Fixing Up the Rate of Return Approach: The Rate of Return on Invested Capital\(^1\)

by

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1. Introduction

In discussing the merits of an investment project, the internal rate of return (IRR) is an extremely useful measure. It is a relative measure of merit that can be compared immediately with other rates of return such as the cost of capital (CC), the minimum attractive rate of return on invested capital (MARR), or the internal rates of return of other projects. The internal rate of return is a well known measure of profitability used in finance and in industry. Functions to compute the IRR of a cash flow stream are available on pocket calculators and in spreadsheet programs. Unfortunately, the IRR measure is seriously flawed. The purpose of this paper is to show how to fix it.

Most users of the IRR approach are well aware of the fact that there exist cash flow streams for which multiple IRR's exist. This problem is mentioned in nearly every text on engineering economy. Fewer users are aware that there exist cash flow streams for which the IRR does not make sense; that is, it's interpretation involves an implicit assumption that most users would regard as unacceptable.

Fortunately, there exists a straightforward correction to the IRR approach that solves both the uniqueness and the interpretation problems in a satisfying fashion. This approach, known as the rate of return on invested capital (RIC) approach has been known in the engineering economy literature since 1965, but it is completely ignored in many texts on engineering economy. In an attempt to correct the uniqueness problem, some texts have advocated *ad hoc* techniques such as the so-called external rate of return (ERR) approach. Unfortunately, such techniques do not solve the problem of interpretation. The texts by Bussey [1978] and, more recently, by Park [1993] are notable exceptions.

The rate of return on invested capital (RIC) is a relative measure of project merit that can be compared to other rates of return such as the MARR. If the user is equipped with a spreadsheet, the RIC is not difficult to calculate. It is unique, provided the cash
flow stream satisfies some minor conditions. The RIC is identical to the IRR in those cases when the IRR has a valid interpretation. In those cases when no IRR has a valid interpretation, the RIC is theoretically correct. Furthermore, the RIC approach is equivalent to the net present value (NPV) approach when determining project profitability. Given all these advantages, it is surprising that more engineering economy texts have not included the RIC approach and that spreadsheet programs and financial calculators do not provide functions for its computation. The text by Park is unique in that it both includes this approach and provides software to compute the RIC. This paper can be viewed as a supplement to the Park text since we integrate the mathematical rationale for the RIC with the presentation.

In this paper, we motivate and present the RIC approach. It is a long paper so we include a summary of the logic, without the mathematical notation or the examples, at the end of the paper (Section 9). The reader may wish to skim the summary frequently to be reminded of the structure of the argument. If the reader is interested only in learning the RIC definition, that is found in Section 7 with an example. The paper is organized as follows. In Section 2, we review the internal rate of return approach and motivate it with a capital budgeting example. In Section 3, we note that rates of return should not be used to rank mutually exclusive projects but instead an incremental rate of return approach should be used. In Section 4, we note that the incremental rate of return approach, as well as projects that involve such cash outflows as environmental cleanup, can give rise to complicated cash flow streams: cash flows which change in sign more than once. It is these cash flow streams which can exhibit multiple internal rates of return. The main problem is not uniqueness, however. In Section 5, we show that the problem is that some internal rates of return do not have a valid interpretation. In Section 6, we note that when an internal rate of return is pure, it has a valid interpretation and we give rules for identifying a pure internal rate of return. In Section 7, we define the rate of return on invested capital. This rate of return by definition always has a valid interpretation. We
compute the RIC for all prior examples. In Section 8, we present important properties of the RIC such as the fact that it equals the pure IRR whenever the latter exists and the RIC criterion is equivalent to the NPV criterion for determining project profitability. In Section 9, we present a summary of the paper in bullet form. The material of this paper is derived from the original articles by Teichroew et al [1965] and from the text by Bussey [1978].

2. THE INTERNAL RATE OF RETURN APPROACH

The Internal Rate of Return

The internal rate of return (IRR) of a project is a breakeven cost of financing the project. It is the value of the interest rate that sets the future value (equivalently, the present value or the annual value) of the project equal to zero.

The internal rate of return criterion states that an investment project is profitable if the IRR > MARR (Minimum Attractive Rate of Return). Unlike the NPV criterion, the IRR criterion is not always valid.

The Capital Budgeting Problem

A good motivation for using the internal rate of return comes from the capital budgeting problem. This refers to the problem of choosing a set of projects in which to invest from a large collection of potential projects when there is a restriction, or budget, on the amount of capital available to finance the initial capital outlays of the projects.

Suppose there are \( N \) projects all of which are suitable for investment in the current budget period. Let \( B \) denote size of the capital budget, that is, the amount of capital available for investment in this budget period. Let the initial capital outlay of the \( i^{th} \) project be denoted by \( P_i \) and let the annual payoff to investment in the \( i^{th} \) project be denoted by \( A_i \). If the cost of capital is \( CC \), then the net present value of the \( i^{th} \) project is \( A_i/CC - P_i \), assuming there is no end to the payoffs for each project (i.e. the payoffs form a perpetual annuity). The decision to invest in the \( i^{th} \) project can be
represented by a decision variable, $x_i$, provided we restrict the variable to the set of values \{0,1\}. The choice $x_i = 1$ will mean the project is selected for investment and the choice $x_i = 0$ will mean that the project is rejected. Thus, depending on the decision, $x_i$, the amount of the budget invested in the $i^{th}$ project is $P_i x_i$ and the net present value of that decision is $(A_i/CC - P_i) x_i$. The capital budgeting problem is the problem of choosing values for all $N$ decision variables in order to maximize the total net present value of the selection subject to investing no more than the available budget. That is, find values for $x_i$, $i=1,...,N$, to

$$\text{maximize} \sum_{i=1}^{N} \left( \frac{A_i}{CC} - P_i \right) x_i$$

subject to

$$\sum_{i=1}^{N} P_i x_i \leq B$$

$x_i \in \{0,1\}$, $i = 1,...,N$.

**The Marginal Analysis Solution**

The capital budgeting problem is a mathematical programming problem commonly known as the knapsack problem. There exist algorithms to find the optimal solution to this type of problem, but here we are interested in a more intuitive approach. Typically, a good solution, although not necessarily an optimal solution, can be found using a technique called marginal analysis.

In the marginal analysis technique, we imagine investing the budget dollar by dollar and we invest each successive dollar in the best possible manner. In fractional terms, if we invested one dollar in the $i^{th}$ project, we would obtain $1/P_i$ of the project and therefore the corresponding net present value would be $(A_i/CC-P_i)/P_i$. Thus, the best way to spend the first dollar of the budget would be to invest in the project with the largest ratio $(A_i/CC-P_i)/P_i$. That is, invest the first dollar in the project that yields the largest net present value per unit of initial capital outlay, or "the most bang for the buck" in colloquial terms. Suppose that it is the $i^{th}$ project that has the largest such ratio.
Then the first dollar should be invested in the $i^{th}$ project, and the second dollar, and the third, and so on until $P_i$ dollars are invested. Once the $i^{th}$ project is fully funded, we look for the next best investment. We invest the incremental budget dollars in the project with the next largest ratio, $(A_i/CC-P_i)/P_i$. We continue in this manner until either all the projects are fully funded or the budget is exhausted. Naturally, it is not possible to fractionally fund a project so the last project being considered would have to dropped if the budget was not sufficient to fund it completely.

Observe that the marginal analysis technique amounts to simply sorting the projects according to the ratio $(A_i/CC-P_i)/P_i$, from largest to smallest, and investing in the projects in that order until the residual budget is insufficient to fund the next project. Also note that the value of the cost of capital, CC, does not affect the order in which the projects are selected. The only role that the cost of capital plays in project selection is in determining which projects are profitable. If CC is so large that the net present value is negative, $A_i/CC-P_i < 0$, then the project is unprofitable and should not be considered in the capital budgeting problem. Thus, all that is important in the ratio is the ratio $A_i/P_i$, which is the internal rate of return for this type of investment. Let $\text{IRR}_i = A_i/P_i$, the internal rate of return of the $i^{th}$ project.

The internal rate of return of the project serves as a guide to investing. By ranking the projects in order of their internal rate of return, not in order of their net present value, we ensure that each successive budget increment is invested to maximum advantage. It is that intuition that serves as the rationale for widespread use of the internal rate of return.

**Example 1.**

Table 1 lists the initial capital outlay and annual returns for a collection of twenty projects. The projects have been pre-sorted and numbered in decreasing order of their internal rate of return, which is simply the ratio of the annual return to the initial outlay. Each project is profitable at a cost of capital of 5%, as evidenced by the positive net present value in each case. The last two columns of the table indicate the cumulative
capital requirements and net present value that would result if the projects were selected for investment in order of their IRR. These two columns are graphed in Figure 1.

Suppose the capital budget is $90,000. The marginal analysis technique would result in selecting the first 18 projects of Table 1, for a total initial capital outlay of $89,754 and a total net present value of $220,506.

<table>
<thead>
<tr>
<th>IRR Sort Sequence</th>
<th>Initial Capital Outlay</th>
<th>Annual Return</th>
<th>Net Present Value @ 5% (NPV)</th>
<th>Internal Rate of Return (IRR)</th>
<th>Cumulative Capital Outlay</th>
<th>Cumulative NPV</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5,153</td>
<td>1,265</td>
<td>20,147</td>
<td>24.55%</td>
<td>5,153</td>
<td>20,147</td>
</tr>
<tr>
<td>2</td>
<td>6,329</td>
<td>1,521</td>
<td>24,091</td>
<td>24.03%</td>
<td>11,482</td>
<td>44,238</td>
</tr>
<tr>
<td>3</td>
<td>9,901</td>
<td>2,262</td>
<td>35,339</td>
<td>22.85%</td>
<td>21,383</td>
<td>79,577</td>
</tr>
<tr>
<td>4</td>
<td>5,772</td>
<td>1,317</td>
<td>20,568</td>
<td>22.82%</td>
<td>27,155</td>
<td>100,145</td>
</tr>
<tr>
<td>5</td>
<td>1,932</td>
<td>426</td>
<td>6,588</td>
<td>22.05%</td>
<td>29,087</td>
<td>106,733</td>
</tr>
<tr>
<td>6</td>
<td>2,907</td>
<td>621</td>
<td>9,513</td>
<td>21.36%</td>
<td>31,994</td>
<td>116,246</td>
</tr>
<tr>
<td>7</td>
<td>6,600</td>
<td>1,192</td>
<td>17,240</td>
<td>18.06%</td>
<td>38,594</td>
<td>133,436</td>
</tr>
<tr>
<td>8</td>
<td>858</td>
<td>152</td>
<td>2,182</td>
<td>17.72%</td>
<td>39,452</td>
<td>135,668</td>
</tr>
<tr>
<td>9</td>
<td>8,669</td>
<td>1,498</td>
<td>21,291</td>
<td>17.28%</td>
<td>48,121</td>
<td>156,959</td>
</tr>
<tr>
<td>10</td>
<td>4,866</td>
<td>716</td>
<td>9,454</td>
<td>14.71%</td>
<td>52,987</td>
<td>166,413</td>
</tr>
<tr>
<td>11</td>
<td>1,910</td>
<td>272</td>
<td>3,530</td>
<td>14.24%</td>
<td>54,897</td>
<td>169,943</td>
</tr>
<tr>
<td>12</td>
<td>3,177</td>
<td>442</td>
<td>5,663</td>
<td>13.91%</td>
<td>58,074</td>
<td>175,606</td>
</tr>
<tr>
<td>13</td>
<td>8,813</td>
<td>1,202</td>
<td>15,227</td>
<td>13.64%</td>
<td>66,887</td>
<td>190,833</td>
</tr>
<tr>
<td>14</td>
<td>7,800</td>
<td>980</td>
<td>11,800</td>
<td>12.56%</td>
<td>74,687</td>
<td>202,633</td>
</tr>
<tr>
<td>15</td>
<td>6,807</td>
<td>789</td>
<td>8,973</td>
<td>11.59%</td>
<td>81,494</td>
<td>211,606</td>
</tr>
<tr>
<td>16</td>
<td>5,285</td>
<td>606</td>
<td>6,835</td>
<td>11.47%</td>
<td>86,779</td>
<td>218,441</td>
</tr>
<tr>
<td>17</td>
<td>1,155</td>
<td>104</td>
<td>925</td>
<td>9.00%</td>
<td>87,934</td>
<td>219,366</td>
</tr>
<tr>
<td>18</td>
<td>1,820</td>
<td>148</td>
<td>1,140</td>
<td>8.13%</td>
<td>89,754</td>
<td>220,506</td>
</tr>
<tr>
<td>19</td>
<td>9,872</td>
<td>678</td>
<td>3,688</td>
<td>6.87%</td>
<td>99,626</td>
<td>224,194</td>
</tr>
<tr>
<td>20</td>
<td>6,286</td>
<td>360</td>
<td>914</td>
<td>5.73%</td>
<td>105,912</td>
<td>225,108</td>
</tr>
</tbody>
</table>

Table 1. Capital Budgeting Projects Sorted by IRR
Quality of the Marginal Analysis Solution

How good is the marginal analysis solution to the capital budgeting problem? It is possible that a different set of selected investments could achieve a higher net present value. In general, it is not easy to find the optimal solution but it is easy to establish a bound on how much better such a solution could be, compared to the marginal analysis solution.

Observe in example 1 that the marginal analysis solution does not completely exhaust the budget. There is $246 (= $90,000 - $89,754) left over. If fractional investments were permitted, then the best project in which to invest this residual budget would be the nineteenth project, where each dollar invested earns a rate of return of 6.87%. Investing in this way would yield an additional $16.90 (= $246 * 0.0687) of annual return. The net present value of this increment would be $92 (= $16.9 / 0.05 - $246) Consequently, the highest total net present value that any solution to the capital
budgeting problem could achieve is $220,598 (= $220,506 + $92). This is less than 0.1% more than the solution found by marginal analysis. In other words, the marginal analysis solution to the capital budgeting problem of example 1 is within 0.1% of the best possible solution.

Whenever the marginal analysis solution results in a very small residual budget, as in example 1, the solution will be extremely good. In particular, when the initial capital outlay of every project is small relative to the size of the capital budget, the residual budget must also be small. Consequently, when all projects are small relative to the capital budget, investing in projects in order of their internal rate of return will lead to very good solutions.

**The Minimum Attractive Rate of Return**

Many corporate finance departments screen requests for capital outlays by requiring that all projects that are submitted for consideration achieve an internal rate of return in excess of a specified hurdle rate, called the Minimum Attractive Rate of Return (MARR). The MARR is an arbitrary rate that should reflect the best rate of return that could be achieved by the finance department using alternative means. The MARR should be at least the cost of capital since the finance department could reduce its acquisition of capital and thereby save the cost of capital.

In relation to the capital budgeting problem, the MARR can be seen as a way to ration capital. In example 1, if the corporate finance department announced a MARR of 15%, then only nine of the twenty projects would have been submitted for consideration. If the MARR were dropped to 10%, then sixteen of the projects would be submitted. Figure 2 graphs the total capital requirements of submitted projects as a function of the announced MARR. Faced with a capital budget of $90,000, the corporate finance department could announce a MARR of anything between 6.87% (the IRR of the 19th project) and 8.13% (the IRR of the 18th project) and exactly the right number of projects.
would be submitted for funding, according to the marginal analysis solution to the capital budgeting problem. In theory, if the MARR is to represent the next best rate of return available, the announced MARR should be 6.87% and the criterion should be to submit only those projects whose IRR is strictly greater than 6.87%.

![Capital Budget Requests as a Function of the MARR](image)

**Figure 2. Capital Budget Requests as a Function of the MARR**

Therefore, one view of the MARR is that it should equal the IRR of the last project to be considered for investment using the marginal analysis technique for solving the capital budgeting problem. It represents the rate of return that could be achieved on the next increment to the capital budget. It is higher than the cost of capital because capital is being rationed. It is the *opportunity cost of capital* because it represents the opportunity to earn a higher rate of return that is foregone because of the capital restriction. If there were no limit to the amount of capital available at the current cost of capital, we would continue to invest in projects provided the internal rate of return
exceeded the cost of capital. In the case of unlimited capital, the MARR would equal the cost of capital.

In practice, the corporate finance department cannot announce a MARR to screen projects and expect that the resulting set of submitted projects will solve the capital budgeting problem exactly. Picking the right MARR to announce would require detailed knowledge of the projects before they are submitted, an impossibility. However, with experience, the finance department can anticipate the total capital requests that will result from a particular MARR and can estimate a value of the MARR to announce that is appropriate for the current budget.

3. THE INCREMENTAL RATE OF RETURN APPROACH

The internal rate of return approach is sometimes used inappropriately to rank mutually exclusive projects. As described in the previous section, the internal rate of return ranking of projects can lead to a good selection of projects when there is a capital budget. However, if faced with a choice between two or more projects and only one of the projects can be selected for reasons other than capital availability, then selecting the project with the highest internal rate of return can lead to an inferior decision. The correct decision can be made by viewing the selection as a series of incremental investment decisions and by applying the internal rate of return approach to this series.

Example 2.

One worker is currently employed manufacturing wooden parts for doll houses using a hand saw. The worker is paid $8.10 an hour and can produce a year's worth of parts in just 8 weeks of 40 hours per week. Total production is 1600 parts. Therefore, he averages 5 parts per hour when working by hand. The doll house manufacturer is considering the purchase of a power band saw with associated fixtures to improve the productivity of this operation. There are three models of power saw that could be purchased: the ECONO, the MIDSZ and the DLUX. The investment cost including the
cost of fixtures for the three models is $4,000, $6,000 and $7,000, respectively. The chief operating difference between these models is their speed of operation. The doll house manufacturer estimates that the worker could average 10 parts per hour with the ECONO version, 15 parts per hour with the MIDSZ, and 20 parts per hour with the DLUX. She also predicts a very long useful life for the power saw and the corresponding labor savings. (For simplicity, assume the annual labor savings form a perpetual annuity.) The power saw has no other potential use. The doll house manufacturing company uses a MARR of 10% to screen projects for investment.

Table 2 summarizes the calculations used to compute the internal rate of return of the three possible investment choices faced by the doll house manufacturer. All three choices have internal rates of return well in excess of the MARR of 10%. This indicates that each choice is profitable (since the MARR is always greater than CC, the cost of capital) and there is sufficient capital in the budget to invest in any one of them. However, only one of the power saw models will be purchased. When ranked by internal rate of return, the ECONO model appears to make the most efficient use of available capital. However, we will argue that the best investment is the DLUX model, the investment with the lowest internal rate of return.

<table>
<thead>
<tr>
<th>Category</th>
<th>Units</th>
<th>By Hand</th>
<th>ECONO</th>
<th>MIDSZ</th>
<th>DLUX</th>
</tr>
</thead>
<tbody>
<tr>
<td>Production requirement</td>
<td>parts/yr.</td>
<td>1,600</td>
<td>1,600</td>
<td>1,600</td>
<td>1,600</td>
</tr>
<tr>
<td>Production rate</td>
<td>parts/hr.</td>
<td>5</td>
<td>10</td>
<td>15</td>
<td>20</td>
</tr>
<tr>
<td>Labor hours required</td>
<td>hrs./yr.</td>
<td>320</td>
<td>160</td>
<td>107</td>
<td>80</td>
</tr>
<tr>
<td>Labor cost @ $8.10/hr.</td>
<td>$/yr.</td>
<td>2,592</td>
<td>1,296</td>
<td>864</td>
<td>648</td>
</tr>
<tr>
<td>Labor savings</td>
<td>$/yr.</td>
<td></td>
<td>1,296</td>
<td>1,728</td>
<td>1,944</td>
</tr>
<tr>
<td>Initial investment</td>
<td>$</td>
<td></td>
<td>4,000</td>
<td>6,000</td>
<td>7,000</td>
</tr>
<tr>
<td>Internal rate of return</td>
<td></td>
<td>32.40%</td>
<td>28.80%</td>
<td>27.77%</td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Internal Rates of Return of Power Saw Alternatives

The reason the internal rate of return is misleading is that in the marginal analysis approach to the capital budgeting problem it is used to add projects to a set whereas in
the case of mutually exclusive projects, the decision is between projects. We can convert
the problem to one of project addition by considering the differences between the
projects. For example, suppose the manufacturer decided to purchase at least the most
inexpensive power saw, the ECONO version, and was contemplating whether or not it
was worthwhile to increase the investment and purchase the MIDSZ version. We could
refer to this decision to upgrade to the MIDSZ as UPTOMIDSZ. The incremental
investment required by this decision is $2000 (=6000 - $4000) but it would result in
increased annual labor savings of $432 (=1728 -$1296). The MIDSZ investment can be
seen as the sum of two investment projects: the ECONO plus the UPTOMIDSZ, both of
which can be funded.

The internal rate of return on the ECONO investment is greater than the MARR,
so it should be funded. The internal rate of return on the UPTOMIDSZ incremental
investment is 21.6% which is greater than the MARR as well. Thus, both ECONO and
UPTOMIDSZ should be funded. In other words, MIDSZ is a better investment than
ECONO. Think of it this way: the doll house manufacturer is willing to invest in projects
that earn only 10%. UPTOMIDSZ presents an opportunity to earn 21.6% on an
incremental investment of $2000, which is better than the best alternative use of this
capital. ECONO would be the preferred investment only if the $2000 saved by not
investing in MIDSZ could be invested elsewhere at more than 21.6% rate of return. The
MARR indicates that this is not the case.

A similar incremental analysis can be performed to compare the DLUX to the
MIDSZ alternative. (If the MIDSZ alternative was inferior to the ECONO alternative,
then we would compare DLUX to ECONO in this analysis.) The incremental investment
of choosing DLUX over MIDSZ is $1000. The corresponding reduction in labor costs is
$216. The internal rate of return on the incremental investment is thus 21.6% which is
greater than the MARR of 10%. The incremental investment is justified. The preferred
alternative is to invest in the DLUX model power saw.
We conclude that the internal rate of return approach must be applied to project increments when selecting from among mutually exclusive alternatives. Begin with the project requiring the lowest capital investment and switch to a project with a larger capital investment if and only if the internal rate of return on the incremental investment exceeds the minimum attractive rate of return. Any two projects can be compared in this manner provided that the project with the lower capital investment meets the investment criterion. That is, the smaller project must have an internal rate of return that exceeds the minimum attractive rate of return. Otherwise, it makes no sense to evaluate the incremental investment to the larger project. With care, the incremental rate of return approach can be used to select among mutually exclusive projects.

4. The Problem of Multiple Internal Rates of Return

Lorie and Savage [1955] use the incremental rate of return approach applied to the choice of pump sizes to create a simple example of an investment with multiple internal rates of return, none of which has a valid interpretation. We will repeat the essence of their example.

Example 3. The Pump Problem

An oil company is considering two different sizes of pump to install in wells in an oil field. The smaller pump will extract 50% of the known crude oil reserve in the first year of operation and the remaining 50% in the second year. The larger pump will extract 100% of the known reserve in the first year. The total oil revenues over two years is the same for both pumps, $20 million. The advantage of the large pump is that it allows 50% of the revenues to be realized a year earlier than with the small pump. Investing in the larger pump would require an additional investment of $1.6 million. There is no limit on the capital budget. The MARR and the cost of capital, CC, are both 15%. Assuming the small pump is profitable to install, the remaining question is whether or not the incremental investment of $1.6 million to acquire the larger pumps is justified.
Table 3 summarizes the calculation of the incremental cash flows for the Pump Problem. Figure 3 is a cash flow diagram of the incremental cash flows. Notice that the cash flows change sign twice: from negative to positive in the first year and from positive to negative in the second year. Figure 4 is a plot of the net present value of the project as a function of the cost of capital. There are two points at which the net present value function crosses the x-axis: 25% and 400%. Thus, the incremental investment has two internal rates of return.

<table>
<thead>
<tr>
<th></th>
<th>With Small Pump</th>
<th>With Large Pump</th>
<th>Incremental Cash Flows</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial cash flow</td>
<td>-1.6</td>
<td>-1.6</td>
<td>-1.6</td>
</tr>
<tr>
<td>Revenue, year 1</td>
<td>10</td>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td>Revenue, year 2</td>
<td>10</td>
<td>0</td>
<td>-10</td>
</tr>
</tbody>
</table>

Table 3. Incremental Cash Flows for Pump Problem ($ millions)

Figure 3. Incremental Cash Flows of Pump Problem
Figure 4. Net Present Value Function of Pump Problem

The net present value of the incremental investment at the cost of capital, 15%, is $(466) thousand. Since this is negative, we can conclude that the incremental investment is not profitable and the investment in the larger pump is not justified. The initial outlay is too large compared to the economic advantage of obtaining the oil revenues sooner. The net present value criterion (NPV > 0) yields a clear and correct evaluation of the profitability of the incremental investment.

The internal rate of return criterion is more complicated for this investment. Referring to Figure 4, we see that the project is profitable whenever the cost of capital falls between the two internal rates of return, 25% and 400%, and it is unprofitable for all other nonnegative costs of capital. Since the cost of capital, 15%, is below 25%, we conclude that the incremental investment is unprofitable. When there are multiple
internal rates of return, we can expect a more complicated internal rate of return
criterion. The internal rates of return divide the range of interest rates into intervals.
Profitability of the project will depend upon the interval into which the cost of capital
falls.

Investments with multiple rates of return can arise naturally in situations other
than incremental analysis. The next example illustrates that projects involving substantial
equipment overhaul or environmental clean-up operations late in the life of the project
are likely candidates for multiple rates of return.

Example 4. The Clean-Up Problem

Table 4 and Figure 5 present the cash flows for a hypothetical investment project
that yields positive cash flows for much of the project life except that a substantial cash
outflow is required at the end of the project life. This cash outflow is associated with de-
commissioning the facility. Examples of the kind of project are common in high
technology industries that involve the use of toxic chemicals or radioactive materials.
The clean-up costs can easily exceed the salvage value of the recovered materials,
equipment or facilities.

<table>
<thead>
<tr>
<th>Year</th>
<th>Cash Flow ($ \times 10^3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-60</td>
</tr>
<tr>
<td>1</td>
<td>40</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>6</td>
<td>-100</td>
</tr>
</tbody>
</table>

Table 4. Cash Flows for Clean-Up Problem
Figure 5. Cash Flows for Clean-Up Problem

Figure 6 is a plot of the net present value of the Clean-Up problem cash flow sequence as a function of the interest rate. It reveals that there are two nonnegative internal rates of return for this problem. The two internal rates of return are 7.4% and 29.4%. As in the Pump Problem, the internal rate of return criterion for profitability is complicated: if the cost of capital is between 7.4% and 29.4%, then the project is profitable; otherwise, it is not. For example, if the cost of capital is 15%, the project is profitable with a net present value of $2,900.
Figure 6. Net Present Value Function for Clean-Up Problem

To conclude, cash flow sequences that give rise to multiple internal rates of return arise naturally in engineering economy studies and the analyst should understand their implications.

There is a well-known result in number theory that states that the number of real roots of a polynomial cannot exceed the number of sign changes in the sequence of coefficients in the polynomial. Given a vector $A=(A_0,A_1,\ldots,A_N)$ representing a cash flow sequence, let $P(A,x)$ denote the net present value of the cash flow sequence $A$ when the interest rate is $x-1$:

$$P(A,x) = \sum_{n=0}^{N} \frac{A_n}{x^n}.$$ 

An interest rate, $i^*$, is an internal rate of return of $A$ if and only if $P(A,x^*)=0$, where $x^*=1+i^*$. Similarly, let $F(A,x)$ denote the future value of that cash flow sequence when the interest rate is $x-1$:

$$F(A,x) = \sum_{n=0}^{N} A_n x^{N-n}.$$
An interest rate, $i^*$, is an internal rate of return of $A$ if and only if $F(A,x^*)=0$, where $x^*=1+i^*$. The future value is a polynomial in $x$. Thus, the number of real internal rates of return cannot exceed the number of sign changes in the cash flow sequence, $A$. For example, the cash flow sequences for the Pump Problem and the Clean-Up Problem each exhibit two sign changes and two nonnegative internal rates of return.

It is easy to construct examples of cash flow sequences with many sign changes but few internal rates of return. That is, do not expect to find multiple internal rates of return simply because there are multiple sign changes in the cash flow sequence. Nevertheless, it is a useful condition to check. We refer to investments whose cash flows change sign exactly once as conventional investments. Those investments with multiple cash flow sign changes are called nonconventional investments. The analyst must be alert to the potential for multiple internal rates of return with nonconventional investments.

**Proposition 1.**

A conventional investment (with $A_0 < 0$) has a unique nonnegative internal rate of return if and only if the sum of the cash flows is nonnegative. The internal rate of return is positive if and only if the sum of the cash flows is positive.

**Proof:**

We present the proof only for the special case of a conventional investment in which $A_0 < 0$ and $A_n \geq 0$, for all $n > 0$. The function $P(A,x)$ is continuously differentiable in $x$ over the range $(0,\infty)$. We are only interested in values of $x \geq 1$, corresponding to nonnegative interest rates. When the interest rate is zero, $x = 1$ and the net present value is the sum of the cash flows:

$$P(A,1) = \sum_{n=0}^{N} A_n.$$  

For large values of the interest rate, the net present value approaches the value of the initial cash flow:

$$\lim_{x \to \infty} P(A,x) = A_0 < 0.$$  

Thus, if the sum of the cash flows is positive, then $P(A,1) > 0$ and $\lim_{x \to \infty} P(A,x) < 0$. Continuity of the function $P(A,\bullet)$ implies that there will exist at least one $x > 1$ such that the function crosses the $x$-axis. That is, there will exist at least one positive internal rate of return. Looking at the slope of the function:
\[
\frac{\partial P(A,x)}{\partial x} = -\sum_{n=1}^{N} A_n x^{-(n+1)} < 0,
\]
for \( x \geq 1 \), since the only negative cash flow is the initial cash flow, \( A_0 \). Thus, the function \( P(A,x) \) is strictly decreasing in \( x \) for \( x \geq 1 \). Thus, it can cross the \( x \)-axis at most once in the range \((1, \infty)\). Whether or not it crosses is determined by the value of \( P(A,1) \), the sum of the cash flows. ♦

**Corollary 1.**

For a conventional investment with a positive cash flow sum, the project is profitable if and only the IRR > MARR.

**Proof:**

Again, for the special case considered in the proof of Proposition 1, the present value function \( P(A,x) \) is strictly decreasing in \( x \) for \( x \geq 1 \). Hence, \( P(A,x) > 0 \) for \( 1 \leq x < 1+i* \) and \( P(A,x) < 0 \) for \( x > 1+i* \), where \( i* \) is the unique positive internal rate of return. The value \( x-1 \) corresponds to the interest rate used to evaluate the investment. The wording of the corollary follows. ♦

The proposition and corollary show that conventional investments with positive cash flow sums have unique positive internal rates of return and simple profitability criteria.

We refer to a cash flow sequence as an *investment* if the initial cash flow is negative \((A_0 < 0)\) and a *loan* if it is positive \((A_0 > 0)\). (If \( A_0 = 0 \), we classify the sequence as loan or investment by the sign of the first nonzero cash flow.) A conventional loan, therefore, is a loan with exactly one change in sign of the cash flow sequence.

**Corollary 2.**

A conventional loan with a nonpositive sum of cash flows has a unique positive internal rate of return and is profitable if and only if the IRR < MARR.

**Proof:**

The proof follows by reversing signs in the arguments for Proposition 1 and Corollary 1. ♦
Note that the comparison between the IRR and the MARR is reversed in determining the profitability of a conventional loan. One way to think of this is to consider the case in which the MARR = CC. In this case a loan with IRR < CC represents a source of capital (recall $A_0 > 0$) that has a lower cost than the firm's cost of capital. That makes it an attractive source of financing. The reverse is true if IRR > CC. In cases in which the MARR > CC, then it is better to think in terms of the MARR as representing the opportunity cost of capital: the rate the firm could earn on additional capital if it could increase the budget. A project with $A_0 > 0$ represents an opportunity to increase the budget. As long as the firm can earn more than the IRR of this project (the loan) it is profitable to commit to the project.

Corollaries 1 and 2 and the explanation of the previous paragraph provide some insight into why the internal rate of return criterion can become more complicated with nonconventional investments. Nonconventional investments can be viewed as a combination of loan and investments. For example, in the Clean-Up Problem we could arbitrarily pick one of the years 1 to 4 and split that cash flow sequence into a conventional investment that runs from year 0 to the year of the split and a conventional loan that runs from the year after split to year 6. Choose the year of the split to ensure that the sum of the cash flows is positive for investment and negative for the loan. Year 2 is the earliest possible split year. The investment portion has a unique positive internal rate of return and the loan portion has a unique positive internal rate of return. Each portion has a simple profitability criterion (Corollaries 1 or 2), but the profitability of the project may depend on tradeoffs between the two portions.

For example, suppose we split the Clean-Up Problem cash flows arbitrarily at year 2. Table 5 shows the resulting cash flows. The internal rate of return for the conventional investment portion is 22%. Thus, the investment portion is profitable if the MARR is less than 22%. The internal rate of return for the conventional loan is 18%. The loan portion is profitable if the MARR is greater than 18%. Clearly, if the MARR falls in
the range 18% to 22% then both portions are profitable and the project as a whole is profitable. Outside that range, one or the other of the portions must be unprofitable and there is a tradeoff. For the project to be profitable, the profitability of the investment portion must outweigh the unprofitability of the loan portion or vice versa. It is not surprising therefore, that the internal rate of return profitability criterion can become more complicated when the investment is nonconventional.

<table>
<thead>
<tr>
<th>Year</th>
<th>Conventional Investment</th>
<th>Conventional Loan</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-60</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>40</td>
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</tr>
<tr>
<td>2</td>
<td>40</td>
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<td>4</td>
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<td>20</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>-100</td>
</tr>
<tr>
<td>IRR</td>
<td>22%</td>
<td>18%</td>
</tr>
</tbody>
</table>

Table 5. Arbitrary Split of Clean-Up Problem and Resulting IRR's

5. THE PROBLEM OF INTERPRETATION

For the reasons cited in the opening sections of this paper, it is common for managers to request to know the internal rate of return of a project. If the project is nonconventional and multiple internal rates of return exist, how should the analyst respond? As we shall see in this section, the dilemma faced by the analyst is not one of choosing among a set of internal rates of return but rather one of not being able to identify any single internal rate of return that is useful to the manager. The real problem is not the multiplicity of internal rates of return, but the interpretation of any of the internal rates of return. The problem will be resolved in the next section.
Balance of Investment Profiles

When studying any interest rate applied to a cash flow sequence, it is useful to construct a balance of investment profile. For a cash flow sequence \( A = (A_0, A_1, \ldots, A_N) \), let \( F_n(A, x) \) denote the future value, in period \( n \), of the partial cash flow sequence \((A_0, A_1, \ldots, A_n)\):

\[
F_n(A, x) = \sum_{k=0}^{n} A_k x^{n-k},
\]

(1)

where the interest rate is given by \( x-1 \) and \( n \) is any of the values \( 0, 1, 2, \ldots, N \). We refer to \( F_n(A, x) \) as the balance of investment in period \( n \). The balance of investment profile is the complete sequence \( F_0(A, x), F_1(A, x), \ldots, F_N(A, x) \).

Observe that the balance of investment profile can be computed recursively:

\[
F_n(A, x) = F_{n-1}(A, x) x + A_n
\]

\[
= F_{n-1}(A, x) + F_{n-1}(A, x)(x - 1) + A_n.
\]

(2)

In terms familiar from bank account transactions, this recursion can be seen as stating that the closing balance of account in period \( n \) (given by \( F_n(A, x) \)) is equal to the opening balance of account (given by \( F_{n-1}(A, x) \), which is the closing balance of period \( n-1 \)) plus the interest on account (given by \( F_{n-1}(A, x)(x - 1) \)) plus any cash deposits (given by positive values of \( A_n \)) and less any cash withdrawals (given by negative values of \( A_n \)) occurring at the end of period \( n \) (i.e. at time \( n \)).

**Example 5.**

Table 6 presents the cash flows of a conventional investment. The cash flows are identical to those of the Clean-Up Problem except in the terminal year, when a positive salvage value is enjoyed rather than the clean-up costs. For an interest rate of 15% (the MARR), Table 7 presents the computation of the balance of investment profile using the balance of account recursion. Note that the opening balance of one period is the closing balance of the previous period and the initial opening balance is zero. The new balance column refers to the sum of the opening balance plus the interest (i.e., \( F_{n-1}(A, x)x \)). The
balance of investment profile is the given by the last column in Table 7, the closing balances.

<table>
<thead>
<tr>
<th>Year</th>
<th>Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-60</td>
</tr>
<tr>
<td>1</td>
<td>40</td>
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<td>3</td>
<td>30</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
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<tr>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
</tr>
</tbody>
</table>

**Table 6. A Conventional Investment**

<table>
<thead>
<tr>
<th>Year</th>
<th>Opening Balance</th>
<th>Interest</th>
<th>New Balance</th>
<th>Cash Flow</th>
<th>Closing Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-60</td>
<td>-60</td>
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</tr>
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</tr>
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<td>93</td>
<td>14</td>
<td>107</td>
<td>30</td>
<td>137</td>
</tr>
</tbody>
</table>

**Table 7. Balance of Investment Computations**

The data of Table 7 can be presented in graphical form, called the *balance of investment diagram*, as shown in Figure 8. Each arrow in the diagram corresponds in length and direction to the magnitude and sign, respectively, of one of the cash flows, $A_n$. The arrow points to the closing balance of account for the corresponding period $n$. It points from the value $F_{n-1}(A, x)x$, which is given by the new balance column of Table 7. The interest in a period is indicated by the vertical difference between the closing balance of one period (the end of one arrow) and the new balance of the next period (the beginning of another arrow). These points are connected by dotted lines.
Figure 7. Balance of Investment Diagram

Table 6 and Figure 7 can be interpreted as follows. The MARR of 15% represents the opportunity cost of capital for the firm. If capital is consumed by a project, that capital is unavailable for other projects and therefore earnings at the rate MARR are foregone on the use of that capital. If capital is released by a project to the firm, it can be invested elsewhere in the firm at the rate MARR. Initially, the project requires an investment of $60,000. This is borrowed from the firm: the balance of account is initially negative. The opportunity cost on that capital is 15% per year. The opportunity cost in the first year is $9,000 (15% of $60,000). At the end of the first year there is a cash inflow from the project of $40,000. This is returned to the firm: at the end of the first year, the project has a net indebtedness to the firm of $29,000. The opportunity cost of that indebtedness in the second year is $4,350 (numbers in Table 6 have been rounded). The cash inflow of $40,000 in the second year eliminates the indebtedness and results in a net balance of $6,650 with the firm. The firm can invest this balance elsewhere and earn 15% per year. The opportunity benefit of this $6,650 balance is $998. From that
year onwards, the balance of investment continues to grow and the opportunity benefits compound at the rate of 15% per year. The project has a future value to the firm of $136,707 after 6 years.

The negative balances of investment are referred to as **underrecovered balances**. The positive balances of investment are referred to as **overrecovered balances**. Underrecovered balances represent capital invested in the project. Overrecovered balances represent capital invested elsewhere in the firm. For a project to be profitable, the terminal balance at the MARR must be overrecovered, as in example 5.

The cash flow sequence of Table 5 has a unique positive internal rate of return of 53%. Table 7 shows the balance of account computations for this rate of interest.

<table>
<thead>
<tr>
<th>Year</th>
<th>Opening Balance</th>
<th>Interest</th>
<th>New Balance</th>
<th>Cash Flow</th>
<th>Closing Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-60</td>
<td>-60</td>
</tr>
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<td>-20</td>
<td>-10</td>
<td>-30</td>
<td>30</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table 8. Balance of Investment Calculations for the IRR**
Figure 8. Balance of Investment Diagram for the IRR

The internal rate of return is chosen so that the terminal balance of investment is zero. Table 8 and Figure 8 show a profile in which all of the intermediate balances of investment are underrecovered. This permits the following interpretation. The firm could charge the project a cost of capital as high as 53% and the project would still manage to completely recover the initial balance of investment by the end of the project life. It is in this sense that we can say the project "earns a rate of return" of 53% per year or "recovers its investment" at a rate of 53% per year.

Such an interpretation is not possible with the internal rates of return of both the Pump Problem and the Clean-Up Problem. To see why, consider the balance of investment profiles for the Clean-Up Problem for each of the IRR’s, 7.4% and 29.4%. Table 9 summarizes the balance of investment computations for smaller IRR, 7.4%. Figure 9 is the balance of investment diagram for this IRR. Figure 10 is the balance of investment diagram for the larger IRR, 29.4%
<table>
<thead>
<tr>
<th>Year</th>
<th>Opening Balance</th>
<th>Interest Balance</th>
<th>New Balance</th>
<th>Cash Flow</th>
<th>Closing Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-60</td>
<td>-60</td>
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<td>6</td>
<td>93</td>
<td>7</td>
<td>100</td>
<td>-100</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 9. Balance of Investment Calculations for the Clean-Up Problem, IRR=7.4%

![Graph showing balance of investment over time](image)

Figure 9. Balance of Investment Diagram for the Clean-Up Problem, IRR=7.4%
Figure 10. Balance of Investment Diagram for the Clean-Up Problem, IRR=29.4%  

Figures 9 and 10 reveal a problem of interpretation for both internal rates of return. In both cases the terminal balance of investment is zero, by construction. The initial balances of investment are underrecovered, but in both cases the balance of investment profile crosses the time axis and exhibits overrecovered balances of investment. Note that these overrecovered balances are earning a rate of return equal to the internal rate of return. Now the interpretation runs as follows. (The proof of some of these statements depends on results developed in Section 8.) The firm could charge the project a cost of capital as high as the IRR and the project would still manage to completely recover the initial balance of investment by the end of the project life, provided that the firm can earn a rate of return at least as high as the IRR on overrecovered balances. The interpretation that a manager might make, that the project "earns" the IRR, is not valid without the qualifying statement in italics. An alternative interpretation is that the firm could earn a rate of return as small as the IRR on overrecovered balances and still completely recover the balance of investment by the end.
of the project life, provided that the opportunity cost of capital (the cost of underrecovered balances) is no greater than the IRR. Observe that neither of these interpretations is appropriate for a MARR of 15%. That is, at the IRR of 29.4% it is not possible to assume that the firm earns a rate of return in excess of 29.4% on overrecovered balances. At the IRR of 7.4%, it is not possible to assume that the cost of capital applied to underrecovered balances is less than 7.4%. Thus, when there are both underrecovered and overrecovered balances of investment at the internal rate of return it is not possible to interpret the internal rate of return as a project rate of return without making some restrictive assumption about the MARR. For example, it is not appropriate for the analyst to make statements such as "the Clean-Up Problem project earns a rate of return of 29.4%", or "the Clean-Up Problem project is a source of capital that only costs 7.4%." Neither IRR has a simple interpretation.

The examples may have led the reader to believe that the interpretation problem is related to the existence of multiple internal rates of return. Example 6 shows that this is not the case.

**Example 6. Overrecovered Balances at a Unique IRR**

Table 10 details a cash flow sequence that is identical to that of the Clean-Up Problem except in the last period. Figure 11 plots the net present value of this cash flow sequence as a function of the interest rate. This cash flow sequence has a unique real internal rate of return in the range \((0, \infty)\), given by \(\text{IRR}=41.3\%\). Figure 12 is the corresponding balance of investment diagram at this IRR.
<table>
<thead>
<tr>
<th>Year</th>
<th>Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-60</td>
</tr>
<tr>
<td>1</td>
<td>40</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
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<tr>
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<tr>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>6</td>
<td>-60</td>
</tr>
</tbody>
</table>

Table 10. A Nonconventional Investment with a Unique Positive IRR

Figure 11. NPV Function for Nonconventional Investment with Unique IRR
Figure 12. Balance of Investment Diagram for Nonconventional Investment with Unique IRR

Figure 11 illustrates the fact that 41.3% is the unique positive IRR and that the internal rate of return profitability criterion is simple for example 6: the project is profitable if and only if the MARR < 41.3%. However, Figure 12 reveals overrecovered investment balances at the IRR. We conclude that the internal rate of return of 41.3% does not have a simple valid interpretation. If the MARR is less than 41.3% then we will argue in the Section 7 that 41.3% overstates the rate of return that this project "earns."

6. Pure Investments and the Minimal Crushing Rate of Return

An internal rate of return is said to be pure if there are no overrecovered balances of investment at that rate of interest. It is said to be non-pure otherwise. A pure IRR has a simple valid interpretation. A non-pure IRR does not; it should not be quoted as "the project's rate of return."

In this section, we develop some understanding of when a pure IRR will exist. We first note that the internal rate of return of a conventional investment must be pure.
PROPOSITION 2.

If a conventional investment has a nonnegative internal rate of return, it must be pure.

Proof:
If \( i^* = x^* - 1 \) is a nonpure internal rate of return, then there must exist a period \( m, 0 < m < N \), such that \( F_m(A, x^*) > 0 \). Since \( F_0(A, x^*) = A_0 \) and \( F_N(A, x^*) = 0 \) it must be that there exists a period \( k, 0 < k \leq m \), with \( A_k > 0 \) and a period \( n, m < n \leq N \), with \( A_n < 0 \). In that case, it cannot be a conventional investment.

It is more difficult, but we will show that if there are multiple nonnegative IRR's, none of them can be pure. The key observation in all the logic to follow is the following simple result:

PROPOSITION 3.

For a given cash flow sequence \( A \), and a particular return factor \( x' > 0 \), if \( A_0 < 0 \) and if \( F_n(A, x') \leq 0 \) for all \( n = 0, 1, 2, ..., m-1 \) for some \( m \leq N \) then \( F_m(A, x) \) is strictly decreasing in \( x \) for all \( x \geq x' \).

Proof:
The proof is by induction on \( m \). For \( m = 1 \), we have \( F_1(A, x) = A_0 x + A_1 \). Since \( A_0 < 0 \), \( F_1 \) is strictly decreasing in \( x \) for all \( x \). Suppose the proposition is true for \( m \). We can show that it is true for \( m+1 \) as follows. If \( F_n(A, x') \leq 0 \) for all \( n = 0, 1, 2, ..., m-1 \) then by the induction hypothesis \( F_m \) is strictly decreasing in \( x \) for all \( x \geq x' \). If, in addition, \( F_m(A, x') \leq 0 \), then \( F_m(A, x) < 0 \) for all \( x > x' \). Therefore, by (2), \( F_m(A, x) \) is strictly decreasing in \( x \) for all \( x \geq x' \).

COROLLARY 3.

If \( A_0 < 0 \) then, for any integer \( m \geq 0 \), there exists an \( x > 0 \) such that \( F_n(A, x) \leq 0 \) for all \( n = 0, 1, 2, ..., m, \).

Proof:
The proof is by induction. It is left as an exercise.
The proposition states that if for a particular interest rate \( i' (=x'-1) \) all balances of investment prior to period \( m \) are underrecovered, then the balance of investment in period \( m \) of an investment will be a strictly decreasing function of the interest rate, \( i (=x-1) \), for \( i \geq i' \). We will say that an interest rate \( i \), and return factor \( x (=1+i) \), is crushing to period \( m \) if all balances of investment prior to period \( m \) are nonpositive: i.e. \( F_n(A,x) \leq 0 \) for all \( n=0,1,2,...,m-1 \). Corollary 3 implies that there exists a crushing rate for every period \( m \), provided only that \( A_0 < 0 \).

**Corollary 4.**

If, for an investment with \( A_0 < 0 \), interest rate \( i' (=x'-1) \) is crushing to period \( N \), and \( F_N(A,x') \geq 0 \), then there exists exactly one pure internal rate of return, \( i^* \), satisfying \( i^* \geq i' \).

**Proof:**

By Corollary 3, there exists a return factor \( x'' \) large enough that \( F_n(A,x'') < 0 \) for all \( n \). By Proposition 2, we have \( F_N(A,x) < 0 \) for all \( x \geq x'' \). Therefore, \( x' < x'' \). Now, \( F_N(A,x) \) is continuous and strictly decreasing in \( x \) for all \( x \geq x' \). There can exist at most one \( x^* \geq x' \) satisfying \( F_N(A,x^*) = 0 \). Since \( F_N(A,x') \geq 0 \), there must exist exactly one such \( x^* \). By proposition 3, since \( x' \) is crushing to period \( N \), and \( x^* \geq x' \), it must be the case that \( x^* \) is crushing to period \( N \). By definition, \( i^* = x^* - 1 \) is a pure internal rate of return. ♦

Proposition 3 implies that if an interest rate \( i' \) is crushing to period \( m \), then all interest rates greater than \( i' \) are also crushing to period \( m \). It makes sense therefore to identify the smallest nonnegative interest rate that is crushing to period \( m \). Denote it by \( MC_m \), the minimal crushing rate to period \( m \). Let \( MC = MC_N \), the minimal crushing rate to period \( N \), or simply, the minimal crushing rate of the investment.

**Corollary 5.**

There can exist at most one pure nonnegative internal rate of return for any investment. It exists if and only if \( F_N(A,MC+1) \geq 0 \). If it exists it must be the largest internal rate of return.
Proof:
If $i^*$ is an internal rate of return satisfying $0 \leq i^* < MC$ then it cannot be crushing to period $N$, since $MC$ is the minimal crushing rate to period $N$. Since $MC$ is crushing to period $N$, $F_N(A,x)$ is strictly decreasing for $x \geq MC+1$. Consequently, if $F_N(A,MC+1) < 0$, there can exist no internal rate of return greater than or equal to $MC$. Corollary 4 completes the proof. ♦

PROPOSITION 4.

If the sum of cash flows is equal to zero and $A_0 < 0$, then there can exist no pure positive internal rate of return.

Proof:
From (2), note that
\[
\sum_{n=0}^{N} A_n = A_0 + \sum_{n=1}^{N} A_n
\]
\[
= A_0 + \sum_{n=1}^{N} \left( F_n^{r}(A,x) - F_{n-1}^{r}(A,x) - F_{n-1}(A,x)(x-1) \right)
\]
\[
= F_N(A,x) - \sum_{n=1}^{N} \left( F_{n-1}(A,x)(x-1) \right).
\]
Since the sum of cash flows is zero, it must be the case that:
\[
F_N(A,x) = \sum_{n=1}^{N} \left( F_{n-1}(A,x)(x-1) \right).
\]
When $x=MC+1$, we have $F_n^{r}(A,x) \leq 0$ for all $n \leq N-1$. Furthermore, $F_0(A,x) = A_0 < 0$. If $MC > 0$, then $x > 1$ and we must have $F_N(A,MC+1) < 0$. Corollary 5 implies the result.
On the other hand, if $MC=0$ then $F_N(A,MC+1) = \sum_{n=0}^{N} A_n = 0$. Hence, 0 must be a pure internal rate of return. By Corollary 4, there can exist no other pure nonnegative internal rate of return. ♦

PROPOSITION 5.

If there exist multiple nonnegative internal rates of return for an investment with $A_0 < 0$, then none of them can be pure.

Proof:
By Corollary 5, if there exist multiple nonnegative internal rates of return, then at most the largest one can be pure. Let $i_0=x_0-1$ be any nonnegative internal rate of return except the largest one. If $i_0=0$, then the sum of the cash flows must be zero. Proposition
4 then implies that there are no nonnegative pure internal rates of return. If \( i_0 > 0 \), define a new cash flow series \( A' \) such that \( A'_n = A_n(x_0)^{N-n} \). Observe that because \( x_0 \) is an IRR:

\[
F_n'(A', 1) = \sum_{m=0}^{N} A'_m = \left( \sum_{m=0}^{N} A_m(x_0)^{N-n} \right) = 0.
\]

Hence, the new cash flows sum to zero. Also note that

\[
F_m(A', x / x_0) = \sum_{n=0}^{m} A'_n(x / x_0)^{m-n} = \left( x_0 \right)^N \sum_{n=0}^{m} A_n(x)^{m-n} = \left( x_0 \right)^N F_m(A, x).
\]

In particular, \( x-1 \) is an IRR of the original series if and only if \( x/x_0 - 1 \) is an IRR of the new series and \( x-1 \) is a pure IRR of the original series if and only if \( x/x_0 - 1 \) is a pure IRR of the new series. Proposition 4 and the fact that the new series sums to zero implies that no IRR of the new series greater than 0 can be pure. Therefore, no IRR of the original series greater than \( x_0 - 1 \) can be pure.

This section has established many ways to identify if a pure IRR exists. First of all is a simple check of the signs of the cash flows. If the investment is conventional, it will have a unique and pure internal rate of return, provided only that the sum of the cash flows is nonnegative. For nonconventional investments, one technique is to find any nonnegative IRR and compute its balance of investment profile. If it is pure, then Corollary 5 assures us that it is unique. If it is not pure, then Proposition 5 asserts that no pure nonnegative IRR exists for this investment. Corollary 5 suggests another approach: find MC and compute \( F_N(A, MC+1) \). If it is nonnegative, then a pure IRR exists to the right of MC; otherwise, no pure nonnegative IRR exists for this investment.

Note that MC is not difficult to find using a spreadsheet and a trial and error procedure: set up formulas to compute the balance of investment profile and increase the interest rate until all intermediate balances are nonpositive. Search for the minimum such interest rate. For example, the minimal crushing rate to period 6 for the Clean-Up
Problem is given by MC₆=49.9%. Figure 13 illustrates the balance of investment diagram at this rate of interest. Note that all intermediate balances are nonpositive. If the interest rate is reduced than the balance of investment in year 5 will be overrecovered. The terminal balance is negative and at any larger interest rates it be even more negative. Consequently, a pure internal rate of return cannot exist for this investment.

![Balance of Investment Diagram](image)

**Figure 13. Balance of Investment at the Minimal Crushing Rate:**

**The Clean-Up Problem**

Investments for which a real pure internal rate of return exists are called *pure investments*. Investments which do not possess a pure internal rate of return are called *mixed investments*. We are left with the question as to what rate of return to quote for a mixed investment.

**7. The Rate of Return on Invested Capital Approach**

In this section, we describe a simple approach that corrects the flaws of the internal rate of return approach. Note in the recursion (2) for computing the balance of
investment profile we apply the same interest rate \((x-1)\) to positive balances as to
negative balances. This assumption has been implicit in all of our rate of return, net
present value, and future value calculations. Suppose, instead, that we apply one factor,
\(x_s\), whenever balances are positive, and a different factor, \(x_b\), whenever they are
negative. Let \(F_n(A,x_b,x_s)\) denote the investment balance that results in period \(n\) as a
result of this rule:

\[
F_0(A,x_b,x_s) = A_0,
\]

and

\[
F_n(A,x_b,x_s) = A_n + \begin{cases} 
F_{n-1}(A,x_b,x_s)x_s, & \text{if } F_{n-1}(A,x_b,x_s) > 0 \\
F_{n-1}(A,x_b,x_s)x_b, & \text{otherwise,}
\end{cases}
\]

\[
= F_{n-1}(A,x_b,x_s) + A_n + \begin{cases} 
F_{n-1}(A,x_b,x_s)(x_s - 1), & \text{if } F_{n-1}(A,x_b,x_s) > 0 \\
F_{n-1}(A,x_b,x_s)(x_b - 1), & \text{otherwise,}
\end{cases}
\]

(3)

for \(n=1,2,\ldots\). In banking terms, the new balance, \(F_n\), is computed as the old balance, \(F_{n-1}\), plus the savings interest, computed at rate \(x_s-1\), if the old balance was positive, or less
the borrowing interest, computed at rate \(x_b-1\), if the old balance was negative, plus the
new deposit, \(A_n\). Observe that we cannot write the balance of investment as a simple sum
such as in (1) unless \(x_s = x_b\). It can be computed only using recursion (3).

The rate of return on invested capital, RIC, is defined as the breakeven cost of
financing unrecovered balances of investment under the condition that overrecovered
balances earn the Minimum Attractive Rate of Return, MARR. To be precise, the RIC
solves the following equation:

\[
F_n(A,RIC+1,MARR+1) = 0.
\]

The fundamental advantage of the RIC, if it exists, is that it has a simple
interpretation: the RIC measures the rate at which the initial investment is recovered over
the life of the project under the reasonable assumption that intermediate overrecovered
balances earn the Minimum Attractive Rate of Return. The RIC exists, uniquely, under
mild conditions. Furthermore, the profitability criterion is simple: if the RIC exceeds the MARR then the project is profitable. These claims are proven in the next section.

The cash flows of examples 3-6 are summarized in Table 11. Table 12 tabulates the RIC for each example under differing values of the MARR. Figure 14 plots the RIC as a function of the MARR for examples 4, 5, and 6.

<table>
<thead>
<tr>
<th>Year from Time of Investment</th>
<th>Example</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>&quot;The Pump Problem&quot;</td>
<td>&quot;The Clean-Up Problem&quot;</td>
</tr>
<tr>
<td>0</td>
<td>-1.6</td>
<td>-60</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>40</td>
</tr>
<tr>
<td>2</td>
<td>-10</td>
<td>40</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>30</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>20</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>-100</td>
</tr>
</tbody>
</table>

Table 11. Summary of Rate of Return Examples
<table>
<thead>
<tr>
<th>MARR %</th>
<th>3 &quot;The Pump Problem&quot;</th>
<th>4 &quot;The Clean-Up Problem&quot;</th>
<th>5 &quot;A Conventional Investment&quot;</th>
<th>6 &quot;Overrecovered Balances at a Unique IRR&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>-70.3</td>
<td>2.1</td>
<td>53.0</td>
<td>31.9</td>
</tr>
<tr>
<td>10</td>
<td>-43.2</td>
<td>12.1</td>
<td>53.0</td>
<td>34.4</td>
</tr>
<tr>
<td>15</td>
<td>-18.5</td>
<td>18.8</td>
<td>53.0</td>
<td>36.3</td>
</tr>
<tr>
<td>20</td>
<td>4.1</td>
<td>23.6</td>
<td>53.0</td>
<td>37.8</td>
</tr>
<tr>
<td>25</td>
<td>25.0</td>
<td>27.0</td>
<td>53.0</td>
<td>38.9</td>
</tr>
<tr>
<td>30</td>
<td>44.2</td>
<td>29.7</td>
<td>53.0</td>
<td>39.8</td>
</tr>
<tr>
<td>35</td>
<td>62.1</td>
<td>31.8</td>
<td>53.0</td>
<td>40.6</td>
</tr>
<tr>
<td>40</td>
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<td>41.2</td>
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<tr>
<td>45</td>
<td>94.0</td>
<td>34.9</td>
<td>53.0</td>
<td>41.7</td>
</tr>
<tr>
<td>50</td>
<td>108.3</td>
<td>36.0</td>
<td>53.0</td>
<td>42.2</td>
</tr>
<tr>
<td>100</td>
<td>212.5</td>
<td>41.6</td>
<td>53.0</td>
<td>44.8</td>
</tr>
<tr>
<td>200</td>
<td>316.7</td>
<td>45.0</td>
<td>53.0</td>
<td>46.3</td>
</tr>
<tr>
<td>300</td>
<td>368.8</td>
<td>45.9</td>
<td>53.0</td>
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<td>400.0</td>
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<td>47.8</td>
</tr>
<tr>
<td>500</td>
<td>420.8</td>
<td>46.9</td>
<td>53.0</td>
<td>48.2</td>
</tr>
</tbody>
</table>

Table 12. RIC for Various Values of MARR
Figure 14. RIC as a Function of the MARR

Observe that the RIC is a non-decreasing function of the MARR. In the case of the conventional investment, example 5, the RIC is independent of the MARR: it is equal to the pure internal rate of return. Examples 4, 5, and 6 are closely related: their cash flows are identical in all but the last period. Consequently, the minimal crushing rate at period 6 is the same for each investment: MC₆ = 49.9%. Observe from Figure 14 that the pure IRR of example 5 lies above this value. The RIC for the mixed investments of examples 4 and 6 lie below this value, no matter how large the MARR is. These observed properties of the RIC are generally true as we shall see in the next section.
8. Properties of the Rate of Return on Invested Capital

Proposition 6.

The balance of investment function \( F_n(A,x_b,x_s) \) is strictly decreasing in \( x_b \geq 0 \) for fixed \( x_s \), provided \( A_0 < 0 \). The balance of investment function \( F_n(A,x_b,x_s) \) is non-increasing in \( x_s \geq 0 \) for fixed \( x_b \).

Proof:
The proof is by induction. It is left as an exercise. ♦

Proposition 7.

The RIC exists and is unique for all projects, pure and mixed, provided the sum of the cash flows is positive.

Proof:
We assume the MARR > 0. Then \( F_N(A,1,1+\text{MARR}) \geq F_N(A,1,1) = \sum_{n=0}^{N} A_n > 0 \).

\( F_N(A,x_b,1+\text{MARR}) \) is strictly decreasing in \( x_b \geq 1 \) for fixed MARR (Proposition 6). It is easily seen that

\[
\lim_{x_b \to \infty} F_N(A,x_b,\text{MARR}+1) = -\infty.
\]

Since \( F_N() \) is a continuous strictly decreasing function that is positive for \( x_b = 1 \) and negative for sufficiently large \( x_b \), there must exist a unique value of \( x_b^* \) satisfying \( F_N(A,x_b^*,\text{MARR}+1) = 0 \). By definition, \( \text{RIC} = x_b^*-1 \). ♦

Proposition 8.

If \( \text{RIC} > \text{MC} \) then the investment is pure and the RIC is equal to the pure IRR, no matter what the value of MARR.

Proof:
For all \( x \geq \text{MC}+1 \), we have \( F_n(A,x,x) \leq 0 \) for all \( n \leq N-1 \). Hence, the value of MARR can have no effect on the value \( F_N(A,\text{RIC}+1,\text{MARR}+1) \). In particular, it must be true that \( 0 = F_N(A,\text{RIC}+1,\text{MARR}+1) = F_N(A,\text{RIC}+1,\text{RIC}+1) \). Therefore, RIC must be an internal rate of return. By Corollary 5, there can be at most one IRR greater than MC and it must be pure. ♦
PROPOSITION 9.

If RIC < MC then a pure IRR does not exist, and the RIC is not equal to any of the IRR's (unless the MARR = IRR). It is increasing in the MARR and it approaches the MC as the MARR goes to infinity.

Proof:

By Proposition 6, the terminal balance is strictly decreasing in the borrowing rate for fixed savings rate. At the RIC, the terminal balance is zero. Therefore, 0 = FN(A,RIC+1,MARR+1) > FN(A,MC+1,MARR+1) = FN(A,MC+1,MC+1). Thus, FN(A,MC+1) < 0. By Corollary 5, there does not exist a pure IRR. The remainder of the proof is left as an exercise. ♦

PROPOSITION 10.

A project is profitable if and only if the RIC > MARR.

Proof:

If portion: By proposition 6, the terminal balance is strictly decreasing in the borrowing rate for fixed savings rate. Therefore, 0 = FN(A,RIC+1,MARR+1) < FN(A,MARR+1,MARR+1) = FN(A,MARR+1). Hence, the future value of the project at the MARR is positive if the RIC > MARR. The only if portion follows similarly. ♦

Proposition 10 states that the RIC profitability criterion is equivalent to the NPV profitability criterion. The theoretically correct rate of return to quote for any investment project is the RIC. The theoretically correct rate of return criterion for project profitability is the RIC criterion. These two statements have been the goal of this paper.

9. SUMMARY

The logic of this paper has been long and involved, with many important concepts developed along the way. The logic of the paper can be summarized as follows in bullet form.

1. Introduction

- It is useful to attach a relative measure of merit to an investment project, one that doesn't depend on the scale of the project. This permits quick comparisons between
various investment and financing opportunities. A natural choice for this relative measure is the internal rate of return. Unfortunately, the internal rate of return approach is seriously flawed. The purpose of this paper is to describe the corrected approach, based on the rate of return of invested capital.

2. The Internal Rate of Return Approach

- The internal rate of return (IRR) of a project is a breakeven cost of financing the project. It is the value of the interest rate that sets the future value (equivalently, the present value or the annual value) of the project equal to zero.

- The internal rate of return criterion states that an investment project is profitable if the IRR > MARR (Minimum Attractive Rate of Return). Unlike the NPV criterion, the IRR criterion is not always valid. One goal of this paper has been to explain the problem with the IRR criterion and to replace it with a rate of return criterion that is always valid.

- It is useful to attach a relative measure of merit to an investment project, one that doesn't depend on the scale of the project. This permits quick comparisons between various investment and financing opportunities.

- For example, when there is a limited budget of capital to invest, there are many investment opportunities and no one investment is large relative to the budget, then ranking the projects by internal rate of return will lead to a good, although not necessarily optimal, collection of investments. Think of allocating the budget dollar by dollar to get the "most bang for the buck." The Minimum Attractive Rate of Return can be thought of as the internal rate of return of the next project to be selected if the budget were increased.

3. The Incremental Rate of Return Approach

- Beware, however, if you are comparing mutually exclusive projects. The internal rate of return ranking can lead to rejecting the most profitable project.

- When comparing mutually exclusive projects, if the smaller project is profitable it is correct to look at the incremental cash flows and ask whether it is profitable to make the incremental investment and invest in the larger project instead. This can be done by computing the internal rate of return of the incremental cash flows and applying the IRR criterion.

- Beware in this circumstance, however, that the IRR criterion may be invalid. It is easy to find examples in which the IRR of an incremental cash flow does not have a valid interpretation (Section 4). The correct approach would be to apply the RIC criterion (Section 7) to the incremental cash flows.
4. The Problem of Multiple Internal Rates of Return

- The Pump Problem and the Clean-Up Problem illustrate that cash flow sequences with multiple internal rates of return are not uncommon.

- If there are multiple IRRs, then the profitability criterion is more complicated. The profitability of the project is determined by the interval of interest rates (defined by adjacent pairs of IRR's) into which the MARR falls.

- The number of internal rates of return is less than or equal to the number of changes in sign in the cash flow sequence.

- A conventional investment is one in which the cash flows change sign at most once.

- A nonconventional investment is an investment (negative initial cash flow) for which the cash flows change sign more than once.

- A conventional investment has a unique positive internal rate of return, provided that the sum of the cash flows is positive. (Proof: Use calculus to show that the present value is strictly decreasing in the interest rate, assuming the initial cash flow is the only negative cash flow. The present value is positive at a zero interest rate and negative for sufficiently large interest rates, so there must be a unique positive interest rate which zeroes the present value.)

- A nonconventional investment could have more than one IRR. Not all nonconventional investments have multiple IRRs but the analyst should be alert to the possibility of multiple IRRs.

- Nonconventional investments arise naturally because investing in a project often entails a contractual or legal responsibility to pay for such major cash outflows as equipment overhauls or environmental restoration (e.g. reforestation after strip mining). These cash outflows occur in the future, after much of the benefit of the project (i.e. positive cash flows) has already occurred. Nonconventional investments also arise when comparing projects using the incremental rate of return approach.

- A conventional investment is attractive if the IRR > MARR.

- A conventional loan is attractive if the IRR < MARR. Think of this as a loan whose cost (IRR) is less than the firm's current cost of capital. However, since the MARR may be larger than the firm's cost of capital, a better interpretation is to say that there exist projects which earn the MARR so any source of capital with a cost of capital below the MARR is attractive.

- (Aside: A nonconventional investment can be thought of as a combination of conventional loans and conventional investments. There is not a unique way to separate a non-conventional investment into conventional loans and conventional investments. However, because the IRR criterion is different for a loan than for an investment, it is not surprising to see more complicated criteria for project acceptance with nonconventional investments. The "loan portion" of the project may look profitable for some MARR's; the "investment portion" may look profitable for others. It is a tradeoff.)
5. The Problem of Interpretation

- Problem: If someone asks what rate of return the project earns, what rate of return do you quote? The goal of this paper has been to find a valid rate of return to quote and a valid rate of return criterion to apply to determine profitability.
- Uniqueness of the IRR is not the issue. The issue is interpretation. The IRR approach involves an unacceptable implicit assumption.
- At some IRR’s, there are overrecovered balances of investment. For such an IRR, there is an implicit assumption that an overrecovered balance earns the IRR. This is a bad assumption. It is one thing to say that this project will breakeven even if the firm charges 22% on underrecovered balances. It is quite another thing to say that the project will breakeven if the firm charges 22% on underrecovered balances provided that it also pays 22% on overrecovered balances. In reality, the firm may be able to earn, say, only 15% on such overrecovered balances. The IRR in that case is internal to the cash flow stream but it is not internal to the project vis à vis the firm.

6. Pure Investments and the Minimal Crushing Rate of Return

- A pure internal rate of return is an IRR such that there are no overrecovered balances at that rate of return.
- If an IRR is not pure, it is not useful: don’t quote it to anyone.
- Let us call a rate of return crushing if all intermediate investment balances are underrecovered at that rate. Let MC denote the minimum crushing interest rate. All interest rates larger than the MC must be crushing. By definition, no interest rate smaller than the MC is crushing. The MC is easy to find using a spreadsheet.
- If an IRR is pure, it must be crushing and, therefore, it must be greater than the MC. Any IRR that is smaller than the MC cannot be pure.
- Look at the terminal balance (future value) of the project when the interest rate is equal to the MC. For interest rates larger than the MC, the terminal balance is a decreasing function of the interest rate. This means that if the terminal balance at the MC is positive, there is exactly one IRR that is larger than the MC. If the terminal balance at the MC is negative, there cannot exist any IRR that is larger than the MC.
- Therefore, an investment project can have at most one pure IRR in the region \((0, \infty)\).
- This means there are two type of investments: those with a unique pure IRR and those that do not have any pure IRR’s. Call the former pure investments. Call the latter mixed investments.
- Conventional investments are pure investments; but some non-conventional investments can be pure investments as well.
- If an investment has multiple real nonnegative internal rates of return, then none of them can be pure. Hence if any nonnegative internal rate of return is not pure, the investment must be mixed. However, there exist mixed investments that have a unique real nonnegative internal rate of return that is not pure.
How can you tell if an IRR is pure? If the investment is conventional, you know it must be pure. Otherwise, look at all the intermediate investment balances at that rate of interest. If none are positive, the IRR is pure, by definition. This is easy to do with a spreadsheet. (Finding an IRR is easy with most spreadsheets, too, since most spreadsheet packages include an IRR function). Alternatively, compare the IRR to the MC. If the IRR > MC, it must be pure.

How can you find the pure IRR of a pure investment, especially since there can be multiple IRR’s? Check the terminal balance of investment at the MC. If this is positive then find the IRR in the region (MC,∞).

What rate of return do you quote for a mixed investment? You cannot quote any of the IRR's. One of the goals of this paper has been to answer this question.

7. The Rate of Return on Invested Capital Approach

- The Rate of Return on Invested Capital (RIC) is the breakeven cost of financing the project given that overrecovered balances earn the MARR.
- Simple search techniques and a spreadsheet can be used to find the RIC for any project. Unfortunately, most spreadsheet packages do not provide a function to find the RIC automatically. The CASH software that accompanies the text by Park does provide a routine to compute the RIC (Park calls the RIC the ROR).
- Note that if you reverse the sign of all cash flows (multiply all cash flows by -1), you do not alter the IRR (or the IRR's if there are more than one), but the RIC could be different. Therefore, unlike computing the IRR, when computing the RIC it is critical to have the correct interpretation of when the project is consuming capital from the firm (a cash outflow) and when it is generating capital for the firm (a cash inflow). As a check, the first cash flow is typically negative.

8. Properties of the Rate of Return on Invested Capital

- The RIC exists and is unique for all projects, pure and mixed, provided the sum of the cash flows is positive.
- If RIC > MC then the investment is pure and the RIC is equal to the pure IRR, no matter what the value of MARR.
- If RIC < MC then a pure IRR does not exist, and the RIC is not equal to any of the IRR's (unless the MARR = IRR). It is something different and its value depends on the MARR. It is increasing in the MARR and it approaches the MC as the MARR goes to infinity. Remember that the overrecovered balances earn the MARR, so the project becomes more attractive (i.e. the RIC increases) as the MARR increases.
- A project is profitable if and only if the RIC > MARR. That is, the RIC criterion agrees with the NPV criterion.
- The theoretically correct rate of return to quote for any investment project is the RIC. The theoretically correct rate of return criterion for project profitability is the RIC criterion. These two statements have been the goal of this paper.
REFERENCES


