infty)$. Also, as $\eta \to 0$ we have

$$J_{\infty}^{(n)}(x) \Rightarrow J_{\infty}(x) := \sum_{j=1}^{\infty} B_j \Gamma_j^{-1/\alpha} e^{-2\pi i x U_j}$$

and so from Theorem 4.2 of (Billingsley, 1968, page 25) it remains to prove for any $\theta > 0$

$$\lim_{\eta \to 0} \limsup_{n \to \infty} P \left[ \|J_{n,z}^{(n)} - J_n z\| > \theta \right] = 0,$$  \hspace{1cm} (2.6)

where $\|x(\cdot) - y(\cdot)\|$ is the $C[0, \infty)$ metric distance between $x, y \in C[0, \infty)$. The method of proof of (2.6) will be amply demonstrated if we show for any $\theta > 0$:

$$\lim_{\eta \to 0} \limsup_{n \to \infty} P \left[ \sup_{0 \leq x \leq 1} \left\| \sum_{j=1}^{n} \frac{Z_j}{b_n} e^{-2\pi i x j/n} 1_{[|Z_j| \leq \eta b_n]} \right\| > \theta \right] = 0.$$ \hspace{1cm} (2.7)

The expression in (2.7) has a bound

$$\lim_{\eta \to 0} \limsup_{n \to \infty} \left[ \sum_{j=1}^{n} \frac{Z_j}{b_n} 1_{[|Z_j| \leq \eta b_n]} > \theta \right] \leq \lim_{\eta \to 0} \limsup_{n \to \infty} n E \left( \left\| \frac{Z_1}{b_n} 1_{[|Z_1| \leq \eta b_n]} \right\|^2 \right) \theta,$$

and applying Karamata’s Theorem in the form given, for example, on p. 579 of Feller (1971) we get the bound

$$\lim_{\eta \to 0} \int_{\{x: |x| \leq \eta\}} |x| \nu(dx) \theta = 0,$$

where

$$\nu(dx) = p\alpha x^{-\alpha-1} dx 1_{[x>0]} + q\alpha |x|^{-\alpha-1} dx 1_{[x<0]}.$$  \hspace{1cm} (2.8)

This completes the proof of Proposition 2.2. \hspace{1cm} \square.

We now continue with the proof of Theorem 2.1. We start with the proof of the first part of (2.1). It is immediate that

$$M_{n,z} \leq \left( b_n^{-1} \sum_{t=1}^{n} |Z_t| \right)^2.$$ \hspace{1cm} (2.8)

It is well known (cf. Feller (1971)) that

$$b_n^{-1} \sum_{t=1}^{n} |Z_t| \Rightarrow Y_\alpha = \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \sim S_\alpha(C_\alpha^{-1/\alpha}, 1, 0).$$ \hspace{1cm} (2.9)

Hence the sequence $(M_{n,z})$ is stochastically bounded and it remains to show the lower bound in the limit for the maximum of the periodogram.

It suffices to show

$$\liminf_{\gamma \to \infty} P[M_{n,z} > \gamma] \geq P[Y_\alpha^2 > \gamma],$$ \hspace{1cm} (2.10)
since then we would have

\[
P[Y^2 > \gamma] \leq \liminf_{n \to \infty} P[M_{n,Z} > \gamma] \leq \limsup_{n \to \infty} P[M_{n,Z} > \gamma]
\]

\[
\leq \limsup_{n \to \infty} P \left[ \left( \sum_{j=1}^{n} \frac{|Z_j|}{b_n} \right)^2 > \gamma \right] = P[Y^2 > \gamma].
\]

But for any \( T > 0 \) and \( n \geq 2T \),

\[
M_{n,Z} = \sup_{x \in [0,1/2]} |J_{n,Z}(x)|^2 = \sup_{x \in [0,n/2]} |J_{n,Z}(x/n)|^2
\]

\[
\geq \sup_{x \in [0,T]} |J_{n,Z}(x/n)|^2 \Rightarrow \sup_{x \in [0,T]} |J_{\infty}(x)|^2,
\]

and therefore

\[
\liminf_{n \to \infty} P[M_{n,Z} > \gamma] \geq \liminf_{n \to \infty} P \left[ \sup_{x \in [0,T]} |J_{n,Z}(x/n)|^2 > \gamma \right]
\]

\[
= P \left[ \sup_{x \in [0,T]} |J_{\infty}(x)|^2 > \gamma \right].
\]

This is true for any \( T \), so we obtain

\[
\liminf_{n \to \infty} P[M_{n,Z} > \gamma] \geq P \left[ \sup_{x \in [0,\infty)} |J_{\infty}(x)|^2 > \gamma \right].
\]

The following lemma identifies the distribution of \( \sup_{x \in [0,\infty)} |J_{\infty}(x)|^2 \).

**Lemma 2.4** We have

\[
\sup_{x \in [0,\infty)} |J_{\infty}(x)|^2 = \sup_{x \in [0,\infty)} \left| \sum_{j=1}^{\infty} B_j \Gamma_j^{-1/\alpha} e^{-2\pi i x U_j} \right|^2 = Y^2 = \left( \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \right)^2. \tag{2.11}
\]

**Proof of Lemma 2.4.** It is enough to prove that the right-hand side of (2.11) is a lower bound for the left-hand side, the other bound being trivial. Define

\[
\Omega_+ = \left\{ \omega : \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha}(\omega) < \infty \text{ and for every } m \geq 1 \right. \]

the numbers \( (U_1(\omega), \ldots, U_m(\omega)) \) are rationally independent. \}

Then \( P[\Omega_+] = 1 \) and for any \( \omega \in \Omega_+ \), by the result of Weyl (1916), the family of the \( m \)-tuples

\[
\left\{ \left( \{xU_1(\omega)\}, \ldots, \{xU_m(\omega)\} \right), x \geq 0 \right\}
\]

(where \( \{z\} \) denotes the fractional part of \( z \)) is dense in \([0,1]^m\).
Fix an $\omega \in \Omega_+$ and take any $\varepsilon > 0$. Choose an $N \geq 1$ such that $\sum_{j=N+1}^{\infty} \Gamma_j^{-1/\alpha}(\omega) < \varepsilon$. There is an $x_0 \in [0, \infty)$ such that
\[
\Re(B_j e^{-2\pi i x_0 U_j}) \geq 1 - \frac{\varepsilon}{N\Gamma_j^{-1/\alpha}}, \quad j = 1, \ldots, N.
\]
We then have
\[
\sup_{x \in [0, \infty)} \left| \sum_{j=1}^{\infty} B_j \Gamma_j^{-1/\alpha} e^{-2\pi i x U_j} \right| 
\geq \sup_{x \in [0, \infty)} \left| \sum_{j=1}^{N} B_j \Gamma_j^{-1/\alpha} e^{-2\pi i x U_j} \right| - \sum_{j=N+1}^{\infty} \Gamma_j^{-1/\alpha}
\geq \sum_{j=1}^{N} \left( 1 - \frac{\varepsilon}{N\Gamma_j^{-1/\alpha}} \right) \Gamma_j^{-1/\alpha} - \varepsilon = \sum_{j=1}^{N} \Gamma_j^{-1/\alpha} - 2\varepsilon.
\]
Letting first $N \to \infty$ and then $\varepsilon \to 0$, we obtain (2.11). This proves the lemma and completes the derivation of the limit distribution for $M_{n, Z}$.

Thus it remains to prove $\widetilde{M}_{n, Z} \Rightarrow Y_\alpha^2$. In view of $M_{n, Z} \Rightarrow Y_\alpha^2$ it suffices to show that for all $\gamma > 0$,
\[
\liminf_{n \to \infty} P[\widetilde{M}_{n, Z} > \gamma] \geq P[Y_\alpha^2 > \gamma]. \tag{2.12}
\]
We observe that for any integer $K$ and sufficiently large $n$
\[
P \left[ \sup_{j=1, \ldots, q} |J_n(j/n)|^2 > \gamma \right] \geq P \left[ \sup_{j=1, \ldots, K} |J_n(j/n)|^2 > \gamma \right].
\]
Now from Proposition 2.2 we have
\[(J_n(j/n), 1 \leq j \leq K) \Rightarrow (J_\infty(j), 1 \leq j \leq K)\]
in $\mathbb{R}^K$, hence
\[
\sup_{j=1, \ldots, K} |J_n(j/n)|^2 \Rightarrow \sup_{j=1, \ldots, K} |J_\infty(j)|^2,
\]
and so
\[
\liminf_{n \to \infty} P \left[ \sup_{j=1, \ldots, q} |J_n(j/n)|^2 > \gamma \right] \geq P \left[ \sup_{j=1, \ldots, \infty} |J_\infty(j)|^2 > \gamma \right].
\]
Finish the proof by adapting Lemma 2.4 to this case (in particular, it follows from Weyl (1916) that the family of the $m$–tuples
\[
\{(jU_1(\omega), \ldots, jU_m(\omega)) \}, \quad j = 1, 2, \ldots \}
\]
is dense in $[0,1]^m$ as well.)

From the previous proof it is clear that the limit random variable $Y_\alpha^2$ appears as the weak limit of the sequence $((b_{n-1}^{-1} \sum_{t=1}^n |Z_t|)^2)$. It also appears as $\sup_{x \geq 0} |J(x)|^2$. When $(Z_t)$ is a sequence of SoS variables, $Y_\alpha^2$ can also appear naturally in a third form which we describe in the following proposition.

**Proposition 2.5** Let $(Z_t)$ be a sequence of iid SoS variables. Then the limit variable $Y_\alpha^2$ in (2.1) has representation

$$Y_\alpha^2 \overset{d}{=} \sup_{x \in I_0} (\alpha^2(x) + \beta^2(x)),$$

where $I_0 \subset (0,0.5)$ is a countable set such that, for an enumeration $(x_n)$ of $I_0$, for every $m \geq 1$ and $x_1, \ldots, x_m \in I_0$, the numbers $1, x_1, \ldots, x_m$ are rationally independent, $I_0$ is dense in $[0,0.5]$, and $((\alpha(x), \beta(x)), x \in I_0)$ is an exchangeable SoS process with finite-dimensional distributions determined by (1.9).

This represents $Y_\alpha^2$ as the result of two different limiting procedures. In Proposition 2.2, we normalize the argument by dividing by $n$ and consider $J_n(x/n)$. It is also possible to proceed using a normalization as in (1.8) which does not divide the argument by $n$. This alternative representation of the limiting random variable tied to (1.8) and (1.9) will be used in Section 3.

**Proof.** Let $I_0$ be a subset of irrational numbers in $(0,0.5)$, having the properties given in the proposition. Let $(\Omega', \mathcal{F}', P')$ be a probability space rich enough to support a family of iid uniform $U(0,1)$ random variables $(U_x, x \in I_0)$. Further, suppose $M$ is an SoS random measure on $(\Omega', \mathcal{F}')$ with control measure $P'$; cf. Samorodnitsky and Taqqu (1994), Section 3.3.

Consider an SoS process $W$ defined by

$$W(x) = \int_{\Omega'} e^{-2\pi i U_\alpha(x')} M(d\omega'), \quad x \in I_0. \quad (2.13)$$

It is immediate that for any distinct numbers $x_1, \ldots, x_m \in I_0$ and $m \geq 1$,

$$B_m = (\Re(W(x_1)), \Im(W(x_1)), \ldots, \Re(W(x_m)), \Im(W(x_m))) \overset{d}{=} A_m, \quad (2.14)$$

where $A_m$ has characteristic function (1.9). Therefore, when $(Z_t)$ is a sequence of SoS variables, we have

$$\liminf_{n \to \infty} P[M_{n,z} > \gamma] \geq P \left[ \sup_{x \in I_0} |W(x)|^2 > \gamma \right] \quad (2.15)$$

for all $\gamma \geq 0$.

The SoS process $W$ has a series representation (cf. Samorodnitsky and Taqqu (1994), Section 3.10)

$$W(x) = \sum_{j=1}^{\infty} \varepsilon_j \Gamma_j^{-1/\alpha} e^{-2\pi i U(x,j)}, \quad x \in I_0, \quad (2.16)$$
where \((U(x, j); j \geq 1, x \in I_0)\) is an array of iid \(U(0,1)\) random variables, \((\varepsilon_j)\) is a sequence of iid Rademacher random variables \((P[\varepsilon_j = 1] = P[\varepsilon_j = -1] = 0.5)\), and \((\Gamma_j)\) are the points of a unit rate Poisson process on \(\mathbb{R}_+\). Moreover, these three sequences of random variables are independent. The representation (2.16) has to be interpreted in distribution, but, since we are interested only in distributional results, we may, as well, assume that it is an almost sure representation of \(W\).

An argument very similar to that of Lemma 2.4 shows that

\[
\sup_{x \in I_0} |W(x)| \overset{d}{=} C_\alpha^{1/\alpha} \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \overset{d}{=} Y_\alpha.
\]  

(2.17)

Therefore, it follows from (2.14) that

\[
\lim_{n \to \infty} \inf P[M_n > \gamma] \geq P[Y_\alpha^2 > \gamma],
\]

thus providing an alternative argument to Theorem 2.1 when \((Z_t)\) are SoS.

Now we consider the periodogram of a general linear process \(X = (X_t)\), given by (1.10), whose coefficients \(\psi_j\) satisfy (1.11) and (1.12), and the \(Z_t\)'s satisfy (1.6) and (1.7) as before. Write

\[
\psi(x) = \sum_{j=-\infty}^{\infty} \psi_j e^{-i2\pi x j}, \quad x \in [0,0.5],
\]

for the \textit{transfer function} of the linear filter \((\psi_j)\). Notice that the \textit{power transfer function}

\[
f_X(x) = |\psi(x)|^2
\]

is the \textit{spectral density} of a Gaussian random process with representation (1.10), where the \(Z_t\)'s have to be replaced by mean-zero iid Gaussian random variables.

**Lemma 2.6** Assume that \(f_X\) is positive on \([0,0.5]\). Then under the assumptions of Theorem 2.1

\[
\max_{x \in [0,0.5]} |I_n, X(x) / f_X(x) - I_n, Z(x)| \overset{P}{\to} 0.
\]  

(2.18)

**Proof.** The following decomposition is standard; see for example Brockwell and Davis (1991), p. 346:

\[
I_n, X(x) = |J_n, X(x)|^2 = f_X(x) |J_n, Z(x)|^2 + R_n(x) = f_X(x) I_n, Z(x) + R_n(x),
\]  

(2.19)
where

\[ R_n(x) = \psi(x)J_{n,2}(x)Y_n(-x) + \psi(-x)J_{n,2}(-x)Y_n(x) + |Y_n(x)|^2, \quad (2.20) \]

\[ J_{n,2}(x) = b_n^{-1} \sum_{t=1}^{n} X_t e^{-i2\pi xt} \]

\[ = b_n^{-1} \sum_{j=-\infty}^{\infty} \psi_j e^{-i2\pi x \psi j} \left( \sum_{t=1}^{n} Z_t e^{-i2\pi xt} + V_{n,j} \right) = \psi(x)J_{n,2}(x) + Y_n(x), \]

\[ V_{n,j} = \sum_{t=1}^{n-j} Z_t e^{-i2\pi xt} - \sum_{t=1}^{n} Z_t e^{-i2\pi xt}, \]

\[ Y_n(x) = b_n^{-1} \sum_{j=-\infty}^{\infty} \psi_j e^{-i2\pi x j} V_{n,j}. \]

The sequence

\[ M_{n,2} = \max_{x \in [0,0.5]} I_{n,2}(x), \quad n \geq 1, \]

is tight and \( f_X \) is bounded away from 0 and \( \infty \). In view of (2.20), it suffices to show that \( \max_{x \in [0,0.5]} |Y_n(x)| \xrightarrow{P} 0 \). In what follows, we indicate the main steps of the proof. For the sake of simplicity we also assume \( \psi_j = 0 \) for \( j < 0 \). Then

\[ Y_n(x) = b_n^{-1} \sum_{j=-n+1}^{\infty} \psi_j e^{-i2\pi x j} V_{n,j} + b_n^{-1} \sum_{j=-1}^{n} \psi_j e^{-i2\pi x j} V_{n,j} = S_1(x) + S_2(x). \]

Furthermore,

\[ S_1(x) = -J_{n,2}(x) \sum_{j=-n+1}^{\infty} \psi_j e^{-i2\pi x j} + b_n^{-1} \sum_{j=-n+1}^{\infty} \psi_j e^{-i2\pi x j} \sum_{t=1}^{n-j} Z_t e^{-i2\pi xt} \]

\[ = S_{11}(x) + S_{12}(x). \]

Now

\[ \max_{x \in [0,0.5]} |S_{11}(x)| \leq \max_{x \in [0,0.5]} |J_{n,2}(x)| \sum_{j=-n+1}^{\infty} |\psi_j| \xrightarrow{P} 0, \]

and for \( S_{12}(x) \) we have

\[ |S_{12}(x)| \leq b_n^{-1} \sum_{t=-n}^{-1} |Z_t| \sum_{j=-n+1}^{n-t} |\psi_j| + b_n^{-1} \sum_{t=-\infty}^{-n-1} |Z_t| \sum_{j=-1}^{n-t} |\psi_j|. \quad (2.21) \]

Notice that

\[ \sum_{t=-n}^{-1} |Z_t| \sum_{j=-n+1}^{n-t} |\psi_j| \leq \sum_{t=1}^{n} |Z_t| \sum_{j=-n+1}^{n+t} |\psi_j| \xrightarrow{P} 0, \]

12
since $b_n^{-1} \sum_{i=1}^{n} |Z_i| \Rightarrow Y_\alpha$ and, by (1.11),

$$\sum_{j=-n+1}^{\infty} |\psi_j| \to 0.$$ 

The second expression in (2.21) is handled as follows. Write

$$\phi_n(\lambda) = E \exp \left\{ -\lambda b_n^{-1} \sum_{t=-\infty}^{-n-1} |Z_t| \sum_{j=1-i}^{n-t} |\psi_j| \right\} = E e^{T_1 + \cdots + T_n}, \quad \text{(2.22)}$$

where $T_j = -\lambda b_n^{-1} \sum_{i=-n+1}^{\infty} |Z_i||\psi_{i+j}|, j = 1, \ldots, n$. Observe that $T_1, \ldots, T_n$ are associated, and hence

$$E e^{T_1 + \cdots + T_n} \geq \prod_{j=1}^{n} E e^{T_j} \geq \left( E e^{T_1} \right)^n. \quad \text{(2.23)}$$

Let

$$\phi(\lambda) = E \exp \{-\lambda |Z_1|\}$$

so that by Karamata’s Tauberian Theorem ((Feller, 1971, page 471))

$$-\log \phi(\lambda) \sim 1 - \phi(\lambda) \sim \Gamma(1 - \alpha) \lambda^\alpha L(1/\lambda), \quad \lambda \uparrow 0. \quad \text{(2.24)}$$

We see that for large $n$ and some $c > 0$

$$E e^{T_n} = \prod_{i=-n+2}^{\infty} \phi(b_n^{-1} Z_i)^{1/\lambda \psi_i}) = \exp \left\{ -\sum_{i=-n+2}^{\infty} \left( -\log \phi(b_n^{-1} \lambda |\psi_i|) \right) \right\}$$

$$\geq \exp \left\{ -c \frac{1}{n} \sum_{i=-n+2}^{\infty} -\log \phi(b_n^{-1} \lambda |\psi_i|) \right\}$$

$$\geq \exp \left\{ -c \frac{1}{n} \sum_{i=-n+2}^{\infty} |\psi_i|\alpha \right\},$$

from (2.24), and so by (2.22) and (2.23) we conclude that $\phi_n(\lambda) \to 1$.

Now we turn to $S_2$ which we decompose as follows:

$$|S_2(x)| = \left| b_n^{-1} \sum_{j=1}^{n} \psi_j e^{-i2\pi x j} \sum_{t=1-j}^{0} Z_t e^{-i2\pi x t} - b_n^{-1} \sum_{j=1}^{n} \psi_j e^{-i2\pi x j} \sum_{t=-n-j+1}^{n} Z_t e^{-i2\pi x t} \right|$$

$$\leq b_n^{-1} \sum_{j=1}^{n} |\psi_j| \left( \sum_{t=1-j}^{0} |Z_t| + \sum_{t=-n-j+1}^{n} |Z_t| \right) = S_{21} + S_{22}.$$
For the first double sum we have  

\[ S_{21} = \sum_{j=1}^{n} |\psi_j| \sum_{i=1}^{n} \frac{|Z_i|}{b_n} = d \sum_{j=1}^{n} |\psi_j| \sum_{k=1}^{j} \frac{|Z_k|}{b_n} \]

\[ = \sum_{k=1}^{k_1} \left( \sum_{j=k}^{n} |\psi_j| \right) \frac{|Z_k|}{b_n} + \sum_{k=k_0+1}^{n} \left( \sum_{j=k}^{n} |\psi_j| \right) \frac{|Z_k|}{b_n} \]

\[ = S_{211} + S_{212} \]

For fixed \( k_0 \),

\[ S_{211} \leq \left( \sum_{k=k_0}^{n} \frac{|Z_k|}{b_n} \right) \sum_{j=1}^{n} |\psi_j| \xrightarrow{p} 0 \]

and if \( k_0 \) is chosen so that \( \sum_{k=k_0}^{n} |\psi_k| < \epsilon \), then

\[ S_{212} \leq \epsilon \sum_{k=k_0+1}^{n} \frac{|Z_k|}{b_n} \leq \epsilon \sum_{k=k_0+1}^{n} \frac{|Z_k|}{b_n} \xrightarrow{p} \epsilon Y_\alpha . \]

Since \( \epsilon > 0 \) is arbitrary, \( S_{212} \xrightarrow{p} 0 \). This proves that \( S_{21} \xrightarrow{p} 0 \).

For the second sum we have

\[ S_{22} = \sum_{i=1}^{n} \frac{|Z_i|}{b_n} \sum_{j=n-i+1}^{n} |\psi_j| = \sum_{i=1}^{n} \frac{|Z_i|}{b_n} \sum_{j=i}^{n} |\psi_j| . \]

The sum on the right-hand side can be treated in a similar way as \( S_{21} \). This completes the proof.

\[ \square \]

The following is now immediate from Lemma 2.6 and Theorem 2.1.

**Corollary 2.7** Assume that the coefficients \( \psi_j \) of the linear process \( (X_i) \) satisfy the condition (1.11) and (1.12), and that the iid sequence \( (Z_i) \) satisfies (1.6) and (1.7). If \( f_X \) is positive on \([0, 0.5]\), then

\[ \max_{x \in [0, 0.5]} I_n, X(x)/f_X(x) \xrightarrow{Y_\alpha^2} . \]

We can also formulate the following limit result for Fisher’s generalised \( g \)-statistic; see (1.5).

**Corollary 2.8** Under the assumptions of Corollary 2.7,

\[ \frac{\max_{j=1, \ldots, q} I_n, X(\omega_j)/f_X(\omega_j)}{\sum_{j=1}^{q} I_n, X(\omega_j)/f_X(\omega_j)} \xrightarrow{Y_\alpha^2 / Y_{\alpha/2}^2} , \]

where the mixed stable random vector \( (Y_\alpha, Y_{\alpha/2}) \) has Laplace transform

\[ E \exp \left\{ -\lambda_1 Y_\alpha - \lambda_2 Y_{\alpha/2} \right\} = \exp \left\{ -E|\lambda_1 C + \lambda_2^{1/2} G|^\alpha \right\} , \quad \lambda_1, \lambda_2 \geq 0 . \]
Here $C$ and $G$ are independent random variables, $C$ is standard Cauchy and $G$ is $N(0,2)$. Alternatively, one can use

$$
(Y_\alpha, Y_{\alpha/2}) \overset{d}{=} \left(\sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha}, \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha}\right),
$$

(2.27)

where $(\Gamma_j, j \geq 1)$ are the standard Poisson arrival times.

Remarks. 1) For $X = Z$, $f_X$ degenerates to a constant. Hence the left-hand side expression of (2.25) coincides with Fisher’s $g$–statistic (1.5).

2) From the proof below it can be seen that $\max_{j=1,\ldots,q} I_{n,X}(\omega_j)/f_X(\omega_j)$ in (2.25) can be replaced with $\max_{x \in [0.0.5]} I_{n,X}(x)/f_X(x)$.

3) A more precise relationship between (2.26) and (2.27) is

$$
(Y_\alpha, Y_{\alpha/2}) \overset{d}{=} \left(b \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha}, b^2 \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha}\right)
$$

for some $b > 0$. However, $b$ is immaterial in (2.25).

Proof. By Lemma 2.6, we have

$$
\sum_{j=1}^{\infty} I_{n,X}(\omega_j)/f_X(\omega_j) = \sum_{j=1}^{Q} I_{n,Z}(\omega_j)(1 + o_P(1)) = \frac{b_n^{-2}}{\sum_{i=1}^{n} Z_i^2}(1 + o_P(1))
$$

and

$$
\max_{j=1,\ldots,q} I_{n,X}(\omega_j)/f_X(\omega_j) = \max_{j=1,\ldots,q} I_{n,Z}(\omega_j)(1 + o_P(1))
$$

Hence it suffices to determine the weak limit of

$$
I_n = \frac{\max_{j=1,\ldots,q} I_{n,Z}(\omega_j)}{b_n^{-2} \sum_{i=1}^{n} Z_i^2}.
$$

(2.28)

We will prove the existence of a weak limit in (2.28) and the representation (2.27). The latter, and some algebra, imply (2.26). It follows from Theorem 2.1 that

$$
M_{n,Z} - \hat{M}_{n,Z} \overset{P}{\rightarrow} 0
$$

as $n \rightarrow \infty$. Therefore, we need to prove that

$$
\max_{x \in [0,0.5]} \frac{b_n^{-1} \sum_{i=1}^{n} Z_i e^{-i2\pi xt}}{b_n^{-2} \sum_{i=1}^{n} Z_i^2} \rightarrow \frac{Y_{\alpha/2}}{Y_{\alpha}},
$$

(2.29)

where $(Y_\alpha, Y_{\alpha/2})$ is given by (2.27).

We observe, first, that the left-hand side of (2.29) is bounded from above by

$$
\frac{(b_n^{-1} \sum_{i=1}^{n} |Z_i|)^2}{b_n^{-2} \sum_{i=1}^{n} Z_i^2},
$$

15
and this self–normalized sum converges as in Resnick (1986). Therefore,
\[
\frac{(b_n^{-1}\sum_{t=1}^{n}|Z_t|)^2}{b_n^{-2}\sum_{t=1}^{n}Z_t^2} = \left(\frac{\sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha}}{\sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha}}\right)^2.
\]  
(2.30)

On the other hand, for every \( T > 0 \) and \( n \geq 2T \), the left–hand side of (2.29) is bounded from below by
\[
\max_{x \in [0, T]} \left| \frac{\int_{\{0 < y \leq 1\} e^{-i2\pi xy} N_n(dy, du)}}{\int_{\{0 < y \leq 1\} e^2N_n(dy, du)}} \right|^2.
\]

We repeat, once again, the truncation argument we used in the argument of Theorem 2.1, and the appeal to Theorem 4.2 in Billingsley (1968), to establish that the above ratio converges weakly to the corresponding ratio with \( N_n \) replaced by \( N \). After letting \( T \to \infty \) and using the result of Weyl (1916), we obtain a lower bound that matches (2.30). We have, therefore, proved the existence of a weak limit in (2.28) and the representation (2.27).

\[\Box\]

3 The case \( \alpha \in [1, 2) \)

In this section we assume \( (Z_t) \) is an iid sequence of SoS random variables. If \( \alpha \in [1, 2) \) then the sequence \( (M_{n, Z}) \) is not stochastically bounded. Indeed, the SoS process in (2.13) is well defined for all \( 0 < \alpha < 2 \), and the argument used to get (2.15) works for any \( 0 < \alpha < 2 \), and shows that for every \( \gamma > 0 \)
\[
\liminf_{n \to \infty} P[M_{n, Z} > \gamma] \geq P \left[ \sup_{x \in l_0} |W(x)|^2 > \gamma \right],
\]
no matter what \( \alpha \) is, and so our claim reduces to showing that
\[
\sup_{x \in l_0} |W(x)| = \infty \text{ with probability 1.} \tag{3.1}
\]

To this end observe that, by symmetry, for every \( N \geq 1 \) and \( \gamma > 0 \),
\[
P \left[ \sup_{x \in l_0} |W(x)| > \gamma \right] \geq \frac{1}{2} P \left[ \max_{j=1}^{N} \sum_{j=1}^{N} \varepsilon_j \Gamma_j^{-1/\alpha} e^{2\pi i U(x, j)} > \gamma \right] = \frac{1}{2} P \left[ \sum_{j=1}^{N} \Gamma_j^{-1/\alpha} > \gamma \right] \to \frac{1}{2},
\]
where now \( (\varepsilon_j) \) are iid symmetric Rademacher random variables taking values \( \pm 1 \). In the last step we used again the argument in the proof of Lemma 2.4. Letting \( N \to \infty \) and recalling that, for \( 1 \leq \alpha < 2 \), \( \sum_{j=1}^{N} \Gamma_j^{-1/\alpha} \to \infty \) with probability 1, we see that
\[
P \left[ \sup_{x \in l_0} |W(x)| > \gamma \right] \geq \frac{1}{2}
\]
for all \( \gamma > 0 \). Since \( W \) is an SoS process, it is bounded with probability 0 or 1 (see e.g. Samorodnitsky and Taqqu (1994), Chapter 10), and so (3.1) follows.
Since the sequence \((M_n, Z)\) is not stochastically bounded when \(\alpha \in [1, 2)\), a normalization for \(M_n, Z\) is needed. This is the content of the following result.

**Proposition 3.1** If \(\alpha \in (1, 2)\), the sequence \((M_n, Z)/(\ln n)^{(2/\alpha)(\alpha - 1)}, n \geq 2\) is tight. If \(\alpha = 1\), the sequence \((M_n, Z)/(\ln \ln n)^{2}, n > e\) is tight.

**Proof.** The sequence \((Z_j)\) has a representation \((A_j^{1/2}G_j)\), where \((G_j)\) are iid \(N(0, 2)\) random variables and \((A_j)\) are iid positive \(\alpha/2\)-stable random variables with Laplace transform \(E \exp \{-\theta A_1\} = \exp \{-\theta^{\alpha/2}\}, \theta \geq 0\). The two sequences are independent. See Samorodnitsky and Taqqu (1994), Section 1.3. Fix a \(\delta > 0\), write \(A_j(\delta) = A_j I_{\{A_j \leq \delta n^{1/\alpha}\}}\) and define

\[
I_{n, Z}^- (\delta) = \max_{x \in [0, 0.5]} \left| n^{-1/\alpha} \sum_{j=1}^{n} G_j(A_j(\delta))^{1/2} e^{-i2\pi x_j} \right|^2. \tag{3.2}
\]

Given \((A_j)\), we introduce a pseudo-metric functional on \([0, 0.5]\):

\[
d(x, y) = \left( \sum_{j=1}^{n} \left| (A_j(\delta))^{1/2} e^{-i2\pi x_j} - (A_j(\delta))^{1/2} e^{-i2\pi y_j} \right|^2 \right)^{1/2}.
\]

Then the pseudo-metric space \(([0, 0.5], d)\) has diameter

\[
D = \max_{x, y \in [0, 0.5]} d(x, y) \leq 2(S_n(A(\delta)))^{1/2}, \tag{3.3}
\]

and

\[
d(x, y) \leq 2(M_n(A(\delta)))^{1/2} |x - y| \left( \sum_{j=1}^{n} (\pi j)^2 \right)^{1/2} \leq c (M_n(A(\delta)))^{1/2} |x - y| n^{3/2}
\]

for some constant \(c > 0\). In what follows, we write \(c\) for any positive constant. Let \(N(\varepsilon) = N([0, 0.5], d; \varepsilon)\) be the \(\varepsilon\)-covering number of \([0, 0.5]\), i.e. the minimal number of open balls of radius \(\varepsilon > 0\) in the pseudo-metric \(d\) which is necessary to cover \([0, 0.5]\). Then

\[
N(\varepsilon) \leq 1 + \varepsilon^{-1} c n^{3/2} (M_n(A(\delta)))^{1/2}. \tag{3.4}
\]
By virtue of (3.3), Proposition 13.8 and Lemma 13.5 of Ledoux and Talagrand (1991), there exists an absolute constant $c > 0$ such that

$$E_G \max_{x \in [0,0.5]} \left| \sum_{j=1}^{n} (A_j(\delta))^{1/2} G_j e^{-i2\pi x j} \right|^2 \leq c E_\varepsilon \max_{x \in [0,0.5]} \left| \sum_{j=1}^{n} (A_j(\delta))^{1/2} \varepsilon_j e^{-i2\pi x j} \right|^2 \leq c \left( D + \int_{0}^{D} (\ln N(\varepsilon))^{1/2} d\varepsilon \right)^2 \leq c \left( 2 \left( S_n(A(\delta)) \right)^{1/2} + \int_{0}^{D} (\ln N(\varepsilon))^{1/2} d\varepsilon \right)^2,$$

(3.5)

where $(\varepsilon_j)$ is a Rademacher sequence, independent of $(A_j)$, and $E_G$ and $E_\varepsilon$ denote expectations with respect to $(G_j)$ and $(\varepsilon_j)$ respectively (with $(A_j)$ fixed). By (3.3) and (3.4) we have

$$\int_{0}^{D} (\ln N(\varepsilon))^{1/2} d\varepsilon \leq \int_{0}^{2 \left( S_n(A(\delta)) \right)^{1/2}} \left( \ln \left( 1 + \varepsilon^{-1} cn^{3/2}(M_n(A(\delta)))^{1/2} \right) \right)^{1/2} d\varepsilon \leq cn^{3/2}(M_n(A(\delta)))^{1/2} \int_{0}^{2^{-1} n^{-3/2} S_n(A(\delta)) / M_n(A(\delta))} \left( \ln(1 + \varepsilon^{-1}) \right)^{1/2} d\varepsilon. \leq c \left( \ln(1 + \varepsilon^{-1}) \right)^{1/2}.$$

Using for example Karamata’s theorem (cf. Bingham et al. (1987)), one can check that for any $\theta > 0$,

$$\int_{0}^{\theta} \left( \ln(1 + \varepsilon^{-1}) \right)^{1/2} d\varepsilon \leq c \theta \left( \ln(1 + \theta^{-1}) \right)^{1/2}.$$

Therefore

$$\int_{0}^{D} (\ln N(\varepsilon))^{1/2} d\varepsilon \leq c \left( S_n(A(\delta)) \right)^{1/2} \left( \ln \left( 1 + cn^{3/2}(M_n(A(\delta)))^{1/2} \frac{S_n(A(\delta))}{(S_n(A(\delta)))^{1/2}} \right) \right)^{1/2} \leq c S_n(A(\delta)) \ln n.$$

(3.6)

We conclude from (3.5) and (3.6) that

$$E_G \max_{x \in [0,0.5]} \left| \sum_{j=1}^{n} G_j(A_j(\delta))^{1/2} e^{-i2\pi x j} \right|^2 \leq c S_n(A(\delta)) \ln n.$$

(3.7)

Now, by (3.7), (3.2) and an application of the Cauchy–Schwartz inequality,

$$E(I_{n,2}(\delta))^{\alpha/8} \leq c (\ln n)^{\alpha/8} E \left( n^{-2/\alpha} S_n(A(\delta)) \right)^{\alpha/8}.$$

We have by Jensen’s inequality,

$$E \left( n^{-2/\alpha} S_n(A(\delta)) \right)^{\alpha/8} \leq \left( n^{1-2/\alpha} E A_1(\delta) \right)^{\alpha/8}.$$
Direct calculation yields

\[ EA_1(\delta) \leq c \delta^{1-\alpha/2}n^{2/\alpha-1}. \]

We conclude that

\[ (\ln n)^{-\alpha/8}E(I^-_{n,z}(\delta))^{\alpha/8} \leq c \delta^{(\alpha/8)(1-\alpha/2)}. \]

Now take \( \delta = \delta_n = (\ln n)^{-\beta} \) for some \( \beta > 0 \) to be chosen later. Then it follows that the sequence

\[ (\ln n)^{\beta(1-\alpha/2)-1}I^-_{n,z}(\delta), \ n \geq 1, \text{ is tight.} \]  

(3.8)

Define now

\[ I^+_{n,z}(\delta) = \max_{x \in [0,1]} \left| n^{-1/\alpha} \sum_{j=1}^{n} G_j(A_j^c(\delta))^{1/2} e^{-i2\pi x j} \right|^2, \]

where

\[ A_j^c(\delta) = A_j^{I_{A_j > \delta n^{2/\alpha}}} . \]

We have

\[ I^+_{n,z}(\delta) \leq \left| n^{-1/\alpha} \sum_{j=1}^{n} \left| G_j(A_j^c(\delta))^{1/2} \right|^2 . \]  

(3.9)

Clearly, there is a \( C > 1 \) such that for all \( M > 1 \)

\[ (M^{-1}A_j | A_j > M) \overset{st}{\leq} CU_j^{-2/\alpha}, \]

where \( \leq \) denotes stochastic domination, \( U_j \) are iid uniform \( U(0,1) \) random variables, independent of \( (G_j) \) and of the binomial variable

\[ N_{n,\delta} = \# \{ j : 1 \leq j \leq n; A_j > \delta n^{2/\alpha} \} . \]

Choose as above \( \delta = \delta_n = (\ln n)^{-\beta} \). Then

\[ n^{-1/\alpha} \sum_{j=1}^{n} |G_j(A_j^c(\delta))^{1/2} \overset{st}{\leq} c (\ln n)^{-\beta/2} \sum_{j=1}^{N_{n,\delta_n}} |G_j|U_j^{-1/\alpha} . \]

Furthermore, by simply taking the means we see that the sequence \( (N_{n,\delta_n} / (\ln n)^{\alpha \beta/2}) \) is tight.

Now assume that \( \alpha \in (1,2) \). By the law of large numbers the sequence

\[ \left( (\ln n)^{-\alpha \beta/2} \sum_{j=1}^{N_{n,\delta_n}} |G_j|U_j^{-1/\alpha} \right) \]
is tight and, therefore, so is the sequence

\[ \left( \ln n \right)^{-\beta(\alpha - 1)} f_{n,Z}^+(\delta) \].

Combining this fact with (3.8) and choosing \( \beta = 2/\alpha \), we conclude that the statement of the proposition holds in the case \( \alpha \in (1, 2) \).

Now we turn to the case \( \alpha = 1 \). We know that

\[
\sum_{j=1}^{n} |G_j| U_j^{-1} \overset{\text{d}}{=} \Gamma_{n+1} \sum_{j=1}^{n} |G_j| U_j^{-1},
\]

where \( \Gamma_j \) are the points of a unit rate Poisson process, independent of \( (G_j) \). Recalling the rate of growth of the \( \Gamma_j \)'s, we easily see that the sequence

\[
(n \ln n)^{-1} \sum_{j=1}^{n} |G_j| U_j^{-1}, \quad n \geq 2
\]

is tight. This implies the tightness of the sequence

\[
(\ln n)^{-\beta/2} (\ln n)^{-1} \sum_{j=1}^{N_n,\delta} |G_j| U_j^{-1}, \quad n > e.
\]

We conclude that

\[
(\ln n)^{-2} f_{n,Z}^+(\delta), \quad n > e
\]

is tight. Choosing now \( \beta = 2 \), we conclude from (3.8) that \( (M_n, Z/(\ln n)^2, n > e) \) is tight. This concludes the proof of the proposition in all cases. \( \Box \)

As in the case \( \alpha < 1 \) we also formulate a result for the linear process \( (X_t) \) given by (1.10). We have the following analogue to Lemma 2.6:

**Lemma 3.2** Assume that the power transfer function \( f_X \) is positive on \([0, 0.5] \) and that the coefficients \( \psi_j \) of the linear process \( (X_t) \) satisfy the condition

\[
\sum_{j \neq 0, \pm 1} |\psi_j| (\ln |j|)^{(1/\alpha) - 0.5} < \infty
\]

if \( 1 < \alpha < 2 \), and

\[
\sum_{j \neq 0, \pm 1} |\psi_j| (\ln |j|)^{1.5 (\ln \ln |j|)} - 1 < \infty
\]

if \( \alpha = 1 \). Then the following relation holds:

\[
\beta_n^{-2} \max_{x \in [0, 0.5]} \left| I_n(x) / f_X(x) - I_n, Z(x) \right| \overset{P}{\to} 0,
\]

where

\[
\beta_n = \begin{cases} 
(\ln n)^{1-1/\alpha} & \text{if } \alpha \in (1, 2), \\
\ln \ln n & \text{if } \alpha = 1.
\end{cases}
\]
Remark. Conditions (3.10) and (3.11) are stronger than (1.11) and (1.12). The latter restriction is needed in order to ensure the convergence of series (1.10). On the other hand, conditions (3.10) and (3.11) imply, in particular, that \( f_X \) is continuous.

Proof. First recall the decompositions (2.19) and (2.20). By virtue of Proposition 3.1 it suffices to show that \( \beta_n^{-1} \max_{x \in [0,0.5]} |Y_n(x)| \overset{P}{\to} 0. \) As in the proof of Lemma 2.6 we assume for convenience that \( \psi_j = 0 \) for \( j < 0. \) Following the lines of the proof of Lemma 2.6 and using the same notation, we conclude from (3.10) or (3.11) and Proposition 3.1 that

\[
\beta_n^{-1} \max_{x \in [0,0.5]} |S_{11}(x)| \leq \left( \beta_n^{-2} M_{n,2} \right)^{1/2} \sum_{j-n+1}^{\infty} |\psi_j| \overset{P}{\to} 0.
\]

Furthermore, the same arguments as for (3.7) lead to the inequalities

\[
E_G \max_{x \in [0,0.5]} |S_{12}(x)| \leq n^{-1/\alpha} \sum_{j-n+1}^{\infty} |\psi_j| E_G \max_{x \in [0,0.5]} \left| \sum_{t=1-j}^{n-j} G_t A_t^{1/2} e^{-2\pi xt} \right|
\]

\[
\leq c \sum_{j-n+1}^{\infty} |\psi_j| \left( n^{-2/\alpha} \ln n \sum_{t=1-j}^{n-j} A_t \right)^{1/2}
\]

\[
\leq c (\ln n)^{1/2} \sum_{k=1}^{(k+1)n} \psi_{n,k} \left( n^{-2/\alpha} \sum_{j-(k+1)n}^{-(k-1)n} A_j \right)^{1/2},
\]

where

\[
\psi_{n,k} = \sum_{j=kn+1}^{(k+1)n} |\psi_j|.
\]

Suppose first that \( 1 < \alpha < 2. \) Observe that the expression under the square root above has the distribution of \((2 + 1/n)^{\alpha/2} A_1.\) Taking another expectation, we obtain by (3.11),

\[
\beta_n^{-1} E \max_{x \in [0,0.5]} |S_{12}(x)| \leq \beta_n^{-1} c (\ln n)^{1/2} \sum_{k=1}^{\infty} \psi_{n,k} \leq c \sum_{j-n+1}^{\infty} (\ln |j|)^{(1/\alpha) - 0.5} |\psi_j| \overset{P}{\to} 0.
\]

Now suppose that \( \alpha = 1, \) and rewrite the expression above as

\[
E_G \max_{x \in [0,0.5]} |S_{12}(x)| \leq c (\ln n)^{1/2} \sum_{k=1}^{\infty} \psi_{n,k} \left( B_k^{1/2} + \tilde{B}_k^{1/2} \right),
\]

where \((B_k, k \geq 1)\) and \((\tilde{B}_k, k \geq 1)\) are two (not independent) sequences of iid \( S_{1/2}(1,1,0) \) random variables. To prove that \( \beta_n^{-1} \max_{x \in [0,0.5]} |S_{11}(x)| \overset{P}{\to} 0 \) it is, then, enough to prove that

\[
D_n = \beta_n^{-1} (\ln n)^{1/2} \sum_{k=1}^{\infty} \psi_{n,k} B_k^{1/2} \overset{P}{\to} 0.
\]

(3.12)
Letting $\gamma_n = |\psi_n| (\ln n)^{1/2}(\ln \ln n)^{-1}$, we have

$$D_n \leq \sum_{k=1}^{\infty} \left( \sum_{j=k+1}^{(k+1)n} \gamma_j \right) B_k^{1/2} \leq \sum_{k=1}^{\infty} \left( \sum_{j=k+1}^{(k+1)n} B_j \gamma_j^2 \right)^{1/2} \leq \sum_{j=n+1}^{\infty} B_j^{1/2} \gamma_j,$$

and so (3.12) follows from (3.11) and the Three Series Theorem.

Using the same conditioning argument as above, one can show that

$$E_G \max_{x \in [0,0.5]} |S_2(x)|$$

$$\leq c \ n^{-1/\alpha} \sum_{j=1}^{n} |\psi_j| (\ln j)^{1/2} \left( (S_{j-1}(B))^{1/2} + (S_{j-1}((\bar{B}))^{1/2} \right)$$

$$\leq c \ n^{-1/\alpha} \sum_{j=1}^{n} |\psi_j| (\ln j)^{1/2} \left( (S_{n-1}(B))^{1/2} + (S_{n-1}((\bar{B}))^{1/2} \right) ,$$

$$\leq c \ \sum_{j=1}^{n} |\psi_j| (\ln j)^{1/2} \left( B_{1/2}^{1/2} + (\bar{B}_1)^{1/2} \right), \quad (3.13)$$

where

$$S_j(B) = \sum_{k=0}^{j} B_k \quad \text{and} \quad S_j(\bar{B}) = \sum_{k=0}^{j} \bar{B}_k, \quad j \geq 0,$$

and $(B_k, k \geq 1)$ and $(\bar{B}_k, k \geq 1)$ are two (this time independent) sequences of iid $S_{\alpha/2}(1,1,0)$ random variables. By virtue of (3.10) and (3.11), it is easy to see that $\beta_n^{-1} \sum_{j=1}^{n} |\psi_j| (\ln j)^{1/2} \to 0$. Therefore and by (3.13), we finally conclude that

$$\beta_n^{-1} E \max_{x \in [0,0.5]} |S_2(x)| \to 0$$

in all cases. This concludes the proof of the lemma. \qed

An argument completely similar to that of Lemma 3.2 proves the following analogue of Corollary 2.7.

**Corollary 3.3** Assume that $f_X$ is positive on $[0,0.5]$ and that the coefficients $\psi_j$ of the linear process $(X_t)$ satisfy the conditions (3.10) if $1 < \alpha < 2$ and (3.11) if $\alpha = 1$. Then the sequence $(\beta_n^{-2} \max_{x \in [0,0.5]} I_{n,x}(x)/f_X(x))$ is tight.

**Acknowledgment.** This project commenced during a visit by Thomas Mikosch to the School of ORIE at Cornell University in the summer of 1997. He would like to express his gratitude to his colleagues for their excellent hospitality and financial support from NSF Grant DMS-97-04982.
References


23


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