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Strictly Local Martingales and Hedge Ratios on Stochastic Volatility Models

by

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STRICTLY LOCAL MARTINGALES AND HEDGE RATIOS ON STOCHASTIC VOLATILITY MODELS

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We study diffusion processes that appear in finance as models of incomplete markets with stochastic volatility, showing that they are strictly local martingales in some cases, contrary to the usual assumption that they are martingales.

We characterize the existence of a probability measure equivalent to the original under which the first component of the process (underlying asset price) becomes a martingale; this is called a martingale measure. We focus on the case in which the drift term of the equation can be removed, so there always exists a local martingale measure. We show that this measure is a martingale measure if and only if a certain 1-dimensional diffusion doesn't explode in finite time, which can be answered using Feller's classification of processes.

We also study the class of price functionals obtained from local martingale measures for general price processes and, using change of numeraire techniques, we show that the existence of a martingale measure is equivalent to the absence of certain arbitrage opportunities providing a new version of the fundamental theorem of asset

pricing.

Finally, we give explicit integral representations for martingales in models with stochastic volatility. In financial terms this gives the hedge ratios of derivative securities with respect to some underlying instruments.

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Chapter 1

Introduction

1.1 Arbitrage pricing

Suppose that $d+1$ different assets are traded at an exchange. At time t, any number of shares of asset k can be bought or sold at a price $S_t^{(k)}$. This assumes no restrictions on short sales, no transactions costs and no taxes. Also, for simplicity, assume for the moment that $S^{(0)} \equiv 1$ for all t. We what to study the class of random payoffs that we can obtain by trading the $d+1$ assets.

Suppose we build a portfolio, keeping $H_i(t)$ shares of asset i at time t. We construct this portfolio starting with some initial investment x_0 and we rebalance it in time without adding or withdrawing any funds until some final time T. At that time we liquidate the portfolio and obtain a random payoff $X = H_0(T)S_0(T) + H_1(T)S_1(T) + \cdots + H_d(T)S_d(T)$.

We consider three possible scenarios:

1. There exist two portfolio strategies $H^{(1)}$ and $H^{(2)}$ that give exactly the same final payoff X but with different initial costs $x_1 > x_2$ respectively. This is called an *arbitrage opportunity* because, if we build now the portfolio given by $H^{(2)} - H^{(1)}$, we start with a negative initial investment $x_2 - x_1$ and obtain a final payoff always equal to zero; in other words, we get somebody to give us $x_1 - x_2$ today, without having to give anything in return.

   This situation is obviously inconsistent with economic equilibrium: it provides a riskless profit so any agent would want to buy as much of this strategy as possible, and since it doesn't cost anything to enter in such a position, the demand for it will be unlimited. Intuitively, an opportunity like that shouldn't last very long in a competitive market; smart agents will take advantage of it driving the prices to equilibrium. On all the models studied here we will assume the absence of arbitrage opportunities, and we will show general results that give both necessary and sufficient conditions for this to happen.

2. There are no arbitrage opportunities, so whenever there's a strategy H with final payoff X, there's a unique value x_0, initial investment required to start strategy H. We call x_0 the *price at time* 0 of the contingent claim X. Suppose further that for every suitably bounded random variable X there exists a strategy H, with initial investment x_0, that gives payoff X at time T.

   This is called the *complete markets* case: every claim X has a uniquely determined price x_0: the initial investment needed to generate it with strategy H. This was the first case studied using martingale methods, and the results are well known (see Harrison and Pliska (1981)). We will restate the main results here, as a preparation for the new developments.

3. There are no arbitrage opportunities, but there is a bounded contingent claim X for which there is no strategy H that gives it as a payoff. We will study this later in more detail; in this scenario we can't obtain unique prices for all claims.

   This case has received significant attention lately (see for example Ansel and Stricker (1994), El Karoui and Quenez (1995), Hofmann et al. (1992), Karatzas et al. (1991) among others). Here we will study specific models of incomplete markets and prove new results on the absence of arbitrage opportunities and existence of hedging strategies for certain claims.

It is clear that, for any market model, exactly one of the previous conditions holds.

The following results are the cornerstone of the recent developments in mathematical finance

Theorem 1.1.1. *There exists an arbitrage opportunity if and only if there is no probability measure Q, with the same null sets as the original, under which the process S is a martingale.*

Theorem 1.1.2. *The market is complete if and only if there exists exactly one probability measure Q, with the same null sets as the original, under which S is a martingale.*

Corollary 1.1.3. *The market is incomplete if and only if there exists more than one probability measure Q, with the same null sets as the original, under which S is a martingale.*

Theorem 1.1.1 is called the *fundamental theorem of asset pricing*, and there are several versions depending on the space of processes chosen and the set of strategies permitted; see for example Delbaen and Schachermayer (1994b), Lakner (1993), Stricker (1990) and references therein. Theorem 1.1.2 is better understood and easier to prove, see Harrison and Pliska (1983). Observe that Corollary 1.1.3 follows immediately from the other two.

1.2 Plan of the chapters

In chapter 2 we give the formal definitions and state the main theorems known in this area. In particular, the definitions needed to state Theorem 1.1.1 in the discrete time case and Theorem 1.1.2 in the general case will be given, along with some recent results in the incomplete markets case.

In chapter 3 we give one of the versions of theorem Theorem 1.1.1, proved by Delbaen and Schachermayer (1994b) in the case of bounded price processes, and we extend it to the class of non-negative price processes. We use change of numéraire techniques, showing that in particular, the existence of a probability measure under which prices are local-martingales is not sufficient to prevent arbitrage opportunities.

In chapter 4 we study some models for stock prices that are proposed by practitioners and finance theorists, focusing especially on models with stochastic volatility, and we show that in some cases there doesn't exist a probability measure that makes the processes martingales, so arbitrage opportunities exist.

In chapter 5 we show how, for a model with stochastic volatility that has an incomplete market, it is still possible to replicate every contingent claim by constructing a hedging strategy on the stock and a traded European call option with positive strike price.

Chapter 2

The general theory

2.1 Basic definitions

Let $(S_t^{(0)}, S_t^{(1)}, \dots, S_t^{(d)})_{t \in I}$ be a stochastic process on a probability space $(\Omega, \mathcal{F}, P)$. We assume the index set I is either a discrete finite set $I = \{0, 1, \dots, T\}$ or a compact interval $I = [0, T]$. Let $(\mathcal{F}_t)_{t \in I}$ be a filtration of σ-algebras on Ω and suppose

1. S_t is adapted: $S_t^{(k)}$ is $\mathcal{F}_t$-measurable for all $t \in I$, $k = 0, 1, \dots, d$.

2. $(\mathcal{F}_t)$ is increasing: $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for all $0 \leq s \leq t \leq T$.

3. $(\mathcal{F}_t)$ is complete: if $t \in I$, $A \in \mathcal{F}_t$ and $P(A) = 0$ then for all $B \subseteq A$, $B \in \mathcal{F}_t$.

4. $(\mathcal{F}_t)$ is right-continuous: If $I = [0, T]$ then, for each $t \in [0, T]$ $\mathcal{F}_t = \cap_{s>t} \mathcal{F}_s$.

5. $(\mathcal{F}_0)$ is trivial: If $A \in \mathcal{F}_0$ then $P(A) = 0$ or $P(A) = 1$.

$S_t^{(k)}$ is the price at time t of asset k; any investor can buy or sell any number of units of asset k at a price $S_t^{(k)}$ each.

Our main objective is to characterize the set of payoffs that can be obtained by trading the given securities. Let $H_t^{(k)}$ be the number of shares of asset k that an agent is holding at time t. We will define a set of *admissible* portfolios $(H^{(0)}, \dots, H^{(d)})_t$ and then traders will have to choose a strategy in this set to produce some desired payoff at the expiration date T. The market value at time t of any of these portfolios is $H_t S_t = H_t^{(0)} S_t^{(0)} + \dots + H_t^{(d)} S_t^{(d)}$.

Assume $S^{(0)} > 0$ with probability 1 and define the *discounted price process* Z_t as

$$Z_t^{(k)} = \frac{S_t^{(k)}}{S_t^{(0)}} \qquad k = 0, 1, \dots, d$$

Definition. Let Q be a probability measure on $(\Omega, \mathcal{F}_T)$, equivalent to P (i.e. $Q(A) = 0$ iff $P(A) = 0$). We say that the process Z_t is a Q-martingale if $Z^{(k)}$ is Q-integrable for each $k = 0, 1, \dots, d$ and

$$Z_u^{(k)} = Q(Z_t^{(k)} | \mathcal{F}_u) \qquad \text{for all } 0 \leq u \leq t$$

We say that Q is an *equivalent martingale measure* for Z_t if Q is equivalent to P and Z_t is a Q-martingale.

Construct the set of equivalent martingale measures for Z_t

$$\mathbb{Q} = \{Q \text{ probability measure on } (\Omega, \mathcal{F}_T) : Z_t \text{ is a } Q\text{-martingale}\}$$

2.2 Discrete time case

Here we observe the prices $S^{(0)}, S^{(1)}, \dots, S^{(d)}$ only at a finite set of times $I = \{0, 1, 2, \dots, T\}$.

Definition. An admissible trading strategy is a process $H_t = (H^{(0)}, \dots, H^{(d)})$ that satisfies

1. $H_t^{(k)}$ is $\mathcal{F}_t$-adapted for $k = 0, 1, \dots, d$.

2. H_t is *self-financing*:

$$H_{t-1}S_t = H_tS_t \qquad \text{for } t = 1, \dots, T$$

   which is equivalent to

$$H_tS_t - H_0S_0 = \sum_{u=0}^{t-1} H_u(S_{u+1} - S_u) \qquad \text{for } t = 1, \dots, T$$

Remark 2.2.1. H_tS_t is the market value of the portfolio acquired at time t, an it is called the *value process*. Condition (ii) ensures that, when the old portfolio H_{t-1} is liquidated at time t and the new portfolio H_t is formed, the market values of the two are the same so the investor doesn't have to add or withdraw funds at any of the intermediate trading dates. This condition is sometimes rephrased by saying that the change in the value process $H_tS_t - H_0S_0$ is equal to the net gains obtained by trading $\sum_{u=0}^{t-1} H_u(S_{u+1} - S_u)$.

Definition. An admissible trading strategy H_t is an *arbitrage opportunity* if $H_0S_0 = 0$, $H_TS_T \geq 0$ and $P(H_TS_T > 0) > 0$.

Then the fundamental theorem of asset pricing can be stated as (see Morton (1988) or Dalang et al. (1990) for a proof)

Theorem 2.2.2. *(Morton) There are no arbitrage opportunities if and only if there exists an equivalent martingale measure Q for $Z^{(1)}, \dots, Z^{(d)}$.*

Also, Willinger and Taqqu (1988) give a characterization of complete markets in the discrete time setting that involves the structure of the filtration $\mathcal{F}_t$.

2.3 Continuous time case.

We start with a brief review of the theory of stochastic integration. From now on we assume all processes considered are right-continuous with left limits (RCLL).

Definition. A process X_t is a *local martingale* if there exist a sequence of stopping times $\tau_n \uparrow \infty$ such that

$$X_t^{(n)} = X_{t \wedge \tau_n}$$

is a martingale for every n.

A process Y_t is a *semimartingale* if it can be expressed as

$$X_t = M_t + A_t$$

where M_t is a local martingale and A_t is a process of bounded variation. In general, this decomposition is not unique.

Definition. Let X_t, Y_t be semimartingales. The *quadratic covariation* process of X_t and Y_t is the unique RCLL, increasing adapted process $[X,Y]_t$ starting at zero such that

$$X_tY_t - [X,Y]_t \qquad \text{is a local martingale}$$

$$(X_t - X_{t-})(Y_t - Y_{t-}) = [X,Y]_t - [X,Y]_{t-} \qquad \text{for all } t$$

see for example Rogers and Williams (1987) Chapter VI for a construction. The *quadratic variation* process for the semimartingale Y_t is denoted $[Y]_t = [Y,Y]_t$.

Proposition 2.3.1. *If we take partitions $\Pi = \{0=t_0, t_1, \dots, t_n=T\}$ of the interval $[0,T]$ we have, as the size $|\Pi|$ tends to zero:*

$$[X,Y]_t = X_0Y_0 + \lim_{|\Pi|\to 0} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})$$

the limit is calculated in the sense of probability: for all $\epsilon > 0$

$$P\left(\left|[X,Y]_t - X_0Y_0 - \sum_i (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})\right| > \epsilon\right) \to 0$$

as $|\Pi| \to 0$.

For a proof see Protter (1990) Theorem II.23.

Now we will reenact the construction of the stochastic integral for predictable process H_t with respect to a semimartingale X_t

Definition. For a process M_t let M_t^* be the running maximum

$$M_t^* = \sup\{|M_u| : 0 \le u \le t\}$$

and let $\mathcal{H}^q$ be the Banach space of local martingales with norm

$$||M||_{\mathcal{H}^q} = P\left((M_T^*)^q\right)^{1/q}$$

Given a local martingale M_t, let L_M^q be the space of predictable integrands H_t under the norm

$$||H||_{L_M^q} = P\left(\left(\int_0^T H_u^2 d[M]_u\right)^{q/2}\right)^{1/q}$$

For an RCLL-local-martingale M_t we define the space of integrands

$$L_{m,loc}(M) = \left\{H \text{ predictable } : \ ||H_t 1_{\{t\leq\tau_n\}}||_{L_M^1} < \infty \quad \text{with } \tau_n \uparrow \infty\right\}$$

suppose H_t is a *simple integrand* of the form

$$H_t = \sum_i \xi_i 1_{\{\tau_i < t \leq \tau_{i+1}\}}$$

for a sequence of stopping times $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n = T$ with each ξ_i bounded and $\mathcal{F}_{t_i}$-measurable define

$$\int_0^t H_u dM_u = \sum_i \xi_i (M_{t\wedge\tau_{i+1}} - M_{t\wedge\tau_i})$$

if this process belongs to $\mathcal{H}^2$ we obtain the equivalence

$$\left|\left|\int H_u dM_u\right|\right|_{\mathcal{H}^2} = ||H||_{L_M^2}$$

The set of simple integrands H_t is dense in L_M^2; so the isometry given above can be extended by continuity and then by stopping to define the stochastic integral $\int H_u dM_u$ for all $H \in L_{m,loc}(M)$(see Jacod (1979) Chapter II).

Now if A_t is a RCLL process of bounded variation we define

$$L_b(A) = \left\{ H \text{ predictable } : \int_0^t |H_u||dA_u| < \infty \right\}$$

and the integral $\int H_u dA_u$ is the usual path-by-path Lebesgue-Stieltjes integral. The use of the same integral notation in the martingale and bounded variation cases is justified by the following result

Proposition 2.3.2. *Let A_t be a local martingale of bounded variation. If $H \in L_{m,loc}(A) \cap L_b(A)$ then the path-by-path Lebesgue-Stieltjes integral coincides with the stochastic integral $\int_0^t H_u dA_u$ a.s. for all $t \in \mathbb{R}^+$.*

(see for example Protter (1990) Theorem IV 20).

For a general RCLL semimartingale X_t define

$$L(X) = \left\{ H \text{ predictable } : \begin{array}{l} \text{there exist } M, A \text{ such that } \quad X = M + A, \\ \text{and } H \in L_{m,loc}(M), \quad H \in L_b(A) \end{array} \right\}$$

and define the stochastic integral as

$$\int_0^t H_u dX_u = \int_0^t H_u dM_u + \int_0^t H_u dA_u$$

the value of the integral doesn't depend on the decomposition chosen (see for example Jacod (1979) Proposition 2.69 and 2.71 or Protter (1990) Chapter IV for an alternative construction). One important tool for calculation is the *integration by parts formula*: for semimartingales X_t, Y_t we have

$$[X, Y]_t = X_t Y_t - X_0 Y_0 - \int_0^t X_{u-} dY_u - \int_0^t Y_{u-} dX_u$$

Definition. For a local martingale X_t we define the stable subspace $\mathcal{L}(X)$ generated by X as the smallest closed subspace of $\mathcal{H}^1$ that contains X and is closed under stopping; that is, if $M_t \in \mathcal{L}(X)$ and τ is a stopping time then $M_{t\wedge\tau} \in \mathcal{L}(X)$.

The generality of the set of integrand permitted is emphasized by the following result (see Jacod (1979) Théorème 4.6 or Protter (1990) Theorem IV 35 in the square-integrable semimartingale case).

Proposition 2.3.3. *For a local martingale M,*

$$\mathcal{L}(M) = \left\{ \int H_u dX_u \in \mathcal{H}^1 \ : \ H \in L_{m,loc}(M) \right\}$$

In many cases of interest in finance we don't have a single real valued process M_t but a finite collection of processes $M^{(1)}, \ldots, M^{(d)}$. We want to characterize the stable subspace generated by the vector-martingale $(M^{(1)}, \ldots, M^{(d)})$:

$$\mathcal{L}(M^{(1)}, \ldots, M^{(d)}) = \overline{\left\{ \sum_{i=1}^{d} \int H_u^{(i)} dM_u^{(i)} \in \mathcal{H}^1 \ : \ H^{(i)} \in L_{m,loc}(M^{(i)}) \text{ for each } i \right\}}$$

However, the space of componentwise stochastic integrals $\sum_i \int H_u^{(i)} dM_u^{(i)}$ is not closed in $\mathcal{H}^1$. Its closure is given by the set of vector stochastic integrals, which are defined in Jacod (1980).

Let $A_t = (A_t^{(1)}, \ldots, A_t^{(d)})$ is an $\mathbb{R}^d$-valued process of bounded variation. There exists an increasing, adapted process C_t and an adapted $\mathbb{R}^d$-valued process c_t such that for each $i = 1, 2, \ldots, d$

$$A_t^{(i)} = \int_0^t c_t^{(i)} dC_t$$

define the set of integrands

$$L_b(A) = \left\{ H \text{ predictable } : \int_0^t |\sum_i H_u^{(i)} c_u^{(i)}||dC_u| < \infty \right\}$$

and define for $H \in L_b(A)$

$$\int H_t dA_t = \int \left(\sum_i H_t^{(i)} c_t^{(i)} \right) dC_t$$

If $(M^{(1)}, \dots, M^{(d)})$ is a vector local martingale it is always possible to find an increasing adapted process C_t and an adapted $R^{d\times d}$-valued $(c_t)^{(i,j)}$ such that for all $i, j = 1, 2, \dots, d$

$$[M^{(i)}, M^{(j)}]_t = \int_0^t c_u^{(i,j)} dC_u$$

and define

$$L_{m,loc}(M) = \left\{ H \text{ predictable } : \begin{array}{c} P\left(\left(\int_0^{\tau_n} \sum_{i,j} H_u^{(i)} c^{(i,j)} H_u^{(j)} dC_u \right)^{1/2} \right) < \infty \\ \text{for some sequence of stopping times } \tau_n \uparrow \infty \end{array} \right\}$$

this set doesn't depend on the choice of $(C_t, c_t^{(i,j)})$. $\int H_u dM_u$ is again defined by a limiting procedure, and it is equal to the vector Lebesgue-Stieltjes integral defined before in the bounded variation case (see Jacod (1980)).

The vector stochastic integral is an extension of the componentwise stochastic integral $\sum_i \int H_u^{(i)} dM_u^{(i)}$; whenever the later exists we have the equality $\int H_u dM_u = \sum_i \int H_u^{(i)} dM_u^{(i)}$.

Proposition 2.3.4. *(See Jacod (1980) equation (2)) For $H \in L_{m,loc}(M)$ we have $\int H_u dM_u$ is the only local martingale such that, for every local martingale N_t*

$$\left[\int H_u dM_u, N \right]_t = \int_0^t H_u d[M, N]_u$$

where the integral on the right is a Lebesgue-Stieltjes vector integral in the sense previously defined.

The representation result now can be written (for a proof see Jacod (1979) Théorème 4.60)

Proposition 2.3.5. *For any vector local martingale* $(M^{(1)}, \ldots, M^{(d)})$ *we have*

$$\mathcal{L}(M) = \left\{ \int H_u dX_u \in \mathcal{H}^1 \;:\; H \in L_{m,loc}(M) \right\}$$

And finally, we can define the stochastic integral with respect to a vector semi-martingale $X_t = (X_t^{(1)}, \ldots, X_t^d)$ the same way as before. Define the set of integrands

$$L(X) = \left\{ H \text{ predictable } : \begin{array}{c} \exists M, A \text{ s.t.} \quad X = M + A, \\ H \in L_{m,loc}(M), \quad H \in L_b(A) \end{array} \right\}$$

and define the stochastic integral as

$$\int_0^t H_u dX_u = \int_0^t H_u dM_u + \int_0^t H_u dA_u$$

Now we have all the technical machinery we need to define admissible strategies. Recall our price process $S_t = (S_t^{(0)}, S_t^{(1)}, \ldots, S_t^{(d)})$ which is a non-negative stochastic processes on a probability space $(\Omega, \mathcal{F}, P)$. The index set is the bounded interval $I = [0, T]$.

Assume $S^{(0)}$ is a special asset with positive price

$$S_t^{(0)} > 0 \qquad \text{for all } t \in [0, T]$$

we then consider $S^{(0)}$ as the *numéraire*, and compute the relative prices with respect to it, obtaining the *discounted price process*

$$Z_t^{(k)} = \frac{S_t^{(k)}}{S_t^{(0)}} \qquad k = 0, 1, \ldots, d; \quad t \in [0, T]$$

Assume $(Z^{(k)}\ ;\ k = 0, 1, \dots, d)$ are RCLL semimartingales, so we can define the stochastic integral with respect to Z.

Definition. An *admissible trading strategy* is a process $H_t = (H_t^{(0)}, H_t^{(1)}, \dots, H_t^{(d)})$ that satisfies

1. $H \in L(Z)$ (we say H_t is Z_t-integrable).

2. H_t is self-financing:

$$H_t Z_t - H_0 Z_0 = \int_{(0,t]} H_t dZ_t$$

3. There exists a constant vector $c = (c_1, \dots, c_d)$ such that

$$H_t Z_t \geq c Z_t \qquad \text{for all } t$$

(for the motivation and the financial interpretation behind this condition see chapter 3).

Now define the set of contingent claims dominated by payoffs of attainable strategies with zero initial investment

$$C = \{\text{r.v. } X \in \mathcal{F}_T \ :\ X \leq H_T Z_T \text{ for admissible } H_t \text{ with } H_0 Z_0 = 0\}$$

The fundamental theorem of asset pricing states that there exists an equivalent martingale measure for Z_t if and only if there are no positive random variables in the closure of C under an appropriate topology on the space of contingent claims. In chapter 3 we give a more precise statement, and we extend the work of Delbaen and Schachermayer (1994b).

2.4 Complete markets.

We consider the continuous time case $I = [0, T]$, S_t an $\mathbb{R}^{d+1}$ valued stochastic process on $(\Omega, \mathcal{F}, P)$ with $S^{(0)} > 0$ and define $Z_t = S_t / S_t^{(0)}$. We assume the existence of a probability measure $Q \sim P$ such that Z_t is a Q-martingale. We keep the definition of admissible strategy from the previous section.

Definition. We say the admissible trading strategy H_t is *Q-optimal* if $H_t Z_t$ is a Q-martingale.

For a justification of this terminology and its properties see chapter 5.

Definition. S_t is complete under Q if for all Q-integrable, $\mathcal{F}_T$-measurable random variable X there exists a Q-optimal admissible H_t such that

$$X = H_T Z_T = H_0 Z_0 + \int_0^T H_u dZ_u$$

Here we apply results from Jacod (1979) that have to do with extremal measures of martingales (Théorème 11.2, 11.3, Corollaire 11.4 and Proposition 11.14) and from them we extract

Theorem 2.4.1. *(Jacod) We have $\mathcal{L}(Z_t) = \mathcal{H}^1$ if and only if Q is the unique equivalent martingale measure.*

From this theorem and Proposition 2.3.5 we obtain

Theorem 2.4.2. *(Harrison and Pliska) S_t is complete under Q if and only if Q is the only martingale measure.*

This theorem was first stated in this context by Harrison and Pliska (1983). For important discussions regarding its interpretation and special cases see Jarrow and Madan (1991) and Chatelain and Stricker (1994).

2.5 Incomplete markets

Define the set of equivalent local martingale measures

$$\mathbb{Q}_{loc} = \left\{ Q \; : \; \begin{array}{c} Q \text{ is a probability measure on } (\Omega, \mathcal{F}_T), \\ Q \sim P \text{ and } Z_t \text{ is a } Q\text{-local-martingale.} \end{array} \right\}$$

and keep the definition of Q-optimal admissible strategy from the previous section. The following theorem characterizes the class of contingent claims that can be replicated using admissible strategies. We present the version given by Ansel and Stricker (1994).

Theorem 2.5.1. *(Ansel and Stricker) Let $Q \in \mathbb{Q}_{loc}$ and let X be an $\mathcal{F}_T$-measurable, non-negative random variable. Then the following two statements are equivalent.*

1. *There exists a Q-optimal admissible trading strategy H_t with $H_T Z_T = X$ a.s.*

2. *For all $R \in \mathbb{Q}_{loc}$*

$$R(X) \leq Q(X) < \infty$$

Observe that, if H_t is a hedging strategy for X, we would like to say $x_0 = H_0 Z_0$ is the fair price for the claim X and we expect $Q(X) = x_0$ for all $Q \in \mathbb{Q}_{loc}$. However, this assertion is false in general. A counterexample can be found by constructing a non-negative stochastic process Z_t such that Y_t is a martingale under some probability measure $Q \sim P$ and a strictly local martingale under another $R \sim P$. Then

$$Q(Z_T) = Z_0$$
$$R(Z_T) < Z_0 \qquad \text{(by Fatou's Lemma)}$$

Then $X = Z_T$ has a trivial Q-optimal trading strategy and $Q(X) > R(X)$. Schachermayer (1993) provides an example especially constructed with this purpose. In chapter 4 we will show that this phenomenon also occurs in some models of stock prices with stochastic volatility.

Nevertheless, if the contingent claim X is bounded then it is possible to obtain the desired equivalence, as shown by Stricker (1984).

Theorem 2.5.2. *(Stricker) Let X be a bounded, $\mathcal{F}_T$-measurable random variable. Then the following two statements are equivalent.*

1. *There exists $Q \in \mathbb{Q}_{loc}$ and a Q-optimal admissible trading strategy H_t with $H_T Z_T = X$ a.s.*

2. *$Q(X)$ is constant as a function of $Q \in \mathbb{Q}_{loc}$.*

Moreover, Delbaen and Schachermayer (1994b) Theorem 5.7 shows a characterization for the upper bound on the price of a bounded contingent claim that may not be replicated exactly using a hedging strategy but that can be dominated by another contingent claim that is hedgeable.

Theorem 2.5.3. *(Delbaen and Schachermayer) For every* $X \in L^\infty(\Omega, \mathcal{F}_T, P)$ *we have*

$$\sup\{Q(X) \;:\; Q \in \mathbb{Q}_{loc}\} = \inf\left\{ x_0 \in \mathbb{R} \;:\; \begin{array}{c} X \leq H_T Z_T \textit{ for admissible } H_t \\ \textit{with } H_0 Z_0 = x_0 \end{array} \right\}$$

Also Delbaen (1992) Theorem 6.1 gives an earlier proof for bounded continuous processes, and Cvitanić and Karatzas (1993) use a stochastic control approach to prove it in the Brownian diffusion case in the more general context of constrained portfolios.

Chapter 3

The fundamental theorem of asset pricing.

In this chapter we provide a new version of the fundamental theorem of asset pricing; which shows the equivalence between the existence of a martingale measure and the absence of arbitrage opportunities. We study the continuous time case $I = [0, T]$. In this setting the proof of the theorem is much more difficult, and it is possible to find different definitions for no-arbitrage, under different assumptions on the process S_t and depending on the topology assigned to the space of trading strategies. Some of these definitions are shown to be equivalent to the existence of an equivalent martingale measure and some others are only equivalent to the existence of a *local* martingale measure. See Delbaen and Schachermayer (1994b), Stricker (1990), Lakner (1993), and references therein.

Our work is mostly based on the results by Delbaen and Schachermayer (1994b).

They propose a very natural condition for no arbitrage in continuous trading, that they call No Free Lunch with Vanishing Risk (NFLVR). This condition is equivalent to the existence of a martingale measure when the discounted price process is bounded, but implies only the existence of a *local* martingale measure when the process is locally bounded.

We show that a simple variation of their NFLVR condition, which we call No Feasible Free Lunch with Vanishing Risk (NFFLVR), is both necessary and sufficient for the existence of an equivalent martingale measure for general positive price processes. This approach also shows an important difference between the existence of a martingale measure and the existence of a strictly local martingale measure: we show that in the second case there may exist arbitrage opportunities (in the sense of the original FLVR definition) under a change of numéraire.

3.1 Basic definitions and terminology

We will study a continuous time economy with a finite horizon T having d assets trading, whose prices at any time $t \in [0, T]$ are given by $S_t^{(k)} : k = 1, \ldots, d$. Suppose the market is frictionless, with no transaction costs or taxes and no restrictions on short sales. Assume all prices are nonnegative: $S_t^{(k)} \geq 0 \ a.s. \ : \ k = 1, \ldots, d, \ t \in [0, T]$, and that $S^{(1)}$ is a special asset with positive price $S_t^{(1)} > 0 \ a.s.$ and define

$$Z_t^{(k)} = S_t^{(k)} / S_t^{(1)} \ : \ k = 1, \ldots, d; \quad t \in [0, T]$$

We assume $Z^{(k)} : k = 1, \ldots, d$ are RCLL semimartingales on a probability space $(\Omega, \mathcal{F}_T, P)$, adapted to a filtration $(\mathcal{F}_t)_{t \in [0,T]}$ that satisfies the usual conditions with

$\mathcal{F}_0$ being trivial (consisting only of sets with probability zero or one).

A *trading strategy* is a $\mathbb{R}^d$-valued predictable process H_t, vector-integrable with respect to S_t; the *gains* vector of trading according to H_t is given by the stochastic integral

$$G_t = \int_0^t H_u dS_u$$

Recall that the trading strategy H_t is *self-financing* if the value of the portfolio $\sum_{k=1}^d H_t^{(k)} S_t^{(k)}$ changes only because of gains in trading

$$H_t S_t - H_0 S_0 = \sum_{k=1}^d H_t^{(k)} S_t^{(k)} - \sum_{k=1}^d H_0^{(k)} S_0^{(k)} = \int_0^t H_u dS_u$$

so that, once the initial cost is paid, the portfolio doesn't require or produce external funds except possibly at date T.

Let $A^0(S^{(1)}, \dots, S^{(d)})$ be the class of self-financing strategies whose value remains nonnegative.

$$A^0(S) = \left\{ H_t : \begin{array}{l} H_t \text{ self-financing trading strategy on } S_t, \\ H_t S_t \geq 0 \ a.s. \text{ for all } t \in [0,T] \end{array} \right\}$$

and $A^b(S^{(1)}, \dots, S^{(d)})$ be the class of self-financing portfolios whose value remains bounded below.

$$A^b(S) = \left\{ H_t : \begin{array}{l} H_t \text{ self-financing trading strategy on } S_t, \\ H_t S_t \geq -a \ a.s. \text{ for all } t \in [0,T] \text{ and some } a \in \mathbb{R}^+ \end{array} \right\}$$

Definition. Let $c_1, c_2, \dots, c_d$ be positive integers equal to the number of shares outstanding at time zero of $S^{(1)}, S^{(2)}, \dots, S^{(d)}$ respectively. Define $g(S_t) = c_1 S^{(1)} + \cdots + c_d S^{(d)}$. We say $g(S_t)$ is the *total wealth in the economy at time t*.

Now let $A^f(S)$ be the set of *feasible self-financing portfolios;* whose losses are bounded by the total wealth in the economy.

$$A^f(S) = \left\{ H_t : \begin{array}{l} H_t \text{ self-financing trading strategy on } S_t, \\ H_t S_t \geq -g(S_t) \ a.s. \text{ for all } t \in [0,T] \end{array} \right\}$$

Remark 3.1.1. There are several financial interpretations for this condition. The first and most obvious is, since any investor can trade at most c_i shares of security i, the value of any portfolio must be bounded by $g(S_t)$. This assumes that the number of shares outstanding remains constant in time and that each transaction implies the physical exchange of the appropriate number of shares of stock: conditions that are rarely satisfied in today's financial markets. However, a second interpretation is also possible: suppose an investor starts with a portfolio of $n_1, \ldots, n_d$ shares of $S^{(1)}, S^{(2)}, \ldots, S^{(d)}$ respectively at time zero. Then he starts some new trades through his account with his friendly broker during the time interval $[0,T]$. If the value of his portfolio of new trades stays above some negative multiple of his old balance $n_1 S_t^{(1)} + \cdots + n_d S_t^{(d)}$ at all times then his broker (if friendly) will consider that a satisfactory margin. On the other hand, if his trades bring his net balance too short, his broker will certainly demand some extra margin. Our process $g(S_t)$ is, from any practical point of view, a very generous upper bound on the losses that any individual or institutional investor can leave uncovered. In any case, the results that follow don't depend on the financial interpretation of the coefficients $c_1, c_2, \ldots, c_d$, any other portfolio with a positive number of shares of each asset will work as well.

3.2 Existing results and their limitations

The main purpose here is to state the equivalence between the existence of a measure Q equivalent to P under which $Z_t^{(1)}, \ldots, Z_t^{(d)}$ are martingales, and the absence of *arbitrage opportunities.* We will state the recent developments in this area carried out by Delbaen (1992), Schachermayer (1994) and Delbaen and Schachermayer (1994b). Similar results also appear in Stricker (1990), Lakner (1993), and others.

We begin by stating several different notions of no-arbitrage.

1. **No Nonnegative-Arbitrage** (NNA)(Harrison and Pliska (1981)):

   If $H_t \in A^0(S)$, $H_0 S_0 = 0$ then $H_T S_T = 0$ a.s.

2. **No Bounded-Below-Arbitrage** (NBBA)

   (McBeth (1991), Delbaen and Schachermayer (1994b)):

   If $H_t \in A^b(S)$, $H_0 S_0 = 0$, $H_T S_T \geq 0$ then $H_T S_T = 0$ a.s.

3. **No Feasible-Arbitrage** (NFA)(new):

   If $H_t \in A^f(S)$, $H_0 S_0 = 0$, $H_T S_T \geq 0$ then $H_T S_T = 0$ a.s.

Property (1) is perhaps the most natural, but it is too weak: we will show examples for which condition (1) is satisfied but condition (2) is not. When at least one of the processes is bounded away from zero condition (3) is stronger than condition (2), and there are examples of unbounded processes which satisfy condition (2) but not (3).

Observe that each of these conditions depends only on the null sets under P and not on the specific measure:

Proposition 3.2.1. *If Q is a probability measure on $(\Omega, \mathcal{F}_T)$ equivalent to P, then the sets $A^0(S)$, $A^b(S)$, $A^f(S)$ obtained under the measure Q are the same as $A^0(S)$, $A^b(S)$, $A^f(S)$ obtained with P.*

Proof: H_t is S_t-integrable under the measure P if and only if it is S_t-integrable under Q, and $\int_0^t H_t dS_t = H_t S_t - H_0 S_0$ is the same (with probability one) under P and Q. See Protter (1990) Theorem IV.25 for a proof when S_t is $\mathbb{R}$-valued. The (d+1)-dimensional case follows as a limit because of the way the vector-integral is defined from the componentwise integrals. $\square$.

The most severe drawback of condition (2) is that it is not preserved under change of numéraire, as conditions (1) and (3) are:

Proposition 3.2.2. *Let $i, j \in \{1, 2, \dots, d\}$ and suppose the relative prices with respect to $S^{(i)}$ and $S^{(j)}$ are RCLL semimartingales (in particular they are finite). Then $A^0(S^{(1)}/S^{(i)}, \dots, S^{(d)}/S^{(i)}) = A^0(S^{(1)}/S^{(j)}, \dots, S^{(d)}/S^{(j)})$ and $A^f(S^{(1)}/S^{(i)}, \dots, S^{(d)}/S^{(i)}) = A^f(S^{(1)}/S^{(j)}, \dots, S^{(d)}/S^{(j)})$.*

Proof: We imitate the proof given by Geman et al. (1995) in the case of continuous S_t.

Suppose H_t is a self-financing trading strategy on $(S/S^{(i)})_t$. Observe that, for $k = 1, \dots, d$,

$$H_t \left(\frac{S^{(i)}}{S^{(j)}} \right)_{t-} \quad \text{is} \quad \frac{S}{S^{(i)}}\text{-vector-integrable.}$$

by assumption and the fact that $\left(\frac{S^{(i)}}{S^{(j)}} \right)_{t-}$ is locally bounded.

Also $H_{t-} \left(\frac{S}{S^{(i)}} \right)_{t-} = H_0 \left(\frac{S}{S^{(i)}} \right)_0 + \int_0^{t-} H_u d \left(\frac{S}{S^{(i)}} \right)_u$ is locally bounded, so it is $\frac{S^{(i)}}{S^{(j)}}$-integrable.

And, by Proposition 2.3.4 we have

$$\int_0^t H_u d\left[\frac{S}{S^{(i)}}, \frac{S^{(i)}}{S^{(j)}}\right]_u \quad \text{exists and is equal to} \quad \left[H\frac{S}{S^{(i)}}, \frac{S^{(i)}}{S^{(j)}}\right]_t$$

These estimates show that all the integrals in the following application of the integration by parts formula are well defined

$$\begin{aligned}
& d\left(H_t\left(\frac{S}{S^{(j)}}\right)_t\right) \\
&= d\left(\left(H_t\left(\frac{S}{S^{(i)}}\right)_t\right)\left(\frac{S^{(i)}}{S^{(j)}}\right)_t\right) \\
&= H_{t-}\left(\frac{S}{S^{(i)}}\right)_{t-} d\left(\frac{S^{(i)}}{S^{(j)}}\right)_t + \left(\frac{S^{(i)}}{S^{(j)}}\right)_{t-} d\left(H\frac{S}{S^{(i)}}\right)_t + d\left[H\frac{S}{S^{(i)}}, \frac{S^{(i)}}{S^{(j)}}\right]_t \\
&= H_t\left(\frac{S}{S^{(i)}}\right)_{t-} d\left(\frac{S^{(i)}}{S^{(j)}}\right)_t + \left(\frac{S^{(i)}}{S^{(j)}}\right)_{t-} H_t d\left(\frac{S}{S^{(i)}}\right)_t + H_t d\left[\frac{S}{S^{(i)}}, \frac{S^{(i)}}{S^{(j)}}\right]_t \\
&= H_t d\left(\left(\frac{S}{S^{(i)}}\right)_t\left(\frac{S^{(i)}}{S^{(j)}}\right)_t\right) \\
&= H_t d\left(\left(\frac{S}{S^{(j)}}\right)_t\right)
\end{aligned}$$

(where the third identity follows from the use of the self-financing property of H_t with respect to $(S/S^{(i)})_t$ on each of the three terms) therefore H_t is $(S/S^{(j)})$-integrable and self-financing.

Finally, if $H_t(S/S^{(i)})_t \geq -g((S/S^{(i)})_t)$ or if $H_t(S/S^{(i)})_t \geq 0$, multiplying on both sides by $(S^{(i)}/S^{(j)})_t$ we get $H_t(S/S^{(j)})_t \geq -g((S/S^{(j)})_t)$ or $H_t(S/S^{(j)})_t \geq 0$ respectively. $\square$

However, all the previous conditions are too weak to imply the existence of an equivalent martingale measure for general processes defined in continuous time, we need to enlarge the set of possible payoffs by taking limits of terminal gains using these strategies. We will follow the approach of Delbaen and Schachermayer (1994b), restricted to a finite horizon, parallel to our new extension.

Let $C^b(Z)$ be the set of contingent claims dominated by payoffs that can be attained by trading on S with strategies bounded-below

$$C^b(Z) = \left\{ X \in L^\infty(\Omega, \mathcal{F}_T, P) \ : \ X \leq \int_0^T H_t dZ_t \text{ for some } H \in A^b(Z) \right\}$$

and let $C^f(S)$ be the one obtained using feasible strategies

$$C^f(Z) = \left\{ X \in L_g(Z_T) \ : \ X \leq \int_0^T H_t dZ_t \text{ for some } H \in A^f(Z) \right\}$$

where

$$L_g(Z_T) = \{ X \text{ r.v. on } (\Omega, \mathcal{F}_T, P), \text{ such that } |X| \leq g(Z_T) \}$$

with the norm $||X||_g = \underset{\omega\in\Omega}{\text{ess sup}} \left| \frac{X}{g(Z_T)} \right|$. We define $\overline{C^b(Z)}$ and $\overline{C^f(Z)}$ as the closure with respect to the norm topology in L^∞ and $L_g(Z_T)$ respectively. This way we can define

Definition. We say that $(Z_t)_{t\in[0,T]}$ satisfies

1. *No Bounded Below Free Lunch with Vanishing Risk* (NBBFLVR) if

   $X \in \overline{C^b(Z)}, X \geq 0 \ \Rightarrow \ X = 0 \ a.s.$

2. *No Feasible Free Lunch with Vanishing Risk* (NFFLVR) if

   $X \in \overline{C^f(Z)}, X \geq 0 \ \Rightarrow \ X = 0 \ a.s.$

It is not difficult to prove the implications EMM $\Rightarrow$ NBBFLVR $\Rightarrow$ NBBA and also EMM $\Rightarrow$ NFFLVR $\Rightarrow$ NFA $\Rightarrow$ NBBA, using the following

Proposition 3.2.3. *Let $H_t \in A^f(Z)$. If Z_t is a martingale then $H_t Z_t$ is a supermartingale.*

Proof: Observe we cannot directly assert that $\int_0^t H_t dZ_t$ is a local martingale because H_t is a general Z_t-integrable predictable process (see Émery (1980) for a counterexample). However,

$$H_t Z_t = \int_0^t H_u dZ_u + H_0 Z_0 \geq -cZ_t$$

where c is the constant vector that defines g, and the integral of $H_t + c$ is bounded below by a constant

$$\int_0^t (H_u + c) dZ_u \geq -(H_0 + c) Z_0$$

then by Corollaire 3.5 in Ansel and Stricker (1994) we get

$$\int_0^t (H_u + c) dZ_u \quad \text{is a local martingale}$$

and it is also bounded below so Fatou's lemma implies $\int_0^t (H_u + c) dZ_u$ is a supermartingale and then

$$H_t Z_t = \int_0^t (H_u + c) dZ_u - c(Z_t - Z_0) + H_0 Z_0$$

is a supermartingale $\square$.

Proposition 3.2.4. *EMM $\Rightarrow$ NFFLVR*

Proof: Suppose Z_t is a martingale under the measure Q on $(\Omega, \mathcal{F}_T)$. Let $X \in \overline{C^f(Z)}$, so there exist $X_k \in C^f(Z)$, $X_k \xrightarrow{L_g(Z_T)} X$, such that for $k = 1, 2, \ldots$

$$X_k \leq \int_0^T H_t^{(k)} dZ_k \quad \text{for some } H^{(k)} \in A^f(Z)$$

if $X \geq 0$ we may assume

$$\int_0^T H_t^{(k)} dZ_t \geq -\frac{g(Z_T)}{k} \qquad k = 1, 2, \ldots$$

then

$$\int_0^T (H_t^{(k)} + \frac{c}{k}) dZ_t \geq -\frac{cZ_0}{k}$$

Let $\widetilde{H}_t^{(k)} = H_t^{(k)} + \frac{c}{k} \in A^f(Z)$. Then Fatou's lemma implies

$$\begin{aligned} Q(X) &= Q\left(\lim_{k\to\infty} X_k + \frac{c}{k}(Z_T - Z_0)\right) \\ &\leq Q\left(\liminf_{k\to\infty} \int_0^T \widetilde{H}_t^{(k)} dZ_t\right) \\ &\leq \liminf_{k\to\infty} Q\left(\int_0^T \widetilde{H}_t^{(k)} dZ_t\right) \\ &\leq 0 \end{aligned}$$

where for the last inequality we use the previous proposition. This, together with $X \geq 0$ imply $X = 0$ a.s. $\square$

Also, when the time index set is finite we get NBBA $\Rightarrow$ EMM and all the definitions are equivalent (see Dalang et al. (1990) and Schachermayer (1994)). Delbaen and Schachermayer (1994b) obtained

Theorem 3.2.5. *(Delbaen-Schachermayer). Suppose that $(Z_t)_{t\in[0,T]}$ is bounded. Then Z_t satisfies NBBFLVR if and only if there exists an equivalent martingale measure for Z_t.*

Delbaen and Schachermayer (1994a) give examples that show that this theorem is not true for general Z_t. If S_t is locally bounded then they are only able to prove the existence of a *local martingale measure*,

Corollary 3.2.6. *(Delbaen-Schachermayer). Suppose that* $(Z_t)_{t\in[0,T]}$ *is locally bounded. Then* Z_t *satisfies NBBFLVR if and only if there exists an equivalent local martingale measure for* Z_t.

In the next section we will prove that, if we replace NBBFLVR with NFFLVR then Theorem 3.2.5 is true for any positive, possibly unbounded, process $(Z_t)_{t\in[0,T]}$.

3.3 Change of numéraire

Here we assume that any process considered as a numéraire is positive a.s.. and that the relative prices are semimartingales. Introduce the notation

$$S_t^{(i,j)} = \frac{S_t^{(j)}}{S_t^{(i)}} \qquad \text{for } i = 1, \dots, d,\ j = 1, \dots, d,\ t \in \mathbb{R}^+.$$

so, in particular, $S^{(i,i)} \equiv 1$ for all $i = 1, \dots, d$.

The following proposition is a well known result. See for example Geman et al. (1995).

Proposition 3.3.1. *Fix a time horizon* $T \in \mathbb{R}^+$ *and define the two classes of probability measures on* $(\Omega, \mathcal{F}_T)$

$$\mathbb{Q}^{(1)} = \left\{ Q^{(1)} \sim P \ : \ S^{(1,1)}, S^{(1,2)}, \dots, S^{(1,d)} \text{ are martingales under } Q^{(1)} \right\}$$

$$\mathbb{Q}^{(2)} = \left\{ Q^{(2)} \sim P \ : \ S^{(2,1)}, S^{(2,2)}, \dots, S^{(2,d)} \text{ are martingales under } Q^{(2)} \right\}$$

then there is a 1–1 correspondence $H : \mathbb{Q}^{(1)} \to \mathbb{Q}^{(2)}$ *given by* $H(Q^{(1)}) = Q^{(2)}$ *where*

$$Q^{(2)}(A) = Q^{(1)}\left(1_A\left(\frac{S_T^{(1,2)}}{S_0^{(1,2)}}\right)\right) \tag{3.1}$$

and, moreover, for every random variable $X \in \mathcal{F}_T$

$$S_0^{(1)}Q^{(1)}\left(\frac{X}{S_T^{(1)}}\right) = S_0^{(2)}Q^{(2)}\left(\frac{X}{S_T^{(2)}}\right) \tag{3.2}$$

whenever either of the two sides exists.

Proof:

- $Q^{(2)} = H(Q^{(1)}) \in \mathbb{Q}^{(2)}$.

  From equation (3.1) we get $Q^{(2)}$ is a positive measure absolutely continuous with respect to $Q^{(1)}$, $S^{(2)} > 0$ implies $Q^{(2)} \sim Q^{(1)} \sim P$ and $S^{(1,2)}$ being a martingale under $Q^{(1)}$ implies $Q^{(2)}(\Omega) = 1$ so it is a probability measure. By the properties of conditional expectations, for $0 \le t \le u \le T$ we have

$$\begin{aligned}
Q^{(2)}(S_u^{(2,k)}|\mathcal{F}_t) &= \frac{Q^{(1)}(S_u^{(2,k)}S_u^{(1,2)}|\mathcal{F}_t)}{S_t^{(1,2)}} \\
&= \frac{Q^{(1)}(S_u^{(1,k)}|\mathcal{F}_t)}{S_t^{(1,2)}} \\
&= \frac{S_t^{(1,k)}}{S_t^{(1,2)}} = S_t^{(2,k)}
\end{aligned}$$

  $\therefore \quad S_t^{(2,k)}$ is a martingale under $Q^{(2)}$ for all $k = 1, 2, \ldots, d$.

- Define $\widehat{H} : \mathbb{Q}^{(2)} \to \mathbb{Q}^{(1)}$ by $\widehat{H}(Q^{(2)}) = \widehat{Q}^{(1)}$ where

$$\widehat{Q}^{(1)}(A) = Q^{(2)}\left(1_A\left(\frac{S_T^{(2,1)}}{S_0^{(2,1)}}\right)\right)$$

then if $Q^{(2)} = H(Q^{(1)})$, $\widehat{Q}^{(1)} = \widehat{H}(Q^{(2)})$ and $A \in \mathcal{F}_T$

$$\begin{aligned}\widehat{Q}^{(1)}(A) &= Q^{(2)}\left(1_A\left(\frac{S_T^{(2,1)}}{S_0^{(2,1)}}\right)\right)\\ &= Q^{(1)}\left(1_A\left(\frac{S_T^{(2,1)}}{S_0^{(2,1)}}\right)\left(\frac{S_T^{(1,2)}}{S_0^{(1,2)}}\right)\right)\\ &= Q^{(1)}(A)\end{aligned}$$

so $\widehat{H} \circ H$ is equal to the identity function, and by symmetry we get the same for $H \circ \widehat{H}$.

- To prove equation (3.2), observe equation (3.1) implies it when X is of the form $X = S_T^{(2)} 1_A$ for any set $A \in \mathcal{F}_T$; and for arbitrary X, if $|X|/S_T^{(1)}$ is $Q^{(1)}$-integrable or if $|X|/S_T^{(2)}$ is $Q^{(2)}$-integrable we approximate X using simple functions and use Lebesgue's dominated convergence theorem. □

This result is false if we replace the word *martingale* by *local martingale*, and that becomes a problem if we want to consider local martingale measure as a natural extension to the concept of martingale measure: the class of local martingale measures is not "preserved" under change of numéraire, we will show later examples of processes that have a local martingale measure under one numéraire but not under another.

3.4 New version of the FTAP

We will prove first that the notion of No Feasible Free Lunch with Vanishing Risk is preserved under numéraire change

Lemma 3.4.1. *Let $i, j \in \{1, 2, \ldots, d\}$ and suppose that the relative prices with respect to $S^{(i)}$ and $S^{(j)}$ are semimartingales. Then there is No Feasible Free Lunch with Vanishing Risk for $S^{(1)}/S^{(i)}, \ldots, S^{(d)}/S^{(i)}$ if and only if there is No Feasible Free Lunch with Vanishing Risk for $S^{(1)}/S^{(j)}, \ldots, S^{(d)}/S^{(j)}$.*

Proof: Suppose there is NFFLVR for $S^{(1)}/S^{(i)}, \ldots, S^{(d)}/S^{(i)}$. For any sequence of trading strategies $H^{(1)}, H^{(2)}, \ldots \in A^f(S/S^{(j)})$, $f_0 \in L_g(S_T/S_T^{(j)})$, $f_0 \geq 0$ such that for $k = 1, 2, \ldots$ $H_0^{(k)} S_0 = 0$ and

$$H_T^{(k)} \frac{S_T}{S_T^{(j)}} \geq f_0 - \frac{g(S_T/S_T^{(j)})}{k}$$

then

$$\begin{aligned} H_T^{(k)} \left(\frac{S_T}{S_T^{(i)}} \right) &\geq \frac{S_T^{(j)}}{S_T^{(i)}} \left(f_0 - \frac{g(S_T/S_T^{(j)})}{k} \right) \\ &= \frac{S_T^{(j)} f_0}{S_T^{(i)}} - \frac{g(S_T/S_T^{(i)})}{k} \end{aligned}$$

for all k and

$$\frac{S_T^{(j)} f_0}{S_T^{(i)}} \in L_g \left(\frac{S_T}{S_T^{(i)}} \right)$$

and by Proposition 3.2.2 $H_T^{(k)} \in A^f(S/S^{(i)})$ for all k so by hypothesis we must have $f_0 = 0$ a.s. and there is NFFLVR for $S^{(1)}/S^{(j)}, \ldots, S^{(d)}/S^{(j)}$. $\square$

Now we are ready to prove the main theorem

Theorem 3.4.2. *Let $(Z_t)_{t \in [0,T]}$ be a non-negative $\mathbb{R}^d$-valued semimartingale with $Z_t^{(1)} \equiv 1$. Then Z_t satisfies No Feasible Free Lunch with Vanishing Risk if and only if there exist an equivalent martingale measure for Z_t.*

Proof: By Lemma 3.4.1, $Z^{(1)}, \ldots, Z^{(d)}$ satisfy NFFLVR if and only if $S^{(1)}/g(S)$, $\ldots$, $S^{(d)}/g(S)$ satisfy NFFLVR, and the last assertion holds if and only if $S^{(1)}/g(S)$, $\ldots$, $S^{(d)}/g(S)$ satisfy NBBFLVR, which, by Delbaen and Schachermayer's Theorem 3.2.5, happens if and only if there exists an equivalent martingale measure for $S^{(1)}/g(S)$, $\ldots$, $S^{(d)}/g(S)$, and by Geman, El Karoui and Rochet Proposition 3.3.1 this is true if and only if there exists an equivalent martingale measure for $Z^{(1)}, \ldots,$ $Z^{(d)}$. □

3.5 Particular cases and examples

3.5.1 Discrete time

Strictly local martingale measures don't occur very naturally in discrete time. In fact, if we assume price processes are bounded below they don't exist, as shown in the next Proposition.

Proposition 3.5.1. *Let $(Z_t)_{t\in\{0,1,2,\ldots\}}$ be a local martingale on the filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\{0,1,2,\ldots\}}, P)$. If $Z_t \geq 0$ for all t then Z_t is a martingale.*

Proof: Let τ_n be a sequence of stopping times that reduces Z. Then Fatou's

Lemma implies, for $0 \leq u \leq v$

$$\begin{aligned} Z_u &= \liminf_{n\to\infty} Z_{u\wedge\tau_n} \\ &= \liminf_{n\to\infty} P\left(Z_{v\wedge\tau_n}|\mathcal{F}_u\right) \\ &\geq P\left(\liminf_{n\to\infty} Z_{v\wedge\tau_n}|\mathcal{F}_u\right) \\ &= P\left(Z_v|\mathcal{F}_u\right) \end{aligned}$$

so Z_t is a supermartingale (in particular, integrable) and for every $t = 1, 2, \ldots$

$$|Z_{t\wedge\tau_n}| \leq Z_1 + Z_2 + \cdots + Z_t$$

so the Lebesgue dominated convergence theorem implies

$$\begin{aligned} P(Z_t) &= \lim_{n\to\infty} P\left(Z_{t\wedge\tau_n}\right) \\ &= Z_0 \end{aligned}$$

so Z_t is a martingale. □

Even for unbounded processes, if there are finitely many securities traded, it follows from a slightly different version of Theorem 2.2.2, or the generalization by Schachermayer (1994), that the existence of a local martingale measure implies the existence of a martingale measure,

Proposition 3.5.2. *Let $(Z_t)_{t\in\{0,1,2,\ldots\}}$ be an $\mathbb{R}^d$-valued adapted stochastic process on the filtered probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\in\{0,1,\ldots T\}}, P)$. If there exists an equivalent local martingale measure for Z_t then there exists a martingale measure.*

Proof: The existence of a local martingale measure implies no bounded below arbitrage for Z_t, and by Morton (1988) Theorem 2.3.1 or the result in Schachermayer (1994) this is sufficient to imply the existence of an equivalent martingale measure. □

McBeth (1991) shows an example of a countable family of discrete time processes for which there exist a unique (strictly) local martingale measure.

3.5.2 Exponential Brownian motion with exploding variance

This is a variation of an example that appears in Delbaen and Schachermayer (1994a). Let B_t be a Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, P)$ and define the process

$$X_t = \exp\left(B_t - \frac{1}{2}t\right)$$

which is the unique solution to the stochastic differential equation

$$dX_t = X_t dB_t, \qquad X_0 = 1$$

and is a martingale with limit $X_t \to 0$ a.s. as $t \to \infty$. Now do a time scale transformation

$$Y_t = \begin{cases} X_{\tan\left(\frac{\pi t}{2}\right)} & \text{if } 0 < t < 1, \\ 0 & \text{if } t \geq 1. \end{cases}$$

then Y_t is a martingale on $[0, 1)$ with respect to the obvious time changed filtration and a continuous supermartingale on $[0, \infty)$. define the stopping times

$$\tau_{n,k} = \inf\left\{t \in \mathbb{R}^+ : Y_t = n \text{ or } Y_t = \frac{1}{k}\right\}$$

then $\tau_{n,k} < 1$ a.s. and $Y_{t\wedge\tau_{n,k}}$ is a martingale for $t \in [0,\infty)$. Define

$$\tau_n = \begin{cases} \lim_{k\to\infty} \tau_{n,k} & \text{if } Y_{\lim_{k\to\infty}\tau_{n,k}} > 0, \\ +\infty & \text{if } Y_{\lim_{k\to\infty}\tau_{n,k}} = 0. \end{cases}$$

and the Lebesgue dominated convergence theorem implies, for $t \in \mathbb{R}^+$

$$\begin{aligned} P\left(Y_{t\wedge\tau_n}\right) &= P\left(\lim_{k\to\infty} Y_{t\wedge\tau_{n,k}}\right) \\ &= Y_0. \end{aligned}$$

and $\tau_n \uparrow \infty$ a.s., so τ_n reduces Y and Y_t is a local martingale on $[0,\infty)$.

Suppose we have a (riskless) security with constant price $S_t^{(1)} = 1$ for all t and another (risky) security given by $S_t^{(2)} = Y_t$. Recall the process of total wealth in the economy $g(S_t) = c_1 S_t^{(1)} + c_2 S_t^{(2)}$ with c_1, c_2 positive integer constants. The discounted price process with respect to the first asset is the same: $Z_t^{(1)} = S_T^{(1)} = 1$, $Z_t^{(2)} = S_T^{(2)} = Y_t$.

P is a local martingale measure for Z so by Corollary 3.2.6 there is No Bounded Below Free Lunch with Vanishing Risk for Z. Now take

$$H_t^{(1)} = 1, \quad H_t^{(2)} = -1$$

to get

$$H_0 Z_0 = 0, \quad H_t Z_t \geq -g(S_t), \quad H_1 Z_1 = 1$$

so there exists a Feasible Free Lunch with Vanishing Risk for S. By Theorem 3.4.2, this implies there's no martingale measure for Z. In fact, the only equivalent local martingale measure is P itself, which is strictly local.

As we claimed in Section 3, this is an example of a process that satisfies NBBA but not NFA. Now if we discount with respect to the sum of the two to get a bounded process

$$U_t = \frac{S_t^{(1)}}{S_t^{(1)} + S_t^{(2)}}, \qquad V_t = \frac{S_t^{(2)}}{S_t^{(1)} + S_t^{(2)}}$$

then (U_t, V_t) satisfy NNA (which is preserved under a numéraire change) but not NBBA.

3.5.3 Constant elasticity of variance = 4, or the inverse of BES(3)

Let B_t be a Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, P)$ and define the process Y_t as the unique (strong) solution to the stochastic differential equation

$$dY_t = Y_t^2 dB_t, \qquad Y_0 = 1$$

We can use a very general result of Engelbert and Schmidt (84) to obtain the existence of a weak solution, unique in probability law (see for example Karatzas and Shreve (1988) Theorems 5.5.4 and 5.5.7). The diffusion coefficient is a locally Lipschitz function of Y_t, and this implies the existence of a unique strong solution.

We can also obtain a solution to this equation using a three dimensional Brownian Motion W_t and the Bessel process $R_t = |W_t|$ as

$$Y_t = R_t^{-1}, \quad B_t = -\sum_{i=1}^{3} \int_0^t \frac{W_s^{(i)}}{R_s} dW_s^{(i)}$$

and it is presented this way in Delbaen and Schachermayer (1994a).

Take a (riskless) security with constant process $S_t^{(1)} = 1$ for all t, and let $S_t^{(2)} = Y_t$. Let $g(S_t) = c_1 S_t^{(1)} + c_2 S_t^{(2)}$ be the total amount of wealth in the economy, with c_1, c_2 positive integer constants.

From the stochastic differential equation for Y_t we see S_t is a local martingale under P, so there doesn't exist any bounded below free lunch with vanishing risk on S.

We will show the existence of feasible free lunch with vanishing risk for S_t by proving there is no equivalent martingale measure, without using the martingale representation theorem. Define the discounted price processes

$$U_t = \frac{S_t^{(1)}}{S_t^{(1)} + S_t^{(2)}} = \frac{1}{1 + Y_t}$$
$$V_t = \frac{S_t^{(2)}}{S_t^{(1)} + S_t^{(2)}} = 1 - U_t$$

and applying Itô's Lemma we get

$$\begin{aligned} dU_t &= \frac{-Y_t^2}{(1+Y_t)^2} dB_t + \frac{Y_t^4}{(1+Y_t)^3} dt \\ &= -(1-U_t)^2 dB_t + \frac{(1-U_t)^4}{U_t} dt \end{aligned}$$

so if there exists an equivalent martingale measure Q equivalent to P under which U_t is a martingale, U_t should satisfy the equation

$$dU_t = -(1-U_t)^2 d\widehat{B}_t, \qquad U_0 = \frac{1}{2}$$

for some Brownian motion $\widehat{B}$ on Q; but the only solution to this equation takes negative values with positive probability, which is a contradiction. So by Theorem 3.2.5 there exist a sequence of trading strategies $H^{(1)}, H^{(2)}, \ldots$ and bounded

$f_0 \geq 0,\ f_0 \not\equiv 0$ such that

$$H_t^{(k,1)} U_t + H_t^{(k,2)} V_t \geq f_0 - \frac{1}{k}$$

so then

$$H_t^{(k)} Z_T \geq \left(f_0 - \frac{1}{k} \right) g(Z_T)$$

and they generate Feasible Free Lunch with Vanishing Risk for Z_t.

Chapter 4

Some diffusion models of stock prices: stochastic volatility

Here we study models for prices of shares of common stock. We will focus on models for price processes with continuous paths, leaving outside many other relevant examples.

We have two assets trading, a "bond" whose price is of bounded variation, and a "stock" whose price is a real semimartingale. We show several models proposed by practitioners and finance theorists and pursue a systematic verification of the properties defined in chapter 2; in particular, we verify the existence or absence of an equivalent martingale measure. [1]

We show that, for some models of stochastic volatility, the natural candidates for

[1]The initial motivation for the work that appears in this chapter and the next came from a master of engineering project in 1993–1994 at the School of Operations Research sponsored by Morgan Stanley, and from the work with the Summer 1994 study group with Peter Carr at the Johnson Graduate School of Management.

martingale measures are only strictly local martingale measures. This way arbitrage opportunities might appear if one of these measures is used for pricing contingent claims.

4.1 General form and call pricing formula

We study an economy with two assets trading. Assume frictionless markets: no restrictions on short sales, transaction costs or taxes.

We consider an underlying probability space $(\Omega, \mathcal{F}, P)$ with a Brownian motion W_t adapted to a given filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ that satisfies the conditions given in chapter 2.

The first asset is called a bond and its price is given by

$$S_t^{(0)} = \exp\left(\int_0^t r(u,\omega)du\right) \tag{4.1}$$

for some $r(t,\omega)$ progressively measurable integrable stochastic process.

The second asset is the stock, whose price S_t is a continuous, adapted stochastic process that satisfies the equation

$$S_t = x + \int_0^t S_u\sigma(u,\omega)dW_u + \int_0^t S_u\mu(u,\omega)du \tag{4.2}$$

with

$$\int_0^t |\sigma(u,\omega)|^2 du < \infty \qquad \int_0^t |\mu(u,\omega)|^2 du < \infty$$

for all $t \in \mathbb{R}^+$. In this case S_t must be given by the stochastic exponential

$$S_t = S_0 \exp\left(\int_0^t \sigma(u,\omega)dW_u + \int_0^t \mu(u,\omega)du - \frac{1}{2}\int_0^t \sigma^2(u,\omega)du\right)$$

(see for example Protter (1990) Theorem II 36).

The assumption of this form for the equation is not as restrictive as it seems, especially if we want the absence of arbitrage opportunities, as the following result shows

Proposition 4.1.1. *Let S_t be any continuous adapted stochastic process on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, P)$. Suppose there exists a probability measure $Q \sim P$ such that $S_t/S_t^{(0)}$ is a Q-martingale, $Q(S_t > 0\ \forall t) = P(S_t > 0\ \forall t) = 1$ and $\mathcal{F}_t$ is the completion of the filtration generated by W_t. Then there exist progressively measurable processes $\sigma(t,\omega)$ and $\mu(t,\omega)$ such that (4.2) holds.*

Proof: Let

$$\rho = \frac{dQ}{dP}$$

be the Radon-Nikodym derivative and define ρ_t as a continuous version of $P(\rho|\mathcal{F}_t)$. Then $\rho_t > 0$ a.s. for all $t \in \mathbb{R}^+$, so if we let

$$\tau = \inf\{t > 0 \ :\ \rho_t = 0\}$$

we apply the optional stopping theorem to get, for $r \geq 0$, $P(\rho_{\tau+r}) = P(\rho_\tau) = 0$ and therefore $P(\rho_t = 0 \text{ on } [\tau, \infty)) = 1$. This implies then $P(\rho_t > 0\ \forall t \in \mathbb{R}^+) = 1$.

By the martingale representation theorem (see for example Karatzas and Shreve (1988) Theorem 3.4.15 and exercise 3.4.16) we get the existence of a progressively measurable $\alpha(t,\omega)$ such that

$$\begin{aligned}\rho_t &= \rho_0 + \int_0^t \alpha(u,\omega)dW_u \\ &= \rho_0 + \int_0^t \rho_u\left(\frac{\alpha(u,\omega)}{\rho_u}\right)dW_u\end{aligned}$$

also observe that ρ_t^{-1} is a.s. locally bounded, so $\widetilde{\alpha}(u,\omega) = \alpha(u,\omega)/\rho_t$ is W_t-integrable and ρ_t must be given by the Doléans-Dade exponential

$$\rho_t = \rho_0 \exp\left(\int_0^t \widetilde{\alpha}(u,\omega) dW_u - \frac{1}{2} \int_0^t \widetilde{\alpha}^2(u,\omega) du \right)$$

Now observe that $S_t/S_t^{(0)}$ is a Q-martingale by assumption, so $\rho_t S_t/S_t^{(0)}$ is a P-martingale and can also be represented as an integral with respect to Brownian motion

$$\frac{\rho_t S_t}{S_t^{(0)}} = \int_0^t \widetilde{\sigma}(u,\omega) dW_u$$

and then, from the integration by parts formula and Itô's lemma

$$\begin{aligned} dS_t &= d\left[\left(\frac{\rho_t S_t}{S_t^{(0)}} \right) \left(\frac{S_t^{(0)}}{\rho_t} \right) \right] \\ &= \left(\frac{\rho_t S_t}{S_t^{(0)}} \right) d\left(\frac{S_t^{(0)}}{\rho_t} \right) + \left(\frac{S_t^{(0)}}{\rho_t} \right) d\left(\frac{\rho_t S_t}{S_t^{(0)}} \right) + d\left\langle \frac{\rho S}{B}, \frac{B}{\rho} \right\rangle_t \\ &= S_t \left(-\widetilde{\alpha}_t + \frac{S_t^{(0)}}{\rho_t S_t} \widetilde{\sigma}_t \right) dW_t + S_t \left(r_t + \widetilde{\alpha}_t^2 - \frac{\widetilde{\alpha}_t S_t^{(0)}}{\rho_t S_t} \widetilde{\sigma}_t \right) dt \end{aligned}$$

let

$$\sigma(t,\omega) = -\widetilde{\alpha}_t + \frac{S_t^{(0)}}{\rho_t S_t} \widetilde{\sigma}_t \qquad\qquad \mu(t,\omega) = r_t + \widetilde{\alpha}_t^2 - \frac{\widetilde{\alpha}_t S_t^{(0)}}{\rho_t S_t} \widetilde{\sigma}_t$$

which are integrable because S_t^{-1} and ρ_t^{-1} are a.s. locally bounded. Therefore S_t must be the Doléans-Dade exponential given in equation (4.2). $\square$

Now we want to find a fair price for a European call option, which gives the buyer the option to buy the stock at a maturity date T for a prespecified price K. Its payoff at time T is then $(S_T - K)^+$.

Definition. Let X be a $\mathcal{F}_T$-measurable random variable, C_t a $\mathcal{F}_t$-adapted stochastic process. We say C_t is a price for X consistent with $(S_t^{(0)}, S_t)$ provided

$$C_T = X \qquad \text{and}$$

$$\left(1, \frac{S_t}{S_t^{(0)}}, \frac{C_t}{S_t^{(0)}}\right) \qquad \text{satisfy NFFLVR}$$

Proposition 4.1.2. *Suppose C_t is a price for the call option with payoff $(S_T - K)^+$, consistent with $(S_t^{(0)}, S_t)$. Then there exists a probability measure Q on $(\Omega, \mathcal{F}_T)$, $Q \sim P$, such that*

$$C_t = S_t^{(0)} Q\left(\left.\frac{(S_T - K)^+}{S_T^{(0)}}\right| \mathcal{F}_t\right) \tag{4.3}$$

moreover, under our assumption that S_t satisfies equation (4.2) we can express

$$C_t = S_t Q^{(1)}(S_T \geq K|\mathcal{F}_t) - KB(t,T)Q^{(2)}(S_t \geq K|\mathcal{F}_t) \tag{4.4}$$

where

$$Q^{(1)} \sim Q, \qquad \frac{dQ^{(1)}}{dQ} = \frac{S_T B_0}{S_T^{(0)} S_0}.$$

$$Q^{(2)} \sim Q, \qquad \frac{dQ^{(2)}}{dQ} = \frac{(S_T^{(0)})^{-1}}{Q((S_T^{(0)})^{-1})}.$$

$$B(t,T) = S_t^{(0)} Q((S_T^{(0)})^{-1}|\mathcal{F}_t).$$

Remark 4.1.3. This is a well known generalization of the Black-Scholes formula (Proposition 4.2.1 in the next section), it appears for example in Geman et al. (1995).

$B(t,T)$ is the Q-price of a pure discount bond that pays 1 at time T. We could have stated the theorem assuming the existence of a process D_t with $D_T = 1$

and impose consistency of C_t with respect to $(S_t^{(0)}, S_t, D_t)$, obtaining Q such that $B(t,T) = D_t$ a.s. and (4.4) holds.

The measure $Q^{(2)}$ is sometimes referred to as the *forward measure*, and the idea of expressing the formulas in term of forward prices goes back to Merton (1973).

Proof: Use Theorem 3.4.2 to obtain the existence of $Q \sim P$ on $\mathcal{F}_T$ such that

$$1, \frac{S_t}{S_t^{(0)}}, \frac{C_t}{S_t^{(0)}}, \text{ are } Q\text{-martingales}$$

so

$$\begin{aligned}\frac{C_t}{S_t^{(0)}} &= Q\left(\left.\frac{C_T}{S_T^{(0)}}\right| \mathcal{F}_t\right) \\ &= Q\left(\left.\frac{(S_T - K)^+}{S_T^{(0)}}\right| \mathcal{F}_t\right)\end{aligned}$$

and then

$$\begin{aligned}C_t &= S_t^{(0)} Q\left(\left.\frac{(S_T - K)^+}{S_T^{(0)}}\right| \mathcal{F}_t\right) \\ &= S_t^{(0)} Q\left(\left.\frac{S_T}{S_T^{(0)}} 1_{\{S_T \geq K\}}\right| \mathcal{F}_t\right) - S_t^{(0)} Q\left(\left.\frac{K}{S_T^{(0)}} 1_{\{S_T \geq K\}}\right| \mathcal{F}_t\right) \\ &= S_t Q^{(1)}(S_T \geq K | \mathcal{F}_t) - K B(t,T) Q^{(2)}(S_T \geq K | \mathcal{F}_t) \quad \square\end{aligned}$$

4.2 Exponential Brownian motion and the Black-Scholes formula

In this section we assume $\sigma(t,\omega)$ and $r(t,\omega)$ are constants. The results in this case are well known, but we state them again here to compare with the new ones. They appeared first in Black and Scholes (1973) and in Merton (1973). We also assume for simplicity that μ is also constant, although it is not needed for the results. Then equation (4.1) and (4.2) become (in differential form)

$$dS_t^{(0)} = S_t^{(0)} r dt, \qquad B_0 = 1$$

$$dS_t = S_t \sigma dW_t + S_t \mu dt, \qquad S_0 = x.$$

There exists a solution to these stochastic differential equations, it is unique and it is given by

$$S_t^{(0)} = \exp(rt)$$
$$S_t = S_0 \exp(\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t)$$

Proposition 4.2.1. *(Black-Scholes formula). There exists exactly one probability measure $Q \sim P$ such that $S_t/S_t^{(0)}$ is a Q-martingale. It is given by*

$$\frac{dQ}{dP} = \exp\left(\left(\frac{r-\mu}{\sigma}\right)W_t - \frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2 t\right)$$

and the only price process C_t for the call option with payoff $(S_T - K)^+$ which is consistent with $(S_t^{(0)}, S_t)$ is given by

$$C_t = S_t^{(0)} Q\left(\frac{(S_T - K)^+}{S_T^{(0)}} \middle| \mathcal{F}_t\right) = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2) \tag{4.5}$$

where

$$d_1 = \frac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = \frac{\log(S_t/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

and Φ is the standard normal distribution function.

4.3 Constant elasticity of variance models

Here we assume that σ_t and μ_t from equation (4.2) exist and are measurable functions of S_t, so that S_t is a solution to the stochastic differential equation

$$dS_t = g(S_t)dB_t + f(S_t)dt, \qquad S_0 = x. \tag{4.6}$$

We also require that the elasticity of g_t^2 with respect to S_t is constant, so g satisfies the differential equation

$$\frac{dg^2/g^2}{dS/S} = \left(\frac{dg^2}{dS}\right)\frac{S}{g^2} = 2\alpha$$

whose general solution is given by $g(x) = Cx^\alpha$. We assume $\alpha > 0$. Notice that, if $\alpha = 1$, then S_t has the same diffusion coefficient as the exponential Brownian motion discussed in the previous section. We also assume f is locally Lipschitz continuous and $h(x) = f(x)/g(x)$ is bounded.

For simplicity suppose the interest rate $r = 0$ so that $S_t^{(0)} \equiv 1$.

Proposition 4.3.1. *There exists a unique strong solution to equation* (4.6) *up to time τ (the first hitting time of zero) defined by*

$$\tau = \inf\{t \in \mathbb{R}^+ : S_t = 0\}$$

Proof: first construct a weak solution. The stochastic differential equation

$$dX_t = \sigma|X_t|^\alpha dB_t, \qquad X_0 = x.$$

has a non-exploding weak solution by a result of Engelbert and Schmidt (1984) (see for example Karatzas and Shreve (1988) Theorem 5.5.4). Then define the process

$$\gamma_t = \exp\left(\int_0^t h(X_u)dB_u - \frac{1}{2}\int_0^t h^2(X_u)du\right)$$

and define the new probability measure Q on $(\Omega, \mathcal{F}_T)$ with

$$Q(A) = P(\gamma_T 1_A), \qquad A \in \mathcal{F}_T$$

recall that h is bounded so $W_t = B_t - \int_0^t h(X_u)du$ is a Brownian motion under Q and

$$dX_t = \sigma|X_t|^\alpha dW_t + f(X_t)dt, \qquad X_0 = x.$$

is a weak solution to equation (4.6).

The law of this solution may not be unique in general. However, $g(x)$ and $h(x)$ are locally Lipschitz on $x \in (0, \infty)$; so we get pathwise uniqueness (see Ikeda and Watanabe (1989) Chapter IV Theorem 3.1) up to time τ. This implies S_t is the unique strong solution to equation (4.6) on the interval $[0, \tau]$. Observe that $g(0) = 0$ and $|f(0)| \leq K|g(0)| = 0$ so $S_t = 0$ on $[\tau, \infty)$ gives a solution for all $\tau \in \mathbb{R}^+$. □

Now fix a time horizon $T \in \mathbb{R}^+$ and define the probability measure $Q \sim P$ on $(\Omega, \mathcal{F}_T)$ by its Radon-Nikodym derivative with respect to P

$$\gamma_T = \exp\left(\int_0^T h(S_t)dB_t - \frac{1}{2}\int_0^T h^2(S_t)dt\right)$$

$$Q(A) = P(\gamma_T 1_A), \qquad A \in \mathcal{F}_T$$

then the Girsanov theorem implies $W_t = B_t + \int_0^t h(S_u)du$ is a Q-Brownian motion and S_t satisfies the equation

$$dS_t = |S_t|^\alpha \sigma dW_t, \qquad S_0 = x. \tag{4.7}$$

Proposition 4.3.2. *If S_t solves equation (4.7), then S_t hits zero eventually with probability 1 if $0 < \alpha < 1$, and with probability zero if $\alpha \geq 1$.*

Remark 4.3.3. This result implies that, when $0 < \alpha < 1$ S_t couldn't be an exponential semimartingale, so there is no representation of the form (4.2). However, Proposition 4.1.2 up to equation (4.3) is still valid and it can be used to price the European call option.

Cox (1975) and Cox and Ross (1976) study the case $0 < \alpha < 1$ and give explicit formulas for the transition probability distribution of S_t. Emanuel and MacBeth (1982) do the same in the case $\alpha > 1$ pointing out that, surprisingly, under their "risk-neutral" probability distribution the discounted conditional expected value of the price at a future time is not equal to today's price; that is, the discounted stock price process is not a martingale under this measure. The explanation to this mysterious phenomenon is given in the following theorem.

Theorem 4.3.4. *S_t is a martingale under P if and only if $\alpha \leq 1$. When $\alpha > 1$ S_t is a strictly local martingale on $(\Omega, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, P)$.*

Corollary 4.3.5.

$(1,\ S_t)$ *satisfy NFFLVR if* $0 < \alpha \leq 1$.

$(1,\ S_t)$ *admit FFLVR if* $\alpha > 1$.

Proof of Proposition 4.3.2: S_t is a time-homogeneous 1 dimensional diffusion with zero drift and diffusion coefficient locally bounded above and away from zero on the interval $(0, \infty)$ so we can apply Feller's test to see if it hits zero in finite time (see for example Karatzas and Shreve (1988) Section 5.5).

The drift coefficient is zero, so we can take the identity as *scale function*

$$p(x) = x, \text{ thus } \quad p(0+) > -\infty, \qquad p(+\infty) = +\infty$$

and this implies (see for example Karatzas and Shreve (1988) Proposition 5.5.22)

$$\lim_{t\to\infty} S_t = 0 \quad a.s. \qquad \text{and} \qquad \sup_{t\leq\infty} S_t < \infty \quad a.s.$$

and to see if it hits zero in finite time we calculate the function u on $(0, \infty)$

$$\begin{aligned} u(x) &= \int_1^x y \int_1^y \frac{2dz}{zz^{2\alpha}\sigma^2} dy \\ &= \int_1^x \frac{2y}{\sigma^2} \int_1^y z^{-(2\alpha+1)} dz dy \\ &= \frac{2}{\sigma^2} \int_1^x y \frac{(1-y^{-2\alpha})}{2\alpha} dy \\ &= \frac{2}{\sigma^2} \left[\frac{x^2}{2} - \frac{x^{2(1-\alpha)}}{2(1-\alpha)} + \frac{\alpha}{2(1-\alpha)} \right] \end{aligned}$$

here we assumed $\alpha \neq 1$. In the case $\alpha = 1$ we get S_t is an exponential Brownian motion, which is always positive, and the result holds. Thus

$$u(0+) = \begin{cases} \frac{\alpha}{2(1-\alpha)} & \text{if } 0 < \alpha < 1, \\ +\infty & \text{if } 1 < \alpha. \end{cases}$$

$$u(+\infty) = +\infty \qquad \text{for all } \alpha.$$

so when $0 < \alpha < 1$, $u(0+) < \infty$ and $p(+\infty) = \infty$ imply (see Proposition 5.5.32 in Karatzas and Shreve (1988)) that $P\{\tau < \infty\} = 1$ and when $1 < \alpha$, $u(0+) = u(+\infty) = +\infty$ imply (see Theorem 5.5.29 in Karatzas and Shreve (1988)) that $P\{\tau < \infty\} = 0$. $\square$

Now Theorem 4.3.4 follows as an immediate consequence of the following two lemmas

Lemma 4.3.6. *If S_t solves*

$$dS_t = S_t^\alpha \sigma dW_t, \qquad S_0 = x.$$

with S_t absorbed after the first hitting time of zero, then S_t is a supermartingale and for every $T \in \mathbb{R}^+$

$$P(S_T) = S_0 P(\widehat{v}_t \text{ doesn't explode on the interval } [0,T])$$

where $\widehat{v}_t$ is the unique solution up to an explosion time to the SDE

$$d\widehat{v}_t = (\alpha - 1)\widehat{v}_t^2 \sigma dW_t + (\alpha - 1)\left(\frac{\alpha}{2}\right)\widehat{v}_t^3 \sigma^2 dt \qquad \widehat{v}_0 = x^{\alpha - 1}.$$

Lemma 4.3.7. *The (unique) solution to the equation*

$$d\widehat{v}_t = (\alpha - 1)\widehat{v}_t^2 \sigma dW_t + (\alpha - 1)\left(\frac{\alpha}{2}\right)\widehat{v}_t^3 \sigma^2 dt \qquad \widehat{v}_0 = x^{\alpha - 1}.$$

explodes in finite time if and only if $\alpha > 1$.

<u>Proof of Lemma 4.3.6:</u>

S_t is the integral of a predictable process with respect to a local martingale, so it is again a local martingale, and it is nonnegative because it's absorbed at zero after time τ (by pathwise uniqueness of the solution to equation (4.7) when $\alpha \geq 1/2$ and by assumption when $\alpha < 1/2$).

Now define the process

$$v_t = S_t^{\alpha-1} \qquad \text{for } t \in [0,\tau).$$

so we get the following equations valid on $[0,\tau)$ (using Itô's formula for v_t)

$$\begin{aligned} dS_t &= S_t v_t \sigma dW_t, \\ dv_t &= (\alpha-1)S_t^{\alpha-2}dS_t + \frac{1}{2}(\alpha-1)(\alpha-2)S_t^{\alpha-3}d\langle S\rangle_t \\ &= (\alpha-1)S_t^{2(\alpha-1)}\sigma dW_t + \frac{1}{2}(\alpha-1)(\alpha-2)S_t^{3(\alpha-1)}\sigma^2 dt \\ &= (\alpha-1)v_t^2\sigma dW_t + \frac{1}{2}(\alpha-1)(\alpha-2)v_t^3\sigma^2 dt \end{aligned} \tag{4.8}$$

and define the sequence of stopping times

$$\tau_n = \inf\left\{t \in \mathbb{R}^+ : |v_t| \geq n\right\}$$

then $S_t^{(n)} = S_{t\wedge\tau_n}$ is a local martingale under P for $n = 1,2,\ldots$ and if $Z_t^{(n)} = \int_0^{t\wedge\tau_n} v_t \sigma dW_t$ then $S_t^{(n)}$ is the exponential semimartingale of $Z_t^{(n)}$ started at x

$$S^{(n)} = x\,\mathcal{E}xp(Z^{(n)}) \qquad \text{and} \qquad \langle Z^{(n)}\rangle_t \leq n^2\sigma^2 t \quad \forall t \in \mathbb{R}^+$$

so $S_t^{(n)}$ is a positive martingale for $n = 1,2,\ldots$ and then it follows that $\tau_n < \tau$ so the stochastic differential equations (4.8) are valid on $[0,\tau_n]$ for all $n = 1,2,\ldots$. Finally, Fatou's Lemma implies, for $0 \leq u < t$,

$$
\begin{aligned}
P(S_t|\mathcal{F}_u) &= P(\liminf_{n\to\infty} S_t^{(n)}|\mathcal{F}_u) \\
&\leq \liminf_{n\to\infty} P(S_t^{(n)}|\mathcal{F}_u) \\
&= S_u.
\end{aligned}
$$

so S_t is a supermartingale under P.

Now fix $T \in \mathbb{R}^+$ and define the new probability measure Q_n on $(\Omega, \mathcal{F}_T)$ as

$$
Q_n(A) = \frac{1}{S_0} P(S_T^{(n)} 1_A) \qquad \text{for all } A \in \mathcal{F}_T
$$

then Lebesgue's monotone convergence Theorem and Girsanov's Theorem imply that for every set Γ in the Borel σ-field on the space of continuous paths in $[0, T]$,

$$
\begin{aligned}
P\left(S_T 1_{\{W\in\Gamma\}}\right) &= \lim_{n\to\infty} P\left(S_T^{(n)} 1_{\{\tau_n > T\}} 1_{\{W\in\Gamma\}}\right) \\
&= S_0 \lim_{n\to\infty} Q_n\left(1_{\{W\in\Gamma, \tau_n > T\}}\right) \\
&= S_0 \lim_{n\to\infty} P\{\widehat{W} \in \Gamma, \widehat{\tau}_n > T\}
\end{aligned}
$$

where $\widehat{W}_t$, $\widehat{v}_t$ and $\widehat{\tau}_n$ are defined as the (unique in law) solutions to the equations

$$
\begin{aligned}
& d\widehat{W}_t = dW_t + \widehat{v}_t \sigma dt, \\
(4.9) \qquad & d\widehat{v}_t = (\alpha-1)\widehat{v}_t^2 \sigma dW_t + \frac{1}{2}(\alpha-1)(\alpha-2)\widehat{v}_t^3 \sigma^2 dt + (\alpha-1)\widehat{v}_t^3 \sigma^2 dt, \\
& \widehat{\tau}_n = \inf\left\{t \in \mathbb{R}^+ : |\widehat{v}_t| \geq n\right\}.
\end{aligned}
$$

with initial condition $\widehat{W}_0 = 0, \quad \widehat{v}_0 = x^{\alpha-1}$. To see that this is true observe that the coefficients on the stochastic differential equation for $\widehat{v}_t$ are locally Lipschitz continuous, so there exists a unique strong solution up to an explosion time $\widehat{\tau}_\infty = \lim_{n\to\infty}\widehat{\tau}_n$ (see Ikeda and Watanabe (1989) Chapter IV, Theorems 2.3 and 3.1); and applying Girsanov's theorem we obtain that the process

$$W_t^{(n)} = W_t - \int_0^{t\wedge T} 1_{\{s\leq\tau_n\}} v_s^\alpha \sigma ds$$

is a Brownian motion under Q_n and W_t and v_t satisfy

$$\begin{aligned} dW_t &= dW_t^{(n)} + 1_{\{t\leq T\wedge\tau_n\}} v_t \sigma dt, \\ dv_t &= (\alpha-1)v_t^2\sigma dW_t^{(n)} + \frac{1}{2}(\alpha-1)(\alpha-2)v_t^3\sigma^2 dt + 1_{\{t\leq T\wedge\tau_n\}}(\alpha-1)v_t^3\sigma^2 dt, \end{aligned}$$

with initial condition $W_0 = 0, \quad v_0 = x^{\alpha-1}$. This is the same as equation (4.9) up to time $T\wedge\tau_n$, and because $\{W\in\Gamma, \tau_n > T\} \in \mathcal{F}_{T\wedge\tau_n}$ the identity follows.

Now let $\Gamma = C[0,T]$ to get

$$\begin{aligned} P(S_T) &= S_0 \lim_{n\to\infty} P(1_{\{\widehat{\tau}_n > T\}}) \\ &= S_0 P(\widehat{\tau}_n > T \quad \text{for some n}) \\ &= S_0 P(\widehat{v}_t \text{ doesn't explode on } [0,T]) \ \Box \end{aligned}$$

Proof of Lemma 4.3.7:

We will apply Feller's test on $\widehat{v}_t$ taking values on $(0,\infty)$ to check if it explodes in finite time. Notice $\widehat{v}_t$ itself is a diffusion with coefficients locally bounded above and away from zero on $(0,\infty)$, so the test is applicable. (see for example Karatzas and Shreve (1988) section 5.5).

In the case $\alpha = 1$ we get $\widehat{v}_t$ is constant and the result is trivially true, so from now on we assume $\alpha \neq 1$.We calculate the *scale function* p, for $x \in (0,\infty)$:

$$\begin{aligned} p(x) &= \int_1^x \exp\left(-2\int_1^y \frac{(\alpha-1)(\alpha/2)z^3\sigma^2}{(\alpha-1)^2 z^4\sigma^2} dz\right) dy \\ &= C\int_1^x \exp\left(\frac{-\alpha\log(y)}{(\alpha-1)}\right) dy \\ &= (1-\alpha)y^{\frac{-1}{\alpha-1}}\Big|_{y=1}^{x} \end{aligned}$$

so

$$p(0+) = \begin{cases} \alpha - 1 & \text{if } 0 < \alpha < 1, \\ -\infty & \text{if } 1 < \alpha. \end{cases}$$

$$p(+\infty) = \begin{cases} +\infty & \text{if } 0 < \alpha < 1, \\ \alpha - 1 & \text{if } 1 < \alpha. \end{cases}$$

In the case $0 < \alpha < 1$ we get (see Karatzas and Shreve (1988) Proposition 5.5.22)

$$\lim_{t\to\widehat{\tau}_\infty} \widehat{v}_t = 0\, a.s. \qquad \text{and} \qquad \sup_{t\leq\widehat{\tau}_\infty} \widehat{v}_t < \infty\, a.s.$$

so $\widehat{v}_t$ never explodes to $+\infty$.

In the case $1 < \alpha$ we get

$$\lim_{t\to\widehat{\tau}_\infty} \widehat{v}_t = +\infty\, a.s. \qquad \text{and} \qquad \inf_{t\leq\widehat{\tau}_\infty} \widehat{v}_t > 0\, a.s.$$

so $\widehat{v}_t$ explodes to $+\infty$ at $\widehat{\tau}_\infty$. We only need to check if $P\{\widehat{\tau}_\infty < \infty\}$ is zero or not to see if $\widehat{v}_t$ explodes in finite time. For that we define the function u on $(0,\infty)$

$$
\begin{aligned}
u(x) &= \int_1^x p'(y) \int_1^y \frac{2dz}{(\alpha-1)^2 z^4 \sigma^2} dy \\
&= \int_1^x y^{\frac{-\alpha}{\alpha-1}} \int_1^y \frac{2z^{-(3\alpha-4)/(\alpha-1)}}{(\alpha-1)^2\sigma^2} dzdy \\
&= \int_1^x \frac{2y^{\frac{-\alpha}{\alpha-1}}}{(\alpha-1)^2\sigma^2} \left(\frac{y^{-(2\alpha-3)/(\alpha-1)} - 1}{-(2\alpha-3)/(\alpha-1)} \right) dy \\
&= \frac{2}{(\alpha-1)\sigma^2(3-2\alpha)} \left[\frac{-x^{-2}}{2} + (\alpha-1)x^{\frac{-1}{\alpha-1}} + \frac{3-2\alpha}{2} \right]
\end{aligned}
$$

if $\alpha \neq 3/2$; and if $\alpha = 3/2$ then

$$
\begin{aligned}
u(x) &= \int_1^x \frac{2y^{-3}}{(1/4)\sigma^2} \int_1^y z^{-1} dzdy \\
&= \frac{8}{\sigma^2} \int_1^x y^{-3} log(y) dy \\
&= \frac{8}{\sigma^2} \left[\frac{-log(x)x^{-2}}{2} - \frac{x^{-2}}{4} + \frac{1}{4} \right]
\end{aligned}
$$

so in any case

$$
u(0+) = +\infty \qquad \text{for all } \alpha
$$

$$
u(+\infty) = \begin{cases} +\infty & \text{if } 0 < \alpha < 1, \\ \frac{1}{\sigma^2(\alpha-1)} & \text{if } 1 < \alpha. \end{cases}
$$

thus, when $\alpha > 1$ we get $u(+\infty) = \infty$ and $p(0+) = -\infty$, and this implies $P\{\widehat{\tau}_\infty < \infty\} = 1$ (see Karatzas and Shreve (1988) Proposition 5.5.32). □

Proof of Corollary 4.3.5: The martingale representation theorem with respect to the Brownian motion W_t implies that for every Q-integrable random variable $X \in \mathcal{F}_{T \wedge \tau_n}$ we can find a Q-optimal trading strategy h_t such that

$$
X = h_T^{(0)} + h_T^{(1)} S_T = \int_0^T h_t^{(1)} dS_t
$$

so by Theorem 2.4.2 Q is the only equivalent local martingale measure for $S_{t\wedge\tau_n}$. If $\alpha > 1$ then $\tau_n \uparrow \tau = \infty$ with probability 1, and then the result follows as an application of Theorem 3.4.2. $\square$

4.4 Stochastic volatility models I: linear diffusion driving the variance

We begin with a fairly general stochastic volatility framework that captures several of the known models. Start with a 2-dimensional Brownian motion $B_t = (B_t^{(1)}, B_t^{(2)})$ on a probability space $(\Omega, \mathcal{F}, P)$, adapted to a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ that satisfies the conditions stated in chapter 2.

Suppose there exist diffusion processes $(S_t^{(0)}, S_t, v_t)$ on $(\Omega, \mathcal{F}, P)$ (see Remark 4.4.2), $v_t \geq 0$ a.s., that satisfy the equations:

$$\begin{aligned} dS_t^{(0)} &= S_t^{(0)} r\,dt, && S_0^{(0)} = 1 \\ dS_t &= S_t v_t^{\alpha}\big(\sigma_t^{(1)} dB_t^{(1)} + \sigma_t^{(2)} dB_t^{(2)}\big) + S_t \mu_t(\omega)\,dt, && S_0 = x \end{aligned} \tag{4.10}$$

and the variance parameter process v_t solves

$$dv_t = v_t\big(a_t^{(1)} dB_t^{(1)} + a_t^{(2)} dB_t^{(2)}\big) + \rho_t\big(L_t - v_t\big)dt, \qquad v_0 = 1$$

Here α is a positive constant, $\sigma_t = (\sigma_t^{(1)}, \sigma_t^{(2)})$ and $a_t = (a_t^{(1)}, a_t^{(2)})$ are deterministic vectors defining the direction and strength of the diffusion component increments of S_t and v_t respectively. $S_t^{(0)}$ is the price of the bond and S_t is the price of the stock. v_t is the process driving the volatility function of the stock; it is a diffusion with linear coefficients and a mean reversion drift term.

If we let $\alpha = 1/2$ we obtain the model proposed by Hull and White (1987) and further studied by Dupire (1992). If $\alpha = 1$ we get the model suggested by Wiggins (1987) and an example from Eisenberg and Jarrow (1994).

Remark 4.4.1. Observe that α is almost a redundant parameter. Its effect can be embedded in the drift and diffusion coefficients of v_t if we assume a slightly more general form of the drift function. We keep α and the simple form of the drift to emphasize the relationship with existing models.

Notation: From now on we will write $\sigma_t dB_t$ to represent $\sigma_t^{(1)} dB_t^{(1)} + \sigma_t^{(2)} dB_t^{(2)}$ and $a_t dB_t$ is defined similarly. For any vectors $x, y \in \mathbb{R}^2$ let $|x| = \sqrt{x_1^2 + x_2^2}$ denote the Euclidean norm and $(x \cdot y) = x_1 y_1 + x_2 y_2$ the inner product.

We also impose the following restrictions on the time-dependent coefficients:

$$
\begin{aligned}
&\sigma : \mathbb{R}^+ \to \mathbb{R}^2, && 0 < |\sigma_t| < M \quad \text{for all } t \\
&a : \mathbb{R}^+ \to \mathbb{R}^2, && 0 < |a_t| < M \quad \text{for all } t \\
&\rho : \mathbb{R}^+ \to \mathbb{R}, && 0 \leq \rho_t < M \quad \text{for all } t \\
&L : \mathbb{R}^+ \to \mathbb{R}, && 0 < L_t < M \quad \text{for all } t \\
&\mu : \mathbb{R}^+ \times \Omega \to \mathbb{R} && \mathcal{F}_t\text{-adapted and a.s. locally bounded.} \\
&\alpha > 0.
\end{aligned}
$$

for some $M \in \mathbb{R}^+$.

Remark 4.4.2. There exists exactly one solution to (4.10). v_t should satisfy the

1-dimensional stochastic differential equation

$$dv_t = v_t|a_t|d\overline{B}_t + \rho_t\big(L_t - v_t\big)dt, \qquad v_0 = 1$$

with respect to the Brownian motion

$$\overline{B}_t = \int_0^t \frac{a_u^{(1)}dB_u^{(1)} + a_u^{(2)}dB_u^{(2)}}{\sqrt{\big(a_u^{(1)}\big)^2 + \big(a_u^{(2)}\big)^2}}$$

and the drift and diffusion coefficients are Lipschitz continuous with linear growth, so we can apply Itô's existence theorem to find v_t (unique by Lipschitz continuity) and obtain S_t as the unique exponential semimartingale of $\int v_t^\alpha \sigma_t dB_t + \int \mu_t dt$ starting at x (see Elliot (1982) Theorem 13.5). We will show later that $v_t > 0$ for all $t \in \mathbb{R}^+$ (see Proposition 4.4.13).

Let $a_t^\perp$ be a vector in $\mathbb{R}^2$ orthogonal to a_t, with the same magnitude, and

$$\delta_t = \frac{\mu_t}{v_t^\alpha(a_t^\perp \cdot \sigma_t)} a_t^\perp$$

if a_t and σ_t are not parallel for any $t \in \mathbb{R}^+$ then then δ_t is finite, and assume also that it is B_t-integrable so we can define

$$G_t = \exp\Big(-\int_0^t \delta_t dB_t + \frac{1}{2}\int_0^t \delta_t^2 dt\Big)$$

Assumption 4.4.3. $|(a_t \cdot \sigma_t)| < |a_t||\sigma_t|$ *for all* t *(the vectors are not parallel) and the exponential local martingale* G_t *exists and is a martingale.*

Assumption 4.4.3 holds if, for example, δ_t is a.s. uniformly bounded on $[0, T]$ for all $T \in \mathbb{R}^+$. We can then define the new measure Q on $(\Omega, \mathcal{F}_t)$ for every $t \in \mathbb{R}^+$ as

$$Q(A) = P(1_A G_t) \qquad \text{for all } A \in \mathcal{F}_t$$

and Girsanov's Theorem (Karatzas and Shreve (1988) Theorem 3.5.1) asserts that under Q the process

$$W_t = B_t + \int_0^t \delta_t dt$$

is a Brownian motion; therefore we can obtain the processes S_t, v_t under the measure Q as the only solutions to the equations

$$\begin{aligned} dS_t &= S_t v_t^\alpha \sigma_t dW_t, && S_0 = x \\ dv_t &= v_t a_t dW_t + \rho_t (L_t - v_t) dt, && v_0 = 1 \end{aligned}$$

4.4.1 Results

We will study the simplest case, ignoring the effect of interest rates and assuming all processes are time homogeneous.

Assumption 4.4.4. *We suppose that $r = 0$ and that the coefficients are constant in time: $\sigma_t \equiv \sigma$, $a_t \equiv a$, $\rho_t \equiv \rho$ and $L_t \equiv L$, for all t.*

The main theorem of this section can be stated as follows

Theorem 4.4.5. *Let W_t be a 2-dimensional Brownian motion on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, Q)$, and let (S_t, v_t) satisfy*

$$\begin{aligned} dS_t &= S_t v_t^\alpha \big(\sigma^{(1)} dW_t^{(1)} + \sigma^{(2)} dW_t^{(2)}\big), && S_0 = x \\ dv_t &= v_t \big(a^{(1)} dW_t^{(1)} + a^{(2)} dW_t^{(2)}\big) + \rho\big(L - v_t\big) dt, && v_0 = 1 \end{aligned}$$

Then S_t is a martingale under Q if and only if $(a \cdot \sigma) \leq 0$.

<u>Remark</u> *4.4.6.* The explicit form of the drift coefficient for v_t is not essential to the proof; the same result can be obtained for most drifts which are linear or bounded functions of v_t. Even for arbitrary bounded drift we obtain one of the implications as the following corollary shows.

Corollary 4.4.7. *Let W_t^b be a 2-dimensional Brownian motion on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, Q^b)$, and (S_t, v_t) satisfy*

$$
\begin{aligned}
dS_t &= S_t v_t^\alpha\big(\sigma^{(1)} dW_t^{b(1)} + \sigma^{(2)} dW_t^{b(2)}\big), && S_0 = x \\
dv_t &= v_t\big(a^{(1)} dW_t^{b(1)} + a^{(2)} dW_t^{b(2)}\big) + \rho\big(L - v_t\big)dt + b_t(\omega)dt, && v_0 = 1
\end{aligned} \tag{4.11}
$$

Assume $(a\cdot\sigma) > 0$ and for all $T \in \mathbb{R}^+$ b_t/v_t is bounded on $[0,T]\times\Omega$. Then S_t is a strictly local martingale under Q^b.

<u>Remark</u> *4.4.8.* Observe that, when the filtration $\mathcal{F}_t$ is generated by the Brownian motion B_t, Girsanov's theorem implies that for every $T \in \mathbb{R}^+$ and every martingale measure Q^b on $\mathcal{F}_T$ equivalent to Q there exist b_t and W_t^b such that (S_t, v_t) satisfy (4.11) on $[0,T]$. The condition b_t/v_t being bounded means Q^b is "close" to Q. Many interesting measures, for example, the *minimal martingale measure* (see Ansel and Stricker (1993) and Hofmann et al. (1992)), if it exists, should satisfy this condition.

Definition. Q^* is a *minimal local martingale measure* if $Q^* \sim P$, S_t is a Q^*-local-martingale, and for all N_t P-local martingale such that $[S,N] = 0$ a.s. we have N_t is a Q^*-local martingale. Similarly, Q^* is a *minimal martingale measure* if, in addition, S_t is a Q^*-martingale.

In our setting, if Q^* exists we should get the equations

$$dS_t = S_t v_t^\alpha \sigma dW_t^*, \qquad S_0 = x$$
$$dv_t = v_t a dW_t^* + \rho(L - v_t) - \frac{v_t(a \cdot \sigma)}{|\sigma|^2 v_t^\alpha}\mu_t dt, \qquad v_0 = 1$$

where W_t^* is a Brownian motion under Q^*. Then from Corollary 4.4.7 we obtain

Corollary 4.4.9. *Suppose $(a \cdot \sigma) > 0$ and (S_t, v_t) satisfy equation (4.10). If μ_t / v_t^α is bounded on $[0, T] \times \Omega$ then S_t admits a minimal local martingale measure but no minimal martingale measure.*

Moreover, if we relax assumption 4.4.3 we can construct a process that admits an equivalent martingale measure but no minimal (even local) martingale measure.

Corollary 4.4.10. *Suppose $(a \cdot \sigma) < 0$ and (S_t, v_t) satisfy*

$$(4.12) \qquad \begin{aligned} dS_t &= S_t v_t^\alpha \sigma dB_t + S_t v_t^{2\alpha} |\sigma|^2 dt, \qquad S_0 = x \\ dv_t &= v_t a dB_t + \rho(L - v_t) dt, \qquad v_0 = 1 \end{aligned}$$

then S_t admits an equivalent martingale measure but no minimal local martingale measure.

Schachermayer (1993) shows a continuous semimartingale which is constructed especially to obtain a process that admits a martingale measure but no minimal martingale measure. Here we just showed another example which also belongs to a class of models proposed by finance practitioners.

In the extreme case when σ is parallel to a all local martingale measures coincide on the filtration generated by the stock and volatility processes and, using the same techniques that appear here, it is possible to show that this measure is not a martingale measure for the stock S_t whenever $(a \cdot \sigma) > 0$. Other examples of unique strictly local martingale measures appeared in Delbaen and Schachermayer (1994a).

However, there still exist martingale measures whenever σ and a are not parallel, as shown in the following theorem

Theorem 4.4.11. *Let W_t be a 2-dimensional Brownian motion on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, Q)$, and (S_t, v_t) solve*

$$dS_t = S_t v_t^\alpha \sigma dW_t, \qquad S_0 = x$$

$$dv_t = v_t a dW_t + \rho(L - v_t)dt, \qquad v_0 = 1$$

For every $T \in \mathbb{R}^+$ there exist a probability measure Q^a equivalent to Q on $\mathcal{F}_T$ with respect to which S_t is a martingale on $[0, T]$. If $(a \cdot \sigma) > 0$ we can find Q^a under which S_t satisfies

$$\begin{aligned} dS_t &= S_t v_t^\alpha \sigma dW_t^a, & S_0 &= x \\ dv_t &= v_t a dW_t^a + (a \cdot \sigma)\left(\frac{\rho L}{(a \cdot \sigma)} - v_t^{1+\alpha}\right) dt, & v_0 &= 1 \end{aligned}$$

on $t \in [0, T]$, where W_t^a is a Brownian motion under Q^a.

Finally, using the machinery developed in chapter 3 we obtain

Corollary 4.4.12.

$$(1,\ S_t) \quad \textit{satisfy NFFLVR}$$

(there are no arbitrage opportunities). However, we might introduce arbitrage by pricing under Q^b; for example if Q^a, Q^b, W_t^a, W_t^b, are as before and

$$d\widetilde{S}_t^a = \widetilde{S}_t^a v_t^\alpha a dW_t^a, \qquad \widetilde{S}_0^a = x$$

$$d\widetilde{S}_t^b = \widetilde{S}_t^b v_t^\alpha a dW_t^b, \qquad \widetilde{S}_0^b = x$$

then

$$(1,\ S_t,\ \widetilde{S}_t^a) \quad \textit{satisfy NFFLVR}$$

$$(1,\ S_t,\ \widetilde{S}_t^b) \quad \textit{admit FFLVR}$$

4.4.2 Proofs

We will show first that $v_t \geq 0$ a.s. for all t, so v_t^α is defined, as promised in the statement of the model.

Proposition 4.4.13. *Suppose v_t satisfies*

$$dv_t = v_t a_t dW_t + \rho_t(L_t - v_t)dt, \qquad v_0 = 1$$

then $v_t > 0$ a.s. for all $t \in \mathbb{R}^+$.

Proof: Let $\widetilde{v}_t$ be the unique solution to the SDE

$$d\widetilde{v}_t = \widetilde{v}_t a_t dW_t - \rho_t \widetilde{v}_t dt, \qquad \widetilde{v}_0 = 1$$

then $\widetilde{v}_t$ is a semimartingale exponential given by

$$\widetilde{v}_t = \exp\left(\int_0^t a_u dW_u - \int_0^t \left(\rho_u + \frac{1}{2}|a_u|^2\right) dt\right)$$

and the comparison theorem (see for example Ikeda and Watanabe (1989) VI, Theorem 1.1) implies $v_t \geq \widetilde{v}_t > 0$ for all t. $\square$

Proof of Corollary 4.4.12: Q^a is a martingale measure for $(1,\ S_t,\ S_t^a)$ so Theorem 3.4.2 implies $(1,\ S_t)$ and $(1,\ S_t,\ S_t^a)$ satisfy NFFLVR. On the other hand, the martingale representation theorem implies Q^b is the only equivalent local martingale measure for $(1,\ S_t,\ S_t^b)$ and it is strictly local, so in this case Theorem 3.4.2 says $(1,\ S_t,\ S_t^b)$ admit FFLVR. $\square$

Theorem 4.4.5 is an immediate consequence of the following two results

Lemma 4.4.14. *If (S_t, v_t) satisfies*

$$dS_t = S_t v_t^{\alpha} \sigma dW_t, \qquad S_0 = x$$

$$dv_t = v_t a dW_t + \rho(L - v_t)dt, \qquad v_0 = 1$$

then S_t is a supermartingale and for every $T \in \mathbb{R}^+$

$$Q(S_T) = S_0 Q(\widehat{v}_t \text{ doesn't explode on the interval } [0,T])$$

where $\widehat{v}_t$ is the unique solution up to an explosion time to the SDE

$$d\widehat{v}_t = \widehat{v}_t a dW_t + \rho(L - \widehat{v}_t)dt + \widehat{v}_t^{\alpha+1}(a \cdot \sigma)dt \qquad \widehat{v}_0 = 1.$$

Lemma 4.4.15. *The (unique) solution to the equation*

$$d\widehat{v}_t = \widehat{v}_t a dW_t + \rho(L - \widehat{v}_t)dt + \widehat{v}_t^{\alpha+1}(a \cdot \sigma)dt \qquad \widehat{v}_0 = 1.$$

explodes to $+\infty$ in finite time with positive probability if and only if $(a \cdot \sigma) > 0$

Proof of Theorem 4.4.5:

S_t is a martingale if and only if it is a supermartingale and $Q(S_t) = S_0$ for all $t \in \mathbb{R}^+$, and by Lemmas 4.4.14 and 4.4.15 this happens if and only if $(a \cdot \sigma) \leq 0$.

□

Proof of Lemma 4.4.14:

S_t is the integral of a predictable process with respect to a local martingale, so it is again a local martingale, and it is also positive because it is a solution to the equation of the semimartingale-exponential of $\int v_t^\alpha \sigma dW_t$ (see for example Karatzas and Shreve (1988) example 3.3.9). Define a sequence of stopping times

$$\tau_n = \inf\left\{t \in \mathbb{R}^+ : |v_t| \geq n\right\}$$

then $S_t^{(n)} = S_{t\wedge\tau_n}$ is a local martingale under Q for $n = 1, 2, \ldots$. Define $Z_T^{(n)} = \int_0^{t\wedge\tau_n} v_t^\alpha \sigma dW_t$. Then $S_t^{(n)}$ is the semimartingale-exponential of $Z_t^{(n)}$ started at x,

$$dS_t^{(n)} = S_t^{(n)} dZ_t^{(n)} \qquad S_0 = x.$$

and $\langle Z^{(n)}\rangle_t \leq n^{2\alpha}|\sigma|^2 t$ for all t, so $S_t^{(n)}$ is a positive martingale for $n = 1, 2, \ldots$ and the sequence $(\tau_n)_{n=1,2,\ldots}$ reduces S_t. Then Fatou's Lemma implies, for $0 \leq u < t$,

$$\begin{aligned} Q(S_t|\mathcal{F}_u) &= Q(\liminf_{n\to\infty} S_t^{(n)}|\mathcal{F}_u) \\ &\leq \liminf_{n\to\infty} Q(S_t^{(n)}|\mathcal{F}_u) \\ &= S_u. \end{aligned}$$

so S_t is a supermartingale under Q.

Now fix $T \in \mathbb{R}^+$ and define the new probability measure Q_n on $(\Omega, \mathcal{F}_T)$ as

$$Q_n(A) = \frac{1}{S_0} Q(S_T^{(n)} 1_A) \qquad \text{for all } A \in \mathcal{F}_T$$

then the Lebesgue Dominated convergence Theorem and Girsanov's Theorem imply that for every set Γ in the Borel sigma field on the space of continuous paths in $[0,T]$,

$$
\begin{aligned}
Q\left(S_T 1_{\{W\in\Gamma\}}\right) &= \lim_{n\to\infty} Q\left(S_T^{(n)} 1_{\{W\in\Gamma,\tau_n>T\}}\right) \\
&= S_0 \lim_{n\to\infty} Q_n\left(1_{\{W\in\Gamma,\tau_n>T\}}\right) \\
&= S_0 \lim_{n\to\infty} Q\left(1_{\{\widehat{W}\in\Gamma,\widehat{\tau}_n>T\}}\right)
\end{aligned}
$$

where $\widehat{W}_t$ and $\widehat{v}_t$ are defined as the (unique in law) solutions to the following stochastic differential equations

$$
\begin{aligned}
d\widehat{W}_t &= dW_t + \widehat{v}_t^{\alpha}\sigma dt, & \widehat{W}_0 &= 0 \\
d\widehat{v}_t &= \widehat{v}_t a dW_t + \rho(L-\widehat{v}_t)dt + \widehat{v}_t^{\alpha+1}(a\cdot\sigma)dt & \widehat{v}_0 &= 1.
\end{aligned}
$$

and $\widehat{\tau}_n$ is given by

$$
\widehat{\tau}_n = \inf\left\{t\in\mathbb{R}^+ : |\widehat{v}_t| \geq n\right\}
$$

The existence of a weak solution $(\widehat{W}_t, \widehat{v}_t)$ to these equations up to an explosion time $\widehat{\tau}_\infty = \lim_{n\to\infty}\widehat{\tau}_n$ follows from the continuity of the drift and diffusion coefficients (see Ikeda and Watanabe (1989) IV.Theorem 2.3); we even get pathwise uniqueness on $\{(x,y) : y > 0\}$ because the coefficients of the SDE are locally Lipschitz (see Ikeda and Watanabe (1989) IV.Theorem 3.1), so strong solutions always exist and are unique up to time $\widehat{\tau}_\infty$. Applying Girsanov's theorem we obtain that the process

$$
W_t^{(n)} = W_t - \int_0^{t\wedge T} 1_{\{u\leq\tau_n\}} v_u^{\alpha}\sigma du
$$

is a Brownian motion under Q_n and W_t and v_t satisfy

$$dW_t = dW_t^{(n)} + 1_{\{t \le \tau_n \wedge T\}} v_t^{\alpha} \sigma dt, \qquad W_0 = 0$$

$$dv_t = v_t a dW_t^{(n)} + \rho(L - v_t)dt + 1_{\{t \le \tau_n \wedge T\}} v_t^{\alpha+1}(a \cdot \sigma)dt \qquad v_0 = 1.$$

which is the same as the previous equation up to time $\tau_n \wedge T$, and because $\{W \in \Gamma, \tau_n > T\} \in \mathcal{F}_{\tau_n \wedge T}$ the identity follows.

Now let $\Gamma = C[0,T]$ to get

$$\begin{aligned} Q(S_T) &= S_0 \lim_{n \to \infty} Q(1_{\{\widehat{\tau}_n > T\}}) \\ &= S_0 Q(\widehat{\tau}_n > T \quad \text{for some n}) \\ &= S_0 Q(\widehat{v}_t \text{ doesn't explode until after time } T) \quad \square \end{aligned}$$

<u>Proof of Lemma 4.4.15:</u>

We will apply Feller's test to $\widehat{v}_t$ to check whether it explodes in finite time. Notice the test is applicable because $\widehat{v}_t$ is a 1-dimensional Ito diffusion with respect to the Brownian motion $(a/|a|)B_t$ and the volatility and drift coefficients are locally bounded above and away from zero on the interval $(0, \infty)$. (see for example Karatzas and Shreve (1988) section 5.5). The careful reader might notice that, in the statement of Feller's test, explosion means that the process escapes from the interval it is defined on, $(0, \infty)$ for $\widehat{v}_t$, whereas here by explosion we mean that the process tends to $+\infty$. However, we will show it's impossible that $\widehat{v}_t$ reaches zero in finite time, but at this point we should be aware of the distinction.

First we need to calculate the *scale function*, for $x \in (0, \infty)$:

$$\begin{aligned} p(x) &= \int_1^x \exp\left(-2\int_1^y \frac{(\rho L - \rho z - (a\cdot\sigma)z^{1+\alpha})}{z^2|a|^2}dz\right)dy \\ &= C\int_1^x \exp\left(\frac{2\rho L}{|a|^2}y^{-1} + \frac{2\rho}{|a|^2}\log(y) - \frac{2(a\cdot\sigma)y^\alpha}{|a|^2\alpha}\right)dy \end{aligned}$$

$$p'(x) = C\exp\left(\frac{2\rho L}{|a|^2}x^{-1} + \frac{2\rho}{|a|^2}\log(x) - \frac{2(a\cdot\sigma)x^\alpha}{|a|^2\alpha}\right)$$

so

$$\begin{aligned} p(0+) &= -C\int_0^1 \exp\left(\frac{2\rho L}{|a|^2}y^{-1} + \frac{2\rho}{|a|^2}\log(y) - \frac{2(a\cdot\sigma)y^\alpha}{|a|^2\alpha}\right)dy \\ &\sim -\int_0^1 \exp\left(\frac{2\rho L}{|a|^2}y^{-1}\right)dy \\ &= -\int_1^\infty \frac{\exp\left(\frac{2\rho L}{|a|^2}u\right)}{u^2}dy \\ &= \begin{cases} -\infty & \text{if } \rho > 0 \\ > -\infty & \text{if } \rho = 0 \end{cases} \end{aligned}$$

$$\begin{aligned} p(+\infty) &= C\int_1^\infty \exp\left(\frac{2\rho L}{|a|^2}y^{-1} + \frac{2\rho}{|a|^2}\log(y) - \frac{2(a\cdot\sigma)y^\alpha}{|a|^2\alpha}\right)dy \\ &\sim \int_1^\infty \exp\left(-\frac{2(a\cdot\sigma)y^\alpha}{|a|^2\alpha}\right)dy \\ &= \begin{cases} +\infty & \text{if } (a\cdot\sigma) \le 0 \\ < +\infty & \text{if } (a\cdot\sigma) > 0 \end{cases} \end{aligned}$$

Now we will distinguish 2 cases:

<u>CASE 1:</u> $(a\cdot\sigma) \le 0$.

Here we have $p(+\infty) = +\infty$ so $\widehat{v}_t$ never explodes to $+\infty$ in finite time (see Karatzas and Shreve (1988) Proposition 5.5.22). Depending on the value of ρ we get

$$\text{If } \rho = 0 \quad \text{then} \quad \lim_{t \to \widehat{\tau}_\infty} \widehat{v}_t = 0 \qquad \sup_{0 \le t < \widehat{\tau}_\infty} \widehat{v}_t < \infty$$

$$\text{If } \rho > 0 \quad \text{then} \quad \inf_{0 \le t < \widehat{\tau}_\infty} \widehat{v}_t = 0 \qquad \sup_{0 \le t \le \widehat{\tau}_\infty} \widehat{v}_t = \infty \qquad \widehat{\tau}_\infty = +\infty \; a.s.$$

<u>CASE 2:</u> $(a \cdot \sigma) > 0$.

Again depending on the value of ρ we get two cases

$$\text{If } \rho = 0 \text{ then} \quad 0 < Q\left(\lim_{t \to \widehat{\tau}_\infty} \widehat{v}_t = \infty\right) = 1 - Q\left(\lim_{t \to \widehat{\tau}_\infty} \widehat{v}_t = 0\right) < 1.$$

$$\text{If } \rho > 0 \text{ then} \quad \inf_{0 \le t < \widehat{\tau}_\infty} \widehat{v}_t > 0 \quad \lim_{t \to \widehat{\tau}_\infty} \widehat{v}_t = +\infty \; a.s.$$

To see if $Q(\widehat{\tau}_\infty < \infty) > 0$ we study the behavior of the following function at the endpoints

$$\begin{aligned}
u(x) &= \int_1^x p'(y) \int_1^y \frac{2}{p'(z)|a|^2 z^2} dz dy \\
&= \frac{2}{|a|^2} \int_1^x \frac{1}{p'(z) z^2} \int_z^x p(y) dy dz \\
&= \frac{2}{|a|^2} \int_1^x \frac{\exp\left(\frac{-2\rho L}{|a|^2} z^{-1} - \frac{2\rho}{|a|^2} \log(z) + \frac{2(a \cdot \sigma) z^\alpha}{|a|^2 \alpha}\right)}{|a|^2 z^2} \\
&\quad \cdot \int_z^x \exp\left(\frac{2\rho L}{|a|^2} y^{-1} + \frac{2\rho}{|a|^2} \log(y) - \frac{2(a \cdot \sigma) y^\alpha}{|a|^2 \alpha}\right) dy dz
\end{aligned}$$

where the second equation is justified by Fubini's Theorem.

Now when $x > 1$

$$u(x) \le \frac{2}{|a|^2}\int_1^x \frac{\exp\left(\frac{-2\rho}{|a|^2}\log(z) + \frac{2(a\cdot\sigma)z^\alpha}{|a|^2\alpha}\right)}{|a|^2 z^2}\int_z^x \exp\left(\frac{2\rho}{|a|^2}\log(y) - \frac{2(a\cdot\sigma)y^\alpha}{|a|^2\alpha}\right)dydz$$
$$\sim \frac{2}{|a|^2}\int_1^x \frac{\exp\left(\frac{(a\cdot\sigma)z^\alpha}{|a|^2\alpha}\right)}{|a|^2 z^2}\int_z^x \exp\left(\frac{-(a\cdot\sigma)y^\alpha}{|a|^2\alpha}\right)dydz$$

because the function $g(x) = \frac{2\rho}{|a|^2}\log(x) - \frac{(a\cdot\sigma)x^\alpha}{|a|^2\alpha}$ is decreasing on $x \ge M$ for some $M \in \mathbb{R}^+$; so

$$\left(\frac{2\rho}{|a|^2}\log(y) - \frac{(a\cdot\sigma)y^\alpha}{|a|^2\alpha}\right) - \left(\frac{2\rho}{|a|^2}\log(z) - \frac{(a\cdot\sigma)z^\alpha}{|a|^2\alpha}\right) \le 0$$

on $M \le z \le y$. Let $\beta = (a\cdot\sigma)/|a|^2$ and n be the smallest positive integer greater than or equal to $\gamma = (1-\alpha)/\alpha$. Using integration by parts several times we obtain the following majorant for the inner integral above,

$$\begin{aligned}\int_z^x e^{-(\beta/\alpha)y^\alpha}dy &= \int_{z^\alpha}^{x^\alpha} e^{-(\beta/\alpha)u}\frac{u^{(1-\alpha)/\alpha}}{\alpha}du \\ &\le \int_{z^\alpha}^{\infty} e^{-(\beta/\alpha)u}\frac{u^\gamma}{\alpha}du \\ &= \sum_{k=0}^{n-1}\frac{\gamma(\gamma-1)\dots(\gamma-k+1)\alpha^k}{\beta^{k+1}}e^{-(\beta/\alpha)z^\alpha}z^{1-\alpha(1+k)} \\ &\quad + \int_{z^\alpha}^{\infty}\frac{\gamma(\gamma-1)\dots(\gamma-n+1)\alpha^{n-1}}{\beta^n}e^{-(\beta/\alpha)u}u^{\gamma-n}du \\ &\le \sum_{k=0}^{n-1}\frac{\gamma(\gamma-1)\dots(\gamma-k+1)\alpha^k}{\beta^{k+1}}e^{-(\beta/\alpha)z^\alpha}z^{1-\alpha(1+k)} \\ &\quad + z^{1-\alpha(1+n)}\int_{z^\alpha}^{\infty}\frac{\gamma(\gamma-1)\dots(\gamma-n+1)\alpha^{n-1}}{\beta^n}e^{-(\beta/\alpha)u}du\end{aligned}$$

so there are positive constants C_i such that

$$\begin{aligned} u(x) &\sim C_1 \int_1^x \frac{\sum_{k=0}^n z^{1-\alpha-\alpha k}}{z^2} dz \\ &\leq C_2 \int_1^\infty z^{-(1+\alpha)} dz \\ &< \infty \end{aligned}$$

for $x > 1$, so we get $u(\infty) < \infty$ and thus Feller's test implies $\widehat{v}_t$ explodes in finite time with positive probability when $(a \cdot \sigma) > 0$.

We can check the value of u at zero as well; if $\rho > 0$ then $p(0+) = -\infty$ so $u(0+) = +\infty$ (see Karatzas and Shreve (1988) Problem 5.5.27) and if $\rho = 0$ then for $x < 1$,

$$\begin{aligned} u(x) &= \frac{2}{|a|^2} \int_x^1 \frac{e^{(\beta/\alpha)z^\alpha}}{z^2} \int_x^z e^{-(\beta/\alpha)y^\alpha} dy dz \\ &\geq \frac{2}{|a|^2} e^{-(|\beta|/\alpha)} \int_x^1 \frac{1}{z^2} \int_x^z dy dz \\ &= e^{-(|\beta|/\alpha)} \left(-\ln(x) + x \left(1 - \frac{1}{x}\right)\right) \\ &\to +\infty \qquad \text{as } x \to 0+ . \end{aligned}$$

so in any case $u(0+) = +\infty$ and this with the previous result on $p(x)$ imply (see Proposition 5.5.32 in Karatzas and Shreve (1988)) that $Q\{\widehat{\tau}_\infty < \infty\} = 1$ if $\rho > 0$ and $0 < Q\{\widehat{\tau}_\infty < \infty\} < 1$ if $\rho = 0$.

Our next step is to show that the event $\{\widehat{\tau}_\infty < \infty, \lim_{t \to \widehat{\tau}_\infty} \widehat{v}_t = 0\}$ has probability zero. Define $\widetilde{v}_t$ as the unique solution to

$$d\widetilde{v}_t = \widetilde{v}_t a dW_t - \rho \widetilde{v}_t dt, \qquad \widetilde{v}_0 = 1.$$

then

$$\widetilde{v}_t = \exp\left(aW_t - \left(\rho + \frac{1}{2}|a|^2\right)t\right)$$

and the comparison theorem (see for example Ikeda and Watanabe (1989) VI, Theorem 1.1) implies that if $(a\cdot\sigma) > 0$ then $\widehat{v}_t \geq \widetilde{v}_t$ a.s., so $Q\{\widehat{\tau}_\infty < \infty, \lim_{t\to\widehat{\tau}_\infty} \widetilde{v}_t = 0\} = 0$ implies $Q\{\widehat{\tau}_\infty < \infty, \lim_{t\to\widehat{\tau}_\infty} \widehat{v}_t = 0\} = 0$; therefore

$$\begin{aligned}
&Q\left\{\widehat{\tau}_\infty < \infty, \lim_{t\to\widehat{\tau}_\infty} \widehat{v}_t = \infty\right\} \\
&= Q\left\{\widehat{\tau}_\infty < \infty\right\} - Q\left\{\widehat{\tau}_\infty < \infty, \lim_{t\to\widehat{\tau}_\infty} \widehat{v}_t = 0\right\} \\
&= Q\left\{\widehat{\tau}_\infty < \infty\right\} \\
&> 0.
\end{aligned}$$

and $\widehat{v}_t$ explodes to $+\infty$ in finite time with positive probability. □

Now we are ready to prove the corollary

Proof of Corollary 4.4.7:

Suppose $(a\cdot\sigma) > 0$ and S_t is a martingale under Q^b. Fix $T \in \mathbb{R}^+$ and define the new equivalent measure $\widetilde{Q}^b$ on $\mathcal{F}_T$ as

$$\widetilde{Q}^b(A) = \frac{Q^b(1_A S_T)}{S_0} \qquad \text{for all } A \in \mathcal{F}_T$$

Girsanov's theorem states that $\widetilde{W}^b_t = W^b_t - \int_0^t 1_{\{u\leq T\}} v_u^\alpha \sigma du$ is a Brownian motion under $\widetilde{Q}^b$ and v_t satisfies

$$dv_t = v_t a d\widetilde{W}^b_t + \rho(L - v_t)dt + b_t dt + 1_{\{t\leq T\}} v_t^{\alpha+1}(a\cdot\sigma)dt, \qquad v_0 = 1.$$

and because b_t/v_t is bounded on $[0,T]\times\Omega$ the process

$$M_t = \exp\left(-\int_0^t \left(\frac{b_t}{v_t|a|^2}\right) a d\widetilde{W}^b_t - \frac{1}{2}\int_0^t \frac{|b_t|^2}{|v_t|^2|a|^2} dt\right)$$

is a martingale under $\widetilde{Q}^b$ and defining the equivalent measure Q^* with

$$\frac{dQ^*}{d\widetilde{Q}^b} = M_T$$

we see

$$W_t^* = \widetilde{W}_t^b + \int_0^t 1_{\{u \le T\}} \left(\frac{b_u}{v_u |a|^2} \right) a du \quad \text{is a Brownian motion under } Q^*$$

and v_t satisfies

$$dv_t = v_t a dW_t^* + \rho(L - v_t)dt + v_t^{\alpha+1}(a \cdot \sigma)dt, \qquad v_0 = 1.$$

on $[0, T]$ for arbitrary T; which is a contradiction because we already proved that the only solution to this equation explodes in finite time with positive probability when $(a \cdot \sigma) > 0$. □

Proof of Corollary 4.4.10:

In this case the minimal (local) martingale measure Q^* on $\mathcal{F}_T$ should satisfy

$$\frac{dQ^*}{dQ} = \exp\left(\int_0^T v_t^\alpha (-\sigma) dB_t - \frac{1}{2} \int_0^T v_t^{2\alpha} |\sigma|^2 dt \right)$$

but the Q expected value of the right hand side is strictly less than 1 when $(a \cdot \sigma) < 0$ by Theorem 4.4.5; so there doesn't exist a minimal local martingale measure.

Now let

$$\frac{dQ^a}{dQ} = \exp\left(\int_0^T \frac{-|\sigma|^2 v_t^\alpha}{(\sigma \cdot a^\perp)} a^\perp dB_t - \frac{1}{2} \int_0^T \frac{v_t^{2\alpha} |\sigma|^4 |a^\perp|^2}{(\sigma \cdot a^\perp)^2} dt \right)$$

and $\left(\frac{-|\sigma|^2}{(\sigma \cdot a^\perp)} a^\perp, a \right) = 0$ so Theorem 4.4.5 shows the Q expected value of the right hand side is 1 and Girsanov theorem then implies

$$B_t^a = B_t + \int_0^t \frac{|\sigma|^2 v_u^\alpha}{(\sigma \cdot a^\perp)} a^\perp du$$

is a $\widetilde{Q}^a$-Brownian motion on $[0,T]$ so

$$dS_t = S_t v_t^\alpha \sigma dB_t^a,$$

$$dv_t = v_t a dB_t^a + \rho(L - v_t)dt,$$

and again by Theorem 4.4.5 and the fact that $(a \cdot \sigma) < 0$ we have S_t is a Q^a-martingale. □

Proof of Theorem 4.4.11:

In light of the previous results, the only case we still need to verify is $(a \cdot \sigma) > 0$. Let $\widetilde{Q}^a$ be given by

$$\frac{d\widetilde{Q}^a}{dQ} = \exp\left(-\int_0^T \frac{v_t^\alpha (a \cdot \sigma)}{(a \cdot \sigma^\perp)} \sigma^\perp dB_t - \frac{1}{2}\int_0^T \frac{v_t^{2\alpha}(a \cdot \sigma)^2 |\sigma^\perp|^2}{(a \cdot \sigma^\perp)^2} dt\right)$$

Observe that $\left(a \cdot \frac{(a\cdot\sigma)}{(a\cdot\sigma^\perp)}\sigma^\perp\right) = (a \cdot \sigma) > 0$ so the fact that the right hand side is a martingale in T is proven the same way as we did for S_t when $(a \cdot \sigma) \leq 0$. Then Girsanov's theorem implies that under $\widetilde{Q}_t^a$ the process $\widetilde{B}_t^a = B_t + \int_0^t \frac{v_t^\alpha (a\cdot\sigma)}{(a\cdot\sigma^\perp)}\sigma^\perp dt$ is a Brownian motion. Then do a new measure change with

$$\frac{dQ^a}{d\widetilde{Q}^a} = \exp\left(\frac{\rho}{(a \cdot \sigma^\perp)}\sigma^\perp \widetilde{B}_T^a - \frac{1}{2}\frac{\rho^2|\sigma^\perp|^2}{(a \cdot \sigma^\perp)^2}T\right)$$

and get the new Brownian Motion $B_t^a = \widetilde{B}_t^a - \int_0^t \frac{\rho}{(a\cdot\sigma^\perp)}\sigma^\perp t$ under Q_t^a. Now (S_t, v_t) satisfy

$$dS_t = S_t v_t^\alpha \sigma dB_t^a, \qquad S_0 = x$$

$$dv_t = v_t a dB_t^a + (a \cdot \sigma)\left(\frac{\rho L}{(a \cdot \sigma)} - v_t^{1+\alpha}\right), \qquad v_0 = 1$$

and following exactly the same argument as in Lemma 4.4.14 we obtain that S_t is a martingale under Q^a iff $\widehat{v}_t$ doesn't explode to $+\infty$ in finite time, where $\widehat{v}_t$

satisfies

$$d\widehat{v}_t = \widehat{v}_t a d\widehat{B}_t + \rho L dt, \qquad v_0 = 1$$

this is an equation with linear coefficients in $\widehat{v}_t$, so it always has a non-exploding solution. □

4.5 Stochastic volatility models II: square root function on the diffusion coefficient of the volatility process

The stock price process is now given as a solution to the equations

$$(4.13) \qquad dS_t = S_t v_t^{\alpha} \sigma dW_t, \qquad S_0 = x$$

$$(4.14) \qquad dv_t = v_t^{1/2} a dW_t + \rho(L - v_t)dt, \qquad v_0 = 1.$$

assume $\rho \geq 0$, $L > 0$, $\alpha > 0$, and $|\sigma| > 0$, $|a| > 0$.

This models is proposed by Hull and White (1988), Johnson and Shanno (1987), Scott (1992) among others.

Let's show first the existence and uniqueness of a strong solution to equation (4.14). Observe that the diffusion and drift coefficients are continuous with linear growth, so by Ikeda and Watanabe (1989) Theorem IV 2.3 and IV 2.4 there exists a non-exploding weak solution. Also observe that the diffusion coefficient is Hölder-1/2 continuous and the drift coefficient is locally Lipschitz, so by Ikeda and Watanabe (1989) Theorem IV 3.2 and its preceding remark we get pathwise uniqueness on (4.14). Finally, pathwise uniqueness plus weak existence implies strong existence (see Ikeda and Watanabe (1989) Theorem IV 1.1), so we can define v_t as the unique strong solution to (4.14) and then obtain S_t as the unique solution to (4.13) given v_t, which can be expressed as

$$S_t = \exp\left(\int_0^t v_u^\alpha \sigma dW_u - \frac{1}{2}\int_0^t v_u^{2\alpha}|\sigma|^2 du\right)$$

Proposition 4.5.1. *$v_t \geq 0$ a.s. for all $t \in \mathbb{R}^+$. Moreover, if $2\rho L \geq |a|^2$ then $P(v_t > 0,\ \text{for all}\ t \in \mathbb{R}^+) = 1$. If $2\rho L < |a|^2$ then $P(v_t = 0,\ \text{for some}\ t \in \mathbb{R}^+) = 1$.*

Proof: Let u_t be the unique solution to

$$du_t = u_t^{1/2} a dW_t - u_t dt, \qquad u_0 = 1$$

observe that if $u_t = 0$ for some t then $u_s = 0$ for all $s > t$, so the solution is absorbed after the first hitting time of zero and $u_t \geq 0$ for all t. Then the comparison theorem (see Ikeda and Watanabe (1989) Theorem VI 1.1) implies $v_t \geq u_t \geq 0$ for all $t \in \mathbb{R}^+$.

Now let

$$\tau = \inf\{t > 0 \ : \ v_t = 0\}$$

we want to calculate $P(\tau < \infty)$. Observe v_t is a time homogeneous Itô diffusion with drift and diffusion coefficients locally bounded above and away from zero on $(0, \infty)$. Then *scale function* $p(x)$, $x \in (0, \infty)$ for v_t exists and is given by

$$p'(x) = \exp\left(-2\int_1^x \frac{\rho(L-z)}{|a|^2 z}dz\right)$$
$$= \exp\left(\frac{-2\rho L}{|a|^2}\log(x) + \frac{2\rho}{|a|^2}(x-1)\right)$$
$$p(x) = \int_1^x \exp\left(\frac{-2\rho L}{|a|^2}\log(y) + \frac{2\rho}{|a|^2}(y-1)\right) dy$$

as $x \to +\infty$

$$p(x) \sim \int_1^x \exp\left(\frac{2\rho}{|a|^2}y\right) dy \to +\infty$$

as $x \to 0$

$$p(x) \sim \int_1^x \exp\left(\frac{-2\rho L}{|a|^2}\log(y)\right) dy$$
$$\sim \begin{cases} 1 - x^{1-2\rho L/|a|^2} & \text{if } 2\rho L \neq |a|^2 \\ \log(x) & \text{if } 2\rho L = |a|^2 \end{cases}$$
$$\to \begin{cases} -\infty & \text{if } 2\rho L \geq |a|^2 \\ > -\infty & \text{if } 2\rho L < |a|^2 \end{cases}$$

so Feller's test implies $P(v_t > 0, \text{ for all } t \in \mathbb{R}^+) = 1$ if $2\rho L \geq |a|^2$ (see for example Karatzas and Shreve (1988) section 5.5).

If $2\rho L < |a|^2$, we need to check the finiteness of the function

$$u(x) = \int_1^x \int_1^y \exp\left(\frac{-2\rho L}{|a|^2}\log\left(\frac{y}{z}\right) + \frac{2\rho}{|a|^2}(y-z)\right) \frac{dzdy}{|a|^2 z}$$

when $x \to 0$

$$u(x) \sim \int_x^1 \int_y^1 \exp\left(\frac{-2\rho L}{|a|^2} \log\left(\frac{y}{z}\right)\right) \frac{dzdy}{|a|^2 z}$$
$$= \int_x^1 \int_x^z y^{-2\rho L/|a|^2} z^{2\rho L/|a|^2} \frac{dydz}{|a|^2 z}$$
$$= \frac{1}{|a|^2} \int_x^1 z^{-(|a|^2-2\rho L)/|a|^2} \frac{|a|^2}{|a|^2 - 2\rho L} \left(z^{(|a|^2-2\rho L)/|a|^2} - x^{-(|a|^2-2\rho L)/|a|^2}\right) dz$$
$$\leq \frac{1}{|a|^2} \int_0^1 \frac{|a|^2}{|a|^2 - 2\rho L} dz < \infty$$

so Feller's test shows in this case $P(\tau < \infty) = 1$. □

Theorem 4.5.2. *Let* (S_t, v_t) *satisfy equations* (4.13), (4.14), *with* $2\rho L \geq |a|^2$. *Then* S_t *is a martingale if and only if* $(\sigma \cdot a) \leq 0$ *or* $\alpha \leq 1/2$. *If* $(\sigma \cdot a) > 0$ *and* $\alpha > 1/2$ *then* S_t *is a strictly local martingale.*

Proof: Define the sequence of stopping times

$$\tau_n = \inf\left\{t \in \mathbb{R}^+ : |v_t| \geq n\right\}$$

then $\tau_n \uparrow +\infty$ and $S_t^{(n)} = S_{t\wedge\tau_n}$ is a martingale under Q. Define

$$Q_n(A) = \frac{1}{S_0} Q\left(S_T^{(n)} 1_A\right), \qquad A \in \mathcal{F}_T$$

then

$$Q(S_T) = \lim_{n\to\infty} Q\left(S_T^{(n)} 1_{\{\tau_n > T\}}\right)$$
$$= S_0 \lim_{n\to\infty} Q_n(\tau_n > T)$$
$$= S_0 \lim_{n\to\infty} Q(\widetilde{\tau}_n > T)$$
$$= S_0 Q(\widetilde{\tau}_\infty > T)$$

where

$$\widetilde{\tau}_\infty = \lim_{n\to\infty} \widetilde{\tau}_n$$
$$\widetilde{\tau}_n = \inf\left\{t \in \mathbb{R}^+ : |\widetilde{v}_t| \geq n\right\}$$
$$d\widetilde{v}_t = \widetilde{v}_t^{1/2} a dW_t + \rho(L - \widetilde{v}_t)dt + \widetilde{v}_t^{1/2+\alpha}(a\cdot\sigma)dt, \qquad \widetilde{v}_0 = 1.$$

Observe that the drift and diffusion coefficients of the equation for $\widetilde{v}_t$ are locally Lipschitz on the set $\{v_t > 0\}$, so the solution is pathwise unique up to the first hitting time of zero.

Applying Girsanov's theorem we obtain that the process

$$W_t^{(n)} = W_t - \int_0^{t\wedge T} 1_{\{u\leq\tau_n\}} v_u^\alpha \sigma du$$

is a Brownian motion under Q_n and for $t \in [0,T]$, W_t and v_t satisfy

$$dW_t = dW_t^{(n)} + 1_{\{t\leq\tau_n\}} v_t^\alpha \sigma dt, \qquad W_0 = 0$$
$$dv_t = v_t^{1/2} a dW_t^{(n)} + \rho(L - v_t)dt + 1_{\{t\leq\tau_n\}} v_t^{\alpha+1/2}(a\cdot\sigma)dt \qquad v_0 = 1.$$

which is the same as the previous equation up to time $\tau_n \wedge T$, and because $\{\tau_n > T\} \in \mathcal{F}_{\tau_n\wedge T}$ the identity follows.

To check whether $\widetilde{v}_t$ explodes we first calculate the scale function:

$$p'(x) = \exp\left(-2\int_1^x \frac{[\rho(L-y) + (a\cdot\sigma)y^{1/2+\alpha}]}{|a|^2 y} dy\right)$$
$$= \exp\left(\frac{-2\rho L}{|a|^2}\log(x) + \frac{2\rho}{|a|^2}(x-1) - \frac{2(a\cdot\sigma)}{|a|^2}\frac{(x^{1/2+\alpha}-1)}{(1/2+\alpha)}\right)$$

and again, as $x \to 0$

$$\begin{aligned} p(x) &\sim \int_1^x \exp\left(\frac{-2\rho L}{|a|^2}\log(y)\right) dy \\ &\to -\infty \end{aligned}$$

when $x \sim +\infty$ we need to consider three cases:

Case 1: $\alpha < 1/2$.

$$p(x) \sim \int_1^x \exp\left(\frac{2\rho}{|a|^2}y\right) dy \to +\infty \qquad \text{as } x \to +\infty$$

Case 2: $\alpha = 1/2$.

$$p(x) \sim \int_1^x \exp\left(\frac{2}{|a|^2}[\rho - (a\cdot\sigma)]y\right) dy \to \begin{cases} +\infty & \text{if } \rho \geq (a\cdot\sigma) \\ < \infty & \text{if } \rho < (a\cdot\sigma) \end{cases}$$

Case 3: $\alpha > 1/2$.

$$p(x) \sim \int_1^x \exp\left(\frac{-2(a\cdot\sigma)}{|a|^2}\frac{y^{1/2+\alpha}}{(1/2+\alpha)}\right) dy \to \begin{cases} +\infty & \text{if } (a\cdot\sigma) \leq 0 \\ < \infty & \text{if } (a\cdot\sigma) > 0 \end{cases}$$

Now we calculate $u(x)$ to check whether $Q(\widetilde{\tau}_\infty < \infty)$ is zero or positive.

$$\begin{aligned} u(x) &= \int_1^x p'(y) \int_1^y \frac{2dzdy}{p'(z)|a|^2 z} \\ &= \int_1^x \int_1^y \exp\left(\frac{-2\rho L}{|a|^2}\log\left(\frac{y}{z}\right) + \frac{2\rho}{|a|^2}(y-z)\right. \\ &\quad \left. - \frac{2(a\cdot\sigma)}{|a|^2}\frac{(y^{1/2+\alpha} - z^{1/2+\alpha})}{(1/2+\alpha)}\right)\frac{dzdy}{|a|^2 z} \end{aligned}$$

From the results on $p(x)$ we know $u(0+) = +\infty$ and $u(+\infty) = +\infty$ if $\alpha < 1/2$.

Now assume $\alpha \geq 1/2$.

<u>Case 1:</u> $\alpha = 1/2$. Assume $\rho < (a \cdot \sigma)$ (if $\rho \geq (a \cdot \sigma)$ then $u(+\infty) = +\infty$ again from the result on $p(x)$). Then

$$\begin{aligned}
u(x) &= \int_1^x \int_1^y \exp\left(\frac{-2\rho L}{|a|^2} \log\left(\frac{y}{z}\right) - \frac{2}{|a|^2}((a \cdot \sigma) - \rho)(y - z) \right) \frac{dzdy}{|a|^2 z} \\
&\geq K_1 \int_1^x \int_1^y \exp\left(\frac{-3}{|a|^2}((a \cdot \sigma) - \rho)(y - z) \right) \frac{dzdy}{z} \\
&= K_2 \int_1^x \frac{e^{3((a\cdot\sigma)-\rho)z/|a|^2}\left(e^{-3((a\cdot\sigma)-\rho)x/|a|^2} - e^{-3((a\cdot\sigma)-\rho)z/|a|^2}\right)}{z(\rho - (a \cdot \sigma))} dz \\
&= K_2 \int_1^x \frac{\left(1 - e^{-3((a\cdot\sigma)-\rho)(x-z)/|a|^2}\right)}{z((a \cdot \sigma) - \rho)} dz \\
&\geq K_2 \frac{\left(1 - e^{-3((a\cdot\sigma)-\rho)/|a|^2}\right)}{((a \cdot \sigma) - \rho)} \int_1^{x-1} \frac{1}{z} dz \\
&= K_3 \log(x - 1) \quad \to \infty \qquad \text{as } x \to \infty.
\end{aligned}$$

so in this case $Q(\widetilde{\tau}_\infty < \infty) = 0$ and S_t is a martingale.

<u>Case 2:</u> $\alpha > 1/2$. Then

$$\begin{aligned}
u(x) &= \int_1^x \int_1^y \exp\left(\frac{-2\rho L}{|a|^2} \log\left(\frac{y}{z}\right) + \frac{2\rho}{|a|^2}(y - z) \right. \\
&\qquad \left. - \frac{2(a \cdot \sigma)(y^{1/2+\alpha} - z^{1/2+\alpha})}{|a|^2(1/2 + \alpha)} \right) \frac{dzdy}{|a|^2 z} \\
&\leq \int_1^x \int_1^y \exp\left(\frac{2\rho}{|a|^2}(y - z) - \frac{2(a \cdot \sigma)(y^{1/2+\alpha} - z^{1/2+\alpha})}{|a|^2(1/2 + \alpha)} \right) \frac{dzdy}{|a|^2 z} \\
&\leq K_1 \int_1^x \int_1^y \exp\left(\frac{-(a \cdot \sigma)(y^{1/2+\alpha} - z^{1/2+\alpha})}{|a|^2(1/2 + \alpha)} \right) \frac{dzdy}{z} \\
&= K_1 \int_1^x \frac{\exp\left(\beta z^{1/2+\alpha}\right)}{z} \int_z^x \exp\left(-\beta y^{1/2+\alpha}\right) dydz
\end{aligned}$$

where $\beta = (a \cdot \sigma)/(|a|^2(1/2+\alpha))$ and then

$$\begin{aligned}
\int_z^x \exp\left(-\beta y^{1/2+\alpha}\right) dy &= \int_{z^{1/2+\alpha}}^{x^{1/2+\alpha}} e^{-\beta u} \frac{u^{(1/2-\alpha)/(1/2+\alpha)}}{(1/2+\alpha)} du \\
&\leq \frac{z^{1/2-\alpha}}{1/2+\alpha} \int_{z^{1/2+\alpha}}^{\infty} e^{-\beta u} du \\
&= \frac{z^{1/2-\alpha}}{(1/2+\alpha)\beta} \exp(-\beta z^{1/2+\alpha})
\end{aligned}$$

so

$$\begin{aligned}
u(x) &\leq K_1 \int_1^x \frac{\exp(\beta z^{1/2+\alpha})}{z} \frac{z^{1/2-\alpha}}{(1/2+\alpha)\beta} \exp(-\beta z^{1/2+\alpha}) dz \\
&= K_2 \int_1^x z^{-(1/2+\alpha)} dz \\
&< K_3 < \infty.
\end{aligned}$$

so Feller's test implies $Q(\widetilde{\tau}_\infty < \infty) = 1$ and for large enough time horizon T, S_t is a strictly local martingale. $\square$

4.6 Stochastic volatility models III: Ornstein-Uhlenbeck volatility

The stock price process is now given as a solution to the equations

$$\begin{aligned}
dS_t &= S_t |v_t|^\alpha \sigma dW_t, \qquad & S_0 &= x \\
dv_t &= a dW_t + \rho(L - v_t) dt, \qquad & v_0 &= 1.
\end{aligned} \tag{4.15}$$

assume $\rho > 0$, $L > 0$, $\alpha > 0$, and $|\sigma| > 0$, $|a| > 0$.

This model is proposed by Scott (1987), Stein and Stein (1991), and Heston (1993) among others. There always exist a unique strong solution to these equations:

we can find first v_t as the Ornstein-Uhlenbeck process which is the unique solution to the given equation with linear coefficients, and then S_t is the usual Doléans-Dade exponential

$$S_t = \exp\left(\int_0^t |v_u|^\alpha \sigma dW_u - \frac{1}{2}\int_0^t |v_u|^{2\alpha}|\sigma|^2 du\right)$$

Theorem 4.6.1. *Let (S_t, v_t) satisfy equation (4.15). Then S_t is a martingale if and only if $(\sigma \cdot a) \leq 0$ or $\alpha \leq 1$. If $(\sigma \cdot a) > 0$ and $\alpha > 1$ then S_t is a strictly local martingale.*

Proof: The proof goes along the same lines as the ones given in the two previous sections.

Define the sequence of stopping times

$$\tau_n = \inf\left\{t \in \mathbb{R}^+ : |v_t| \geq n\right\}$$

then $\tau_n \uparrow +\infty$ and $S_t^{(n)} = S_{t\wedge\tau_n}$ is a martingale under Q. Define

$$Q_n(A) = \frac{1}{S_0} Q\left(S_T^{(n)} 1_A\right), \qquad A \in \mathcal{F}_T$$

then

$$\begin{aligned} Q(S_T) &= \lim_{n\to\infty} Q\left(S_T^{(n)} 1_{\{\tau_n > T\}}\right) \\ &= S_0 \lim_{n\to\infty} Q_n\left(\tau_n > T\right) \\ &= S_0 \lim_{n\to\infty} Q\left(\widetilde{\tau}_n > T\right) \\ &= S_0 Q\left(\widetilde{\tau}_\infty > T\right) \end{aligned}$$

where

$$\widetilde{\tau}_\infty = \lim_{n\to\infty} \widetilde{\tau}_n$$
$$\widetilde{\tau}_n = \inf\left\{t \in \mathbb{R}^+ \,:\, |\widetilde{v}_t| \geq n\right\}$$
$$d\widetilde{v}_t = adW_t + \rho(L - \widetilde{v}_t)dt + \widetilde{v}_t^\alpha(a\cdot\sigma)dt, \qquad \widetilde{v}_0 = 1.$$

Observe that the drift and diffusion coefficients of the equation for $\widetilde{v}_t$ are locally Lipschitz continuous, so the solution is pathwise unique.

Applying Girsanov's theorem we obtain that the process

$$W_t^{(n)} = W_t - \int_0^{t\wedge T} 1_{\{u\leq\tau_n\}} v_u^\alpha \sigma du$$

is a Brownian motion under Q_n and, for $t \in [0,T]$ W_t and v_t satisfy

$$dW_t = dW_t^{(n)} + 1_{\{t\leq\tau_n\}} v_t^\alpha \sigma dt, \qquad W_0 = 0$$
$$dv_t = adW_t^{(n)} + \rho(L - v_t)dt + 1_{\{t\leq\tau_n\}}|v_t|^\alpha(a\cdot\sigma)dt \qquad v_0 = 1.$$

which is the same as the previous equation up to time $\tau_n \wedge T$, and because $\{\tau_n > T\} \in \mathcal{F}_{\tau_n\wedge T}$ the identity follows.

To check whether $\widetilde{v}_t$ explodes we first calculate the scale function:

$$p^{'}(x) = \exp\left(-2\int_0^x \frac{[\rho(L-y) + (a\cdot\sigma)|y|^\alpha]}{|a|^2}dy\right)$$
$$= \exp\left(\frac{-2\rho L}{|a|^2}x + \frac{\rho}{|a|^2}x^2 - \frac{2(a\cdot\sigma)}{|a|^2}\frac{|x|^{\alpha+1}}{(\alpha+1)}\right)$$
$$p(x) = \int_0^x \exp\left(\frac{-2\rho L}{|a|^2}y + \frac{\rho}{|a|^2}y^2 - \frac{2(a\cdot\sigma)}{|a|^2}\frac{|y|^{\alpha+1}}{(\alpha+1)}\right)dy$$

so when $x \sim +\infty$ we get the following cases

1. $\alpha + 1 < 2$ then $p(+\infty) = +\infty$.

2. $\alpha+1=2$ then $p(+\infty)=+\infty$ iff $\rho>(a\cdot\sigma)$.

3. $\alpha+1>2$ then $p(+\infty)=+\infty$ iff $(a\cdot\sigma)\leq 0$.

and when $x\sim-\infty$ we obtain

1. $\alpha+1<2$ then $p(-\infty)=-\infty$.

2. $\alpha+1=2$ then $p(-\infty)=-\infty$ iff $\rho\geq(a\cdot\sigma)$.

3. $\alpha+1>2$ then p$(-\infty)=-\infty$ iff $(a\cdot\sigma)\leq 0$.

So in the case $\alpha<1$ we get $\widetilde{v}_t$ never explodes and therefore S_t is a martingale. Otherwise we define

$$\begin{aligned} u(x) &= \int_0^x p'(y)\int_0^y \frac{2dzdy}{p'(z)|a|^2} \\ &= \int_0^x\int_0^y \exp\left(\frac{-2\rho L}{|a|^2}(y-z)+\frac{\rho}{|a|^2}(y^2-z^2)\right. \\ &\quad \left.-\frac{2(a\cdot\sigma)}{|a|^2}\frac{(y^{\alpha+1}-z^{\alpha+1})}{(\alpha+1)}\right)\frac{dzdy}{|a|^2} \end{aligned}$$

so, to obtain the result, we just need to show $u(-\infty)=u(+\infty)=+\infty$ if $\alpha=1$ and $u(+\infty)<\infty$ if $\alpha>1$ and $(a\cdot\sigma)>0$.

Case 1: $\alpha=1$. If $(a\cdot\sigma)<\rho$ then $p(-\infty)=-\infty$ and $p(+\infty)=+\infty$ so $u(-\infty)=u(+\infty)=+\infty$, $\widetilde{v}_t$ never explodes and S_t is a martingale.

If $(a\cdot\sigma)=\rho$ then

$$\begin{aligned}u(x)&=\frac{2}{|a|^2}\int_0^x\int_0^y\exp\left(\frac{-2\rho L}{|a|^2}(y-z)\right)\frac{dzdy}{|a|^2}\\&=\frac{1}{\rho L}\int_0^x\left(1-\exp\left(\frac{-2\rho L}{|a|^2}y\right)\right)dy\\&\to+\infty\qquad\text{as }|x|\to+\infty\end{aligned}$$

and if $(a\cdot\sigma)>\rho$ then

$$\begin{aligned}u(x)&=\frac{2}{|a|^2}\int_0^x\int_0^y\exp\left(\frac{-2\rho L}{|a|^2}(y-z)-\frac{(a\cdot\sigma)-\rho}{|a|^2}(y^2-z^2)\right)dzdy\\&\geq K_1\int_0^x\int_0^y\exp\left(\frac{-2((a\cdot\sigma)-\rho)}{|a|^2}(y^2-z^2)\right)dzdy\end{aligned}$$

observe that the last integral is symmetric in x, so it is enough to prove that it tends to $+\infty$ as $x\to+\infty$. If $x\sim+\infty$

$$\begin{aligned}u(x)&\geq K_1\int_1^x\int_0^y\exp\left(\frac{-2((a\cdot\sigma)-\rho)}{|a|^2}(y^2-z^2)\right)dzdy\\&\geq K_2\int_1^x\exp\left(\frac{-2((a\cdot\sigma)-\rho)}{|a|^2}y^2\right)\left(\exp\left(\frac{2((a\cdot\sigma)-\rho)}{|a|^2}y^2\right)-1\right)2y^{-1}dy\\&=K_2\int_1^x2y^{-1}dy-\int_1^x\exp\left(\frac{-2((a\cdot\sigma)-\rho)}{|a|^2}y^2\right)2y^{-1}dy\\&\to+\infty\qquad\text{as }x\to+\infty\end{aligned}$$

<u>Case 2: $\alpha>1$ and $(a\cdot\sigma)>0$.</u> In this case

$$\begin{aligned}u(x)&\leq\frac{2}{|a|^2}\int_0^x\int_0^y\exp\left(\frac{-(a\cdot\sigma)}{|a|^2}\frac{(|y|^{\alpha+1}-|z|^{\alpha+1})}{(\alpha+1)}\right)dzdy\\&=\frac{2}{|a|^2}\int_0^x\int_z^x\exp\left(\frac{-(a\cdot\sigma)}{|a|^2}\frac{(|y|^{\alpha+1}-|z|^{\alpha+1})}{(\alpha+1)}\right)dydz\end{aligned}$$

If we let $\beta = (a \cdot \sigma)/(|a|^2(\alpha+1))$ we can get an upper bound the same way as in the previous sections

$$\begin{aligned} u(x) &\leq K_1 \int_1^x \exp(\beta z^{\alpha+1}) \frac{z^{-\alpha}}{(\alpha+1)\beta} \exp(-\beta z^{\alpha+1}) dz \\ &= K_2 \int_1^x z^{-\alpha} dz \\ &< K_3 < +\infty \quad \square \end{aligned}$$

4.7 Summary of results on stochastic volatility

Putting together Theorems 4.4.5, 4.5.2 and 4.6.1 we obtain

Theorem 4.7.1. *Let W_t be an $\mathbb{R}^2$-valued Brownian motion on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, Q)$, and suppose (S_t, v_t) satisfy*

$$(4.16) \qquad \begin{aligned} dS_t &= S_t |v_t|^\alpha \sigma dW_t, & S_0 &= x \\ dv_t &= |v_t|^\beta a dW_t + \rho(L - v_t)dt, & v_0 &= 1 \end{aligned}$$

where $\alpha > 0$, $\beta = 0, 1/2, 1$, $\rho \geq 0$, $L > 0$, $|\sigma| > 0$, $|a| > 0$ and $|(a \cdot \sigma)| < |a||\sigma|$. Also assume $2\rho L \geq |a|^2$ if $\beta = 1/2$ and $\rho > 0$ if $\beta = 1$.

Then S_t is a martingale if and only if $(\sigma \cdot a) \leq 0$ or $\alpha + \beta \leq 1$. If $(\sigma \cdot a) > 0$ and $\alpha + \beta > 1$ then S_t is a strictly local martingale.

Moreover, imitating the proof of Theorem 4.4.11 we can show

Theorem 4.7.2. *Suppose (S_t, v_t) satisfy equation (4.16). Then for every $T \in \mathbb{R}^+$ there exist a probability measure Q^a equivalent to Q on $\mathcal{F}_T$ with respect to which S_t*

is a martingale on $[0, T]$. *If* $(a \cdot \sigma) > 0$ *we can find* Q^a *under which* S_t *satisfies*

$$dS_t = S_t |v_t|^\alpha \sigma dW_t^a,$$

$$dv_t = |v_t|^\beta a dW_t^a + \rho(L - v_t)dt - (a \cdot \sigma) v_t^{\alpha+\beta} dt,$$

on $t \in [0, T]$, *where* W_t^a *is a Brownian motion under* Q^a.

Chapter 5

Hedge ratios on Stochastic Volatility Models

We study a model in which the market is incomplete for the underlying basic securities, and we will give explicit formulas to determine a hedge strategy for the payoff of a European call option with respect to a process $(X^{(1)}(t), X^{(2)}(t))$ where $X^{(1)} \equiv S$ and $X^{(2)}$ is an additional asset that completes the market. We will also show that, in some cases, the second component of the hedge portfolio of the call option is positive with probability one, which implies that we can hedge any other contingent claim on $[0, T]$ using the stock and a traded European call option.

5.1 General theory

Let $(S^{(0)}(t), S^{(1)}(t), \ldots, S^{(n)}(t))$ be a RCLL stochastic processes, and again $S^{(k)}(t)$ is the price of a share of asset k at time t. They live on a probability space $(\Omega, \mathcal{F}, P)$

and we also have a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ that satisfies the usual conditions and with respect to which $S(t)$ is adapted. Again we consider prices $S(t)$ only on a finite time interval $t \in [0, T]$

As before, we assume $P(S^{(0)}(t) > 0) = 1$ for all t and define the discounted price process

$$Z_t^{(k)} = \frac{S_t^{(k)}}{S_t^{(0)}} \qquad k = 0, 1, \dots, n; \quad t \in [0, T].$$

Assumption 5.1.1. *Z_t admits NFFLVR, so there exists a probability measure $Q \sim P$ on $(\Omega, \mathcal{F}_T)$ such that Z_t is a Q-martingale.*

Let X be a nonnegative, $\mathcal{F}_T$-measurable random variable and H_t be a Z_t-integrable predictable process. We say H_t is an *admissible trading strategy* for X provided

$$H_T Z_T = X \qquad \textit{and}$$

$$H_t Z_t = H_0 Z_0 + \int_0^t H_u dZ_u \geq c Z_t \qquad \text{(self-financing bounded below)}$$

and we say that an admissible trading strategy H_t is Q-optimal if

$$H_t Z_t = H_0 Z_0 + \int_0^t H_u dZ_u \qquad \text{is a } Q\text{-martingale.}$$

Proposition 5.1.2. *(Q-optimal strategies are unique) Assume Z_t is locally square-integrable, and suppose H_t and G_t are Q-optimal admissible trading strategies for X. Then*

$$\int_0^T |H_t - G_t|^2 d[Z]_t = 0 \quad a.s.$$

Proof: Let (τ_n) be an increasing sequence of stopping times tending to $+\infty$ such that $Z_{\tau_n \wedge t}$ is in $\mathcal{H}^2$ (by integrability assumption). By Q-optimality, we have that for every finite stopping time τ

$$\int_0^\tau H_u dZ_u = Q(X|\mathcal{F}_\tau) - Q(X) = \int_0^\tau G_u dZ_u$$

so

$$\int_0^{\tau_n} (H_u - G_u) dZ_u = 0 \quad \text{a.s. for } n = 1, 2, \ldots$$

and then

$$\begin{aligned} 0 &= Q\left(\left(\int_0^{\tau_n} (H_u - G_u) dZ_u\right)^2\right) \\ &= Q\left(\int_0^{\tau_n} (H_u - G_u)^2 d[Z]_u\right) \end{aligned}$$

thus

$$\int_0^{\tau_n} (H_u - G_u)^2 d[Z]_u = 0 \quad \text{a.s.}$$

and letting $n \to \infty$ the result follows. □

Proposition 5.1.3. *Let G_t be a Q-optimal admissible trading strategy for X. Then, for every other admissible strategy H_t for X we have*

$$G_0 Z_0 \leq H_0 Z_0$$

Proof: The admissibility lower bound $\int_0^t H_u dZ_u \geq c Z_t$ implies $\int_0^t H_u dZ_u$ is a Q-supermartingale (see Lemma 3.2.3) and $\int_0^t G_u dZ_u$ is a Q-martingale so

$$\int_0^t (H_u - G_u) dZ_u \quad \text{is a } Q\text{-supermartingale}$$

and therefore

$$\begin{aligned}0 &= E(H_T Z_T - G_T Z_T)\\ &= H_0 Z_0 - G_0 Z_0 + Q\left(\int_0^t (H_u - G_u) dZ_u\right)\\ &\leq H_0 Z_0 - G_0 Z_0 \quad \square\end{aligned}$$

5.2 The stochastic volatility case

Let $B(t) = (B^{(1)}(t), B^{(2)}(t))$ be a 2-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, adapted to a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ that satisfies the conditions stated in chapter 2.

Let $(S(t), v(t))$ be diffusion processes on $(\Omega, \mathcal{F}, P)$, $v(t) \geq 0$ a.s., that satisfy the equations:

$$\begin{aligned}dS(t) &= S(t)v(t)\big(\sigma^{(1)}(t)dB^{(1)}(t) + \sigma^{(2)}(t)dB^{(2)}(t)\big) + S(t)\mu(t,\omega)dt,\\ dv(t) &= v(t)\big(a^{(1)}(t)dB^{(1)}(t) + a^{(2)}(t)dB^{(2)}(t)\big) + m(t,\omega)dt,\end{aligned} \tag{5.1}$$

With $S_0 = x$ and $v_0 = 1$. We have then a model that belongs to the first class of stochastic volatility discussed in the previous chapter (check equation (4.10)), where the diffusion and drift coefficients of $v(t)$ are linear in v. The vectors $\sigma(t) = (\sigma^{(1)}(t), \sigma^{(2)}(t))$ and $a(t) = (a^{(1)}(t), a^{(2)}(t))$ may depend on time but are nonrandom. Assume also zero interest rate for simplicity, so there is a security traded at a constant price $S^{(0)}(t) = 1$ for all t.

Recall the restrictions we impose on the time-dependent coefficients:

$$\sigma : \mathbb{R}^+ \to \mathbb{R}^2, \qquad 0 < |\sigma(t)| < M \quad \text{for all } t$$

$$a : \mathbb{R}^+ \to \mathbb{R}^2, \qquad 0 < |a(t)| < M \quad \text{for all } t$$

$$\mu : \mathbb{R}^+ \times \Omega \to \mathbb{R} \qquad \mathcal{F}_t\text{-adapted and a.s. locally bounded.}$$

$$m : \mathbb{R}^+ \times \Omega \to \mathbb{R} \qquad \mathcal{F}_t\text{-adapted and a.s. locally bounded.}$$

for some $M \in \mathbb{R}^+$.

5.3 Hedging using the stock and a European call.

Fix a time horizon $T \in (0, \infty)$ and suppose there exists an economy with a security traded at price $S(t)$ for all times $t \in [0, T]$ and another instrument whose price is constant equal to 1. We assume the existence of a probability measure Q on $(\Omega, \mathcal{F}_T)$ under which $S(t)$, $v(t)$ satisfy

$$\begin{aligned} dS(t) &= S(t)v(t)\sigma(t)dW(t), & S_0 &= x \\ dv(t) &= v(t)a(t)dW(t) + \rho(t)(L(t) - v(t))dt, & v_0 &= 1 \end{aligned} \tag{5.2}$$

for $\rho(t)$ and $L(t)$ bounded, non-negative real functions and $W(t)$ a Brownian Motion on $(\Omega, (\mathcal{F}_t)_{\{t\in[0,T]\}}, Q)$. A direct application of Girsanov's theorem provides an example of when this is possible.

Proposition 5.3.1. *Suppose S_t satisfies equation* (5.1) *and* $\gamma(t)$

$$\begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \end{bmatrix} = \begin{bmatrix} \sigma_1(t) & \sigma_2(t) \\ a_1(t) & a_2(t) \end{bmatrix}^{-1} \begin{bmatrix} -\mu(t)/v(t) \\ -m(t)/v(t) + \rho(t)(L(t)/v(t) - 1) \end{bmatrix}$$

exists and satisfies

$$P\left(\exp\left(\int_0^T \gamma(t)dB(t) - \frac{1}{2}\int_0^T |\gamma(t)|^2 dt\right)\right) = 1$$

Define the probability measure Q *on* $(\Omega, \mathcal{F}_T)$ *as*

$$Q(A) = P\left(1_A \exp\left(\int_0^T \gamma dB - \frac{1}{2}\int_0^T |\gamma|^2 dt\right)\right), \qquad A \in \mathcal{F}_T$$

then $W_t = B_t - \int_0^t \gamma(t)dt$ *is a Brownian motion under* Q *and* S_t *satisfies* (5.2).

Assume also that $(\mathcal{F}_t)_{\{t\in[0,T]\}}$ is the completion of the filtration generated by $W(t)$. We define a price functional on the set of nonnegative random variables $X \in \mathcal{F}_T$ as a continuous version of the conditional expected value under Q

$$x_t = Q(X|\mathcal{F}_t)$$

x_t is the price of the claim X at time t. Observe that this choice of price functional is not unique: there exist other probability measures $\widetilde{Q}$ with their respective Brownian motions $\widetilde{W}(t)$ such that the corresponding stochastic differential equation of $S(t)$ with respect to $\widetilde{W}(t)$ has a zero drift term and then $v(t)$ might have a different drift coefficient. The choice we make at this point affects the prices we obtain an also the hedge strategies produced: observe the result in Corollary 5.3.7 depends on the values of ρ and L.

Then the price of a European put option on $S(t)$, which gives the buyer the option to sell the stock at the expiration time T at a prespecified price K is given by

$$D(t) = Q((K - S(T))^+|\mathcal{F}_t)$$

Observe also that $(S(t), v(t))$ give a pathwise-unique solution to the stochastic differential equation (5.2), so $(S(t), v(t))$ is a Markov process and using Itô's formula (assuming that $D(t)$ is a smooth function of $(t, S(t), v(t))$) we can get a partial differential equation that D must satisfy.

Also notice that, using the martingale representation theorem, we can find a Q-optimal admissible trading strategy χ_t for $D(t)$ with respect to $B(t)$

$$D(t) = D(0) + \int_0^t \chi^{(1)}(t)dB^{(1)}(t) + \int_0^t \chi^{(2)}(t)dB^{(2)}(t)$$

We want first to obtain an explicit formula for $\chi(t)$ as a function of the paths of $S(t)$ and $v(t)$, and then use this representation to generate a hedging strategy with respect to some traded assets. To accomplish the first step we will use Clark's formula, and its extension, Haussmann's formula [1]. A direct proof can be found in Haussmann (1979). The version we present is from Occone (1984), which he proves using Malliavin's calculus. (we state it for dimension 2 and use stronger assumptions, which is all we need here. The result is valid for general $\mathbb{R}^n$ valued diffusion). For a simple introduction to this kind of explicit martingale representation see Rogers and Williams (1987) IV.41. First some preliminary results

Proposition 5.3.2. *Let $(Y(t))_{\{t\in[0,T]\}}$ be an $\mathbb{R}^2$-valued diffusion that solves*

$$dY(t) = g(t, Y(t))dt + h(t, Y(t))dW(t), \qquad y(0) = y_0.$$

where $g(t, y)$ and $h(t, y)$ are Borel measurable in (t, y), continuously differentiable

[1] The idea of applying Clark's theorem in this context was suggested by Robert Jarrow.

in y for each t, and

$$
\text{(5.3)} \qquad \begin{aligned}
&\sup_{(t,y)\in[0,T]\times\mathbb{R}^2} \left|\frac{\partial g_i(t,y)}{\partial y_j}\right| < \infty \qquad 1 \le i,j \le 2 \\
&\sup_{(t,y)\in[0,T]\times\mathbb{R}^2} \left|\frac{\partial h_{i,j}(t,y)}{\partial y_k}\right| < \infty \qquad 1 \le i,j,k \le 2
\end{aligned}
$$

Then $Q((Y_T^*)^p) < \infty$ *for all* $p \in \mathbb{R}^+$ *(recall* $Y_T^* = \sup\{|Y(t)| : 0 \le t \le T\}$*).*

Proof: Observe

$$
Y_T^* \le |y_0| + \int_0^T |g(t,Y)|dt + \left(\int_0 h(t,Y)dW(t)\right)_t^*
$$

and then Jensen's inequality implies, for $p \ge 1$

$$
(Y_T^*)^p \le 3^{p-1}\left(|y_0|^p + \left[\int_0^T |g(t,Y)|dt\right]^p + \left[\left(\int_0 h(t,Y)dW(t)\right)_t^*\right]^p\right)
$$

now Jensen's inequality again shows

$$
\left[\int_0^T |g(t,Y)|dt\right]^p \le T^{p-1}\int_0^T |g(t,Y)|^p dt
$$

and the Burkholder-Davis-Gundy inequality gives

$$
\begin{aligned}
Q\left(\left[\left(\int_0 h(t,Y)dW(t)\right)_t^*\right]^p\right) &\le C_p Q\left(\left[\int_0^T |h(t,Y)|^2 dt\right]^{p/2}\right) \\
&\le C_p t^{p/2-1} Q\left(\int_0^T |h(t,Y)|^p dt\right)
\end{aligned}
$$

with another use of Jensen's inequality, now assuming $p \ge 2$. So, putting the pieces together

$$
Q((Y_T^*)^p) \le C\left(|y_0|^p + Q\left(\int_0^T |g(t,Y)|^p dt\right) + Q\left(\int_0^T |h(t,Y)|^p dt\right)\right)
$$

and assumption (5.3) implies the existence of a constant $K \in \mathbb{R}^+$ such that $|g(t,y)| + |h(t,y)| < K|y|$ for all $t \in [0,T]$, $y \in \mathbb{R}^2$. This way we get

$$Q((Y_T^*)^p) \leq C\left(|y_0|^p + 2KQ\left(\int_0^T (Y_T^*)^p dt\right)\right)$$

and Gronwall's lemma then implies

$$Q((Y_T^*)^p) \leq C|y_0|^p e^{2CKT} < \infty$$

for all $p \geq 2$, and this implies $(Y_T^*)^p$ is Q-integrable for all $p \in \mathbb{R}^+$. □

Let $Z(t)$ satisfy the equation of first variation associated with $Y(t)$:

$$dZ(t) = \partial_Y g(t, Y(t))Zdt + \partial_Y h_1(t, Y(t))ZdW_1(t) + \partial_y h_2(t, Y(t))ZdW_2(t)$$

$$Z(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where

$$(\partial_y g(t,y))_{i,j} = \frac{\partial g_i(t,y)}{\partial y_j} \qquad (\partial_y h_k(t, Y(t)))_{i,j} = \frac{\partial h_{i,k}(t,y)}{\partial y_j}$$

Definition. Let X, Y be Banach spaces, $F : X \to Y$. We say F is Fréchet differentiable if for each $x \in X$ there exist a continuous linear functional $DF(x) : X \to Y$ such that for all $y \in X$, $\epsilon \in \mathbb{R}$

$$\frac{||F(x + \epsilon y) - F(x) - DF(x)(\epsilon y)||}{|\epsilon|} \to 0 \quad \text{as } \epsilon \to 0$$

If $F : C([0,T], \mathbb{R}^2) \to \mathbb{R}$ we can use the Riesz representation theorem to find, for each $y \in X$, a signed measure $\lambda^F(y)$ that represents $DF(y)$.

Theorem 5.3.3. *(Haussmann) Let $F : C([0,T];\mathbb{R}^2) \to \mathbb{R}$ be bounded, Fréchet differentiable with derivative $\lambda^F(b)$ weakly continuous in b, and Y satisfies* (5.3). *Then for $0 \le t \le T$*

$$Q(F(Y)|\mathcal{F}_t) = Q(F(Y)) + \int_0^t \chi(t)dW(t)$$

where $\chi(t)$ is given by

$$\chi(t) = Q\left(\int_{(t,T]} \lambda^F(Y,du)Z(u)Z^{-1}(t)\Big|\mathcal{F}_t\right) h(t,Y(t))$$

In our setting we define $Y(t) = [\log(S(t)), v(t)]$ so

$$\begin{bmatrix} dY_1(t) \\ dY_2(t) \end{bmatrix} = \begin{bmatrix} Y_2(t)\sigma_1(t) & Y_2(t)\sigma_2(t) \\ Y_2(t)a_1(t) & Y_2(t)a_2(t) \end{bmatrix} \begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \rho(t)(L(t) - Y_2(t)) \end{bmatrix} dt$$

and let Z_t satisfy the equation of first variation associated with $Y(t)$:

$$\begin{aligned} \begin{bmatrix} dZ_{1,1}(t) & dZ_{1,2}(t) \\ dZ_{2,1}(t) & dZ_{2,2}(t) \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & -\rho(t) \end{bmatrix} \begin{bmatrix} Z_{1,1}(t) & Z_{1,2}(t) \\ Z_{2,1}(t) & Z_{2,2}(t) \end{bmatrix} dt \\ &\quad + \sum_{i=1}^{2} \begin{bmatrix} 0 & \sigma_i(t) \\ 0 & \alpha_i(t) \end{bmatrix} \begin{bmatrix} Z_{1,1}(t) & Z_{1,2}(t) \\ Z_{2,1}(t) & Z_{2,2}(t) \end{bmatrix} dW_i(t) \\ &= \begin{bmatrix} 0 & 0 \\ -Z_{2,1}\rho & -Z_{2,2}\rho \end{bmatrix} dt \\ &\quad + \begin{bmatrix} Z_{2,1}\sigma_1 & Z_{2,2}\sigma_1 \\ Z_{2,1}a_1 & Z_{2,2}a_1 \end{bmatrix} dW_1 + \begin{bmatrix} Z_{2,1}\sigma_2 & Z_{2,2}\sigma_2 \\ Z_{2,1}a_2 & Z_{2,2}a_2 \end{bmatrix} dW_2 \end{aligned}$$

$$\begin{bmatrix} dZ_{1,1}(0) & dZ_{1,2}(0) \\ dZ_{2,1}(0) & dZ_{2,2}(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is a system of linear stochastic differential equations with solution

$$\begin{aligned} Z_{1,1}(t) &= 1 \\ Z_{1,2}(t) &= \int_0^t Z_{2,2}(u)(\sigma(u)dW(u)) \\ Z_{2,1}(t) &= 0 \\ Z_{2,2}(t) &= \exp\left(\int_0^t a(u)dW(u) - \rho(u)du - \frac{1}{2}|a(u)|^2 du\right) \end{aligned}$$

Using Haussmann's result we can obtain the explicit representation for the value of the European put option. First some technical results

Assumption 5.3.4. *There is $K \in \mathbb{R}^+$ such that for all $x \in \mathbb{R}^2$, $t \in [0,T]$ we have*

$$||x|| \leq K \left\| \begin{bmatrix} x^{(1)} & x^{(2)} \end{bmatrix} \begin{bmatrix} \sigma_t^{(1)} & \sigma_t^{(2)} \\ a_t^{(1)} & a_t^{(2)} \end{bmatrix} \right\|$$

that is,

$$\left\| \begin{bmatrix} \sigma_t^{(1)} & \sigma_t^{(2)} \\ a_t^{(1)} & a_t^{(2)} \end{bmatrix}^{-1} \right\| \leq K$$

Lemma 5.3.5. *The transition probability function under Q of the Markov process $(S(t), v(t))$ admits a density with respect to Lebesgue measure; for all $0 \leq u < t \leq T$*

$$\begin{aligned} Q((S(t), v(t)) \in A \times B | \mathcal{F}_u) &= Q(u, S(u), v(u); t, A, B) \\ &= \int_A \int_B q(u, S(u), v(u); t, s, v) dv ds \end{aligned}$$

moreover, there's a lower-semicontinuous version such that $q(u, s_1, v_1; t, s_2, v_2) > 0$ for all $s_1, s_2, v_1, v_2 \in \mathbb{R}^+$.

Proof: Define the process $Y(t) = (\log(S(t)), \log(v(t)))$ so

$$\begin{bmatrix} dY_1(t) \\ dY_2(t) \end{bmatrix} = \begin{bmatrix} \sigma_1(t)\exp\left(Y_2(t)\right) & \sigma_2(t)\exp\left(Y_2(t)\right) \\ a_1(t) & a_2(t) \end{bmatrix} \begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix} + \begin{bmatrix} \exp(2Y_2(t))|\sigma(t)|^2/2 \\ \rho(t)(L(t)\exp(-Y_2(t)) - 1) - |a(t)|^2/2 \end{bmatrix} dt$$

recall that $Q(S(t) > 0, v(t) > 0 \text{ for all } t \in [0,T]) = 1$ so a non-exploding solution to this equation exists and is unique because the coefficients are locally Lipschitz continuous. Then, from Assumption 5.3.4 it follows that the probability transition function of $Y(t)$ is absolutely continuous with respect to the Lebesgue measure, with some density function $q_l(u, y_{11}, y_{21}; t, y_{12}, y_{22})$ (see Stroock and Varadhan (1979) Corollary 10.1.4).

We will construct a version of q_l which is positive and lower-semicontinuous. Define the process Y_n by killing Y at the first exit time of $(-\infty, \infty) \times (-n, n)$, that is

$$\tau_n = \inf\{t > 0 \;:\; |Y_2(t)| = n\}$$

$$Y_n(t) = \begin{cases} Y(t) & \text{if } t < \tau_n \\ \infty & \text{if } t \geq \tau_n \end{cases}$$

then, for each n, Y_n is a diffusion with bounded and smooth coefficients on $\mathbb{R} \times (-n, n)$, and we can use a classical result on parabolic differential equations (see for example Dynkin (1965) Theorem 0.4) to obtain a positive transition density $q_{l,n}$ for Y_n, which is jointly continuous in all its variables, smooth in its "backward" variables and satisfies the Kolmogorov backward equation. Then define

$$q_l = \lim_{n\to\infty} q_{l,n}$$

and obtain, for $n > \max(y_{21}, y_{22})$

$$0 < q_{l,n}(u, y_{11}, y_{21}; t, y_{12}, y_{22}) \leq q_l(u, y_{11}, y_{21}; t, y_{12}, y_{22})$$

Then a version for the transition density function for $(S(t), v(t))$ is

$$q(u, s_1, v_1; t, s_2, v_2) = \frac{1}{s_2 v_2} q_l(u, \log(s_1), \log(v_1); T, \log(s_2), \log(v_2))$$

for any $s_1, v_1, s_2, v_2 \in \mathbb{R}^+$ and the result follows. □

Theorem 5.3.6. *Let $(S(t), v(t))$ satisfy equation (5.2). Then our price for the European put option with strike K and expiration T can be expressed as*

$$\begin{aligned} D(t) &= Q((K - S(T))^+ | \mathcal{F}_t) \\ &= Q((K - S(T))^+) \\ &\quad + \int_0^t Q(1_{\{S(T)\leq K\}} S(T) | \mathcal{F}_u) v(u)\sigma(u) dW(u) \\ &\quad + \int_0^t Q\left(1_{\{S(T)\leq K\}} S(T) \frac{(Z_{1,2}(T) - Z_{1,2}(u))}{Z_{2,2}(u)} \Big| \mathcal{F}_u\right) v(u) a(u) dW(u) \\ &\quad - \int_0^t Q\left(1_{\{S(T)\leq K\}} S(T) \int_u^T v(r)|\sigma(r)|^2 \frac{Z_{2,2}(r)}{Z_{2,2}(u)} dr \Big| \mathcal{F}_u\right) v(u) a(u) dW(u) \end{aligned}$$

<u>Proof:</u> Define the weakly continuous functional $F : C[0,T] \to \mathbb{R}$ as

$$F(Y) = D(t) = \left[K - \exp\left(Y_1(T) - \frac{1}{2}\int_0^T Y_2^2(t)|\sigma(t)|^2 dt\right)\right]^+$$

now, for $n = 1, 2, \ldots$, let $H_n : \mathbb{R} \to \mathbb{R}$ be continuously differentiable, bounded, and satisfying

$$H_n(x) = (K - x)^+ \quad \text{for all } x \in [0, K - 1/n] \cup [K + 1/n, \infty)$$

$$H_n'(x) = -1_{\{x<K\}} \quad \text{for all } x \in [0, K - 1/n] \cup [K + 1/n, \infty)$$

$$|H_n(x) - (K - x)^+| + |H_n'(x) + 1_{\{x<K\}}| \leq 2 \qquad \text{for all } x \in [0, \infty)$$

and let

$$F_n(Y) = H_n\left(\exp\left(Y_1(T) - \frac{1}{2}\int_0^T Y_2^2(t)|\sigma(t)|^2 dt\right)\right)$$

then F_n is Fréchet differentiable; its derivative can be identified with a measure in $C([0, T]; \mathbb{R}^2)$

$$\lambda^{F_n}(Y, dt) = \begin{bmatrix} H_n'(S(T))S(T)\delta_T(dt) \\ -H_n'(S(T))S(T)Y_2(t)|\sigma(t)|^2 dt \end{bmatrix}^T$$

Observe that $F_n(Y)$ satisfies the conditions of Theorem 5.3.3 so we can conclude that

$$H_n(S(T)) = F_n(Y) = Q(H_n(S(T))) + \int_0^T \chi_n(t) dW(t)$$

where

$$\chi_n(t) = Q(\gamma_n(t)|\mathcal{F}_t) \begin{bmatrix} Y_2(t)\sigma_1(t) & Y_2(t)\sigma_2(t) \\ Y_2(t)a_1(t) & Y_2(t)a_2(t) \end{bmatrix}$$

and

$$\gamma_n(t) = \int_{(t,T]} \lambda^{F_n}(Y,du) Z(u) Z^{-1}(t)$$
$$= \int_{(t,T]} \lambda^{F_n}(Y,du) \begin{bmatrix} 1 & Z_{1,2}(u) \\ 0 & Z_{2,2}(u) \end{bmatrix} \begin{bmatrix} Z_{2,2}(t) & -Z_{1,2}(t) \\ 0 & 1 \end{bmatrix} \frac{1}{Z_{2,2}(t)}$$
$$= \int_{(t,T]} \lambda^{F_n}(Y,du) \begin{bmatrix} 1 & (Z_{1,2}(u) - Z_{1,2}(t))/Z_{2,2}(t) \\ 0 & Z_{2,2}(u)/Z_{2,2}(t) \end{bmatrix}$$
$$= \begin{bmatrix} H_n' S(T) \\ H_n' S(T) \left(Z_{1,2}(T) - Z_{1,2}(t) - \int_t^T v(u)|\sigma(u)|^2 Z_{2,2}(u) du \right) Z_{2,2}^{-1}(t) \end{bmatrix}^T$$

so

$$\chi_n(t) = Q(H_n'(T)S(T)|\mathcal{F}_t)v(t)\sigma(t)$$
$$+ Q\left(H_n' S(T) \frac{(Z_{1,2}(T) - Z_{1,2}(t))}{Z_{2,2}(t)} \Big| \mathcal{F}_t \right) v(t)a(t)$$
$$- Q\left(H_n'(S(T) \int_t^T v(u)|\sigma(u)|^2 \frac{Z_{2,2}(u)}{Z_{2,2}(t)} du \Big| \mathcal{F}_t \right) v(t)a(t)$$

Let $M_n(t)$ be the continuous martingale obtained by taking conditional expected values of $F(Y)$ with respect to $\mathcal{F}_t$

$$M_n(t) = M_n^{(0)}(t) + M_n^{(1)}(t) + M_n^{(2)}(t) - M_n^{(3)}(t)$$

where

$$M_n^{(0)}(t) = Q(H_n(S(T)))$$
$$M_n^{(1)}(t) = \int_0^t Q(H_n' S(T)|\mathcal{F}_u) v(u)\sigma(u) dW(u)$$
$$M_n^{(2)}(t) = \int_0^t Q\left(H_n' S(T) \frac{(Z_{1,2}(T) - Z_{1,2}(u))}{Z_{2,2}(u)} \Big| \mathcal{F}_u \right) v(u)a(u) dW(u)$$
$$M_n^{(3)}(t) = \int_0^t Q\left(H_n' S(T) \int_u^T v(r)|\sigma(r)|^2 \frac{Z_{2,2}(r)}{Z_{2,2}(u)} dr \Big| \mathcal{F}_u \right) v(u)a(u) dW(u)$$

and similarly define

$$M(t) = M^{(0)}(t) + M^{(1)}(t) + M^{(2)}(t) - M^{(3)}(t)$$

where

$$M^{(0)}(t) = Q(H(S(T)))$$
$$M^{(1)}(t) = \int_0^t Q(H^{'}S(T)|\mathcal{F}_u)v(u)\sigma(u)dW(u)$$
$$M^{(2)}(t) = \int_0^t Q\left(H^{'}S(T)\frac{(Z_{1,2}(T) - Z_{1,2}(u))}{Z_{2,2}(u)}\Big|\mathcal{F}_u\right) v(u)a(u)dW(u)$$
$$M^{(3)}(t) = \int_0^t Q\left(H^{'}S(T)\int_u^T v(r)|\sigma(r)|^2\frac{Z_{2,2}(r)}{Z_{2,2}(u)}dr\Big|\mathcal{F}_u\right) v(u)a(u)dW(u)$$

we will show that, for $i = 1, 2, 3$, $Q(\langle M^{(i)} - M_n^{(i)}\rangle_T) \to 0$ and $|M^{(0)} - M_n^{(0)}| \to 0$ as $n \to \infty$, which in particular implies that $M(t)$ is a square-integrable martingale on $[0, T]$ and that

$$M_n(T) = H_n(S(T)) \xrightarrow{L^2} M(T)$$

but $F_n(Y) \to F(Y)$ on the set $\{S(T) \neq K\}$, which has probability 1 by Lemma 5.3.5. Thus

$$\begin{aligned} M(T) &= \lim_{n\to\infty} M_n(T) \\ &= \lim_{n\to\infty} F_n(Y) \\ &= F(Y) \\ &= (S(T) - K)^+ \quad \text{a.s.} \end{aligned}$$

and we obtain the desired representation. Now lets prove that the expected value of the quadratic variation of $M(T) - M_n(T)$ tends to zero. We have

$$
\begin{aligned}
|M_n^{(0)} - M^{(0)}| &\le Q(|H_n(S(T)) - H(S(T))|) \\
&\le 2Q\left(1_{\{K-\frac{1}{n}<S(T)<K+\frac{1}{n}\}}\right) \\
&\downarrow 0 \quad \text{as} \quad n \to \infty
\end{aligned}
$$

by Lemma 5.3.5 and

$$
\begin{aligned}
Q(\langle M_n^{(1)} - M^{(1)}\rangle_T) &= Q\left(\int_0^T [Q((H_n^{'} - H^{'})S(T)|\mathcal{F}_t)]^2 v^2(t)|\sigma(t)|^2 dt\right) \\
&\le Q\left(\int_0^T Q((H_n^{'} - H^{'})^2 S^2(T)|\mathcal{F}_t) v^2(t)|\sigma(t)|^2 dt\right) \\
&\le Q\left(\int_0^T 4Q(1_{\{K-\frac{1}{n}<S(T)<K+\frac{1}{n}\}} S^2(T)|\mathcal{F}_t) v^2(t)|\sigma(t)|^2 dt\right)
\end{aligned}
$$

the inner conditional expectation is bounded by $4(K+1/n)^2$ so the whole integral is bounded by $4(K+1/n)^2 T(\sigma^*(t))^2 Q((v^*(T))^2) < \infty$ and using the dominated convergence theorem twice and the fact that $Q\{S(T) = K\} = 0$ we obtain that it decreases to zero as $n \to \infty$. For the next component

$$
\begin{aligned}
&Q(\langle M_n^{(2)} - M^{(2)}\rangle_T) \\
&= Q\left(\int_0^T \left[Q\left((H_n^{'} - H^{'})S(T)\frac{(Z_{1,2}(T) - Z_{1,2}(t))}{Z_{2,2}(t)}\middle|\, \mathcal{F}_t\right)\right]^2 v^2(t)|a(t)|^2 dt\right) \\
&\le Q\left(\int_0^T Q\left((H_n^{'} - H^{'})^2 S^2(T)\frac{(Z_{1,2}(T) - Z_{1,2}(t))^2}{Z_{2,2}^2(t)}\middle|\, \mathcal{F}_t\right) v^2(t)|a(t)|^2 dt\right) \\
&\le A_1 Q\left(\int_0^T Q\left(1_{\{-\frac{1}{n}<S(T)-K<\frac{1}{n}\}}\frac{(Z_{1,2}(T) - Z_{1,2}(t))^2}{Z_{2,2}^2(t)}\middle|\, \mathcal{F}_t\right) v^2(t)|a(t)|^2 dt\right) \\
&\downarrow 0 \quad \text{as} \quad n \to \infty
\end{aligned}
$$

if we can show the integrand is bounded above by an integrable process and use dominated convergence theorem. Observe that $Z_{2,2}^{-1}$ satisfies an equation with coefficients linear in Z, so Proposition 5.3.2 implies $Q(((Z_{2,2}^{-1})_t^*)^p) < \infty$ for all $p > 0$; and using this fact we can obtain a bound of the form

$$
\begin{aligned}
&Q(\langle M_n^{(2)} - M^{(2)}\rangle_T)\\
&\leq A_1 Q\left(\int_0^T Q\left(\frac{(Z_{1,2}(T) - Z_{1,2}(t))^2}{Z_{2,2}^2(t)}\middle|\mathcal{F}_t\right) v^2(t)|a(t)|^2 dt\right)\\
&= 4\left(K + \frac{1}{n}\right)^2 Q\left(\int_0^T Q\left(\int_t^T \frac{Z_{2,2}^2(u)}{Z_{2,2}^2(t)}|\sigma(u)|^2 du\middle|\mathcal{F}_t\right) v^2(t)|a(t)|^2 dt\right)\\
&\leq A_1 \int_0^T Q\left(\left(\int_t^T \frac{Z_{2,2}^2(u)}{Z_{2,2}^2(t)}|\sigma(u)|^2 du\right)^2\right)^{1/2} Q(v^4(t))^{1/2} a^*(T)^2 dt\\
&\leq A_1 \int_0^T Q\left(\left(\int_t^T (Z_{2,2}^*(T))^2 (Z_{2,2}^{-1*}(T))^2 (\sigma^*(T))^2 du\right)^2\right)^{1/2} Q(v^4(t))^{1/2} a^*(T)^2 dt\\
&\leq A_1 T^2 (\sigma^*(T))^2 a^*(T)^2 Q((Z_{2,2}^*(T))^8)^{1/4} Q((Z_{2,2}^{-1*}(T))^8)^{1/4} Q((v^*(T))^4)^{1/2} < \infty
\end{aligned}
$$

and the conclusion above holds. Now, for the third component of $M(t)$:

$$
\begin{aligned}
&Q(\langle M^{(3)} - M_n^{(3)}\rangle_T)\\
&= Q\left(\int_0^T \left[Q\left((H_n' - H')S(T)\int_t^T v(u)|\sigma(u)|^2 \frac{Z_{2,2}(u)}{Z_{2,2}(t)} du\middle|\mathcal{F}_t\right)\right]^2 v^2(t)|a(t)|^2 dt\right)\\
&\leq Q\left(\int_0^T Q\left((H_n' - H')^2 S^2(T)(T-t)\int_t^T v^2|\sigma|^4 \frac{Z_{2,2}^2(u)}{Z_{2,2}^2(t)} du\middle|\mathcal{F}_t\right) v^2|a|^2 dt\right)\\
&\leq A_2 Q\left(\int_0^T Q\left(1_{\{-\frac{1}{n} < S(T) - K < \frac{1}{n}\}}\int_t^T v^2|\sigma|^4 \frac{Z_{2,2}^2(u)}{Z_{2,2}^2(t)} du\middle|\mathcal{F}_t\right) v^2|a|^2 dt\right)\\
&\downarrow 0 \quad \text{as} \quad n \to \infty
\end{aligned}
$$

provided again an upper bound on the integral to apply the dominated convergence theorem. In this case

$$A_2 Q\left(\int_0^T Q\left(\int_t^T v^2(u)|\sigma(u)|^4 \frac{Z_{2,2}^2(u)}{Z_{2,2}^2(t)} du \Big| \mathcal{F}_t\right) v^2(t)|a(t)|^2 dt\right)$$

$$\leq A_2 \int_0^T Q\left(\left(\int_t^T v^2(u)|\sigma(u)|^4 \frac{Z_{2,2}^2(u)}{Z_{2,2}^2(t)} du\right)^2 \mathcal{F}_t\right)^{1/2} Q(v^4(t))^{1/2}(a^*(T))^2 dt$$

$$\leq A_2 T^2(\sigma^*(T))^4(a^*(T))^2 Q((Z_{2,2}^*(T))^{16})^{1/8} Q((Z_{2,2}^{-1*}(T))^8)^{1/4} Q((v^*(T))^{16})^{1/4}$$

this concludes the proof □.

Now, assuming that S_t is a martingale under Q, we can find a corresponding expression for the hedge ratio of a call option, and we can greatly simplify its form. Recall Theorem 4.4.5 that shows S_t is a martingale whenever σ and a are constants and $(\sigma \cdot a) \leq 0$.

Let $\widetilde{Z}_{1,2}(t)$ be given by

$$\widetilde{Z}_{1,2}(t) = Z_{1,2}(t) - \int_0^t v(u)|\sigma(u)|^2 Z_{2,2}(u) du$$

$$= \int_0^t Z_{2,2}(u)\sigma(u) dW(u) - Z_{2,2}(u)v(u)|\sigma(u)|^2 du$$

then we can express

$$C(t) = Q((S(T) - K)^+|\mathcal{F}_t)$$

$$= S(t) - K + D(t)$$

$$= Q((S(T) - K)^+)$$

$$+ \int_0^t Q(1_{\{S(T)\geq K\}} S(T)|\mathcal{F}_u) v(u)\sigma(u) dW(u)$$

$$+ \int_0^t Q\left(1_{\{S(T)\geq K\}} S(T) \frac{(\widetilde{Z}_{1,2}(T) - \widetilde{Z}_{1,2}(u))}{Z_{2,2}(u)} \Big| \mathcal{F}_u\right) v(u) a(u) dW(u)$$

where we used the fact that $[S, \widetilde{Z}_{1,2}]_t \equiv 0$ so

$$Q\left(S(T)\frac{(\widetilde{Z}_{1,2}(T)-\widetilde{Z}_{1,2}(u))}{Z_{2,2}(u)}\Big|\mathcal{F}_u\right)=0$$

Now define the new probability measure $\widetilde{Q}$ on $(\Omega, \mathcal{F}_T)$ as

$$\widetilde{Q}(A)=Q\left(\frac{S(T)}{S(0)}1_A\right), \qquad A\in\mathcal{F}_T$$

then Girsanov's theorem states that the process

$$\widetilde{W}(t)=W(t)-\int_0^t v(u)\sigma(u)du$$

is an $\mathbb{R}^2$-valued Brownian motion on $(\Omega, \mathcal{F}_T, \widetilde{Q})$, $0\le t\le T$.and

$$\begin{aligned}
dS(t)&=S(t)v(t)\sigma(t)d\widetilde{W}(t)+S(t)v^2(t)|\sigma(t)|^2dt,\\
dv(t)&=v(t)a(t)d\widetilde{W}(t)+\rho(t)(L(t)-v(t))dt+v^2(t)(a\cdot\sigma)(t)dt,\\
d\widetilde{Z}_{1,2}(t)&=Z_{2,2}(t)\sigma(t)d\widetilde{W}(t),\\
dZ_{2,2}(t)&=Z_{2,2}(t)a(t)d\widetilde{W}(t)-Z_{2,2}(t)\rho(t)dt+Z_{2,2}(t)v(t)(a\cdot\sigma)(t)dt,
\end{aligned}$$

with $S(0)=x_0$, $v(0)=1$, $\widetilde{Z}_{1,2}(0)=0$, $Z_{2,2}(0)=1$; and we conclude

Corollary 5.3.7. *Suppose S_t is a Q-martingale. Then the Q-minimal hedging strategy for the European call option with strike K and expiration T is given by*

$$\begin{aligned}
C(t)&=Q((S(T)-K)^+|\mathcal{F}_t)\\
&=Q((S(T)-K)^+)\\
&\quad+\int_0^t\widetilde{Q}(1_{\{S(T)\ge K\}}|\mathcal{F}_u)dS(u)\\
&\quad+\int_0^t\widetilde{Q}\left(1_{\{S(T)\ge K\}}\frac{(\widetilde{Z}_{1,2}(T)-\widetilde{Z}_{1,2}(u))}{Z_{2,2}(u)}\Big|\mathcal{F}_u\right)d\widetilde{S}(u)
\end{aligned}$$

where $\widetilde{S}(t)$ *satisfies*

$$d\widetilde{S}(t) = S(t)v(t)a(t)dW(t), \qquad \widetilde{S}(0) = x_0,$$

Observe that the first integral has exactly the same form as the Black-Scholes hedge (but the hedge portfolios don't coincide because the distributions under the two models are different), and we get an extra second term with diffusion component in the direction of $a(t)$.

Now we want to show that the hedge on the $a(t)$ direction is positive a.s.. First an intermediate result

Lemma 5.3.8. *Suppose* $(a \cdot \sigma) = 0$ *and define* f_t *as a continuous version of*

$$\begin{aligned} f_t &= \widetilde{Q}(S(T) \geq K | \mathcal{F}_t) \\ &= \widetilde{Q}(t, S(t), v(t); T, [K, \infty), (-\infty, \infty)) \end{aligned}$$

then

$$f_t = f_0 + \int_0^t g^{(1)}(t) dS(t) + \int_0^t g^{(2)}(t) d\widetilde{S}(t) \tag{5.4}$$

where $g^{(1)}$ *and* $g^{(2)}$ *are progressively measurable processes,*

$$g^{(1)}(t) = \int_{-\infty}^{\infty} \frac{K}{S(T)} \widetilde{q}(t, S(t), v(t); T, K, v) dv$$

and $\widetilde{q}$ *is the density obtained in Lemma 5.3.5 if we replace* Q *by* $\widetilde{Q}$.

<u>Proof:</u> Use the martingale representation theorem with respect to Brownian motion, and Assumption 5.3.4 together with the positivity of S_t and v_t, to obtain progressively measurable $g^{(1)}$ and $g^{(2)}$ such that equation (5.4) holds. To show the explicit

expression for $g^{(1)}$ define the sequence of stopping times

$$\tau_n = \inf\{t > 0 \; : \; |\log(v(t))| = n \text{ or } |\log(S(t))| = n\}$$

Let $\widetilde{q}_n$ be the smooth transition density function for the process $(S(t), v(t))$ killed at the stopping time τ_n (see the proof of Lemma 5.3.5) and define $\widetilde{q}$ as the limit

$$\widetilde{q}(t, s_1, v_1; T, s_2, v_2) = \lim_{n\to\infty} \widetilde{q}_n(t, s_1, v_1; T, s_2, v_2)$$

then the Lebesgue monotone convergence theorem implies $\widetilde{q}$ is a version of the transition density for $(S(t), v(t))$.

Now, for $n, m = 1, 2, \ldots$ let $\phi_{n,m} : \mathbb{R}^2 \to \mathbb{R}$ be C^∞ with

$$\begin{aligned} &0 \le \phi_{n,m} \le 1 && \text{for all } s, v \in \mathbb{R} \\ &\phi_{n,m}(s, v) = 1 && \text{on } K \le s \le e^{n-1},\, e^{-(n-1)} \le v < e^{n-1} \\ &\phi_{n,m}(s, v) = 0 && \text{on } s \le K - 1/m,\, s \ge e^{n-1} + 1/m \text{ or } |\log(v)| \ge n \end{aligned}$$

and define

$$\Gamma_{n,m}(t, s_1, v_1) = \int\int \widetilde{q}_n(t, s_1, v_1; T, s_2, v_2)\phi_{n,m}(s_2, v_2)ds_2dv_2$$

then $\Gamma(t, s, v)$ is $C^{1,2,2}$ and satisfies the Kolmogorov backward equation, so by Itô's lemma

$$\begin{aligned} &\Gamma_{n,m}(t, S(t), v(t)) - \Gamma_{n,m}(0, S(0), v(0)) \\ &= \int_0^t \left(\frac{\partial \Gamma_{n,m}}{\partial s}\right) dS_u + \int_0^t \left(\frac{\partial \Gamma_{n,m}}{\partial v}\right) dv_u + \int_0^t \left(\left(\frac{\partial}{\partial u} + L_u\right)\Gamma_{n,m}\right) du \end{aligned}$$

where L_u is the generator of the process (S, v). The Kolmogorov backward equation is

$$\left(\frac{\partial}{\partial u} + L_u\right)\Gamma_{n,m}(u, S(u), v(u)) = 0$$

for $u \le \tau_n$ so we have

$$\Gamma_{n,m}(t \wedge \tau_n, S(t \wedge \tau_n), v(t \wedge \tau_n)) - \Gamma_{n,m}(0, S(0), v(0))$$
$$= \int_0^{t\wedge\tau_n} \left(\frac{\partial \Gamma_{n,m}}{\partial s}\right) dS_u + \int_0^{t\wedge\tau_n} \left(\frac{\partial \Gamma_{n,m}}{\partial v}\right) dv_u$$

Define the set

$$A = \{K - 1/m < S(T) < K\} \cup \{\log(S(T)) > n-1\}$$
$$\cup \{|\log(v(T))| > n-1\} \cup \{\tau_n > T\}$$

then from the definition of $\phi(s, v)$ we obtain

$$f_{t\wedge\tau_n} = \Gamma_{n,m}(t \wedge \tau_n, S(t \wedge \tau_n), v(t \wedge \tau_n))$$
$$+ \widetilde{Q}\left(\left(1_{\{S(T)\ge K\}} - \phi_{n,m}(S(T), v(T))\right) 1_A \middle| \mathcal{F}_t\right) + \widetilde{Q}(S(T) \ge K,\ \tau_n \le T | \mathcal{F}_t)$$

so

$$\widetilde{Q}\Bigg(\Big(f_0 - \widetilde{Q}\left(\left(1_{\{S(T)\ge K\}} - \phi_{n,m}\right) 1_A \middle| \mathcal{F}_t\right)$$
$$-\widetilde{Q}(S(T) \ge K,\ \tau_n \le T|\mathcal{F}_t) - \Gamma_{n,m}(0, S(0), v(0))\Big)^2\Bigg)$$
$$= \widetilde{Q}\left(\left(\int_0^{t\wedge\tau_n} \left(\frac{\partial \Gamma_{n,m}}{\partial s} - g^{(1)}\right) dS_u + \int_0^{t\wedge\tau_n} \left(\frac{\partial \Gamma_{n,m}}{\partial v} - g^{(2)}\right) dv_u\right)^2\right)$$
$$= \widetilde{Q}\left(\int_0^{t\wedge\tau_n} \left(\frac{\partial \Gamma_{n,m}}{\partial s} - g^{(1)}\right)^2 d[S]_u + \int_0^{t\wedge\tau_n} \left(\frac{\partial \Gamma_{n,m}}{\partial v} - g^{(2)}\right)^2 d[v]_u\right)$$

and the first expression tends to zero as $n, m \to \infty$ because $\Gamma_{n,m}(0, S(0), v(0)) \to f_0$, $\widetilde{Q}(A) \to 0$ and $\widetilde{Q}(\tau_n \le T) \to 0$. Also, by the translation invariance of $\log(S)$ we get (using the notation from Lemma 5.3.5)

$$\widetilde{q}_l(t, x_1 + \delta, y_1; T, x_2, y_2) = \widetilde{q}_l(t, x_1, y_1; T, x_2 - \delta, y_2)$$

with Lebesgue measure 1 so

$$\frac{\partial}{\partial s_1}\widetilde{q}(t,s_1,v_1;T,s_2,v_2) = -\frac{\partial}{\partial s_2}\left(\frac{s_2}{s_1}\widetilde{q}(t,s_1,v_1;T,s_2,v_2)\right)$$

and then

$$\begin{aligned}
&\frac{\partial}{\partial S(t)}\Gamma_{n,m}(t,S(t),v(t)) \\
&= \int\int\left(\frac{\partial}{\partial S(t)}\widetilde{q}_n(t,S(t),v(t);T,s,v)\right)\phi_{n,m}(s,v)dsdv \\
&= -\int\int\left(\frac{\partial}{\partial s}\frac{s}{S(t)}\widetilde{q}_n(t,S(t),v(t);T,s,v)\right)\phi_{n,m}(s,v)dsdv \\
&= \int\int\frac{s}{S(t)}\widetilde{q}_n(t,S(t),v(t);T,s,v)\left(\frac{\partial}{\partial s}\phi_{n,m}(s,v)\right)dsdv \\
&= \int_{-\infty}^{\infty}\int_{K-1/m}^{K}\frac{s}{S(t)}\widetilde{q}_n(t,S(t),v(t);T,s,v)\left(\frac{\partial}{\partial s}\phi_{n,m}(s,v)\right)dsdv \\
&\quad + \int_{-\infty}^{\infty}\int_{\exp(n-1)}^{\exp(n-1)+1/m}\frac{s}{S(t)}\widetilde{q}_n(t,S(t),v(t);T,s,v)\left(\frac{\partial}{\partial s}\phi_{n,m}(s,v)\right)dsdv \\
&\to \int_{-\infty}^{\infty}\frac{K}{S(t)}\widetilde{q}_n(t,S(t),v(t);T,K,v)dv - \int_{-\infty}^{\infty}\frac{e^{n-1}}{S(t)}\widetilde{q}_n(t,S(t),v(t);T,e^{n-1},v)dv \\
&\to \int_{-\infty}^{\infty}\frac{K}{S(t)}\widetilde{q}(t,S(t),v(t);T,K,v)dv
\end{aligned}$$

letting $m \to \infty$ first and then $n \to \infty$. Now apply the Lebesgue dominated convergence theorem to get, for every stopping time $\tau \leq \tau_n$ for some n

$$\begin{aligned}
&\widetilde{Q}\left(\int_0^{t\wedge\tau}\left(\int_{-\infty}^{\infty}\frac{K}{S(t)}\widetilde{q}(t,S(t),v(t);T,K,v)dv - g^{(1)}\right)^2 d[S]_u\right) \\
&= \lim_{m,n\to\infty}\widetilde{Q}\left(\int_0^{t\wedge\tau}\left(\frac{\partial\Gamma_{n,m}(t,S(t),v(t))}{\partial S(t)} - g^{(1)}\right)^2 d[S]_u\right) \\
&= 0
\end{aligned}$$

and the result follows. □

Theorem 5.3.9. *Suppose* $(a \cdot \sigma) \equiv 0$. *Then, for every* Q*-integrable contingent claim* $X \in \mathcal{F}_{T-\epsilon}$, $\epsilon > 0$, *there exists a* Q*-optimal admissible* $(H^{(0)}, H^{(1)}, H^{(2)})$ *such that*

$$X = x_0 + \int_0^{T-\epsilon} H_t^{(1)} dS(t) + \int_0^{T-\epsilon} H_t^{(2)} dC(t)$$

Proof: fix $u \in [0, T)$ and let

$$f_t = \widetilde{Q}(S(T) \geq K | \mathcal{F}_t) \qquad Z_t = \frac{Z_{1,2}(t) - Z_{1,2}(u)}{Z_{2,2}(u)}$$

then f_t and Z_t are continuous $\widetilde{Q}$-martingales on $[u, T]$ and, using the integration by parts formula

$$d(f_t Z_t) = f_t dZ_t + Z_t df_t + d[Z, f]_t$$

furthermore, $|f_t| \leq 1$ so $\int f_t dZ_t$ is a $\widetilde{Q}$-martingale and also

$$\begin{aligned} \widetilde{Q}\left(\int_0^T Z_t^2 d[f]_t\right) &\leq \widetilde{Q}\left((Z_t^*)^2 [f]_T\right) \\ &\leq C\widetilde{Q}((Z_t^*)^4)^{1/2} \widetilde{Q}((f_t^*)^4)^{1/2} \\ &< \infty \end{aligned}$$

so $\int Z_t df_t$ is a $\widetilde{Q}$-martingale; and recalling the hedge coefficients for the European call option from Corollary 5.3.7 we have

$$\begin{aligned}
\chi^{(1)}(u) &= \widetilde{Q}(1_{\{S(T)\geq K\}}|\mathcal{F}_u) \\
\chi^{(2)}(u) &= \widetilde{Q}\left(1_{\{S(T)\geq K\}}\frac{(\widetilde{Z}_{1,2}(T)-\widetilde{Z}_{1,2}(u))}{Z_{2,2}(u)}\bigg|\mathcal{F}_u\right) \\
&= \widetilde{Q}(f_T Z_T|\mathcal{F}_u) \\
&= f_u Z_u + \widetilde{Q}\left(\int_u^T d[Z,f]_t\bigg|\mathcal{F}_u\right) \\
&= \widetilde{Q}\left(\int_u^T \frac{Z_{2,2}(t)}{Z_{2,2}(u)} g^{(1)}(t)v(t)|\sigma(t)|^2 dt\bigg|\mathcal{F}_u\right) \\
&> 0 \quad \text{a.s.} \qquad \text{for } 0\leq u < T
\end{aligned}$$

where $g^{(1)}$ in defined in Lemma 5.3.8.

Observe that, for $0 \leq u \leq T-\epsilon$ we have

$$\chi^{(2)}(u) \geq \frac{1}{Z_{2,2}(u)}\widetilde{Q}\left(\int_{T-\epsilon}^T Z_{2,2}(t)g^{(1)}(t)v(t)|\sigma(t)|^2 dt\bigg|\mathcal{F}_u\right)$$

and as in the proof of Proposition 4.1.1 the continuous version of the martingale on the right hand side is bounded away from zero with probability 1.

Also, by the martingale representation theorem for Brownian motion we can obtain a predictable $h(t)$ such that

$$\begin{aligned}
X &= x_0 + \int_0^{T-\epsilon} h^{(1)}(t)\sigma(t)dW(t) + \int_0^{T-\epsilon} h^{(2)}(t)a(t)dW(t) \\
&= \int_0^{T-\epsilon} \frac{h^{(2)}(t)}{\chi^{(2)}(t)S(t)v(t)}\left(\chi^{(2)}(t)S(t)v(t)a(t)dW(t) + \chi^{(1)}(t)S(t)v(t)\sigma(t)dW(t)\right) \\
&\quad - \int_0^{T-\epsilon} \frac{h^{(2)}(t)\chi^{(1)}(t)}{\chi^{(2)}(t)S(t)v(t)}S(t)v(t)\sigma(t)dW(t) + \int_0^{T-\epsilon} \frac{h^{(1)}(t)}{S(t)v(t)}S(t)v(t)\sigma(t)dW(t) \\
&= x_0 + \int_0^{T-\epsilon} H^{(1)}(t)dS(t) + \int_0^{T-\epsilon} H^{(2)}(t)dC(t)
\end{aligned}$$

where

$$H^{(1)}(t) = \frac{1}{S(t)v(t)}\left(h^{(1)}(t) - \frac{h^{(2)}(t)\chi^{(1)}(t)}{\chi^{(2)}(t)}\right)$$
$$H^{(2)}(t) = \frac{h^{(2)}(t)}{\chi^{(2)}(t)S(t)v(t)}$$

and $\chi(t)$, $S(t)$, $v(t)$ are continuous in t with $\chi^{(2)}(t) > 0$, $S(t) > 0$, $v(t) > 0$ for all $t \in [0, T)$ with probability 1; so $H^{(1)}$ and $H^{(2)}$ are integrable, $H^{(0)}$ is determined by the self-financing condition and the result follows. □

Remark *5.3.10.* The fact that the diffusion matrix of the process (S_t, v_t) is non-singular a.s., together with Theorem 2.4.2, imply the existence of a *vector* integrable H_t that hedges every $X \in \mathcal{F}_T$. However, we have $\chi^{(2)}(t) \to 0$ as $t \to T$ so this strategy may not be componentwise integrable.

5.4 Concluding remarks

We have shown an explicit integral representation for a European call option on a specific stochastic volatility model, and we have used it to prove that the call option competes the market. This representation is very particular of the model chosen, and we prove completeness with the call only in the case when $(a \cdot \sigma) = 0$.

It might be of interest to characterize the class of models for which one or several European call options complete the market. It isn't true that these options always complete the market, as the following example shows

Let $S(t)$ be given by the the stochastic volatility model discussed before (equation (5.2)). Define the process $Y(t)$ as a continuous version of

$$Y(t) = Q(S(T) > 1|\mathcal{F}_t)$$

and suppose $0 < Q(S(T) > 1) = 1 - Q(S(T) \leq 1) < 1$. We have an economy with 2 assets trading: one with constant price equal to 1 and the other with price $Y(t)$. Assume NFFLVR and suppose the European call option on Y with strike price $0 < K < 1$ and expiration T is traded at price $C(t)$. Then there exist an equivalent martingale measure $\widetilde{Q}$ such that

$$Y(t) = \widetilde{Q}(Y(T)|\mathcal{F}_t)$$
$$C(t) = \widetilde{Q}((Y(T) - K)^+|\mathcal{F}_t)$$

but then

$$\begin{aligned} C(t) &= \widetilde{Q}((1-K)1_{\{Y(T)=1\}}|\mathcal{F}_t) \\ &= (1-K)Q(Y(T)|\mathcal{F}_t) \\ &= (1-K)Y(t) \end{aligned}$$

so the price of the call option is a constant multiple of the underlying security: a clearly redundant asset that doesn't complete the market.

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