TAILS OF LÉVY MEASURE OF GEOMETRIC STABLE RANDOM VARIABLES

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Abstract. The explicit form of Lévy measure for geometric stable (GS) random variables follows from the general Lévy–Kchintchine representation of a subordinated infinitely divisible process. Through this form, asymptotic properties of Lévy measure are studied. In particular, logarithmic asymptotics around the origin imply exponential rate of convergence in series representation of GS random variables which stands in sharp contrast with much slower power rate for stable variables.

1. Introduction and preliminaries

Geometric stable (GS) distributions in $\mathbb{R}^d$ are obtained as limiting laws of appropriately normalized random sums of i.i.d. d-dimensional random vectors,

$$\mathbf{S}_N = \mathbf{X}_1 + \cdots + \mathbf{X}_N, \tag{1}$$

where the number of terms $N = N_p$ is geometrically distributed with mean $1/p$, and $p \to 0$ [see, e.g., Mittnik and Rachev (1991)]. Like an α-stable distribution, a GS law is best described in terms of its characteristic function (ch.f.), which in the symmetric one dimensional case, has a particularly simple form:

$$\psi(t) = \frac{1}{1 + \sigma^\alpha |t|^\alpha}, \tag{2}$$

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and it is known in the literature as Linnik's characteristic function [see, e.g., Kotz et al. (1995) and references therein]. In the general case, as shown in Mittnik and Rachev (1991), there is a one-to-one correspondence between ch.f.'s of GS and stable distributions. Namely, the distribution of $\mathbf{Y}$ is GS if and only if its ch. f. ψ has the form

$$\psi(\mathbf{t}) = E\exp(i\mathbf{t}\cdot\mathbf{Y}) = \frac{1}{1-\log\phi(\mathbf{t})}, \tag{3}$$

where ϕ is a ch.f. of some stable distribution. We use $GS_\alpha(\Gamma,\mathbf{m})$ for a GS law corresponding through (3) to a multivariate stable distribution $S_\alpha(\Gamma,\mathbf{m})$ with Γ as its spectral measure and $\mathbf{m}$ as a location parameter [see, e.g., Samorodnitsky and Taqqu (1994)]. In the one-dimensional case we have

$$\psi(t) = \begin{cases} \left[1+\sigma^\alpha|t|^\alpha\left(1-i\tan\frac{\pi\alpha}{2}\beta\ \mathrm{sign}(t)\right)-i\mu t\right]^{-1}, & \text{if } \alpha\neq 1,\\ \left[1+\sigma^\alpha|t|^\alpha\left(1+i\frac{2}{\pi}\beta\log|t|\,\mathrm{sign}(t)\right)-i\mu t\right]^{-1}, & \text{if } \alpha=1. \end{cases} \tag{4}$$

Here, $\alpha\in(0,2]$ is the index of stability, determining in particular the tail of the distribution, $\beta\in[-1,1]$ is the skewness parameter, σ is the scale parameter, while $\mu\in\mathbb{R}$ controls the location. We shall use $GS_\alpha(\sigma,\beta,\mu)$ and $S_\alpha(\sigma,\beta,\mu)$ to denote the corresponding one dimensional GS and stable distributions, respectively.

As shown in Kozubowski and Panorska (1996), the tail behavior of GS and corresponding stable distributions is identical. The left and the right plots of Figure 1 illustrate, correspondingly, stable and GS densities in the symmetric case. We note that, unlike their stable counterparts, GS densities are unbounded for $\alpha\leq 1$.

The following distributional relation between a $S_\alpha(1,\beta,0)$ random variable X and a $GS_\alpha(\sigma,\beta,\mu)$ random variable Y:

$$Y \stackrel{d}{=} T(X,W), \tag{5}$$

where

$$T(x,w) = \begin{cases} \mu w + w^{1/\alpha}\sigma x, & \text{if } \alpha\neq 1,\\ \mu w + w\sigma x + \frac{2}{\pi}\sigma w\beta\log(w\sigma), & \text{if } \alpha = 1 \end{cases} \tag{6}$$

and W is a standard exponential random variable independent of X, reduces, to great extent, studies of GS distributions to those of stable distributions [see, e.g. Kozubowski (1994)]. The Lévy-Khintchine representation and its consequences belong to those few

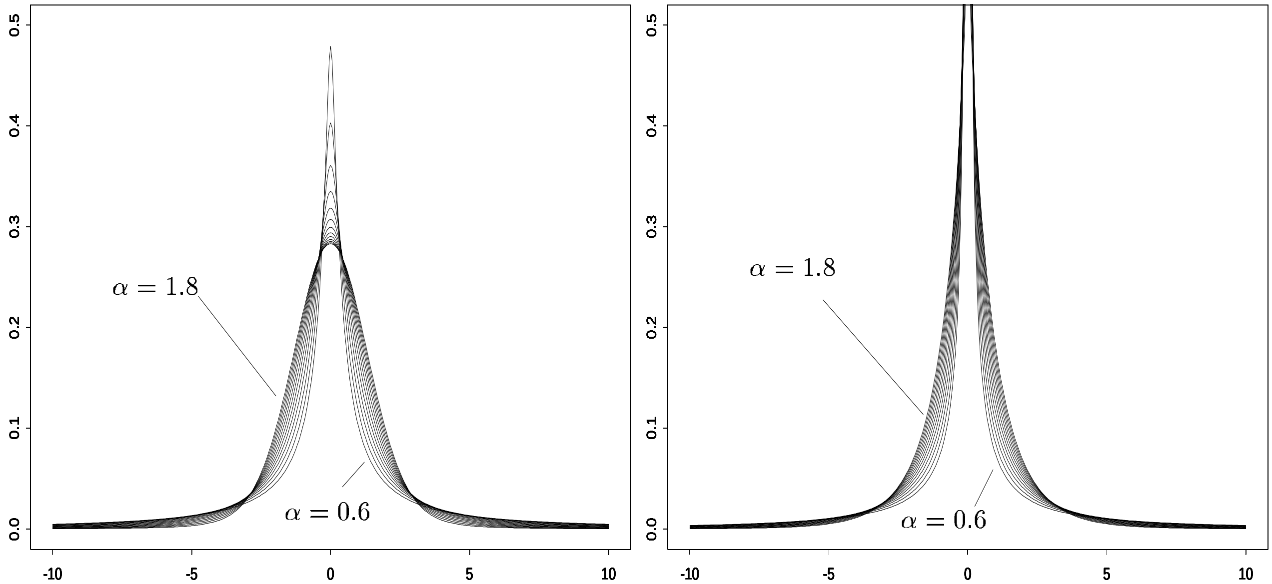

FIGURE 1. Stable vs. geometric stable densities, symmetric case, $\alpha = 0.6, 0.7, \ldots, 1.8$.

properties of GS distributions, that do not follow directly from (5) and corresponding properties of stable laws. In this work we discuss this representation and, in Section 2, derive the asymptotic behavior of GS Lévy measure. The type of asymptotics at zero is entirely different than that of stable laws. For example in the symmetric case, we have: $\lim_{u\to 0} \Lambda(u)/\ln u = -\alpha/2$, where $\Lambda(u)$ is GS Lévy measure of the interval (u, ∞), while stable Lévy measure of (u, ∞) is proportional to $u^{-\alpha}$.

Recall that a GS distribution has a series representation which follows from the general series representation of infinitely divisible distributions. As shown in Section 3, the asymptotic behavior of Λ at zero leads to the exponential rate of almost sure convergence of such series representation. In the symmetric case (for the general case see Section 3), the series

$$\sum_{i=1}^{\infty} \delta_i \Lambda^{-1}(\Gamma_i), \tag{7}$$

is absolutely convergent with probability one, where $\Lambda^{-1}(y) = \inf\{x \geq 0 : \Lambda(x) \leq y\}$, and (δ_i) is a sequence of symmetric independent signs, independent of arrival times (Γ_i) of a Poisson process with rate 2 [see LePage (1981)]. By Theorem 2, for some $K > 0$ the following remainder

$$e^{n/K}\sum_{i=n}^{\infty}\delta_i\Lambda^{-1}(\Gamma_i)$$

is bounded with probability one. This stands in sharp contrast with rate of almost sure convergence of the series representation $\sum_{i=1}^{\infty}\delta_i\Gamma_i^{-1/\alpha}$ of stable laws for which we have with probability one

$$\sum_{i=n}^{\infty}\delta_i\Gamma_i^{-1/\alpha}=O(n^{1/2-1/\alpha}\sqrt{\log\log n})$$

and the remainder converges almost surely to zero slower than $n^{1/2-1/\alpha}$. See Appendix.

Let us conclude this section with an explicit form of the Lévy-Kchintchine representation of GS laws. This particular form follows from Lemma 7, VI.2, Bertoin (1996) and from the fact that a GS random variable Y can be written as $Y=Z(W)$, where $(Z(t))$ is a stable process with independent increments, independent of exponentially distributed W, and $Z(1)\stackrel{d}{=}S_\alpha(\sigma,\beta,\mu)$ [see also Huff (1969)]. In the one-dimensional case we have the following representation.

Proposition 1. *Let* $Y\stackrel{d}{=}GS_\alpha(\sigma,\beta,\mu)$, $f_{\alpha,\beta,\sigma}$ *denote the density of* $S_\alpha(\sigma,\beta,0)$. *Then*

$$\psi(t)=\exp\left(\int_{\mathbb{R}}(e^{itu}-1)d\Lambda(u)\right), \tag{8}$$

where

$$\begin{aligned}\frac{d\Lambda}{du}(u) &= \int_0^\infty f_{\alpha,\beta,\sigma w^{1/\alpha}}(u-\mu w)\frac{e^{-w}}{w}dw\\ &= \frac{1}{\sigma}\int_0^\infty f_{\alpha,\beta,1}\left(\frac{u-\mu w}{\sigma w^{1/\alpha}}\right)\frac{e^{-w}}{w^{1+1/\alpha}}dw \ \ \textit{if } \alpha\neq 1,\ \textit{ or } \alpha=1,\beta=0.\end{aligned}$$

In the symmetric case we have

$$\frac{d\Lambda}{du}(u)=\frac{1}{\sigma}\int_0^\infty f_{\alpha,0,1}\left(\frac{u}{\sigma w^{1/\alpha}}\right)\frac{e^{-w}}{w^{1+1/\alpha}}dw=\frac{\alpha}{2|u|}E\exp\left(-\left|\frac{u}{\sigma X}\right|^\alpha\right),$$

where X has a $S_\alpha(1,0,0)$ distribution with the ch.f. $e^{-|t|^\alpha}$.

Proposition 1 extends naturally to multivariate case.

Proposition 2. *Let* $\mathbf{Y}$ *have a truly* d*-dimensional* $GS_\alpha(\Gamma, \mathbf{m})$ *law and* $f_{\alpha,\Gamma}(\cdot)$ *be the density of* $S_\alpha(\Gamma, \mathbf{0})$ *with respect to the* d*-dimensional Lebesgue measure. Then*

$$\psi(\mathbf{t}) = \exp\left(\int_{R^d} \left(e^{i\mathbf{t}\cdot\mathbf{x}} - 1\right) \Lambda(d\mathbf{x})\right),$$

where

$$\frac{d\Lambda}{d\mathbf{x}}(\mathbf{x}) = \int_0^\infty g(w, \mathbf{x}) \frac{e^{-w}}{w} dw,$$

with

$$g(w, \mathbf{x}) = \begin{cases} f_{\alpha,\Gamma}\left(w^{-1/\alpha}(\mathbf{x} - w\mathbf{m})\right) w^{-d/\alpha}, & \alpha \neq 1, \\ f_{\alpha,\Gamma}\left(w^{-1}(\mathbf{x} - w\mathbf{m}) - \frac{2}{\pi}\int_{S^d} \mathbf{s}\Gamma(d\mathbf{s}) \log w\right) w^{-d}, & \alpha = 1. \end{cases}$$

2. Asymptotics of Lévy measure

It follows directly from (5) that the right and left tails of a $GS_\alpha(\sigma, \beta, \mu)$ distribution can be written as

$$H_\pm(u) = P(\pm Y > u) = E(P(\pm T(X, W) > u|W)), \ u > 0,$$

where X is $S_\alpha(\sigma, \beta, \mu)$ and independent of a standard exponential random variable W. The tails of the Lévy measure Λ of Y have an analogous form.

Corollary 1. *Let* $\Lambda_+(u) = \Lambda\big((u, \infty)\big)$ *and* $\Lambda_-(u) = \Lambda\big((-\infty, -u)\big), u > 0$. *Then,*

$$\Lambda_+(u) = E\left(\frac{P(T(X, W) > u|W)}{W}\right), \quad \Lambda_-(u) = E\left(\frac{P(T(X, W) < -u|W)}{W}\right). \tag{9}$$

Proof. It is obvious that given $W = w > 0$, the conditional distribution of $T(X, W)$ is $S_\alpha(w^{1/\alpha}\sigma, \beta, \mu w)$. Thus,

$$\Lambda_+(u) = \int_0^\infty \int_u^\infty f_{\alpha,\beta,w^{1/\alpha}\sigma}(x - \mu w) dx \frac{e^{-w}}{w} dw = \int_0^\infty P(T(X, w) > u) \frac{e^{-w}}{w} dw.$$

The same arguments give the second equality in (9). □

The asymptotic behavior of Λ_+ and Λ_- is provided in the next result.

Theorem 1. *(i) The tails* $\Lambda_\pm$ *of a GS Lévy measure are regularly varying at infinity:*

$$\lim_{u\to\infty} u^\alpha \Lambda_\pm(u) = \sigma^\alpha C_\pm(\alpha, \beta), \tag{10}$$

where $C_{\pm}(\alpha,\beta)=\lim_{x\to\infty}x^{\alpha}P(\pm X>x)$*, and* X *is a* $S_{\alpha}(\sigma,\beta,0)$ *random variable.*
(ii) The limit $L_{\pm}(\alpha,\sigma,\beta,\mu)=-\lim_{u\to 0}\Lambda_{\pm}(u)/\ln u$ *exists, and*

$$L_{+}(\alpha,\sigma,\beta,\mu)=\begin{cases}\frac{\alpha}{2}+\frac{1}{\pi}\arctan\Big(\beta\tan\frac{\pi\alpha}{2}\Big), & \text{if } 1<\alpha<2,\\ 0, & \text{if } \alpha=1 \text{ and } \beta>0,\\ \frac{1}{2}+\frac{1}{\pi}\arctan(\frac{\mu}{\sigma}), & \text{if } \alpha=1 \text{ and } \beta=0,\\ 1, & \text{if } \alpha=1 \text{ and } \beta<0,\\ 0, & \text{if } 0<\alpha<1 \text{ and } \mu<0,\\ \frac{\alpha}{2}+\frac{1}{\pi}\arctan\Big(\beta\tan\frac{\pi\alpha}{2}\Big), & \text{if } 0<\alpha<1 \text{ and } \mu=0,\\ 1, & \text{if } 0<\alpha<1 \text{ and } \mu>0.\end{cases} \tag{11}$$

Further, $L_{-}(\alpha,\sigma,\beta,\mu)=L_{+}(\alpha,\sigma,-\beta,-\mu)$*.*

The proof of this theorem is given in Section 4.

Remark 1. In the symmetric case the asymptotic behavior at zero reduces to

$$\lim_{u\to 0}\frac{\Lambda_{\pm}(u)}{\ln u}=-\frac{\alpha}{2}. \tag{12}$$

On Figure 2, the densities of the Lévy measures for stable and geometric stable symmetric laws are compared. Recall here that in the symmetric stable case these densities are equal to $\sigma^{\alpha}C_{\alpha}x^{-1-\alpha}$, $x>0$, where $C_{\alpha}=\alpha(1-\alpha)/(2\Gamma(2-\alpha)\cos(\pi\alpha/2))$ if $\alpha\neq 1$, and $C_{\alpha}=1/\pi$ for $\alpha=1$.

3. Convergence rate in series representation

Theorem 2 establishes the exponential rate of convergence of the almost sure series representation of a GS law. Csörgő (1995) gave an example of a particularly fast (in Lévy distance) convergence of the series representation of an infinitely divisible law by constructing a Lévy measure with the same asymptotics at the origin as that of a GS law. Thus, our result establishes GS distributions as a large class of more natural examples of measures with this fast convergence rate of the series representation.

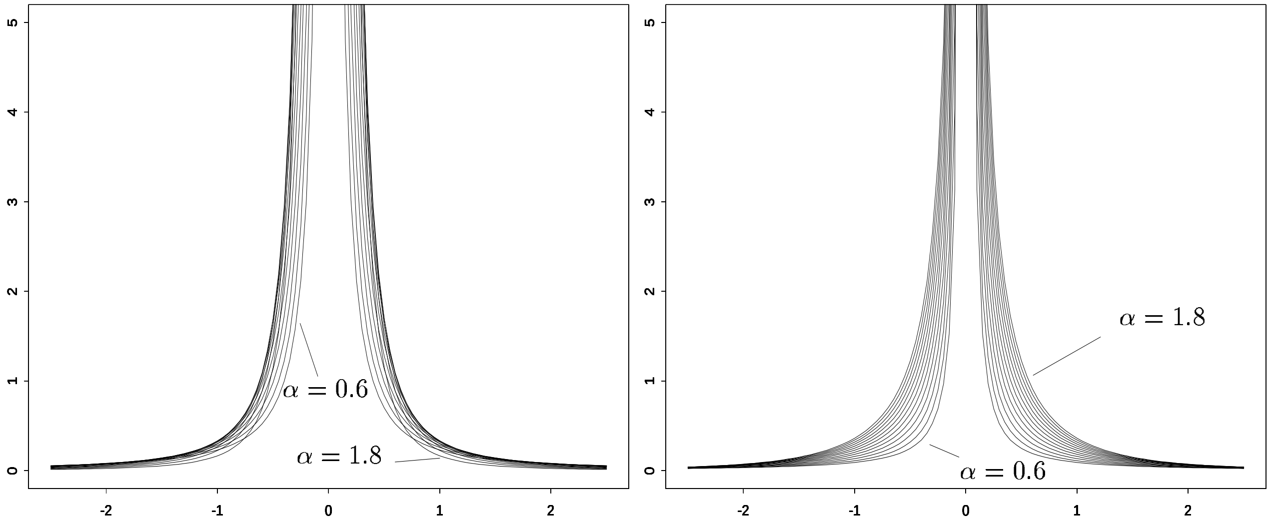

FIGURE 2. Stable vs. geometric stable densities of Lévy measures, symmetric case, $\alpha = 0.6, 0.7, \ldots, 1.8$.

Theorem 2. *Let Λ be the Lévy measure of a GS law with the ch. f. (8). Then, the following series is absolutely convergent with probability one*

$$Z = \sum_{i=1}^{\infty} \delta_i \Lambda_{\delta_i}^{-1}(\Gamma_i),$$

and there exists a positive constant K such that, with probability one,

$$|Z - \sum_{i=1}^{N} \delta_i \Lambda_{\delta_i}^{-1}(\Gamma_i)| = O(e^{-N/K}) \quad as \quad N \to \infty. \tag{13}$$

Moreover, Z has a $GS_\alpha(\sigma, \beta, \mu)$ law.

Proof. By Theorem 1, there exists a $C > 0$ such that for sufficiently small $u > 0$ we have

$$\Lambda_{\pm}(u) \leq -C \ln u,$$

which implies that for sufficiently large $\gamma > 0$ we have

$$\Lambda_{\pm}^{-1}(\gamma) \leq e^{-\gamma/C}. \tag{14}$$

Thus,

$$\sum_{i=1}^{\infty} \Lambda_{\delta_i}^{-1}(\Gamma_i) \leq \sum_{i=1}^{N-1} \Lambda_{\delta_i}^{-1}(\Gamma_i) + \sum_{i=N}^{\infty} e^{-\Gamma_i/C},$$

where N, selected pointwisely, is such that (14) holds with $\gamma = \Gamma_i$ for all $i \geq N$.

On the set of probability one where $\Gamma_n/n \to 1$, we have

$$\sum_{i=N}^{\infty} e^{-\Gamma_i/C} \leq \sum_{i=N}^{\infty} e^{-i/(2C)} = \frac{e^{-N/(2C)}}{1 - e^{-1/(2C)}},$$

if N, again taken pointwisely, is large enough. Thus, (13) holds with probability one.

Moreover, the ch. f. of Z has the form

$$\psi(t) = \exp\left(\int_{\mathbb{R}} (e^{itu} - 1) d\Lambda(u)\right)$$

[cf., Ferguson and Klass (1972)], which implies the remaining part of the result. □

4. Proof of Theorem 1

Proof of Theorem 1. Clearly, it is enough to consider the right tail Λ_+.

(i) This part follows easily from the well-known fact that $\Lambda_\pm$ have the same limiting behavior at infinity as $H_\pm$ [cf. Embrechts and Goldie (1981)]. The latter were studied in Kozubowski and Panorska (1996), where the asymptotics of type (10) was established for $H_\pm$.

(ii) Through the rest of the proof let $f = f_{\alpha,\beta,1}$. Assume first that $\alpha \neq 1$. By the l'Hospital rule we need to compute

$$L_+(\alpha, \sigma, \beta, \mu) = \frac{1}{\sigma} \lim_{x \to 0+} x \int_0^\infty f\left(\frac{x - \mu w}{\sigma w^{1/\alpha}}\right) \frac{e^{-w}}{w^{1+1/\alpha}}\, dw.$$

Since f is bounded, thus there exists $M > 0$ such that, for every $\epsilon > 0$,

$$\int_\epsilon^\infty f\left(\frac{x - \mu w}{\sigma w^{1/\alpha}}\right) \frac{e^{-w}}{w^{1+1/\alpha}}\, dw \leq M \int_\epsilon^\infty \frac{e^{-w}}{w^{1+1/\alpha}}\, dw.$$

We immediately conclude that for each fixed $\epsilon > 0$,

$$L_+(\alpha, \sigma, \beta, \mu) = \frac{1}{\sigma} \lim_{x \to 0+} x \int_0^\epsilon f\left(\frac{x - \mu w}{\sigma w^{1/\alpha}}\right) \frac{1}{w^{1+1/\alpha}}\, dw,$$

provided, of course, that the last limit exists and is the same for every $\epsilon > 0$. (We will not specifically mention such disclaimers for the remainder of the proof, even though similar situations will arise several times in the sequel.) The last integral $\int_0^\epsilon$ can be split in the two ways: $I_1(x) + I_2(x) = \int_{\epsilon x}^\epsilon + \int_0^{\epsilon x}$ and $I_3(x) + I_4(x) + I_5(x) = \int_0^{(1-\epsilon)x/\mu} + \int_{(1-\epsilon)x/\mu}^{(1+\epsilon)x/\mu} + \int_{(1+\epsilon)x/\mu}^\epsilon$.

Case: $1 < \alpha < 2$**.** Note that

$$xI_1(x) \le Mx \int_{\epsilon x}^{\infty} w^{-1-1/\alpha}\, dw = M\alpha x^{1-1/\alpha}$$

which converges to zero as $x \to 0$ and thus $L_+(\alpha, \sigma, \beta, \mu) = \frac{1}{\sigma} \lim_{x\to 0+} xI_2(x)$. Now, by the continuity, positivity and regular variation at infinity of the stable density, there is a nonincreasing function $K : (0, \infty) \to [1, \infty)$ with $K(\rho) \to 1$ as $\rho \to 1$ such that for every $\rho > 0$ and $x > 0$,

$$K(\rho)^{-1} \le f(\rho x)/f(x) \le K(\rho). \tag{15}$$

Since for every $w \in (0, \epsilon x)$ we have $(x - \mu w) \in \Big(x(1 - \epsilon|\mu|), x(1 + \epsilon|\mu|)\Big)$, (15) implies that

$$\begin{aligned} L_+(\alpha, \sigma, \beta, \mu) &= \frac{1}{\sigma} \lim_{x\to 0+} x \int_0^{\epsilon x} f\left(\frac{x}{\sigma w^{1/\alpha}}\right) \frac{1}{w^{1+1/\alpha}} dw \\ &= \alpha \lim_{x\to 0+} \int_{\sigma^{-1}\epsilon^{-1/\alpha}x^{1-1/\alpha}}^{\infty} f(y)\, dy \\ &= \alpha \int_0^{\infty} f(y)\, dy = \frac{\alpha}{2} + \frac{1}{\pi} \arctan\Big(\beta \tan \frac{\pi\alpha}{2}\Big), \end{aligned}$$

where the last equality is the well known probability that a stable random variable is positive. See e.g. (8) in Bingham (1973). This proves the first line in (11).

Case: $0 < \alpha < 1,\ \mu > 0$**.** We claim that

$$\lim_{x\to 0+} xI_3(x) = 0. \tag{16}$$

If $\beta = -1$, there is nothing to prove, for the stable density is then 0 on the positive half line. If $-1 < \beta \le 1$, then by the continuity, unimodality and regular variation at infinity of the stable density, there is an $M_1 = M_1(\epsilon) < \infty$ such that for all $\epsilon \le \rho \le 1$ and $x > 0$ we have

$$\frac{f(\rho x)}{f(x)} \le M_1 \tag{17}$$

By this we immediately conclude that

$$xI_3(x) \le M_1 x \int_0^{(1-\epsilon)x/\mu} f\left(\frac{x}{\sigma w^{1/\alpha}}\right) \frac{1}{w^{1+1/\alpha}}\, dw = M_1 \sigma\alpha \int_{\sigma^{-1}(1-\epsilon)^{-1/\alpha}\mu^{1/\alpha}x^{1-1/\alpha}}^{\infty} f(y)\, dy$$

which proves (16) as the last expression tends to zero when $x \to 0+$.

For all $w > (1+\epsilon)x/\mu$ we have

$$\left|\frac{x-\mu w}{\sigma w^{1/\alpha}}\right| \geq \frac{\epsilon}{1+\epsilon}\frac{\mu w}{\sigma w^{1/\alpha}} = \frac{\epsilon}{1+\epsilon}\frac{\mu}{\sigma}w^{1-1/\alpha}$$

and there exists an $M_2 < \infty$ such that for all $y \in R$ we have

$$f(y) \leq M_2|y|^{-1-\alpha}. \tag{18}$$

Consequently,

$$f\left(\frac{x-\mu w}{\sigma w^{1/\alpha}}\right) \leq C w^{-(1+\alpha)(1-1/\alpha)},$$

where C is a finite positive constant, that in the sequel may change from time to time. We conclude that

$$xI_5(x) \leq Cx\int_{(1+\epsilon)x/\mu}^{\infty} w^{-(1+\alpha)}\,dw \leq Cx^{1-\alpha} \to 0$$

as $x \to 0+$. This together with (16) implies that

$$L_+(\alpha,\sigma,\beta,\mu) = \frac{1}{\sigma}\lim_{x\to 0+} x\int_{(1-\epsilon)x/\mu}^{(1+\epsilon)x/\mu} f\left(\frac{x-\mu w}{\sigma w^{1/\alpha}}\right)\frac{1}{w^{1+1/\alpha}}\,dw. \tag{19}$$

Suppose first of all that $-1 < \beta < 1$. Then the stable density is still positive everywhere, so that (15) (and its extension to the negative half line) holds. Since ϵ can be taken as close to 0 as we wish, we may, therefore, replace $w^{1/\alpha}$ in the argument of f in (19) by $(x/\mu)^{1/\alpha}$. It does not require an argument that the same can be done to the integrand $1/w^{1+1/\alpha}$. We conclude that

$$\begin{aligned}
L_+(\alpha,\sigma,\beta,\mu) &= \frac{1}{\sigma}\lim_{x\to 0+} x\int_{(1-\epsilon)x/\mu}^{(1+\epsilon)x/\mu} f\left(\frac{x-\mu w}{\sigma(x/\mu)^{1/\alpha}}\right)\frac{1}{(x/\mu)^{1+1/\alpha}}\,dw \\
&= \mu^{1+1/\alpha}\frac{1}{\sigma}\lim_{x\to 0+} x^{-1/\alpha}\int_{(1-\epsilon)x/\mu}^{(1+\epsilon)x/\mu} f\left(\frac{x-\mu w}{\sigma\mu^{-1/\alpha}x^{1/\alpha}}\right)\,dw \\
&= \mu^{1+1/\alpha}\frac{1}{\sigma}\lim_{x\to 0+} x^{-1/\alpha}\int_{-\epsilon x}^{\epsilon x} f\left(\frac{y}{\sigma\mu^{-1/\alpha}x^{1/\alpha}}\right)\,dy \qquad (20) \\
&= \frac{1}{\sigma}\lim_{x\to 0+}\int_{-\epsilon\mu^{1/\alpha}x^{1-1/\alpha}}^{\epsilon\mu^{1/\alpha}x^{1-1/\alpha}} f(z/\sigma)\,dz = \int_{-\infty}^{\infty} f(z)\,dz = 1.
\end{aligned}$$

This proves the last line in (11) in the case $-1 < \beta < 1$.

Suppose now that $\beta = 1$ (the case $\beta = -1$ is similar). Then the stable density vanishes on the negative half line, and so by the above

$$L_+(\alpha,\sigma,1,\mu) = \frac{1}{\sigma}\lim_{x\to 0+} x\int_{(1-\epsilon)x/\mu}^{x/\mu} f\left(\frac{x-\mu w}{\sigma w^{1/\alpha}}\right)\frac{1}{w^{1+1/\alpha}}\,dw.$$

For $\epsilon_1 > 0$, it is convenient to split the last integral as a sum of two integrals $I_6(x)+I_7(x) = \int_{(1-\epsilon)x/\mu}^{(x/\mu)-\epsilon_1 x^{1/\alpha}/\mu^{1+1/\alpha}} + \int_{(x/\mu)-\epsilon_1 x^{1/\alpha}/\mu^{1+1/\alpha}}^{x/\mu}$.

Now, by the boundedness of the stable density we have for all $x > 0$ small enough

$$xI_7(x) \le Cx\int_{(x/\mu)-\epsilon_1 x^{1/\alpha}/\mu^{1+1/\alpha}}^{x/\mu} (x/2\mu)^{-1-1/\alpha}\,dw \le C\epsilon_1 \tag{21}$$

(and ϵ_1 can be taken as small as we wish).

Observe that the following analog of (15) holds in the present case. For any $\delta > 0$ there is a function K as in (15) such that (15) holds as long as both $x \ge \delta$ and $\rho x \ge \delta$. Since for every $(1-\epsilon)x/\mu < w < (x/\mu) - \epsilon_1 x^{1/\alpha}/\mu^{1+1/\alpha}$ we have

$$1 \ge \frac{x-\mu w}{\sigma w^{1/\alpha}} \ge \frac{\epsilon_1}{\sigma}$$

uniformly in x, we conclude that we may, once again, replace w under the integral in $I_6(x)$ by x/μ. Therefore, we obtain as in (20)

$$\lim_{x\to 0+} xI_6(x) = \frac{1}{\sigma\mu^{1+1/\alpha}}\lim_{x\to 0+} x^{-1/\alpha}\int_{(1-\epsilon)x/\mu}^{(x/\mu)-\epsilon_1 x^{1/\alpha}/\mu^{1+1/\alpha}} f\left(\frac{x-\mu w}{\sigma(x/\mu)^{1/\alpha}}\right)dw = \int_{\epsilon_1}^{\infty} f(z)\,dz.$$

We now let $\epsilon_1 \to \infty$, and we obtain, once again, the last line in (11) by the above, (21) and by recalling that the stable density vanishes, in the present case, on the negative half line.

Case: $0 < \alpha < 1$, $\mu < 0$. We start, once again, with the case $-1 < \beta < 1$. By the positivity, continuity and eventual monotonicity of the stable density there is an $M_3 < \infty$ such that for all $0 \le y_1 \le y_2$ we have

$$\frac{f(y_2)}{f(y_1)} \le M_3. \tag{22}$$

By this and by (18) we have

$$xI_1(x) \le Cx\int_0^{\epsilon x} f\left(\frac{x}{\sigma w^{1/\alpha}}\right)\frac{1}{w^{1+1/\alpha}}\,dw = C\int_{\epsilon^{-1/\alpha}\sigma^{-1}x^{1-1/\alpha}}^{\infty} f(y)\,dy$$

with the right hand side converging to zero, and

$$
\begin{aligned}
xI_2(x) &\leq Cx\int_{\epsilon x}^{\epsilon} f(|\mu|w/\sigma w^{1/\alpha})1/w^{1+1/\alpha}\,dw \\
&\leq Cx\int_{\epsilon x}^{\epsilon} w^{-(1-1/\alpha)(1+\alpha)}1/w^{1+1/\alpha}\,dw \\
&= Cx\int_{\epsilon x}^{\epsilon} w^{-(1+\alpha)}\,dw \leq Cx^{1-\alpha}.
\end{aligned}
$$

These prove the corresponding line in (11), in the case $-1 < \beta < 1$. The case $\beta = -1$ does not require an argument since then the stable density vanishes on the positive half line.

Consider now the case $\beta = 1$, in which case, the stable density vanishes on the negative half line. By (17) (whose validity does not depend on the sign of μ) we have

$$
\begin{aligned}
xI_1(x) &\leq Cx\int_0^{\epsilon x} f\left(\frac{x(1+|\mu|\epsilon)}{\sigma w^{1/\alpha}}\right)\frac{1}{w^{1+1/\alpha}}\,dw \\
&= C\int_{\sigma^{-1}\epsilon^{-1/\alpha}x^{1-1/\alpha}(1+|\mu|\epsilon)^{-1/\alpha}}^{\infty} f(y)\,dy
\end{aligned}
$$

and thus convergence of $xI_1(x)$ to zero. Further, by (18) we have

$$
xI_2(x) \leq Cx\int_{\epsilon x}^{x}\left(\frac{\mu w}{\sigma w^{1/\alpha}}\right)^{-1-\alpha}\frac{1}{w^{1+1/\alpha}}\,dw \leq Cx\int_{\epsilon x}^{\infty} w^{-1-\alpha}\,dw = Cx^{1-\alpha}
$$

which altogether proves the third from the bottom line in (11) in all cases.

Case: $0 < \alpha < 1,\ \mu = 0$. Here

$$
\begin{aligned}
L_+(\alpha,\sigma,\beta,0) &= \frac{1}{\sigma}\lim_{x\to 0+} x\int_0^{\epsilon} f\left(\frac{x}{\sigma w^{1/\alpha}}\right)\frac{1}{w^{1+1/\alpha}}\,dw = \alpha\lim_{x\to 0+}\int_{\sigma^{-1}\epsilon^{-1/\alpha}x}^{\infty} f(y)\,dy \\
&= \alpha\int_0^{\infty} f(y)\,dy = \frac{\alpha}{2} + \frac{1}{\pi}\arctan\Big(\beta\tan\frac{\pi\alpha}{2}\Big)
\end{aligned}
$$

as before. This proves the second from the bottom line in (11).

Let us consider now the case $\alpha = 1$. Here, once again by the l'Hospital rule we need to compute

$$
L_+(1,\sigma,\beta,\mu) = \frac{1}{\sigma}\lim_{x\to 0+} x\int_0^{\infty} f\left(\frac{x-\mu w-(2/\pi)\beta\sigma w\log(\sigma w)}{\sigma w}\right)\frac{e^{-w}}{w^2}\,dw.
$$

The same argument as in the case $\alpha \neq 1$ shows that for every $\epsilon > 0$,

$$
L_+(1,\sigma,\beta,\mu) = \frac{1}{\sigma}\lim_{x\to 0+} x\int_0^{\epsilon} f\left(\frac{x-\mu w-(2/\pi)\beta\sigma w\log(\sigma w)}{\sigma w}\right)\frac{1}{w^2}\,dw.
$$

Case: $\alpha = 1,\ \beta > 0$. Observe that in the case $\alpha = 1$ the stable density is everywhere positive thus (22) still holds. Moreover, there is a $\theta > 0$ such that for all ϵ small enough and all $0 < w < \epsilon$ we have

$$-\mu w - (2/\pi)\beta\sigma w \log(\sigma w) \geq \theta w |\log w|.$$

Therefore, by (22) we have

$$\int_0^\epsilon f\left(\frac{x - \mu w - (2/\pi)\beta\sigma w \log(\sigma w)}{\sigma w}\right)\frac{1}{w^2}\,dw \leq C\int_0^\epsilon f\left(\frac{x + \theta w|\log w|}{\sigma w}\right)\frac{1}{w^2}\,dw.$$

Write now for a $k > 1$

$$\int_0^\epsilon f\left(\frac{x + \theta w|\log w|}{\sigma w}\right)\frac{1}{w^2}\,dw = \int_0^{kx/|\log x|} + x\int_{kx/|\log x|}^{kx} + \int_{kx}^{\epsilon} = I_8(x) + I_9(x) + I_{10}(x).$$

Again using (22), we obtain

$$xI_8(x) \leq Cx\int_0^{kx/|\log x|} f\left(\frac{x}{\sigma w}\right)\frac{1}{w^2}\,dw = C\int_{k^{-1}\sigma^{-1}|\log x|}^{\infty} f(z)\,dz$$

with the latter converging to zero as $x \to 0+$. Furthermore, by (22) and (18) we have

$$\begin{aligned} xI_9(x) &\leq Cx\int_{kx/|\log x|}^{kx} f(\theta\sigma^{-1}|\log w|)\frac{1}{w^2}\,dw \leq Cx(\log x)^{-2}\int_{kx/|\log x|}^{kx}\frac{1}{w^2}\,dw \\ &\leq Cx(\log x)^{-2}\int_{kx/|\log x|}^{\infty}\frac{1}{w^2}\,dw = C|\log x|^{-1} \end{aligned}$$

ensuring convergence of $xI_9(x)$ to zero. Finally, by the boundedness of the stable density

$$xI_{10}(x) \leq Cx\int_{kx}^{\infty}\frac{1}{w^2}\,dw = Ck^{-1},$$

and since we may take k as big as we wish, the second line in (11) follows.

Case: $\alpha = 1,\ \beta < 0$. Write

$$\begin{aligned} &\int_0^\epsilon f\left(\frac{x - \mu w - (2/\pi)\beta\sigma w \log(\sigma w)}{\sigma w}\right)\frac{1}{w^2}\,dw \\ = &\int_0^{\frac{1-\epsilon}{(2/\pi)|\beta|\sigma}\frac{x}{|\log x|}} + \int_{\frac{1-\epsilon}{(2/\pi)|\beta|\sigma}\frac{x}{|\log x|}}^{\frac{1+\epsilon}{(2/\pi)|\beta|\sigma}\frac{x}{|\log x|}} + \int_{\frac{1+\epsilon}{(2/\pi)|\beta|\sigma}\frac{x}{|\log x|}}^{\epsilon} \\ = &\ I_{11}(x) + I_{12}(x) + I_{13}(x). \end{aligned}$$

First, we write $I_{12}(x)$ as

$$\begin{aligned}
&\int_{\frac{1-\epsilon}{(2/\pi)|\beta|\sigma}\frac{x}{|\log x|}}^{\frac{1+\epsilon}{(2/\pi)|\beta|\sigma}\frac{x}{|\log x|}} f\left(\frac{1}{\sigma}\Big(\frac{x}{w}-\frac{2}{\pi}|\beta|\sigma|\log w|-(\mu+\frac{2}{\pi}\beta\sigma\log\sigma)\Big)\right)\Big(\frac{x}{w^2}-\frac{2}{\pi}|\beta|\sigma\frac{1}{w}\Big)\,dw \\
+\quad & \frac{2}{\pi}|\beta|\sigma\int_{\frac{1-\epsilon}{(2/\pi)|\beta|\sigma}\frac{x}{|\log x|}}^{\frac{1+\epsilon}{(2/\pi)|\beta|\sigma}\frac{x}{|\log x|}} f\left(\frac{1}{\sigma}\Big(\frac{x}{w}-\frac{2}{\pi}|\beta|\sigma|\log w|-(\mu+\frac{2}{\pi}\beta\sigma\log\sigma)\Big)\right)\frac{1}{w}\,dw \\
=\quad & I_{14}(x)+I_{15}(x).
\end{aligned}$$

Observe that by the boundedness of the stable density,

$$I_{15}(x)\le C\log\frac{1+\epsilon}{1-\epsilon}. \tag{23}$$

We now concentrate on $I_{14}(x)$. Change the variable of integration by defining

$$y=y(w)=\frac{x}{w}-\frac{2\sigma}{\pi}|\beta||\log w|.$$

Then

$$y'(w)=-\frac{x}{w^2}+\frac{2\sigma}{\pi}|\beta|\sigma\frac{1}{w}<0$$

for all

$$\frac{\pi(1-\epsilon)}{2\sigma|\beta|}\frac{x}{|\log x|}<w<\frac{\pi(1+\epsilon)}{2\sigma|\beta|}\frac{x}{|\log x|}$$

when $x>0$ is small enough. Therefore,

$$I_{14}(x)=\int_{A(x)}^{B(x)} f\left(y/\sigma\right)\,dy,$$

with

$$\begin{aligned}
A(x) &= y\Big(\frac{1+\epsilon}{(2/\pi)|\beta|\sigma}\frac{x}{|\log x|}\Big)-(\mu+\frac{2}{\pi}\beta\sigma\log\sigma),\\
B(x) &= y\Big(\frac{1-\epsilon}{(2/\pi)|\beta|\sigma}\frac{x}{|\log x|}\Big)-(\mu+\frac{2}{\pi}\beta\sigma\log\sigma).
\end{aligned}$$

Note that $A(x)\to-\infty$ and $B(x)\to\infty$ as $x\to 0+$. We conclude that

$$\lim_{x\to 0+} I_{14}(x)=\int_{-\infty}^{\infty} f(y/\sigma)\,dy=\sigma. \tag{24}$$

Concerning I_{13}, observe that there is a $\gamma=\gamma(\epsilon)>0$ such that for all $\frac{1+\epsilon}{(2/\pi)|\beta|\sigma}\frac{x}{|\log x|}<w<\epsilon$ we have

$$\Big|\frac{x-\mu w-(2/\pi)\beta\sigma w\log(\sigma w)}{\sigma w}\Big|\ge\gamma|\log w|.$$

Therefore, by (18) we have

$$\begin{aligned} xI_{13}(x) &\le Cx\int_{\frac{1+\epsilon}{(2/\pi)|\beta|\sigma}\frac{x}{|\log x|}}^{\epsilon} (\log w)^{-2} 1/w^2\,dw \\ &\le Cx(\log x)^{-2}\int_{\frac{1+\epsilon}{(2/\pi)|\beta|\sigma}\frac{x}{|\log x|}}^{\epsilon} 1/w^2\,dw \\ &\le C|\log x|^{-1}\to 0 \qquad (25)\end{aligned}$$

as $x\to 0+$.

Finally for I_{11}, we have by (22) that

$$xI_{11}(x)\le Cx\int_0^{\frac{1-\epsilon}{(2/\pi)|\beta|\sigma}\frac{x}{|\log x|}} f\left(\frac{x}{\sigma w}\right)\frac{1}{w^2}\,dw = C\int_{\frac{(2/\pi)|\beta|\sigma}{1-\epsilon}|\log x|}^{\infty} f(z)\,dz\to 0 \qquad (26)$$

as $x\to 0+$.

Now the fourth line in (11) follows from (23), (24), (25) and (26) after letting $\epsilon\to 0$.

Case: $\alpha=1,\ \beta=0$. We have

$$\begin{aligned} L_+(1,\sigma,0,\mu) &= \frac{1}{\sigma}\lim_{x\to 0+} x\int_0^{\epsilon} f\left(\frac{x-\mu w}{\sigma w}\right)\frac{1}{w^2}\,dw = \frac{1}{\sigma}\lim_{x\to 0+} x\int_0^{\epsilon} f\left(\frac{x}{\sigma w}-\frac{\mu}{\sigma}\right)\frac{1}{w^2}\,dw \\ &= \lim_{x\to 0+}\int_{x/(\sigma\epsilon)}^{\infty} f\left(y-\frac{\mu}{\sigma}\right)dy = \int_0^{\infty} f\left(y-\frac{\mu}{\sigma}\right)dy \\ &= \int_{-\mu/\sigma}^{\infty}\frac{1}{\pi(1+y^2)}\,dy = \frac{1}{2}+\frac{1}{\pi}\arctan\left(\frac{\mu}{\sigma}\right),\end{aligned}$$

thus proving the only remaining line in (11) and completing the proof. □

Appendix

There exists numerous papers on the rate of convergence for the series representation of stable laws. However, most of the work dealt with convergence in distribution with respect to suitable metrics and, surprisingly, we could not find suitable reference for the rate of almost sure convergence. By the result below, with probability one the remainder R_n of the series in the symmetric case is of order $O(n^{1/2-1/\alpha}\sqrt{\log\log n})$. On the other hand, $R_n/(n^{1/2-1/\alpha})$ is not convergent almost surely to zero as the sequence $\{(R_n/n^{1/2-1/\alpha})^2\}$ is uniformly integrable and $\lim_{n\to\infty} E(R_n/(n^{1/2-1/\alpha}))^2=\alpha/(2-\alpha)>0$.

Theorem 3. *Let* (δ_i) *be a Rademacher sequence and* Γ_i*'s are arrival times of a standard Poisson process and* $\alpha \in (0,2)$*. Then with probability one*

$$\limsup_{n\to\infty} \frac{\sum_{i=n}^{\infty} \delta_i \Gamma_i^{-1/\alpha}}{n^{1/2-1/\alpha}\sqrt{\log\log n}} \le \frac{4-\alpha}{2-\alpha}.$$

Proof. Let $R_n = \sum_{i=n}^{\infty} \delta_i \Gamma_i^{-1/\alpha}$, $A_n = -\sum_{i=1}^{n-1} \delta_i / \Gamma_n^{1/\alpha}$, $B_n = -\sum_{i=n+1}^{\infty} (\sum_{k=1}^{i-1} \delta_k)(\Gamma_i^{-1/\alpha} - \Gamma_{i-1}^{-1/\alpha})$, $c_n = n^{1/2-1/\alpha}\sqrt{\log\log n}$. Then, by a discrete version of integration by parts,

$$R_n = A_n + B_n$$

and thus

$$\limsup_{n\to\infty} \frac{R_n}{c_n} \le \limsup_{n\to\infty} \frac{A_n}{c_n} + \limsup_{n\to\infty} \frac{B_n}{c_n}.$$

By Law of Large Numbers and Law of Iterated Logarithm $\limsup_{n\to\infty} \frac{A_n}{c_n} = 1$. To deal with B_n/c_n note that since $\limsup_{i\to\infty} \sum_{k=1}^{i-1} \delta_k / \sqrt{\Gamma_{i-1} \log\log \Gamma_{i-1}} = 1$, we have

$$\begin{aligned}
\limsup_{n\to\infty} \frac{B_n}{c_n} &\le \limsup_{n\to\infty} \frac{1}{c_n} \sum_{i=n}^{\infty} \sqrt{\Gamma_{i-1} \log\log \Gamma_{i-1}} (\Gamma_{i-1}^{-1/\alpha} - \Gamma_i^{-1/\alpha}) \\
&\le \frac{1}{\alpha} \limsup_{n\to\infty} \frac{1}{c_n} \int_{\Gamma_{n-1}}^{\infty} x^{-1/\alpha - 1/2} \sqrt{\log\log x}\, dx \\
&= \frac{1}{\alpha} \frac{2\alpha}{2-\alpha} \limsup_{n\to\infty} \frac{\Gamma_{n-1}^{1/2-1/\alpha} \sqrt{\log\log \Gamma_{n-1}}}{c_n} \\
&= \frac{2}{2-\alpha}
\end{aligned}$$

which concludes the proof. □

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