
with

\[ \kappa(\theta) = \mu \left[ \int_0^\infty \left( \frac{x^2}{\theta^2} \land 1 \right) \rho(dx) + 2 \int_\theta^\infty \log \frac{x}{\theta} \rho(dx) \right], \]

and the conditions of Theorem 5.1 are easily checked. Therefore, a two-sided Ornstein-Uhlenbeck process with respect to any infinitely divisible random measure with respect to which it is well defined is in \( AC^p[a,b] \) for all \( p \geq 1 \).

References


where \( \{D(t), a \leq t \leq b\} \) is a symmetric infinitely divisible process given by
\[
D(t) = \int_S \hat{h}(t, s) M(ds), \ a \leq t \leq b, \tag{5.10}
\]
and \( \hat{h}(\cdot, s) \) is the derivative of \( h(\cdot, s) \) in the sense of absolute continuity. In particular, \( h(\cdot, s) \) is absolutely continuous. Now the required properties of \( \hat{h} \) follow by applying Theorem 3.1 with \( \psi(x) = x^p \) to the process \( \{D(t), a \leq t \leq b\} \) in (5.10).

In the opposite direction, we define \( \{D(t), a \leq t \leq b\} \) by (5.10). Then by Theorem 3.1 almost all sample paths of \( \{D(t), a \leq t \leq b\} \) are in \( L^p[a, b] \), and the process \( \{\bar{X}(t), a \leq t \leq b\} \) defined by
\[
\bar{X}(t, \omega) = \int_S h(a, s) M(ds, \omega) + \int_a^t D(u, \omega) du, \ a \leq t \leq b
\]
if \( \{D(t, \omega), a \leq t \leq b\} \) is in \( L^p[a, b] \), and \( \bar{X}(t) = 0 \), all \( t \), otherwise, is the required version of \( \{X(t), a \leq t \leq b\} \) with all sample paths in \( AC^p[a, b] \). ■

Of course, any simplification to \( L^p[a, b] \) of the conditions in Theorem 3.1 (e.g. the cases discussed in the previous section) will lead to a corresponding simplification in the conditions of Theorem 5.1.

**Example 5.1 Ornstein-Uhlenbeck processes** Let
\[
X(t) = \int_{-\infty}^{t} e^{-\mu |t-s|} M(ds), \ -\infty < t < \infty, \tag{5.11}
\]
where \( \mu > 0 \), the control measure \( \lambda \) of the infinitely divisible random measure \( M \) is equivalent to the Lebesgue measure, and \( \rho(dx, s) = g(s)^{-1} \rho(dx) \), where \( \lambda(ds) = g(s) ds \). We will call this process a (two-sided) Ornstein-Uhlenbeck process with respect to the infinitely divisible random measure \( M \). The process is well defined if and only if
\[
\int_1^{\infty} \log x \rho(dx) < \infty \tag{5.12}
\]
(see (1.3)). In this case it is a stationary mixing process. Letting \( \hat{h}(t, s) = e^{-\mu |t-s|}, a \leq t \leq b, -\infty < s < \infty \), we see immediately that for every \( s \in (-\infty, \infty) \) the function \( h(\cdot, s) \) is in \( AC^p[a, b] \) with
\[
\hat{h}(t, s) = \mu \text{sign}(t-s) e^{-\mu |t-s|}, \ a \leq t \leq b, -\infty < s < \infty.
\]
Therefore, we have
\[
||\hat{h}(\cdot, s)|| = \begin{cases} 
\mu^{1-1/p} p^{-1/p} \left( e^{-\mu p(a-s)} - e^{-\mu p(b-s)} \right)^{1/p} & \text{if } s < a \\
\mu^{1-1/p} p^{-1/p} \left( 2 - e^{-\mu p(s-a)} - e^{-\mu p(b-s)} \right)^{1/p} & \text{if } a < s < b, \\
\mu^{1-1/p} p^{-1/p} \left( e^{-\mu p(b-s)} - e^{-\mu p(a-s)} \right)^{1/p} & \text{if } s > b
\end{cases}
\]
and for every \( a \leq t \leq b \) we have
\[
\hat{\sigma}(t) = \sigma = \inf \{ \theta > 0 : \kappa(\theta) \leq \delta' \},
\]
23
Theorem 5.1 A symmetric infinitely divisible process given by (1.1) has a version with all sample paths in \( AC^p[a, b] \) if and only if there is a function \( h : [a, b] \times S \to R \), such that

\[
\lambda\{s \in S : h(t, s) \neq f(t, s)\} = 0 \text{ for every } t \in [a, b] \tag{5.1}
\]

and

\[
h(t, s) = h(a, s) + \int_a^t \hat{h}(u, s) \, du,
\]

where \( \hat{h} : [a, b] \times S \to R \) is a (product) measurable function such that \( \hat{h} \in L^p[a, b] \) for every \( s \in S \), and satisfying for some (equivalently, all) \( c > 0 \) and \( \delta' > 0 \),

\[
\int_S \rho \left( \frac{e}{||h(\cdot, s)||}, \infty \right) (ds) < \infty, \tag{5.3}
\]

\[
\int_a^b \sigma(t)^p \, dt < \infty, \tag{5.4}
\]

where

\[
\sigma(t) = \inf \left\{ \theta > 0 : n(\hat{h}(t, \cdot)/\theta) \leq \delta' \right\}
\]

and

\[
\int_a^b \left( \int_S \left( \frac{e}{||h(t, s)||} \int_{\sigma(t)}^{\infty} x^p \rho(x) \, dx \right) \lambda(ds) \right) \, dt < \infty. \tag{5.6}
\]

Proof: The idea is to use the necessary and sufficient conditions of part (i) of Theorem 3.1 with \( \psi(x) = x^p \). We follow the lines of Theorem 11.7.4 of Samorodnitsky and Taqqu [ST94]. Assume first that \( \{X(t), a \leq t \leq b\} \) has a version with all sample paths in \( AC^p[a, b] \). In particular, this version has continuous sample paths, and so there is a function \( h : [a, b] \times S \to R \), such that \( h(\cdot, s) \) is continuous for all \( s \in S \), and (5.1) holds. Then the process

\[
Z(t) = \sum_{j=1}^{\infty} \epsilon_j R(\Gamma_j, \tau_j) h(t, \tau_j) \tag{5.7}
\]

also has a version with all sample paths in \( AC^p[a, b] \). Replacing \( \epsilon_j \) by \(-\epsilon_j\) for each \( j \geq 2 \) in (5.7) and adding to (5.7), we conclude by Proposition 11.7.3 of Samorodnitsky and Taqqu [ST94] that for \( \lambda \)-almost every \( s \in S \),

\[
h(\cdot, s) \in AC^p[a, b]. \tag{5.8}
\]

Redefining \( h \) to be equal to 0 for every \( s \) in the exceptional null set in (5.8) and every \( t \), and noting that (5.1) still holds for the new \( h \), we may and will assume that (5.8) holds for every \( s \in S \).

Let \( \{Y(t), a \leq t \leq b\} \) be a version of \( \{X(t), t \in T\} \) with continuous sample paths. We use once again Proposition 11.7.3 of Samorodnitsky and Taqqu [ST94] to conclude that, on an event of full probability, the function \( \{Y(\cdot, \omega), a \leq t \leq b\} \) is in \( AC^p[a, b] \) and, moreover,

\[
Y(t, \omega) = Y(a, \omega) + \int_a^t D(u, \omega) \, du, \ a \leq t \leq b, \tag{5.9}
\]
Proposition 4.1 Let \( p > 0 \), and assume that there is a \( B > 0 \) such that either
\[
B \int_0^a x^p \rho(dx, s) \leq a^p \rho\left(\left(a, \infty\right), s\right)
\]  
for all \( s \in S \) and \( a > 0 \) or
\[
B \int_a^\infty x^p \rho(dx, s) \leq a^p \rho\left(\left(a, \infty\right), s\right)
\]  
for all \( s \in S \) and \( a > 0 \). Then the process \( X \) has almost all sample paths in \( L^p(T, m) \), \( p \geq 2 \) if and only if \( f(\cdot, s) \in L^p(T, m) \) for \( \lambda \)-almost every \( s \in S \), and for some (equivalently, all) \( c > 0 \) and \( \delta^t > 0 \), (3.1) and (3.2) hold.

Remark The assumptions of Proposition 4.1 hold, for example, in the \( \alpha \)-stable case, \( \alpha \neq p \).

Proof: We only need to show that (3.1) and (3.2) imply (3.4). Suppose first that (4.17) holds. Then we have
\[
\frac{1}{T} \left( \int \left( \int_{\sigma(t)}^{\frac{1}{\lambda(f(\cdot, s))}} x^p f(t, s) \rho(dx, s) \lambda(ds) \right) m(dt) \right)
\]
\[
\leq \int_S \left( \int_{\sigma(t)}^{\frac{1}{\lambda(f(\cdot, s))}} x^p \rho(dx, s) \lambda(ds) \right) m(dt) \leq B^{-1} \int_S \rho\left(\left(\frac{1}{\lambda(f(\cdot, s))}, \infty\right), s\right) \lambda(ds) < \infty
\]
by (3.1) with \( c = 1 \). This checks (3.4) with \( c = 1 \).

Suppose, on the other hand, that (4.18) holds. Then we have
\[
\frac{1}{T} \left( \int \left( \int_{\sigma(t)}^{\frac{1}{\lambda(f(\cdot, s))}} x^p f(t, s) \rho(dx, s) \lambda(ds) \right) m(dt) \right)
\]
\[
\leq B^{-1} \int_T \sigma(t)^p \int_S \rho\left(\left(\frac{\sigma(t)}{\lambda(f(\cdot, s))}, \infty\right), s\right) \lambda(ds) m(dt)
\]
\[
\leq B^{-1} \int_T \sigma(t)^p m(f(t, \cdot)/\sigma(t)) m(dt) \leq B^{-1} \delta^t \int_T \sigma(t)^p m(dt) < \infty
\]
by (3.2). This checks (3.4) with \( c = 1 \) in this case. ■

5 Absolutely continuous infinitely divisible processes

Suppose that \( T = [a, b] \) for some \(-\infty < a < b < \infty\). For a \( p \geq 1 \) we denote by \( AC^p[a, b] \) the space of all absolutely continuous functions \( \phi \) on \([a, b]\) for which the derivative \( \phi' \in L^p[a, b] \). In this section we give necessary and sufficient conditions for an infinitely divisible process \( X \) given by (1.1) to have a version with all sample paths in \( AC^p[a, b] \). The corresponding result in the symmetric \( \alpha \)-stable case was proven by Rosińński [Ros86]. See also Section 11.7 of Samorodnitsky and Taqqu [ST94], which also contains extra details on the argument that follows.
Therefore, for every $M > 1$ we have
\[
\int_T \left[ \int_S \left( f(t, s)^2 \int_{\mathbb{R}^d} x^2 \rho(dx, s) \right) \lambda(ds) \right]^{p/2} \, m(dt)
\]
\[
\leq M^p \int_T \left[ \int_S f(t, s)^2 \|f(\cdot, s)\|^{-2} \rho\left( \left( \frac{1}{\|f(\cdot, s)\|}, \infty \right), s \right) \lambda(ds) \right]^{p/2} \, m(dt)
\]
\[
\leq M^p a^{\frac{p}{2} - 1} \int_S \rho\left( \left( \frac{1}{\|f(\cdot, s)\|}, \infty \right), s \right) \lambda(ds) < \infty,
\]

implying by (4.8) that for every $M > 0$ we have
\[
\int_T \left[ \int_S \left( f(t, s)^2 \int_{\mathbb{R}^d} x^2 \rho(dx, s) \right) \lambda(ds) \right]^{p/2} \, m(dt) < \infty.
\] (4.15)

Now use (4.7) to choose an $M > 0$ big enough so that
\[
\int_S \rho\left( \left( \frac{M}{\|f(\cdot, s)\|}, \infty \right), s \right) \lambda(ds) \leq \frac{\delta'}{2}.
\]

We have then with an $M$ as above,
\[
\delta' = n\left( f(t, \cdot)/\sigma(t) \right) \leq \int_S \left( \int_0^{\|f(\cdot, s)\|} x^2 f(t, s)^2 \frac{\rho(dx, s)}{\sigma(t)^2} \lambda(ds) + \int_S \rho\left( \left( \frac{M}{\|f(\cdot, s)\|}, \infty \right), s \right) \lambda(ds) \right.
\]
\[
\leq \frac{1}{\sigma(t)^2} \int_S \left( f(t, s)^2 \int_{\mathbb{R}^d} x^2 \rho(dx, s) \right) \lambda(ds) + \frac{\delta'}{2}.
\]

Therefore,
\[
\sigma(t)^2 \leq \frac{2}{\delta'} \int_S \left( f(t, s)^2 \int_{\mathbb{R}^d} x^2 \rho(dx, s) \right) \lambda(ds).
\] (4.16)

Now (3.2) follows from (4.15) and (4.16). Finally,
\[
\int_S \left( \int_T \left( |f(t, s)|^p \int_{\frac{1}{\sigma(t)}}^M x^p \rho(dx, s) \right) m(dt) \right) \lambda(ds)
\]
\[
\leq \int_S \left( \int_T \left( |f(t, s)|^p \int_{\frac{1}{\sigma(t)}}^M x^p \rho(dx, s) \right) m(dt) \right) \lambda(ds)
\]
\[
= \int_S \left( \frac{1}{\|f(\cdot, s)\|^p} \int_{\frac{1}{\sigma(t)}}^M x^p \rho(dx, s) \right) \lambda(ds) < \infty
\]

by (4.6). This checks (3.4), and so completes the proof of the theorem. \[\end{proof}\]

If the left and the right tails of the pointwise Lévy measures $\rho$ in (1.1) are balanced in a particular way, only two of the conditions in part (i) of Theorem 3.1 is necessary.
Now observe that for every \( t \in T \),
\[
\delta' = n(f(t, \cdot)/\sigma(t)) \geq \int_S \left( \frac{1}{\|f(\cdot, s)\|} > \frac{\sigma(t)}{\|f(t, s)\|} \right) \int \frac{1}{\|f(\cdot, s)\|^p} x^2 \rho(d(x, s)) \lambda(ds)
\]
\[
= \int_S \left( \frac{1}{\|f(\cdot, s)\|} > \frac{\sigma(t)}{\|f(t, s)\|} \right) \rho \left( \int \frac{1}{\|f(\cdot, s)\|^p} x^2 \rho(d(x, s)) \lambda(ds) \right) \lambda(ds).
\]
Therefore, by Hölder’s inequality, for every \( t \in T \),
\[
\int_S \left( f(t, s)^2 \left( \frac{1}{\|f(\cdot, s)\|} > \frac{\sigma(t)}{\|f(t, s)\|} \right) \int \frac{1}{\|f(\cdot, s)\|^p} x^2 \rho(d(x, s)) \right) \lambda(ds)
\]
\[
\leq (\delta')^{1-2/p} \left[ \int S \left( \left( f(t, s)^p \left( \frac{1}{\|f(\cdot, s)\|} > \frac{\sigma(t)}{\|f(t, s)\|} \right) \int \frac{1}{\|f(\cdot, s)\|^p} x^2 \rho(d(x, s)) \right) \lambda(ds) \right]^{2/p}
\]
implies that
\[
\int_T \left[ \int S \left( f(t, s)^2 \left( \frac{1}{\|f(\cdot, s)\|} > \frac{\sigma(t)}{\|f(t, s)\|} \right) \int \frac{1}{\|f(\cdot, s)\|^p} x^2 \rho(d(x, s)) \right) \lambda(ds) \right]^{p/2} m(dt)
\]
\[
\leq (\delta')^{p/2-1} \int S \left( \left( f(t, s)^p \int \frac{1}{\|f(\cdot, s)\|^p} x^2 \rho(d(x, s)) \right) \lambda(ds) \right) m(dt) < \infty
\]
by (3.4). Additionally, observe that the same argument as in (4.10) shows that for every \( t \in T \),
\[
\sigma(t)^2 \geq \frac{1}{\delta'} \int_S \left( f(t, s)^2 \int_0^{\|f(\cdot, s)\|^p} x^2 \rho(d(x, s)) \right) \lambda(ds),
\]
which implies that
\[
\int_T \left[ \int S \left( f(t, s)^2 \int_0^{\|f(\cdot, s)\|^p} x^2 \rho(d(x, s)) \right) \lambda(ds) \right]^{p/2} m(dt)
\]
\[
\leq \delta' \int_T \sigma(t)^p m(dt) < \infty
\]
by (3.2). Now (4.8) follows from (4.11), (4.12) and (4.14).

Suppose now that (4.6), (4.7) and (4.8) hold. Then (3.1) with \( c = 1 \) is, once again, just (4.7).

Observe further that if
\[
a = \int_S \rho \left( \left( \frac{1}{\|f(\cdot, s)\|}, \infty \right), s \right) \lambda(ds),
\]
then by Hölder’s inequality we have for every \( t \in T \),
\[
\left[ \int_S f(t, s)^2 \|f(\cdot, s)\|^{-2} \rho \left( \left( \frac{1}{\|f(\cdot, s)\|}, \infty \right), s \right) \lambda(ds) \right]^{p'/2}
\]
\[
\leq a^{p-1} \int_S |f(t, s)|^p \|f(\cdot, s)\|^{-p} \rho \left( \left( \frac{1}{\|f(\cdot, s)\|}, \infty \right), s \right) \lambda(ds).
\]
Suppose first that (3.1), (3.2) and (3.4) hold. The conditions (4.4), (4.5) and (4.6) are obviously independent of $\gamma$, and so we may assume that $\gamma = 1$. Clearly, (4.7) is just (3.1) (with $c = 1$). Furthermore,

$$
\int_S \left( \|f(\cdot, s)\|^p \int_0^{\|\cdot\|_{\ast \gamma}} x^p \rho(d_x, s) \right) \lambda(ds)
$$

$$
\leq \int_S \left( \int_T \left( |f(t, s)|^p \int_0^{\|\|f(\cdot, s)\|_{\ast \gamma}} x^p \rho(d_x, s) \right) m(dt) \right) \lambda(ds)
$$

$$
+ \int_S \left( \int_T \left( |f(t, s)|^p \int_0^{\|\|f(\cdot, s)\|_{\ast \gamma}} x^p \rho(d_x, s) \right) m(dt) \right) \lambda(ds).
$$

Taking into account (3.4) with $c = 1$ we see that (4.6) will follow once we establish that

$$
\int_S \left( \int_T \left( |f(t, s)|^p \int_0^{\|\|f(\cdot, s)\|_{\ast \gamma}} x^p \rho(d_x, s) \right) m(dt) \right) \lambda(ds) < \infty.
$$

Observe that, since $p \geq 2$,

$$
n(f(t, \cdot) / \theta) \geq \int_S \left( \int_0^\infty \left( \frac{|f(t, s)|}{\theta} \wedge 1 \right)^p \rho(dx, s) \right) \lambda(ds)
$$

$$
\geq \int_S \left( \frac{|f(t, s)|^p}{\theta^p} \int_0^{\|\|f(\cdot, s)\|_{\ast \gamma}} x^p \rho(dx, s) \right) \lambda(ds).
$$

Therefore, for every $t \in T$,

$$
\delta' = n(f(t, \cdot) / \sigma(t)) \geq \int_S \left( \frac{|f(t, s)|^p}{\sigma(t)^p} \int_0^{\|\|f(\cdot, s)\|_{\ast \gamma}} x^p \rho(dx, s) \right) \lambda(ds),
$$

so that

$$
\sigma(t)^p \geq \frac{1}{\delta'} \int_S \left( \frac{|f(t, s)|^p}{\sigma(t)^p} \int_0^{\|\|f(\cdot, s)\|_{\ast \gamma}} x^p \rho(dx, s) \right) \lambda(ds),
$$

and we have by (3.2)

$$
\infty > \int_T \sigma(t)^p m(dt) \geq \frac{1}{\delta'} \int_S \left( \int_T \left( |f(t, s)|^p \int_0^{\|\|f(\cdot, s)\|_{\ast \gamma}} x^p \rho(dx, s) \right) m(dt) \right) \lambda(ds),
$$

thus proving (4.9). To prove (4.8) write

$$
\int_T \left[ \int_S \left( f(t, s)^2 \int_0^{\|\|f(\cdot, s)\|_{\ast \gamma}} x^2 \rho(dx, s) \right) \lambda(ds) \right]^{p/2} m(dt)
$$

$$
\leq 2^{p/2} \int_T \left[ \int_S \left( f(t, s)^2 \int_0^{\|\|f(\cdot, s)\|_{\ast \gamma}} x^2 \rho(dx, s) \right) \lambda(ds) \right]^{p/2} m(dt)
$$

$$
+ 2^{p/2} \int_T \left[ \int_S \left( f(t, s)^2 \left( \frac{1}{\|f(\cdot, s)\|} \right) \int_0^{\|\|f(\cdot, s)\|_{\ast \gamma}} x^2 \rho(dx, s) \right) \lambda(ds) \right]^{p/2} m(dt).
$$

18
4 \( L^p \) spaces

A result of Yurinski [Yur74] describes the Lévy measures on an \( L^p \) space with \( p \geq 2 \) as follows. A \( \sigma \)-finite measure \( \mu \) on \( L^p \) with \( \mu(\{0\}) = 0 \) is a Lévy measure if and only if

\[
\int_{L^p} \min(1, ||x||^p) \mu(dx) < \infty \tag{4.1}
\]

and

\[
\sum_{j=1}^{\infty} \left[ \int_{||x|| \leq 1} x_j^2 \mu(dx) \right]^{p/2} < \infty. \tag{4.2}
\]

We generalize this result to the following framework. Let \((T, T, m)\) be an arbitrary \( \sigma \)-finite measure space. Under what conditions does a measurable symmetric infinitely divisible process \( X \) given by (1.1) belong to \( L^p = L^p(T, m) \) with \( p \geq 2 \)? Observe that the Lévy measure of \( X \) (on \( R^T \)) is given by

\[
\mu = F \circ V^{-1}, \tag{4.3}
\]

where \( F \) is a measure on \( S \times R \) given by \( F(ds, dx) = \rho(dx, s) \lambda(ds) \), and \( V : S \times R \to R^T \) is given by \( V(s, x) = \{ xf(t, s), t \in T \} \). If \( f(\cdot, s) \in L^p \) for \( \lambda \)-almost every \( s \in S \), then \( \mu \) in (4.3) is a \( \sigma \)-finite measure on \( L^p \), and we will show that the necessary and sufficient conditions for \( X \) to have almost all sample paths in \( L^p \) are the direct generalization of Yurinski’s conditions (4.1) and (4.2).

**Theorem 4.1** The process \( X \) has almost all sample paths in \( L^p(T, m), p \geq 2 \) if and only if \( f(\cdot, s) \in L^p(T, m) \) for \( \lambda \)-almost every \( s \in S \), and for \( \mu \) given by (4.3) we have

\[
\int_{L^p} \min(1, ||x||^p) \mu(dx) < \infty \tag{4.4}
\]

and

\[
\int_{T} \left[ \int_{||x|| \leq 1} x(t)^2 \mu(dx) \right]^{p/2} m(dt) < \infty. \tag{4.5}
\]

**Proof:** Obviously, (4.4) and (4.5) are equivalent to the following conditions:

\[
\int_{S} \left( ||f(\cdot, s)|| ||f(\cdot, s)|| \int_{0}^{\gamma ||f(\cdot, s)||/p} x^p \rho(dx, s) \right) \lambda(ds) < \infty, \tag{4.6}
\]

\[
\int_{S} \rho \left( \left( \frac{1}{\gamma ||f(\cdot, s)||}, \infty \right), s \right) \lambda(ds) < \infty \tag{4.7}
\]

and

\[
\int_{T} \left[ \int_{S} \left( f(t, s)^2 \int_{0}^{\gamma ||f(\cdot, s)||/p} x^2 \rho(dx, s) \right) \lambda(ds) \right]^{p/2} m(dt) < \infty. \tag{4.8}
\]

Consequently, the only thing we need to do is to check that (3.1), (3.2) and (3.4) are equivalent, when \( \psi(x) = x^p \) with \( p \geq 2 \) to (4.6), (4.7) and (4.8). Note that in the latter 3 conditions the norm \( || \cdot || \) is the one given by (1.7) (it differs by a constant factor from the usual \( L^p \) norm).
Therefore, the condition (3.1) (with $c = 1$) becomes

$$
\infty > \int_0^1 \rho \left( \left( \psi^{-}(\frac{\gamma}{m((s,1])}), \infty \right) \right) ds
$$

$$
= \int_0^\infty \text{Leb} \left\{ 0 < s < 1 : \psi^{-}(\frac{\gamma}{m((s,1])}) < z \right\} \rho(dz)
$$

$$
= \int_0^\infty \text{Leb} \left\{ 0 < s < 1 : \psi(z) > \frac{\gamma}{m((s,1])} \right\} \rho(dz)
$$

$$
= \int_0^\infty \text{Leb} \left\{ 0 < s < 1 : s < Q\left( \frac{\gamma}{\psi(z)} \right) \right\} \rho(dz) = \int_0^\infty Q\left( \frac{\gamma}{\psi(z)} \right) \rho(dz).
$$

Therefore, in this case (3.1) is equivalent to (3.29). Furthermore, it is elementary to see that in our case

$$
\sigma(t) = G\left( \frac{\delta^l}{t} \right).
$$

Therefore, (3.2) is equivalent, in the present case, to (3.30). Finally, we consider the condition (3.4) (once again, with $c = 1$). We have

$$
= \int_0^1 \left( \int_0^1 \left( \int_{\sigma_{t/\pi}(m(\delta^l/n))}^{\sigma_{t/\pi}(m(\delta^l/n))} \psi(z) \rho(dz) \right) ds \right) m(dt)
$$

$$
= \int_0^\infty \left( \int_0^1 \left( \int_{G(\delta^l/t)}^{G(\delta^l/t)} \psi(z) \rho(dz) \right) ds \right) m(dt)
$$

$$
= \int_0^\infty \psi(z) \left( \int_0^1 \left( \int_0^1 1(s < t, G(\delta^l/t) \leq z, \psi^{-}(\gamma/m((s,1])) \geq z) ds \right) m(dt) \right) \rho(dz)
$$

$$
= \int_0^\infty \psi(z) \left( \int_{Q(\gamma/\psi(z))}^{(\delta^l/n(z)) \wedge 1} \left( \int_0^1 1(s < t, s \geq Q(\gamma/\psi(z)) ds \right) m(dt) \right) \rho(dz)
$$

$$
= \int_0^\infty \psi(z) \left( \int_{Q(\gamma/\psi(z))}^{(\delta^l/n(z)) \wedge 1} \left( t - Q(\gamma/\psi(z)) \right) m(dt) \right) \rho(dz),
$$

and so in this case (3.4) is equivalent to (3.31).

Some other cases where the necessary and sufficient conditions of Theorem 3.1 simplify are presented in the next section.
Conditions in part (i) of Theorem 3.1 simplify in particular cases. As an example, we consider the Lévy motions.

**Example 3.1 Lévy motions** These are the simplest infinitely divisible processes—those with stationary and independent increments. Let \( \{X(t), 0 \leq t \leq 1\} \) be a symmetric Lévy process with Lévy measure \( \mu \). That is,

\[
E e^{i \theta X(t)} = \exp \left( -t \int_0^\infty (1 - \cos \theta x) \, \mu(dx) \right), \quad 0 \leq t \leq 1.
\]

Let \( m \) be a \( \sigma \)-finite Borel measure on \([0,1]\). Define

\[
Q(u) = \sup \{ s > 0 : m((s,1]) > u \}, \quad u > 0,
\]

\[
n(\theta) = \int_0^\infty \left( \frac{x^2}{2} \wedge 1 \right) \, \mu(dx), \quad \theta > 0,
\]

and

\[
G(u) = \inf \{ \theta > 0 : n(\theta) \leq u \}, \quad u > 0.
\]

Let \( \psi \) be an \( M \)-function satisfying the \( \Delta_2 \) condition. Then \( \{X(t), 0 \leq t \leq 1\} \) has almost all sample paths in \( L_\psi \) if and only if \( m((\epsilon,1]) < \infty \) for all \( \epsilon > 0 \), and for some (equivalently, all) \( \delta' > 0 \), we have

\[
\int_0^\infty Q\left( \frac{\gamma}{\psi(z)} \right) \, \mu(dz) < \infty,
\]

\[
\int_0^1 \psi\left( G(\delta'/t) \right) m(dt) < \infty
\]

and

\[
\int_0^\infty \psi(z) \left( \int_{Q(\gamma/\psi(z))}^{(\delta'/m(z)) \wedge 1} (t - Q(\gamma/\psi(z))) \, m(dt) \right) \, \mu(dz) < \infty,
\]

where the inner integral in (3.31) is taken to be equal to 0 if its lower limit exceeds its upper limit. Note that we are, in fact, studying only the behavior of the Lévy motion at the origin, for the process has sample paths that are bounded on compact intervals.

To show the necessity and sufficiency of conditions (3.29), (3.30) and (3.31), observe that the Lévy process \( \{X(t), t \geq 0\} \) can be represented in the form (1.1) with \( S = [0,1] \), \( \lambda \) the Lebesgue measure on \([0,1]\), \( \rho(s) = \rho(\cdot) \) for all \( s \in [0,1] \), and \( f(t,s) = 1(s < t) \), \( s,t \in [0,1] \). We, therefore, only need to verify that the above conditions are equivalent to those of part (ii) of Theorem 3.1. We start with observing that to say that \( f(\cdot,s) \in L_\psi(T,m) \) for \( \lambda \)-almost every \( s \in S \) is, in this case, equivalent to saying that \( m((\epsilon,1]) < \infty \) for all \( \epsilon > 0 \). Furthermore, let \( \psi^- (w) = \sup \{ r > 0 : \psi(r) \leq w \} \). Then for every \( s \in [0,1] \),

\[
||f(\cdot,s)||_\psi = \inf \{ \theta > 0 : \int_0^\infty \psi\left( \frac{1(t > s)}{\theta} \right) m(dt) \leq \gamma \}
\]

\[
= \inf \{ \theta > 0 : \psi\left( \frac{1}{\theta} \right) m((s,1]) \leq \gamma \} = \frac{1}{\psi^- (m((s,1]))}.
\]
where \( \{X^i(t), t \in T\}, i = 1, \ldots, K \) are i.i.d. copies of \( \{X_1(t), t \in T\}. \) Since \( L_{\psi} \) is a linear space, and \( P(\{X_1(t), t \in T\} \in L_{\psi}) = 1, \) we conclude that \( P(\{X(t), t \in T\} \in L_{\psi}) = 1. \) This completes the proof of part (i) of the theorem in all cases.

(ii) This part follows from part (iii) of Theorem 2.1 in the same way as the sufficiency part of part (i) of this theorem followed from part (ii) of Theorem 2.1.

(iii) Suppose that almost all sample paths of \( X \) are in \( L_{\psi}. \) In the notation of the necessity part in (i) above, it follows that for \( P_1 \)-almost every \( \omega_1 \in \Omega_1, \) the series \( (2.4) \) converges \( P_2 \)-a.s. in \( L_{\psi}. \) We now use part (iv) of Theorem 2.1 to conclude that for the same fixed \( \omega_1 \in \Omega_1, \) the series

\[
\sum_{j=1}^{\infty} e_j R(\Gamma_j, \tau_j) g(\cdot, \tau_j)
\]

converges \( P_2 \)-a.s. in \( L_{\psi}. \) By Fubini’s theorem this series converges a.s. in \( L_{\psi}, \) and so almost all sample paths of \( Y \) are in \( L_{\psi}. \)

(iv) Define a symmetric infinitely divisible process by

\[
W(t) = \int_S f(t, s) M(t) ds, \; t \in T,
\]

where \( M_2 \) is a symmetric infinitely divisible random measure with the same control measure \( \lambda \) as \( M \) in (1.1), and pointwise \( \text{Lévy measures} \{B \rho(\cdot, s), s \in S\}. \) The process \( \{W(t), t \in T\} \) is the \( B \)th convolution power of \( \{X(t), t \in T\}, \) and since \( L_{\psi} \) is a linear space, we conclude that almost all sample paths of \( \{W(t), t \in T\} \) are in \( L_{\psi}. \) Now compare the series expansions corresponding to \( \{Z(t), t \in T\} \) and \( \{W(t), t \in T\} \) in the same way as in the proof of part (iii), and use part (iv) of Theorem 2.1.

Parts (iii) and (iv) of Theorem 3.1 can be regarded as comparison principles, and more of those can be thought of. In part (iv), if the pointwise \( \text{Lévy measures are, actually, independent of s (i.e. if } \rho(\cdot, s) = \rho(\cdot) \text{ and } \rho_1(\cdot, s) = \rho_1(\cdot) \text{ for some fixed \( \text{Lévy measures } \rho \text{ and } \rho_1 \text{), then we can reformulate (3.9) as follows. For some } B > 0,

\[
\rho_1((x, \infty)) \leq B \rho((x/B, \infty))
\]

for all \( 0 < x \leq x_0. \) As an example, consider the symmetric stable case. That is, take \( \rho(dx) = a_2^{-1+\alpha_2} dx, \) and \( \rho_1(dx) = a_1^{-1+\alpha_1} dx, \) with \( 0 < \alpha_1 < \alpha < 2, \) and \( a, a_1 > 0. \) Then (3.25) holds trivially, and we immediately obtain the following corollary. In its formulation we understand the expression control measure in the way that is conventional in the literature on stable processes (and which is different from the one used in the present paper). See Samorodnitsky and Taqqu [ST94] for details.

**Corollary 3.1** Let \( 0 < \alpha_2 < \alpha_1 < 2 \) and

\[
X_i(t) = \int_S f(t, s) M_i(ds), \; t \in T,
\]

\( i = 1, 2, \) where \( M_i \) is a symmetric \( \alpha_i\)-stable random measure, with a finite control measure \( \mu, \) and \( f(t, s), s \in S, t \in T \) is a product measurable function such that for every \( t \in T, \ f(t, \cdot) \in L^{\alpha_i}(\mu) \) for both \( i = 1 \) and \( i = 2. \) Let \( \psi \) be an \( M \)-function satisfying the \( \Delta_2 \) condition. If almost all sample paths of \( \{X_1(t), t \in T\} \) are in \( L_{\psi}, \) then so are almost all sample paths of \( \{X_2(t), t \in T\}. \)
We divide the interval \((0, \infty)\) into two parts: \((0, h(s))\) and \([h(s), \infty)\), where

\[
h(s) = \rho \left( \frac{c}{\|f(\cdot, s)\|_{\psi}}, \infty, s \right). \tag{3.22}
\]

Hence, the innermost integral in (3.21) is the sum of the integrals over these intervals. Therefore, we obtain the right hand side of (3.21) as a sum of two integrals: \(\frac{1}{2}I_1 + \frac{1}{2}I_2\).

By (3.16) and (3.17) we have

\[
I_2 = \int_T m(dt) \int_S \lambda(ds) \int_{\frac{\sigma(t)}{\|f(\cdot, s)\|_{\psi}}}^{\frac{\xi f(t, s)}{c}} \psi(x) \rho(dx, s) < \infty
\]

by (3.4) and the \(\Delta_2\) condition.

Finally,

\[
I_1 = \int_T m(dt) \int_S \lambda(ds) \int_0^{\frac{\xi f(t, s)}{c}} \psi \left( \frac{f(t, s)}{\|f(\cdot, s)\|_{\psi}} \right) \mathbf{1} \left( \frac{|f(t, s)|}{\|f(\cdot, s)\|_{\psi}} > \sigma(t) \right) dx
\]

\[
\leq \int_S \lambda(ds) \rho \left( \frac{c}{\|f(\cdot, s)\|_{\psi}}, \infty, s \right) \int_T \psi \left( \frac{|f(t, s)|}{\|f(\cdot, s)\|_{\psi}} \right) m(dt)
\]

\[
\leq \gamma \int_S \rho \left( \frac{c}{\|f(\cdot, s)\|_{\psi}}, \infty, s \right) \lambda(ds) < \infty
\]

by (3.1). This checks all the conditions of part (ii) of Theorem 2.1, and so proves that the series (3.19) converges a.s. in \(L_\psi\). Therefore, we have proved the sufficiency part of the theorem in the case when \(\delta' \leq 1/24C^2\).

Let now \(\delta'\) be general. Let \(K\) be a positive integer big enough so that \(\delta'' = \delta'/K \leq 1/24C^2\). Consider a symmetric infinitely divisible stochastic process defined by

\[
X(t) = \int_S f(t, s) M_1(ds), \quad t \in T, \tag{3.23}
\]

with \(M_1\) being a symmetric infinitely divisible random measure with the same control measure \(\lambda\) as \(M\) in (1.1), and pointwise Lévy measures \(\rho(\cdot, s) = \rho(\cdot, s)/K, s \in S\). If \(n_1\) is the function in (1.3) applied to \(\{X_1(t), t \in T\}\), we immediately see that \(n_1(t) = n(t)/K, t \in T\). It is obvious that the process \(\{X_1(t), t \in T\}\) satisfies conditions (3.1), (3.2) and (3.4) with \(\delta''\) instead of \(\delta'\). Since \(\delta'' \leq 1/24C^2\), we conclude by the already taken first step that with probability 1 the sample path of \(\{X_1(t), t \in T\}\) is in \(L_\psi\). However,

\[
\{X(t), t \in T\} \overset{d}{=} \{X_1^1(t) + \ldots + X_1^K(t), t \in T\},
\]

13
it remains to prove that the difference of the expressions in the left hand sides of (3.4) and (3.18) is finite. However, this difference does not exceed

\[
\frac{1}{T} \left( \int_{[T/2,T]} \left( \int_{[T/2,T]} \psi(\xi f(t,s)) \rho(d\xi, s) \lambda(ds) \right) m(dt) \right)
\]

\[
\leq \left( \int_{T} \left( \int_{[T/2,T]} \psi(\sigma_1(t)) \rho(d\xi, s) \lambda(ds) \right) m(dt) \right)
\]

\[
\leq \delta' (1 + \frac{1}{c^2}) \int_{T} \psi(\sigma_1(t)) m(dt) < \infty
\]

by the already proven (3.2). This completes the proof of (3.4) in all cases, and so finishes the proof of the necessity part of the theorem.

* Sufficiency. As the first step, we will prove the sufficiency part of the theorem under the assumption that \( \delta' \leq 1/24C^2 \).

In the same way as above, we use part (iv) of Theorem 2.1 to conclude that the series (2.4) converges a.s. in \( L_\psi \) if the series

\[
\sum_{i=1}^{\infty} \epsilon_i R \left( \frac{1}{2} \Gamma_i, \tau_i \right) f(\cdot, \tau_i)
\]

converges a.s. in \( L_\psi \). Therefore, it is enough to prove that (3.1), (3.2) and (3.4) imply a.s. convergence of the series (3.19). We apply part (ii) of Theorem 2.1 with

\[
X_i = \epsilon_i R \left( \frac{1}{2} \Gamma_i, \tau_i \right) f(\cdot, \tau_i), \quad i = 1, 2, \ldots
\]

Moreover, we use \( \delta = \delta'/2 \).

Repeating the arguments in (3.12), we conclude that

\[
\sum_{i=1}^{\infty} P \left( ||X_i||_\psi > c \right) = 2 \int_{S} \rho \left( \frac{c}{||f(\cdot, s)||_\psi}, \infty \right) \lambda(ds) < \infty
\]

by (3.1). Therefore, (2.7) holds.

Similarly to what we have checked above, in Theorem 2.1 we have \( \beta(t) = \sigma(t) \). Therefore, (2.8) follows from (3.2).

It remains to prove (2.9). We have, as in (3.16),

\[
\int_{T} m(dt) \sum_{i=1}^{\infty} E \left( \psi([X_i(t)]) 1 \left( ||X_i(t)|| > \beta(t) \right) \right)
\]

\[
= \frac{1}{2} \int_{T} m(dt) \int_{S} \lambda(ds) \int_{0}^{\infty} \psi \left( \left( R(x, s)f(\cdot, s) \right)(t) \right) 1 \left( \left| [R(x, s)f(\cdot, s)](t) \right| > \beta(t) \right) dx.
\]
Suppose first that $\delta' < 1/96C^2$, with $C$ being the constant from the $\Delta_2$ condition (1.4). Let $\delta = 2\delta'$. Then $\delta$ satisfies (2.5). If $\beta(t)$ is defined by (2.6), then we have $\beta(t) = \sigma(t)$, and so (3.2) follows from (2.9). Since the condition (3.2) does not become stricter as $\delta'$ increases, this proves (3.2) for all $\delta' > 0$.

Finally, we prove (3.4). We start, once again, with the case $\delta' < 1/96C^2$. Since, with $\delta = 2\delta'$, we have $\beta(t) = \sigma(t)$, we conclude that

$$
\infty > \frac{1}{T} \int m(dt) \sum_{i=1}^{\infty} E \left( \psi([X_i(t)] \mathbf{1}(|[X_i(t)] > \beta(t)) \right)
$$

$$
= 2 \frac{1}{T} \int m(dt) \int_S \lambda(ds) \int_0^\infty \psi \left( |R(x, s) f(\cdot, s)| \right) \mathbf{1} \left( |R(x, s) f(\cdot, s)| > \sigma(t) \right)
$$

$$
\geq 2 \frac{1}{T} \int m(dt) \int_S \lambda(ds) \int_0^\infty \psi \left( \frac{R(x, s) f(t, s)}{c} \right) \mathbf{1} \left( \frac{|R(x, s) f(t, s)|}{c} > \sigma(t) \right) dx.
$$

(3.16)

We have for a fixed $t \in T$ and $s \in S$ with $\xi = R(x, s)$

$$
\int_0^\infty \psi \left( \frac{R(x, s) f(t, s)}{c} \right) \mathbf{1} \left( \frac{|R(x, s) f(t, s)|}{c} > \sigma(t) \right) dx
$$

$$
\rho \left( \frac{\|R(x, s)\|}{\|f(\cdot, s)\|}, \sigma(t) \right)
$$

$$
= \int_0^\infty \psi \left( \frac{\xi f(t, s)}{c} \right) \mathbf{1} \left( \frac{|\xi f(t, s)|}{c} > \sigma(t) \right) \rho(d\xi, s)
$$

(3.17)

where the last integral is set to be equal to 0 if

$$
\frac{\sigma(t)}{|f(t, s)|} > \frac{1}{\|f(\cdot, s)\|}.
$$

Substituting (3.17) into (3.16) and using the $\Delta_2$ condition, we obtain (3.4).

Suppose now that $\delta' \geq 1/96C^2$, and take an arbitrary positive $\delta'' < 1/96C^2$. Let $\sigma_1(t)$ be defined as in (3.3) but with $\delta''$ replacing $\delta'$. Since we have already proved that

$$
\int_T \left( \int_S \left( \int_{\sigma_1(t)}^{\infty} \psi \left( \frac{\xi f(t, s)}{c} \right) \rho(d\xi, s) \right) \lambda(ds) \right) m(dt) < \infty,
$$

(3.18)
\[
\frac{1}{2} \int_S \left( \rho \left( \left( \frac{x}{\|f\|}, \infty \right), s \right) \int_0^{\infty} \frac{e^{-x} x^{i-1}}{(i-1)!} \, dx \right) \lambda(ds)
\]
\[
= \frac{1}{2} \int_S \rho \left( \left( \frac{c}{\|f\|}, \infty \right), s \right) \lambda(ds).
\]
This proves (3.1).

Furthermore, for every \( t \in T \) we have, as above,
\[
\sum_{i=1}^{\infty} E \left[ (\theta^{-1} X_i(t))^2 \wedge 1 \right]
\]
\[
= \sum_{i=1}^{\infty} E \left[ (\theta^{-2} R(2 \tau_i, \tau_i)^2 f(t, \tau_i)^2) \wedge 1 \right]
\]
\[
= 2 \int_0^\infty \int_S \left[ (\theta^{-2} R(x, s)^2 f(t, s)^2) \wedge 1 \right] dx \lambda(ds)
\]
\[
= 2 \int_S \frac{f(t, s)^2}{\theta^2} \int_0^{\infty} R(x, s)^2 dx + \rho \left( \left( \frac{\theta}{\|f\|}, \infty \right), s \right) \lambda(ds)
\]
(3.13)

Now, for every \( a > 0 \) we have
\[
\int_{\rho((a, \infty), s)}^\infty R(x, s)^2 dx = \int_0^\infty \text{Leb} \left\{ x : x \geq \rho((a, \infty), s), R(x, s) > u^{1/2} \right\} du
\]
\[
= \int_0^\infty \text{Leb} \left\{ x : x \geq \rho((a, \infty), s), \rho((u^{1/2}, \infty), s) > x \right\} du
\]
\[
= \int_0^{a^2} \rho ((u^{1/2}, a), s) du = \int_0^a x^2 \rho(dx, s) .
\]
(3.14)

Therefore,
\[
\sum_{i=1}^{\infty} E \left[ (\theta^{-1} X_i(t))^2 \wedge 1 \right]
\]
\[
= 2 \int_S \frac{f(t, s)^2}{\theta^2} \int_0^{\|f(t, s)\|} x^2 \rho(dx, s) + \rho \left( \left( \frac{\theta}{\|f(t, s)\|}, \infty \right), s \right) \lambda(ds)
\]
\[
= 2\pi(f(t, \cdot) / \theta).
\]
(3.15)
where $M_1$ a symmetric infinitely divisible random measure with the same control measure $\lambda$ as $M$ in (1.1), and pointwise Lévy measures $\{\rho_t(\cdot, s), s \in S\}$ such that the corresponding function $R_1(u, s)$ satisfies the following. For some $B > 0$, all $u \geq u_0 > 0$ and all $s \in S$,

$$R_1(u, s) \leq BR(u/B, s). \quad (3.9)$$

If almost all sample paths of $X$ are in $L_\psi$, then so are almost all sample paths of $Z$.

**Proof:** (i) As we know from the discussion in the previous section, the requirement that $f(\cdot, s) \in L_\psi(T, m)$ for $\lambda$-almost every $s \in S$ is necessary if almost all sample paths of $X$ are to be in the Orlicz space, and when this is the case, the theorem can be reformulated as saying that (3.1), (3.2) and (3.4) are necessary and sufficient for (2.4).

**Necessity.** We start with the following statement: if the series (2.4) converges a.s. in $L_\psi$, then the series

$$\sum_{i=1}^{\infty} \epsilon_i R(\bar{\Gamma}_i, \tau_i) f(\cdot, \tau_i) \quad (3.10)$$

converges a.s. in $L_\psi$, where $(\bar{\Gamma}_i, i \geq 1)$ is a sequence independent of $\epsilon_1, \epsilon_2, \ldots$ and $\tau_1, \tau_2, \ldots$ and such that $\bar{\Gamma}_1, \bar{\Gamma}_2, \ldots$ are independent and for every $i \geq 1$, $\bar{\Gamma}_i \equiv \Gamma_i$.

Indeed, we may assume that $(\Gamma_i, i \geq 1)$ and $(\bar{\Gamma}_i, i \geq 1)$ are independent and live on a probability space $(\Omega_1, \mathcal{F}_1, P_1)$, while $(\epsilon_i, i \geq 1)$ and $(\tau_i, i \geq 1)$ live on a probability space $(\Omega_2, \mathcal{F}_2, P_2)$. Let

$$\Omega_+^{(1)} = \{ \omega_1 \in \Omega_1 : \text{the series} \sum_{i=1}^{\infty} \epsilon_i R(\bar{\Gamma}_i, \tau_i) f(\cdot, \tau_i) \text{ converges} P_2 - \text{a.s. in} L_\psi, \text{ and} \quad \Gamma_i < \frac{3}{2} i \text{ eventually, and} \quad \bar{\Gamma}_i > \frac{3}{4} i \text{ eventually} \}.$$

Then $P_1(\Omega_+^{(1)}) = 1$, and for every $\omega_1 \in \Omega_+^{(1)}$, the series (2.4) converges $P_2$-a.s. in $L_\psi$.

We now apply part (iv) of Theorem 2.1 to conclude that the series (3.10) converges $P_2$-a.s in $L_\psi$ for every $\omega_1 \in \Omega_+^{(1)}$, and by Fubini’s theorem, the series (3.10) converges a.s.

Now, the series (3.10) is a series of independent symmetric $L_\psi$-valued random variables, and so we may apply part (ii) of Theorem 2.1 with

$$X_i = \epsilon_i R(2\bar{\Gamma}_i, \tau_i) f(\cdot, \tau_i), \ i \geq 1. \quad (3.11)$$

Since (2.7) must hold, we have

$$\infty > \sum_{i=1}^{\infty} P \left( ||X_i||_{\psi} > c \right) = \sum_{i=1}^{\infty} P \left( R(2\bar{\Gamma}_i, \tau_i) > \frac{c}{||f(\cdot, \tau_i)||_{\psi}} \right)$$

$$= \sum_{i=1}^{\infty} P \left( \bar{\Gamma}_i < \frac{1}{2} \rho \left( \frac{c}{||f(\cdot, \tau_i)||_{\psi}}, \infty \right), \tau_i \right)$$

$$= \int \left( \sum_{i=1}^{\infty} P \left( \bar{\Gamma}_i < \frac{1}{2} \rho \left( \frac{c}{||f(\cdot, \psi)||_{\psi}}, \infty \right), s \right) \right) \lambda(ds) \quad (3.12)$$

9
**Theorem 3.1** (i) Let \( \{X(t), t \in T\} \) be a measurable symmetric infinitely divisible process given by (1.1). Then the process has a version with all sample paths in \( L_\psi(T, m) \) if and only if \( f(\cdot, s) \in L_\psi(T, m) \) for \( \lambda \)-almost every \( s \in S \), and the following conditions hold: for some (equivalently, all) \( c > 0 \) and \( \delta' > 0 \),

\[
\int_S \rho\left(\frac{c}{\|f(\cdot, s)\|_\psi}, \infty\right) \lambda(ds) < \infty, \tag{3.1}
\]

\[
\int_T \psi(\sigma(t)) m(dt) < \infty, \tag{3.2}
\]

where

\[
\sigma(t) = \inf \{\theta > 0 : n(f(t, \cdot) | \theta) \leq \delta'\} \tag{3.3}
\]

(see (1.3)) and

\[
\int_T \left( \int_S \left( \int_{\sigma(t)}^{\sigma(t) + \delta'} \psi(\xi f(t, s)) \rho(\xi ds) \right) \lambda(ds) \right) m(dt) < \infty, \tag{3.4}
\]

where the inner integral in (3.4) is set to be zero, if its lower limit of integration exceeds the upper limit.

(ii) Let \( \bar{\sigma}(t), t \in T \) be a measurable function satisfying \( \sigma(t) \leq \bar{\sigma}(t), t \in T \). If \( f(\cdot, s) \in L_\psi(T, m) \) for \( \lambda \)-almost every \( s \in S \), (3.1) holds,

\[
\int_T \psi(\bar{\sigma}(t)) m(dt) < \infty, \tag{3.5}
\]

and

\[
\int_T \left( \int_S \left( \int_{\sigma(t)}^{\sigma(t) + \delta'} \psi(\xi f(t, s)) \rho(\xi ds) \right) \lambda(ds) \right) m(dt) < \infty, \tag{3.6}
\]

then almost all sample paths of \( X \) are in \( L_\psi \).

(iii) Let \( \mathbf{Y} = \{Y(t), t \in T\} \) be a symmetric measurable infinitely divisible process given in the form

\[
Y(t) = \int_S g(t, s) M(ds), t \in T, \tag{3.7}
\]

where \( g(t, s), t \in T, s \in S \) is a product measurable function such that for some \( B > 0 \),

\[
g(t, s) \leq B f(t, s), t \in T, s \in S.
\]

If almost all sample paths of \( X \) are in \( L_\psi \), then so are almost all sample paths of \( \mathbf{Y} \).

(iv) Let \( \mathbf{Z} = \{Z(t), t \in T\} \) be a symmetric measurable infinitely divisible process given in the form

\[
Z(t) = \int_S f(t, s) M_1(ds), t \in T, \tag{3.8}
\]
Now, by the contraction principle we have
\[
P \left( \psi \left( \sum_{i=1}^{\infty} |X_i(t)| \right) > \psi(\gamma(t)) \right) \leq 2P \left( \left| \sum_{i=1}^{\infty} X_i(t) \right| > \gamma(t) \right) \\
\leq 2P \left( \sum_{i=1}^{\infty} X_i(t) \mathbf{1}[|X_i(t)| \leq \gamma(t)] > \gamma(t) \right) + 2 \sum_{i=1}^{\infty} P(|X_i(t)| > \gamma(t)) \\
\leq 2E\left[ \sum_{i=1}^{\infty} X_i(t) \mathbf{1}[|X_i(t)| \leq \gamma(t)] \right] + 2 \sum_{i=1}^{\infty} P(|X_i(t)| > \gamma(t)) \\
\leq 2 \sum_{i=1}^{\infty} E \left( \left( \frac{X_i(t)}{\gamma(t)} \right)^2 \wedge 1 \right) \leq 2\delta \leq C^2 \frac{\delta}{24}. \tag{2.22}
\]

Now,
\[
E \sup_{t \geq 1} \psi(|X_i(t)|) \leq \psi(\gamma(t)) + E \sup_{t \geq 1} \psi\left( |X_i(t)| \mathbf{1}[|X_i(t)| > \gamma(t)] \right). \tag{2.23}
\]

Therefore, (2.20) follows from (2.21), (2.22) and (2.23).

For part (iv), replacing \( X_i \) by \( BX_i \) for every \( i \geq 1 \), we may assume that \( B = 1 \). With the obvious meaning for \( \beta_X(t) \) and \( \beta_Y(t) \), we have by the part (ii) of the theorem
\[
\sum_{i=1}^{\infty} P \left( |X_i|_\psi > c \right) < \infty, \tag{2.24}
\]
\[
\int_T \psi(\beta_X(t)) m(\text{d}t) < \infty \tag{2.25}
\]
and
\[
\sum_{i=1}^{\infty} \int_T E \left( \psi(|X_i(t)|) \mathbf{1}[|X_i(t)| > \beta_X(t)] \right) m(\text{d}t). \tag{2.26}
\]

Now, it follows from (2.12) that for all \( i \geq 1 \)
\[
|Y_i|_\psi \leq |X_i|_\psi \ \text{a.s.}, \tag{2.27}
\]
and, therefore, (2.7) for \( Y_i \)'s follows from (2.24). Furthermore, it follows from (2.12) that for all \( t \) in a subset of \( T \) of a full measure,
\[
\beta_Y(t) \leq \beta_X(t). \tag{2.28}
\]

Now our claim follows by applying to \( Y_i \)'s part (iii) of the theorem with \( \gamma(t) = \beta_X(t), t \in T. \)

The above ideas are applied to infinitely divisible processes in the next section.

3 Processes with sample paths in Orlicz spaces

The following theorem gives a complete answer to the question when a symmetric infinitely divisible process has almost all sample paths in \( L_\psi(T, m) \).
Therefore, for every $k \geq 1$

$$P \left( \int_T \psi \left( \sum_{i=1}^{n_k} X_i(t) - Z(t) \right) m(dt) \geq \frac{\theta}{2} \psi(\epsilon) \right) \geq \frac{\theta}{2 - \theta},$$

 contradicting (2.13).

That is, (2.14) holds, and so

$$\psi \left( \sum_{i=1}^{n_k} X_i(t) - Z(t) \right) \to 0 \text{ as } n \to \infty \text{ in measure } P \times m. \quad (2.16)$$

Therefore, there is a sequence $(n_k, k \geq 1)$ such that

$$\psi \left( \sum_{i=1}^{n_k} X_i(t) - Z(t) \right) \to 0 \text{ as } k \to \infty \text{ in measure } P \times m - \text{a.e.} \quad (2.17)$$

Then, there is a set $T_+ \subset T$ with $m(T_+) = 0$ such that for every $t \in T_+$

$$\psi \left( \sum_{i=1}^{n_k} X_i(t) - Z(t) \right) \to 0 \text{ as } k \to \infty \text{ a.s.,}$$

which is the same as

$$\sum_{i=1}^{n_k} X_i(t) \to Z(t) \text{ a.s.}, \quad (2.18)$$

and we may assume, without loss of generality, that $|Z(t)| < \infty$ a.s. Observe, that (2.18) implies, in particular, that the sequence $(\sum_{i=1}^{n_k} X_i, k \geq 1)$ is tight. But then, by symmetry, for every $n_k < n \leq n_{k+1}$ we have

$$P \left( \left| \sum_{i=1}^{n} X_i(t) \right| > \epsilon \right) \leq 4P \left( \left| \sum_{i=1}^{n_{k+1}} X_i(t) \right| > \epsilon \right), \quad (2.19)$$

and so the sequence $(\sum_{i=1}^{n} X_i, n \geq 1)$ is tight as well. By the Ito-Nisio theorem, then

$$\sum_{i=1}^{n} X_i(t) \to Z(t) \text{ a.s}$$

as $n \to \infty$.

Part (ii) is, basically, Theorem 2.4.1 of Kwapień and Wołczyński [KW92]. Both its sufficiency part and part (iii) of the present theorem will follow once we prove that

$$E\Psi \left( \sum_{i=1}^{\infty} [X_i] \right) = \int_T E\psi \left( \sum_{i=1}^{\infty} [X_i](t) \right) m(dt) < \infty. \quad (2.20)$$

We have by Proposition 2.1 of [KW87] with $a = 2C^2\psi(\gamma(t))$:

$$E\psi \left( \sum_{i=1}^{\infty} [X_i](t) \right) \leq \frac{CE\sup_{t \geq 1} \psi([X_i](t)) + 8C^2\psi(\gamma(t))}{1/3 - 4C^2P(\psi(\sum_{i=1}^{\infty} [X_i](t)) > \psi(\gamma(t)))}. \quad (2.21)$$
\[
\int_T \psi(\beta(t)) m(dt) < \infty \tag{2.8}
\]
and
\[
\sum_{i=1}^\infty \int_T E\left(\psi([X_i](t)) 1([X_i](t) > \beta(t))\right) m(dt) < \infty. \tag{2.9}
\]

(iii) Under assumptions of part (ii), let \(\gamma(t), t \in T\) be a measurable function such that \(\beta(t) \leq \gamma(t)\) for all \(t \in T\). If (2.7) holds,
\[
\int_T \psi(\gamma(t)) m(dt) < \infty \tag{2.10}
\]
and
\[
\sum_{i=1}^\infty \int_T E\left(\psi([X_i](t)) 1([X_i](t) > \gamma(t))\right) m(dt) < \infty, \tag{2.11}
\]
then the series \(\sum_{i=1}^\infty X_i\) converges a.s. in \(L_\psi\).

(iv) Let \(Y_1, Y_2, \ldots\) be independent symmetric \(L_\psi\)-valued random variables. Assume that there is a constant \(B > 0\) such that for almost every \(\omega \in \Omega\) we have
\[
|Y_i(t)| \leq B|X_i(t)| m - \text{almost everywhere}. \tag{2.12}
\]

If the series \(\sum_{i=1}^\infty X_i\) converges a.s. in \(L_\psi\), then so does the series \(\sum_{i=1}^\infty Y_i\).

\textbf{PROOF:} (i) By splitting \(T\) into pieces, if necessary, we may assume that \(m(T) \leq 1\).

Let \(\sum_{i=1}^n X_i \to Z\) in \(L_\psi\). Then
\[
\int_T \psi\left(\sum_{i=1}^n X_i(t) - Z(t)\right) m(dt) \to 0 \text{ as } n \to \infty \quad \text{a.s..} \tag{2.13}
\]

For a fixed \(\epsilon > 0\), let
\[
A_{\epsilon}^{(n)} = \left\{ (\omega, t) \in \Omega \times T : \psi\left(\sum_{i=1}^n X_i(t) - Z(t)\right) \geq \epsilon \right\}.
\]

Then \(A_{\epsilon}^{(n)}\) is product measurable, and we claim that
\[
(P \times m)\left(A_{\epsilon}^{(n)}\right) \to 0 \text{ as } n \to \infty. \tag{2.14}
\]

Indeed, suppose, to the contrary, that there is a \(\theta > 0\) and a sequence \((n_k, k \geq 1)\) such that
\[
(P \times m)\left(A_{\epsilon}^{(n_k)}\right) \geq \theta \text{ for all } k.
\]

Then for each \(k \geq 1\) there is an \(\Omega_k \subset \Omega\) with \(P(\Omega_k) \geq \frac{\theta}{2-\theta}\), such that for every \(\omega \in \Omega_k\)
\[
m\left(t : (\omega, t) \in A_{\epsilon}^{(n_k)}\right) \geq \frac{\theta}{2}. \tag{2.15}
\]

But then for every \(\omega \in \Omega_k\)
\[
\int_T \psi\left(\sum_{i=1}^{n_k} X_i(t) - Z(t)\right) m(dt) \geq \frac{\theta}{2} \psi(\epsilon).
\]
$\Gamma_j = \epsilon_1 + \ldots + \epsilon_j$, with $\epsilon_1, \epsilon_2, \ldots$ being i.i.d. standard exponential random variables. Then the series

$$Y(t) = \sum_{j=1}^{\infty} \epsilon_j R(\Gamma_j, \tau_j) f(t, \tau_j)$$

(2.2)

converges a.s. for every $t \in T$ and, moreover $X \overset{d}{=} Y$ in terms of equality of finite dimensional distributions, where $Y = \{Y(t), t \in T\}$. Such series expansions originate with LePage [LeP80], and have been completely developed by Rosiński [Ros90b]. Since $Y$ is a measurable version of $X$, we conclude that (1.5) is equivalent to

$$\int_T \psi(|Y(t)|) \, m(dt) < \infty \text{ a.s.},$$

(2.3)

and we can use Propositions 2.2 and 2.4 of Norvaisa and Samorodnitsky [NS92] to conclude that (1.5) is equivalent to $f(\cdot, s) \in L_\psi(T, m)$ for $\lambda$-almost every $s \in S$, and

$$\text{the series } \sum_{j=1}^{\infty} \epsilon_j R(\Gamma_j, \tau_j) f(\cdot, \tau_j) \text{ converges a.s. in } L_\psi(T, m).$$

(2.4)

See also the argument on page 1916 of Norvaisa and Samorodnitsky [NS92].

We need, therefore, certain basic facts about convergent random series in Orlicz spaces. The following result is a slightly modified version of Theorem 2.4.1 of Kwapień and Woyczyński [KW92]. We will give only few details in the proof. Fix a $c > 0$ and define for an $f \in L_\psi(T, m)$

$$[f] = \frac{f}{\|f\|_\psi \vee c}.$$

Both in the theorem below and in the rest of the paper we will use the same notation $\psi$ to denote both the original $M$-function and its symmetric extension to the whole of $R$.

**Theorem 2.1** Let $X_1, X_2, \ldots$ be a sequence of independent symmetric $L_\psi$-valued random variables, where $\psi$ satisfies the $\Delta_2$ condition. Let $\delta$ be an arbitrary fixed number satisfying

$$\delta \in (0, \frac{1}{48C^2}),$$

(2.5)

where $C$ is the constant from the $\Delta_2$ condition (1.4). For a $t \in T$ define

$$\beta(t) = \inf \left\{ \theta > 0 : \sum_{i=1}^{\infty} E \left( \left( \theta^{-1} X_i(t) \right)^2 \wedge 1 \right) \leq \delta \right\}.$$

(2.6)

(i) If the series $\sum_{j=1}^{\infty} X_j$ converges a.s. in $L_\psi$ then for $m$-almost every $t \in T$ the series $\sum_{i=1}^{\infty} X_i(t)$ converges a.s., which is the same as $\beta(t) < \infty$.

(ii) Assume that for for every $t \in T$ the series $\sum_{i=1}^{\infty} X_i(t)$ converges a.s. Then the series $\sum_{i=1}^{\infty} X_i$ converges a.s. in $L_\psi$ if and only if

$$\sum_{i=1}^{\infty} P \left( \|X_i\|_{\psi} > c \right) < \infty,$$

(2.7)
$(T, T, m)$ is. For the above and other facts on Orlicz spaces the reader is once again referred to Rao and Ren [RR91]. The functional $||f||_\psi$ will be used repeatedly in this paper.

The question whether or not \((1.5)\) holds falls, according to the above discussion, into the general area of describing infinitely divisible laws on functional vector spaces, which amounts to describing the Lévy measures on functional vector spaces. This is often a difficult problem. Such a description is available only in a handful of cases. It is easy to describe all infinitely divisible laws on a Hilbert space, but apart from that the available literature deals mostly with the relationship between possible Lévy measures on, say, a Banach space and the geometry of the space, expressed, typically, through its type and cotype. See Linde [Lin86].

In this paper we completely solve the question under what conditions \((1.5)\) holds. That is, we completely solve the question when a measurable symmetric infinitely divisible process given in the form \((1.1)\) belongs to a generalized Orlicz space $L_\psi(T, m)$, with $\psi$ satisfying the $\Delta_2$ condition. If, for example, $(T, r)$ is a separable metric space, and $\mathcal{T}$ is the Borel $\sigma$-field on that space, then the map $t \to X(t)$, from $T$ to $L^0(\Omega, P)$ is separable because the process $X$ is measurable (see Hoffmann-Jørgensen [Hof73] or Section 9.4 of Samorodnitsky and Taqqu [ST94]), and so by the results of Section V of Rajput and Rosiński [RR89] we conclude that every symmetric measurable infinitely divisible process has an integral representation \((1.1)\). Therefore, our results completely describe all symmetric measurable infinitely divisible processes with sample paths in $L_\psi$.

When the process is symmetric $\alpha$-stable, the necessary and sufficient conditions for \((1.5)\) have been established in Norvaiša and Samorodnitsky [NS92]. The present paper extends their results to the general infinitely divisible case. We use an idea developed in the above paper. That is, we work with a series representation of the process $X$, and apply the technique of random series in Orlicz spaces. A nice presentation of this technique is in Kwapień and Woyczyński [KW92]. We present a somewhat modified version of the results in Kwapień and Woyczyński [KW92] that we need in the next section, and this section also contains the necessary preliminaries on the series representation of infinitely divisible processes. Section 3 contains the main result and some immediate applications. Another application is given in Section 4, where we generalize a result of Yurinski [Yur74] and describe the infinitely divisible processes with sample paths in $L^p$ spaces with $p \geq 2$. Finally, in Section 5 we give necessary and sufficient conditions for a symmetric infinitely divisible process to have a version with absolutely continuous sample paths.

## 2 Preliminaries

Let $X$ be a symmetric infinitely divisible process given by \((1.1)\). For $s \in S$ and $u > 0$ define

$$R(u, s) = \inf \{x > 0 : \rho((x, \infty), s) \leq u\}. \quad (2.1)$$

Let $\{\epsilon_j, j \geq 1\}$, $\{\tau_j, j \geq 1\}$ and $\{\Gamma_j, j \geq 1\}$ be three independent sequences of random variables, such that the first sequence is a sequence of i.i.d. Rademacher random variables, the second one is a sequence of i.i.d. $S$-valued random variables with the common law $\lambda$, and, finally, the third sequence is a sequence of arrival times of a time homogeneous Poisson process on $(0, \infty)$ with unit rate (that is,
a Gaussian component). If $A_1, A_2, \ldots$ are pairwise disjoint sets in $\mathcal{G}$, then $M(A_1), M(A_2), \ldots$ are independent, and if, in addition, $\bigcup_{n=1}^{\infty} A_n \in \mathcal{G}$, then $M(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} M(A_n)$ a.s. Finally, for every $A \in \mathcal{G}$ the Lévy measure $\eta_A$ of $M(A)$ is given by

$$\eta_A(B) = \int_A \rho(B, s) \lambda(ds). \quad (1.2)$$

The function $f(t, s), t \in T, s \in S$ in (1.1) is a (product) measurable deterministic function such that for every $t \in T$

$$n(f(t, \cdot)) := \int_S \left( \int_0^{\infty} (x^2 f(t, s)^2 \wedge 1) \rho(dx, s) \right) \lambda(ds) < \infty. \quad (1.3)$$

We refer the reader to Rajput and Rosiński [RR89] for more details on infinitely divisible random measures and stochastic integrals with respect to those measures.

A function $\psi : R^+ \to R^+$ is called an $M$-function if it is continuous, nondecreasing, $\psi(x) = 0$ if and only if $x = 0$, and $\psi(x) \to \infty$ and $x \to \infty$. An $M$-function $\psi$ is said to satisfy the $\Delta_2$ condition if there is a $C > 0$ such that

$$\psi(2x) \leq C \psi(x), \text{ all } x > 0. \quad (1.4)$$

Given an $M$-function $\psi$ satisfying the $\Delta_2$ condition the collection of (classes of equivalence of) measurable functions $g$ on $T$ such that $\int_T \psi(|g(t)|) m(dt) < \infty$ is, clearly, a linear space, which we denote $L_\psi = L_\psi(T, m)$. Our goal is to find out whether or not $X \in L_\psi(T, m)$ with probability 1. In other words, under what conditions do we have

$$\int_T \psi(|X(t)|) m(dt) < \infty \text{ a.s.} \quad (1.5)$$

We remark in passing that in many cases the event $\int_T \psi(|X(t)|) m(dt) < \infty$ must have the probability 0 or 1. This is the case, for instance, when $\rho((0, \infty), s) = \infty$ on an $\mathcal{S}$-set of full measure. We refer the reader to Janssen [Jan84] and Rosiński [Ros90a] for this and other 0-1 laws. The discussion in the present paper, however, does not depend on the absence or presence of a 0-1 law.

We define a metric in $L_\psi(T, m)$ by

$$d(f, g) = \inf\{u > 0 : \int_T \psi(|f(t) - g(t)|/u) m(dt) \leq u\}. \quad (1.6)$$

This metric makes $L_\psi(T, m)$ a complete linear metric space (the so called generalized Orlicz space). See Rao and Ren [RR91]. Another useful functional on $L_\psi(T, m)$ is defined by

$$\|f\|_\psi = \inf\{u > 0 : \int_T \psi(|f(t)|/u) m(dt) \leq \gamma\}, \quad (1.7)$$

where $\gamma$ is an arbitrary fixed positive number. If the $M$-function $\psi$ is convex, then $\|f\|_\psi$ defines a norm on $L_\psi(T, m)$ which is equivalent to the metric $d$, and $L_\psi(T, m)$ becomes a Banach space. In the general case (i.e. when $\psi$ is not necessarily convex), $f_n \to f$ in the metric $d$ if and only if $\int_T \psi(|f(t) - f_n(t)|) m(dt) \to 0$, and $f_n \to f$ implies also that $\|f - f_n\|_\psi \to 0$, but the converse may be false. We finally remark that the generalized Orlicz space $L_\psi(T, m)$ is separable if the measure space
Symmetric infinitely divisible processes with sample paths in Orlicz spaces and absolute continuity of infinitely divisible processes

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Abstract

We give necessary and sufficient conditions under which a symmetric measurable infinitely divisible process has sample paths in an Orlicz space $L_\psi$ with a function $\psi$ satisfying the $\Delta_2$ condition and, as an application, obtain necessary and sufficient conditions for a symmetric infinitely divisible process to have a version with absolutely continuous paths.

1 Introduction

Consider a symmetric measurable infinitely divisible process $X = \{X(t), t \in T\}$ on a $\sigma$-finite measure space $(T, T, m)$, given in the form

$$X(t) = \int_S f(t, s) M(ds), \ t \in T.$$  

(1.1)

Here $(S, \mathcal{S})$ is a measurable space, and $M$ a symmetric infinitely divisible random measure with control measure $\lambda$ and pointwise Lévy measures $\{\rho(\cdot, s), s \in S\}$. That is, $\lambda$ is a probability measure on $(S, \mathcal{S})$, and $\{\rho(\cdot, s), s \in S\}$ is a measurable family of Lévy measures on $(0, \infty)$, such that

$$\lambda\{s \in S : \rho((0, \infty)) = 0\} = 0.$$  

If we let

$$\mathcal{G} = \left\{ A \in \mathcal{S} : \int_A \left( \int_0^\infty \left( z^2 \wedge 1 \right) \rho(dx, s) \right) \lambda(ds) < \infty \right\},$$

then $\mathcal{G}$ is a $\delta$-ring of $\mathcal{S}$-sets, such that $\{M(A), A \in \mathcal{G}\}$ is a stochastic process with the following properties. For every $A \in \mathcal{G}$, $M(A)$ is a symmetric infinitely divisible random variable (without

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