SECOND ORDER REGULAR VARIATION,
CONVOLUTION
AND THE CENTRAL LIMIT THEOREM

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Abstract. Second order regular variation is a refinement of the concept of regular variation which is useful for studying rates of convergence in extreme value theory and asymptotic normality of tail estimators. For a distribution tail $1 - F$ which possesses second order regular variation, we discuss how this property is inherited by $1 - F^2$ and $1 - F^{q2}$. We also discuss the relationship of central limit behavior of tail empirical processes, asymptotic normality of Hill's estimator and second order regular variation.

1. Introduction.
In this paper we assume that the distribution function $F$ is concentrated on $[0, \infty)$. The tail $1 - F(x)$ is regularly varying with index $-\alpha$, $\alpha > 0$ (written $1 - F \in RV_{-\alpha}$) if

$$
\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \quad x > 0.
$$

The distribution tail $1 - F$ is second order regularly varying with first order parameter $-\alpha$ and second order parameter $\rho$ (written $1 - F \in 2RV(-\alpha, \rho)$) if there exists a function $A(t) \to 0$, $t \to \infty$ which ultimately has constant sign such that the following refinement of (1.1) holds:

$$
\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha} A(t) = H(x) := cx^{-\alpha} \int_1^x u^{\rho-1} du, \quad x > 0
$$

for $c \neq 0$. Note that for $x > 0$

$$
H(x) = \begin{cases} 
    cx^{-\alpha} \log x, & \text{if } \rho = 0, \\
    cx^{-\alpha} x^{\rho-1}, & \text{if } \rho < 0.
\end{cases}
$$

It is well known (Geluk and de Haan (1987), Theorem 1.9) that if (1.2) holds with $H(x)$ not a multiple of $x^{-\alpha}$, then $H$ satisfies the above representation, $|A| \in RV_{\rho}$ and no other choices of $\rho$ are consistent with $A(t) \to 0$. Moreover convergence in (1.2) is uniform in $x$ on compact intervals of $(0, \infty)$. See de Haan and Stadtmüller (1995) for a related discussion.

Key words and phrases. regular variation, second order behavior, tail empirical measure, extreme value theory, convolution, maxima, Hill estimator.

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Second order regular variation has proven very useful in establishing asymptotic normality of extreme value statistics and also for the study of rates of convergence to extreme value and stable distributions (de Haan and Resnick, 1995; de Haan and Peng, 1995a,b,c; Smith, 1982).

An example of the statistical uses of second order regular variation is as follows: Suppose \( Z_1, \ldots, Z_n \) are iid random variables with common distribution \( F \) satisfying (1.1). A commonly used estimator of \( \alpha^{-1} \) is Hill’s estimator
\[
H_{k,n} := \frac{1}{k} \sum_{i=1}^{k} \log \frac{Z_{(i)}}{Z_{(k+1)}}
\]
where \( Z_{(1)} > \cdots > Z_{(n)} \) are the order statistics of the sample. Under the assumption that the number of upper order statistics \( k \) used in the estimation satisfies \( k \to \infty, \ k/n \to 0, \)
\[
H_{k,n} \xrightarrow{P} \alpha^{-1},
\]
and under a von Mises condition there exists constants \( \alpha_{k,n}^{-1} \) such that
\[
\sqrt{k}(H_{k,n} - \alpha_{k,n}^{-1}) \to N,
\]
where \( N \) is a normal random variable (Mason, 1982; Hall, 1982; Dekkers and de Haan, 1989; Resnick and Stărică, 1995a; Davis and Resnick, 1984; Csorgo and Mason, 1985). In order to construct confidence statements for the inference, one needs to replace \( \alpha_{k,n} \) by \( \alpha \) in the central limit theorem and for this one needs to know \( \sqrt{k}(\alpha_{k,n}^{-1} - \alpha^{-1}) \to 0 \) as \( n \to \infty \). A convenient way to assure this is by assuming \( 1 - F \in 2RV(-\alpha, \rho) \).
(See also Resnick and Stărică, 1995b and Kratz and Resnick, 1995.)

A related statistical problem assumes that one observes \( X_1, \ldots, X_n \) where \( \{X_n\} \) is a stationary infinite order moving average process of the form
\[
X_t = \sum_{j=0}^{\infty} c_j Z_{t-j}, \quad t = 0, \pm 1, \pm 2, \ldots
\]
where \( \{Z_t\} \) are iid with common distribution satisfying (1.1) and (1.2). Resnick and Stărică (1995a) have proven that the Hill estimator applied to \( X_1, \ldots, X_n \) is a consistent estimator of \( \alpha^{-1} \). In order to assess the performance mathematically of this estimator and to compare it to competing procedures, the asymptotic normality must be investigated. In order to do this successfully, one must understand how second order regular variation behaves under convolution and this was the strongest motivation for the present investigation.

First order behavior of regularly varying tails under convolution is fairly tame: If \( 1 - F \) satisfies (1.1), then the convolution tail \( 1 - F*2 \) satisfies
\[
1 - F*2(t) \sim 2(1 - F(t)), \quad (t \to \infty).
\]
Feller (1971) has a straightforward and clear analytical proof and Resnick (1986, 1987) proves this probabilistically using point processes. However, second order regularly varying tails behave rather more complexly.

In Section 2 we prepare the way by discussing behavior of distribution tails of maxima \( Z_1 \lor Z_2 \) of iid random variables having common distribution \( F \) satisfying (1.2). The behavior turns out to depend on how
\[
\lim_{t \to \infty} \frac{1 - F(t)}{A(t)}
\]
behaves. Section 3 gives some results for convolution tails and Section 4 discusses a probabilistically equivalent statement to (1.2) involving the central limit theorem and shows that in a manner to be made precise in Theorem 4.2, asymptotic normality of Hill’s estimator is equivalent to second order regular variation.
We end this introduction with two examples.

**Example 1.1, log gamma distribution:** Suppose $X_1, X_2$ are iid with standard exponential density. The log gamma distribution is the distribution of $\exp\{X_1 + X_2\}$. For $x > 1$,

$$P[\exp\{X_1 + X_2\} > x] = P[X_1 + X_2 > \log x] = \exp\{-\log x\} + \exp\{-\log x\} \log x = x^{-1}(1 + \log x) := 1 - F(x).$$

Thus for $x > 1$

$$\frac{1 - F(tx)}{1 - F(t)} - x^{-1} = x^{-1} \left( \frac{\log x}{1 + \log t} \right) \sim x^{-1} \frac{\log x}{\log t},$$

and thus

$$\lim_{t \to \infty} \frac{1 - F(tx) - x^{-1}}{1 - F(t) - 1/\log t} = x^{-1} \log x,$$

and with $A(t) = 1/\log t$ we have $\alpha = 1, \rho = 0$ and

$$\lim_{t \to \infty} \frac{1 - F(t)}{A(t)} = 0.$$

**Example 1.2, Hall/Weiss class:** Suppose for $x \geq 1, \alpha > 0, \rho < 0$ that

$$1 - F(x) = \frac{1}{2} x^{-\alpha}(1 + x^\rho).$$

Then as $t \to \infty$

$$\frac{1 - F(tx) - x^{-\alpha} \left( \frac{1 + (tx)^\rho}{1 + t^\rho} - 1 \right)}{1 - F(t) - 1/\log t} \sim x^{-\alpha}(x^\rho - 1)t^\rho$$

and so we may set $A(t) = \rho t^\rho$ and

$$\lim_{t \to \infty} \frac{1 - F(t)}{|A(t)|} = \begin{cases} 0, & \text{if } |\rho| < \alpha, \\ |\rho|^{-1}, & \text{if } |\rho| = \alpha, \\ \infty, & \text{if } |\rho| > \alpha. \end{cases}$$

2. **Maxima.**

As preparation for further work, we begin with a two dimensional result.

**Theorem 2.1.** Let $Z_1, Z_2$ be non-negative iid random variables with common distribution $F$ satisfying (1.2). Then for $x > 0, y > 0$

$$\lim_{t \to \infty} \frac{P[\{Z_1 > tx\} \cup \{Z_2 > ty\}]}{1 - F(t)} - (x^{-\alpha} + y^{-\alpha}) = H(x) + H(y) - H(xy)^{-\alpha}$$

(2.1)
provided

\begin{equation}
\lim_{t \to \infty} \frac{1 - F(t)}{A(t)} = l, \quad |l| < \infty.
\end{equation}

If $|l| = \infty$, then

\begin{equation}
\lim_{t \to \infty} \frac{P\{[Z_1 > tx] \cup [Z_2 > ty]\} - (x^{-a} + y^{-a})}{1 - F(t)} = -(xy)^{-a}
\end{equation}

Proof. We have

\[ P\{[Z_1 > tx] \cup [Z_2 > ty]\} = 1 - F(tx)F(ty) = 1 - F(tx) + 1 - F(ty) - (1 - F(tx))(1 - F(ty)) \]

and so

\begin{equation}
\frac{P\{[Z_1 > tx] \cup [Z_2 > ty]\}}{1 - F(t)} - (x^{-a} + y^{-a}) = \left( \frac{1 - F(tx)}{1 - F(t)} - x^{-a} \right) + \left( \frac{1 - F(ty)}{1 - F(t)} - y^{-a} \right) - \left( \frac{(1 - F(tx))(1 - F(ty))}{1 - F(t)} \right)
\end{equation}

and the stated results follow. \qed

By letting $x = y$, we immediately get the Corollary about tail behavior of distribution of maxima.

**Corollary 2.2.** Let $Z_1, Z_2$ be non-negative iid random variables with common distribution $F$ satisfying (1.2). Then for $x > 0$

\begin{equation}
\lim_{t \to \infty} \frac{P\{Z_1 \vee Z_2 > tx\} - 2x^{-a}}{A(t)} = 2H(x) - lx^{-2a}
\end{equation}

if

\begin{equation}
\lim_{t \to \infty} \frac{1 - F(t)}{A(t)} = l, \quad |l| < \infty,
\end{equation}

and if $|l| = \infty$

\begin{equation}
\lim_{t \to \infty} \frac{P\{Z_1 \vee Z_2 > tx\} - 2x^{-a}}{1 - F(t)} = -x^{-2a}
\end{equation}

**Remarks:**

1. Changing normalizations in (2.5) yields

\begin{equation}
\lim_{t \to \infty} \frac{P\{Z_1 \vee Z_2 > tx\} - x^{-a}}{A(t)} = (1 + \frac{1}{2}\alpha l)H(x), \quad x > 0
\end{equation}

so that $P\{Z_1 \vee Z_2 > t\} \in 2RV(-\alpha, \rho)$. Note that for $l \neq 0$ we have $-\alpha = \rho$ since $1 - F(t) \sim lA(t)$. Similarly, modifying (2.7) yields

\begin{equation}
\lim_{t \to \infty} \frac{P\{Z_1 \vee Z_2 > tx\} - x^{-a}}{1 - F(t)} = \frac{1}{2}x^{-a}(1 - x^{-a}), \quad x > 0
\end{equation}
so that in this case, \( P[Z_1 \vee Z_2 > t] \in 2RV(-\alpha,-\alpha) \). Thus \( P[Z_1 > t] \) and \( P[Z_1 \vee Z_2 > t] \) may have different second order parameters.

(2) Applying Theorem 2.1 with \( x \) replaced by \( x/c_1 \) and \( y \) replaced by \( y/c_2 \) where \( x > 0, \, y > 0, \, c_i > 0, \, i = 1, 2 \) yields

\[
\frac{P[\{c_1 Z_1 > tx\} \cup \{c_2 Z_2 > ty\}]}{1-F(t)} - \left(c_1 \alpha x^{-\alpha} + c_2 \alpha y^{-\alpha}\right) = H(x c_1^{-1}) + H(y c_2^{-1}) - l c_1 \alpha c_2 \alpha x^{-2\alpha} \quad x > 0,
\]

if (2.6) is satisfied. The second order behavior of \( P[c_1 Z_1 \vee c_2 Z_2 > t] \) under condition (2.5) is obtained by setting \( x = y \) as follows

\[
\frac{P[c_1 Z_1 \vee c_2 Z_2 > tx]}{1-F(t)} - \left(c_1^\alpha + c_2^\alpha\right) x^{-\alpha} = H(x c_1^{-1}) + H(x c_2^{-1}) - l c_1^\alpha c_2^\alpha x^{-2\alpha} \quad x > 0,
\]

extending (2.5). A similar modification of (2.7) holds.

The following proposition asserts that the relations (2.1) and (2.3) characterize second order regular variation of the underlying distribution tail function. Moreover the normalizations \( A(t) \) and \( 1 - F(t) \) are the only possible.

**Proposition 2.3.** Define for \( x > 0, \, y > 0 \)

\[
LHS(t, x, y) := \frac{P[\{Z_1 > tx\} \cup \{Z_2 > ty\}]}{1-F(t)} - (x^{-\alpha} + y^{-\alpha}).
\]

Suppose further there exists a function \( \psi(t) > 0 \) such that for all \( x > 0, \, y > 0 \)

\[
\lim_{t \to \infty} \frac{LHS(t, x, y)}{\psi(t)} = c(x, y)
\]

where \( \psi(t) \to 0 \) \( (t \to \infty) \) and the function \( c(x, y) \) satisfies \( |c(x, y)| < \infty \) for all \( x, y \) and \( c(x, x) \neq a_1 x^{-\alpha} + a_2 x^{-2\alpha} \) for any choice of real \( a_1, a_2 \). Then \( 1 - F \) satisfies (1.2) with \( A(t) \sim c \psi(t) \) for some \( c \neq 0 \) and

\[
\lim_{t \to \infty} \frac{1-F(t)}{|A(t)|} = 0 \quad \text{exists}.
\]

**Proof.** Since \( \psi(t) \to 0 \) and \( |c(x, y)| < \infty \) it follows that

\[
LHS(t, x, x) = 2 \left(\frac{1 - F(tx)}{1 - F(t)} - x^{-\alpha}\right) - \frac{(1 - F(tx))^2}{1 - F(t)} \to 0 \quad (t \to \infty).
\]

Obviously for \( x > 1 \) we have \( (1 - F(tx))^2/(1 - F(t)) \to 0 \) as \( t \to \infty \) and hence it follows that \( 1 - F \in RV_{-\alpha} \). Observe that

\[
\frac{LHS(t, 1, 1)}{\psi(t)} = \frac{-(1 - F(t))}{\psi(t)} \to c(1, 1).
\]

Hence if \( c(1, 1) = 0 \) we have \( 1 - F(t) = o(\psi(t)) \) implying \( (1 - F(tx))^2/(1 - F(t)) = o(\psi(t)) \), as \( t \to \infty \) for \( x > 0 \). As a consequence, for \( x > 0 \)

\[
\frac{LHS(t, x, x)}{\psi(t)} = \frac{1-F(tx)}{1-F(t)} - x^{-\alpha} + \frac{c(1) \to c(x, x)}{\psi(t)/2} \quad \text{as} \quad t \to \infty.
\]
Since $c(x, x)$ is not a multiple of $x^{-\alpha}$, it follows from Geluk and de Haan (1987), Theorem 1.9 that $c(x, x) = cH(x)$ with $c \neq 0$ and $H$ as in (1.2) and hence $1 - F \in 2RV(-\alpha, \rho)$. Therefore, there exists a function $A$ such that (1.2) is satisfied and

$$\frac{LHS(t, x, x)}{\psi(t)} = \frac{2H(x) + o(1)}{\psi(t)} A(t) \frac{1}{\psi(t)} - (x^{-2\alpha} + o(1)) \frac{1 - F(t)}{\psi(t)}$$

$$= (2H(x) + o(1)) \frac{A(t)}{\psi(t)} + o(1).$$

Let $t \to \infty$ and we conclude

$$c(x, x) = 2H(x) \lim_{t \to \infty} \frac{A(t)}{\psi(t)},$$

and hence we get

$$\lim_{t \to \infty} \frac{A(t)}{\psi(t)} = \frac{c}{2}.$$ 

Thus, if $c(1, 1) = 0$, then $1 - F(t) = o(A(t)), t \to \infty$.

In case $c(1, 1) \neq 0$ we have $1 - F(t) \sim -c(1, 1)\psi(t)$, and therefore

$$\frac{LHS(t, x, x)}{\psi(t)} = \frac{\frac{1 - F(tx)}{1 - F(t)} - x^{-\alpha}}{\psi(t)/2} - \frac{(1 - F(tx))^2}{(1 - F(t))^2} \frac{1 - F(t)}{\psi(t)}$$

$$= \frac{\frac{1 - F(tx)}{1 - F(t)} - x^{-\alpha}}{\psi(t)/2} + x^{-2\alpha}c(1, 1) + o(1) \to c(x, x) \quad \text{as} \quad t \to \infty.$$

It follows that for $x > 0$

$$\frac{\frac{1 - F(tx)}{1 - F(t)} - x^{-\alpha}}{\psi(t)/2} \to c(x, x) - c(1, 1)x^{-2\alpha} \quad (t \to \infty).$$

By assumption the right hand side is not a multiple of $x^{-\alpha}$, hence (1.2) holds with some function $A(t) \sim c_1\psi(t)$ $(t \to \infty)$, where $c_1 \neq 0$ is a constant. □

**Remark:**

(1) In case $c(x, x) = cx^{-2\alpha}$ for some $c \neq 0$ it follows that $c = c(1, 1)$ and

$$\frac{\frac{1 - F(tx)}{1 - F(t)} - x^{-\alpha}}{(\psi(t)/2)} \to 0, \quad x > 0$$

as $t \to \infty$. Hence $\frac{1 - F(tx)}{1 - F(t)} - x^{-\alpha} = o(1 - F(t))$. Thus, in this case if $F$ satisfies (1.2), then the limit in (2.13) is infinite.

**3. Convolution.**

In the sequel we denote $1 - F$ by $\tilde{F}$. The results for convolution are more complex than for maxima. In order to prove the main result Theorem 3.2 we first need a lemma.
Lemma 3.1. Suppose for \( i = 1, 2 \) that \( F_i \in 2RV(-\alpha_i; \rho_i) \), i.e., \( F_i \) satisfies (1.2) with auxiliary function \( A_i \). Further suppose
\[
\bar{F}_{i+2}(x) - k_i \bar{F}_i(x) = (d_i + o(1))A_i(x)\bar{F}_i(x) \quad (x \to \infty)
\]
and
\[
\bar{F}_i(x - b) - \bar{F}_i(x) = o(A_i(x)\bar{F}_i(x)) \quad (x \to \infty),
\]
where \( k_i > 0, b, d_i \in \mathbb{R} \). Then as \( x \to \infty \)
\[
\overline{F}_3 * F_4(x) - \bar{F}_4(x)
= k_1 k_2 (\bar{F}_1(x) - \bar{F}_2(x)) + o(\sum_{i=1}^{2} A_i(x)\bar{F}_i(x)).
\]

Proof. By assumption, for \( \varepsilon > 0 \) there exists \( a > 0 \) such that for \( i = 1, 2 \)
\[
\bar{F}_i(x) < \varepsilon, \quad \bar{F}_{i+2}(x) - k_i \bar{F}_i(x) \leq (d_i + \varepsilon)A_i(x)\bar{F}_i(x), \quad x > a.
\]
It follows that for \( x > a \)
\[
(\text{3.1}) \quad \overline{F}_3 * F_4(x) - \bar{F}_4(x) = \int_{0}^{x} \bar{F}_3(x - u)dF_4(u)
\leq \int_{0}^{x-a} \bar{F}_3(x - u)dF_4(u) + (\bar{F}_4(x - a) - \bar{F}_4(x))
\leq k_1 \int_{0}^{x-a} \bar{F}_1(x - u)dF_4(u) + (d_1 + \varepsilon) \int_{0}^{x-a} A_1(x - u)\bar{F}_1(x - u)dF_4(u)
+ k_2 (\bar{F}_2(x - a) - \bar{F}_2(x)) + o(A_2(x)\bar{F}_2(x))
=: k_1 I_1 + (d_1 + \varepsilon) I_2 + o(A_2(x)\bar{F}_2(x)) \quad (x \to \infty).
\]
Now \( I_1 \) is estimated as follows.
\[
(\text{3.2}) \quad I_1 = \int_{0}^{x-a} \bar{F}_1(x - u)dF_4(u) = \int_{0}^{x} \bar{F}_1(x - u)dF_4(u) + o(A_2(x)\bar{F}_2(x))
= \int_{0}^{x-a} \bar{F}_4(x - u)dF_1(u) + \bar{F}_1(x) - \bar{F}_4(x) + o(\sum_{i=1}^{2} A_i(x)\bar{F}_i(x))
\leq k_2 \int_{0}^{x-a} \bar{F}_2(x - u)dF_1(u) + (d_2 + \varepsilon) \int_{0}^{x-a} A_2(x - u)\bar{F}_2(x - u)dF_1(u)
+ \bar{F}_1(x) - \bar{F}_4(x) + o(\sum_{i=1}^{2} A_i(x)\bar{F}_i(x))
\leq k_2 (\overline{F}_1 * F_2(x) - \bar{F}_1(x)) + (d_2 + \varepsilon) \int_{0}^{x-a} A_2(x - u)\bar{F}_2(x - u)dF_1(u)
+ \bar{F}_1(x) - \bar{F}_4(x) + o(\sum_{i=1}^{2} A_i(x)\bar{F}_i(x)) \quad (x \to \infty).
\]
Since \( A_2 \in RV_{\rho_2} \) and for \( i = 1, 2 \) we have \( \bar{F}_i \in RV_{-\alpha_i} \), we have
\[
\int_{0}^{x-a} A_2(x - u)\bar{F}_2(x - u)dF_1(u) \sim A_2(x)\bar{F}_2(x)
\]
and since a lower inequality for $I_1$ can be proved similarly, combination with (3.2) gives

$$I_1 = k_2(\bar{F}_1 * F_2(x) - \bar{F}_1(x)) + \bar{F}_1(x) - \bar{F}_4(x) + d_2 A_2(x) \bar{F}_2(x) + o(\sum_{i=1}^{2} A_i(x) \bar{F}_i(x)).$$

Substituting $\bar{F}_4(x) = k_2 \bar{F}_2(x) + (d_2 + o(1)) A_2(x) \bar{F}_2(x)$ we find

(3.3) $$I_1 = k_2(\bar{F}_1 * \bar{F}_2(x) - \bar{F}_1(x)) + \bar{F}_1(x) - k_2 \bar{F}_2(x) + o(\sum_{i=1}^{2} A_i(x) \bar{F}_i(x)).$$

Similarly regular variation of $A_1$, $\bar{F}_1$ and $\bar{F}_4$ implies

(3.4) $$I_2 \sim A_1(x) \bar{F}_1(x) \ (x \to \infty).$$

Since a corresponding lower inequality for (3.1) can be proved similarly, combination of (3.1), (3.3) and (3.4) gives an expression for $\bar{F}_2 * \bar{F}_1(x) - \bar{F}_4(x)$. Subtracting $\bar{F}_3(x) = k_1 \bar{F}_1(x) + d_1 A_1(x) \bar{F}_1(x) + o(A_1(x) \bar{F}_1(x))$ then gives the desired result. □

**Theorem 3.2.** Suppose $Z_1$, $Z_2$ are iid non-negative random variables with common distribution function $F$ satisfying (1.2) and suppose $c_1 > 0$, $c_2 > 0$. There exist for each case two functions $A$ and $\bar{H}$ such that

(3.5) $$\lim_{t \to \infty} \frac{P(c_1 Z_1 + c_2 Z_2 > tx) - (c_1^2 + c_2^2) x^{-\alpha}}{A(t)} = \bar{H}(x)$$

for $x > 0$. Define $\xi_\alpha = -\Gamma^2(1 - \alpha)/\Gamma(1 - 2\alpha)$ for $\alpha < 1$. If

I. $\alpha < 1$

1. $\rho \leq -\alpha$ and $\bar{F}(t)/A(t) \to \infty$ as $t \to \infty$, then $\bar{A}(t) = \bar{F}(t)$ and $\bar{H}(x) = c_1^2 + c_2^2 \xi_\alpha x^{-2\alpha} + H(c_1^{-1} x) + H(c_2^{-1} x),$

2. $\rho \geq -\alpha$ and $\bar{F}(t)/A(t) \to l < \infty$ as $t \to \infty$, then $\bar{A}(t) = A(t)$ and $\bar{H}(x) = \{l(c_1^2 + c_2^2) \xi_\alpha x^{-2\alpha} + H(c_1^{-1} x) + H(c_2^{-1} x),$

II. $\alpha = 1$ and the mean $\mu$ is finite

1. $\rho \leq -1$ and $tA(t) \to l < \infty$ as $t \to \infty$, then $\bar{A}(t) = t^{-1}$ and $\bar{H}(x) = 2\mu + H(c_1^{-1} x) + H(c_2^{-1} x),$

2. $\rho \geq -1$ and $tA(t) \to \infty$ as $t \to \infty$, then $\bar{A}(t) = A(t)$ and $\bar{H}(x) = H(c_1^{-1} x) + H(c_2^{-1} x),$

III. $\alpha > 1$

1. $\rho \leq -1$ and $tA(t) \to l < \infty$ as $t \to \infty$ then $\bar{A}(t) = t^{-1}$ and $\bar{H}(x) = \alpha \mu (c_1^2 + c_2^2) x^{-\alpha - 1} + l(H(c_1^{-1} x) + H(c_2^{-1} x),$

2. $\rho \geq -1$ and $tA(t) \to \infty$ as $t \to \infty$ then $\bar{A}(t) = A(t)$ and $\bar{H}(x) = H(c_1^{-1} x) + H(c_2^{-1} x).

**Proof.** The proof for I is based on Lemma 3.1 and Lemma 2.1 (Geluk (1995)). It is known (de Haan (1995)) that there exists a distribution function $F_0$ with $F_0 \in RV(-\alpha, \rho)$ and $F_0(0) = 0$ which has a differentiable density $f_0 \in RV(-\alpha - 1)$ such that

(3.6) $$\bar{F}(t) - \bar{F}_0(t) = o(A(t) \bar{F}(t)) \ (t \to \infty).$$

Let $Z_1^*, Z_2^*$ be iid non-negative random variables with common distribution $F_0$. First we prove I.2. We intend to apply Lemma 3.1 to

$$F_1(t) = P(Z_1^* \leq t), \quad F_2(t) = P(Z_2^* \leq t),$$

$$F_3(t) = P(c_1 Z_1 \leq t), \quad F_4(t) = P(c_2 Z_2 \leq t).$$
In order to apply the Lemma, we first verify its hypotheses. From (3.6) and the fact that $F$ satisfies (1.2) it is clear that $F_1$ and $F_2$ satisfy (1.2) as well (with the same function $A$). Note that as $t \to \infty$ for $i = 1, 2$

$$\frac{P(c_i Z_i > t) - c_i^2 P(Z_i^2 > t)}{A(t) \bar{F}_0(t)} = \frac{\bar{F}(t/c_i) - c_i^2 \bar{F}_0(t)}{A(t) \bar{F}_0(t)}$$

$$= \frac{\bar{F}(t/c_i) - \bar{F}_0(t/c_i)}{A(t/c_i) \bar{F}(t/c_i)} \frac{A(t/c_i) \bar{F}(t/c_i)}{A(t) \bar{F}_0(t)}$$

$$+ \frac{\bar{F}_0(t/c_i) - c_i^2 \bar{F}_0(t)}{A(t) \bar{F}_0(t)}$$

$$- H(c_i^{-1}).$$

So the first condition of Lemma 3.1 is verified with $k_1 = c_1^2$, $k_2 = c_2^2$, $d_1 = H(c_1^{-1})$, $d_2 = H(c_2^{-1})$. The second condition reads

$$\frac{\bar{F}_0(t-b) - \bar{F}_0(t)}{\bar{F}_0(t) A(t)} = \frac{\bar{F}_0(t(1-b/t)) - (1-b/t)^{-\alpha} \bar{F}_0(t)}{\bar{F}_0(t) A(t)} + \frac{(1-b/t)^{-\alpha} - 1}{A(t)}$$

The first term tends to 0 for $t \to \infty$ due to the uniform convergence in the second order condition (1.2).

Since $(1-b/t)^{-\alpha} - 1 \sim ab/t$ and $-\rho \leq \alpha < 1$ the second term clarifies as

$$\frac{(1-b/t)^{-\alpha} - 1}{A(t)} \sim \frac{ab}{tA(t)} \to 0$$

as $t \to \infty$. Since the hypotheses of the Lemma are verified it follows that as $t \to \infty$

$$\frac{P(c_1 Z_1 + c_2 Z_2 > t) - P(c_1 Z_1 > t) - P(c_2 Z_2 > t)}{A(t) \bar{F}(t)} = c_1^2 c_2^2 \frac{\bar{F}_0^2(t) - 2 \bar{F}_0(t) \bar{F}(t)}{A(t)} + o(1)$$

or

$$\frac{P(c_1 Z_1 + c_2 Z_2 > t) - P(c_1 Z_1 > t) - P(c_2 Z_2 > t)}{A(t) \bar{F}(t)} \to \xi_{c_1^2} c_2^2$$

since

$$\frac{\bar{F}_0^2(t) - 2 \bar{F}_0(t)}{\bar{F}^2(t)} \to \xi_{c_2^2}$$

(Omey and Willekens (1986)) and $\bar{F}(t)/A(t) \to t$ from the assumptions of I.2. To finish the proof for this case one applies (3.7) with $c_i$ replaced by $c_i/x \ (i = 1, 2)$ and adds

$$\frac{P(c_1 Z_1 > t x) - c_1^2 x^{-\alpha} \bar{F}(t)}{A(t) \bar{F}(t)} + \frac{P(c_2 Z_2 > t x) - c_2^2 x^{-\alpha} \bar{F}(t)}{A(t) \bar{F}(t)} \to H(c_1^{-1} x) + H(c_2^{-1} x).$$

The proof of I.1 follows the same path, the only difference being that instead of Lemma 3.1 we will employ Lemma 2.1 (Geluk (1995)). The choice of $\alpha$ from Lemma 2.1 is $\alpha = 2$. Defining $F_i$ to $F_4$ as above it follows from (1.2) and (3.6) that the first condition in the Lemma is verified with $k_1 = c_1^2$, $k_2 = c_2^2$, $d_1 = 0$, $d_2 = 0$. The second condition reads

$$\frac{\bar{F}_0(t-b) - \bar{F}_0(t)}{\bar{F}_0^2(t)} = \frac{\bar{F}_0(t(1-b/t)) - (1-b/t)^{-\alpha} \bar{F}_0(t)}{A(t) \bar{F}_0(t) A(t)} \frac{A(t) \bar{F}_0(t) A(t)}{\bar{F}_0(t)}$$

$$+ \frac{(1-b/t)^{-\alpha} - 1}{\bar{F}_0(t)}.$$
The first term on the right hand side tends to 0 since both factors tend to zero (by uniform convergence in (1.2) and by assumption respectively). The second behaves like $ab(tF_{0}(t))^{-1}$. Since we are under the assumption that $	ilde{F} \in RV_{-\alpha}$ with $\alpha < 1$, we have that $tF_{0}(t) \to \infty$ as $t \to \infty$. Therefore the hypotheses of Lemma 2.1 are verified. Thus combination of Lemma 2.1 with (3.8) above gives

$$
P(c_{1}Z_{1} + c_{2}Z_{2} > t) - \frac{P(c_{1}Z_{1} > t) - P(c_{2}Z_{2} > t)}{F^{2}(t)} \to \xi_{0}c_{1}^{\alpha}c_{2}^{\alpha}.
$$

(3.10)

Since in this case $A(t) = a(t)F(t)$ (3.9) implies

$$
P(c_{1}Z_{1} > t x) - \frac{c_{1}x^{-\alpha}F(t)A(t)}{F(t)} + P(c_{2}Z_{2} > t x) - \frac{c_{2}x^{-\alpha}F(t)A(t)}{F(t)} \to 0.
$$

(3.11)

The proof of I.1 is finished if we replace $c_{i}$ by $c_{i}/x$ $(i = 1, 2)$ in (3.10) and add (3.11).

For cases II and III a different approach is needed. Decompose $P(c_{1}Z_{1} + c_{2}Z_{2} > t)$ as follows

$$
P(c_{1}Z_{1} + c_{2}Z_{2} > t) = P(c_{1}Z_{1} + c_{2}Z_{2} > t, c_{1}Z_{1} < c_{2}Z_{2} > t) + P(c_{1}Z_{1} + c_{2}Z_{2} > t, c_{1}Z_{1} < c_{2}Z_{2} \leq t)
$$

$$
+ P(c_{1}Z_{1} < c_{2}Z_{2} > t, c_{1}Z_{1} < c_{2}Z_{2} \leq t) + P(c_{1}Z_{1} < c_{2}Z_{2} > t, c_{1}Z_{1} \land c_{2}Z_{2} \leq t/2)
$$

$$
+ P(c_{1}Z_{1} < c_{2}Z_{2} > t, c_{1}Z_{1} \land c_{2}Z_{2} \leq t/2)
$$

$$
= P(c_{1}Z_{1} > t) + P(c_{2}Z_{2} > t) - P(c_{1}Z_{1} > t)P(c_{2}Z_{2} > t)
$$

$$
+ \int_{0}^{t/2} (\tilde{F}_{3}(t - u) - \tilde{F}_{3}(t))dF_{3}(u) + \int_{0}^{t/2} (\tilde{F}_{4}(t - u) - \tilde{F}_{4}(t))dF_{4}(u)
$$

$$
+ (\tilde{F}_{1}(t/2) - \tilde{F}_{1}(t))(\tilde{F}_{2}(t/2) - \tilde{F}_{2}(t)).
$$

where $F_{3}$ and $F_{4}$ are defined as above. Therefore

$$
\frac{(P(c_{1}Z_{1} + c_{2}Z_{2} > t) - P(c_{1}Z_{1} > t) - P(c_{2}Z_{2} > t))}{\tilde{A}(t)\tilde{F}(t)}
$$

$$
= \frac{A(t/c_{2})}{\tilde{A}(t)} A(t/c_{2}) \frac{\tilde{F}(t/c_{2})}{\tilde{F}(t)} \int_{0}^{t/2} \frac{\tilde{F}(t/c_{2})(1 - u/t)}{\tilde{F}(t/c_{2})A(t/c_{2})} dF_{3}(u)
$$

$$
+ \frac{A(t/c_{1})}{\tilde{A}(t)} A(t/c_{1}) \frac{\tilde{F}(t/c_{1})}{\tilde{F}(t)} \int_{0}^{t/2} \frac{\tilde{F}(t/c_{1})(1 - u/t)}{\tilde{F}(t/c_{1})A(t/c_{1})} dF_{4}(u)
$$

$$
+ \frac{\tilde{F}(t)}{\tilde{A}(t)} \left( \frac{\tilde{F}(t/(2c_{1}))}{\tilde{F}(t)} - \frac{\tilde{F}(t/c_{1})}{\tilde{F}(t)} \right) \left( \frac{\tilde{F}(t/(2c_{2}))}{\tilde{F}(t)} - \frac{\tilde{F}(t/c_{2})}{\tilde{F}(t)} \right) - \frac{\tilde{F}(t)}{A(t/c_{1})A(t/c_{2})} A(t/c_{1}) A(t/c_{2})
$$

$$
= \frac{A(t)}{\tilde{A}(t)} \frac{A(t/c_{2})}{A(t)} \frac{\tilde{F}(t/c_{2})}{\tilde{F}(t)} \int_{0}^{t/2} \frac{\tilde{F}(t/c_{2})(1 - u/t)}{\tilde{F}(t/c_{2})} dF_{3}(u)
$$

$$
+ \frac{1}{tA(t)} \frac{\tilde{F}(t/c_{2})}{\tilde{F}(t)} \int_{0}^{t/2} t((1 - u/t)^{-\alpha} - 1)dF_{3}(u)
$$

$$
+ \frac{A(t)}{\tilde{A}(t)} \frac{A(t/c_{1})}{A(t)} \frac{\tilde{F}(t/c_{1})}{\tilde{F}(t)} \int_{0}^{t/2} \frac{\tilde{F}(t/c_{1})(1 - u/t)}{\tilde{F}(t/c_{1})} dF_{4}(u)
$$

$$
+ \frac{1}{tA(t)} \frac{\tilde{F}(t/c_{1})}{\tilde{F}(t)} \int_{0}^{t/2} t((1 - u/t)^{-\alpha} - 1)dF_{4}(u)
$$

$$
+ \frac{\tilde{F}(t)}{\tilde{A}(t)} \left( \frac{\tilde{F}(t/(2c_{1}))}{\tilde{F}(t)} - \frac{\tilde{F}(t/c_{1})}{\tilde{F}(t)} \right) \left( \frac{\tilde{F}(t/(2c_{2}))}{\tilde{F}(t)} - \frac{\tilde{F}(t/c_{2})}{\tilde{F}(t)} \right) - \frac{\tilde{F}(t/c_{1})\tilde{F}(t/c_{2})}{\tilde{F}^{2}(t)}
$$

$$
= \frac{A(t)}{\tilde{A}(t)} I + \frac{1}{tA(t)} II + \frac{A(t)}{\tilde{A}(t)} III + \frac{1}{tA(t)} IV + \frac{\tilde{F}(t)}{A(t)} V.
$$
The expression of interest becomes
\[
P(c_1 Z_1 + c_2 Z_2 > t) - (c_1^a + c_2^a) \bar{F}(t) = \frac{P(c_1 Z_1 + c_2 Z_2 > t) - P(c_1 Z_1 > t) - P(c_2 Z_2 > t)}{\bar{F}(t)A(t)}
\]
\[
= \frac{A(t) \left( \frac{P(c_1 Z_1 > t)}{A(t)} - c_1^a \frac{P(c_2 Z_2 > t)}{A(t)} \right)}{A(t)} + \frac{A(t)}{A(t)} \left( \frac{P(c_1 Z_1 > t)}{A(t)} - c_1^a \frac{P(c_2 Z_2 > t)}{A(t)} \right)
\]
\[
= \frac{A(t)}{A(t)} \left( I + II + VI \right) + \frac{1}{tA(t)} \left( II + IV \right) + \frac{\bar{F}(t)}{A(t)}V.
\]
where $VI$ denotes the expression between brackets in the middle term. The second order variation assumption (1.2) implies that $V$ and $VI$ converge as $t \to \infty$. Under the assumption of finite mean we prove that $I, II, III, IV$ also converge. The argument, based on Lebesgue’s dominated convergence theorem follows. Due to symmetry we consider only $I$ and $II$. Define
\[
G_t(u) = \frac{\bar{F}(t/c_2)u - u^{-a} \bar{F}(t/c_2)}{\bar{F}(t/c_2)A(t/c_2)}.
\]
Since (1.2) holds locally uniformly, it follows that for any $\epsilon > 0$, there exists a $t_0$ such that, for $t > t_0$ and all $x \in [1/2, 1]$
\[
H(x) - \epsilon \leq G_t(x) \leq H(x) + \epsilon.
\]
The limits of integration in $I$ assure that $1/2 \leq 1 - u/t \leq 1$ and therefore $G_t(1 - u/t)$ can be bounded as follows
\[
2^a \frac{2^{-a} - 1}{\rho} - \epsilon \leq (1 - u/t)^{-a} \left( \frac{(1 - u/t)^{-a} - 1}{\rho} \right) - \epsilon
\]
\[
\leq G_t(1 - u/t) \leq (1 - u/t)^{-a} \left( \frac{(1 - u/t)^{-a} - 1}{\rho} \right) + \epsilon
\]
\[
< \epsilon.
\]
The previous bound together with the fact that $G_t(1 - u/t) \to 0$ as $t \to \infty$ implies by Lebesgue’s dominated convergence theorem that $I \to 0$ as $t \to \infty$. For $II$ notice that as $t \to \infty$
\[
t((1 - u/t)^{-a} - 1) \to \alpha u
\]
and that
\[
0 \leq t((1 - u/t)^{-a} - 1) \leq 2(2^a - 1) u
\]
for $0 \leq u \leq t/2$ since $s \to ((1 - s)^{-a} - 1)/s$ is non-decreasing on $(0, 1)$. Therefore
\[
II = \frac{\bar{F}(t/c_2)}{\bar{F}(t)} \int_0^{t/2} t((1 - u/t)^{-a} - 1)dF_2(u) \to \alpha c_1 c_2 \mu.
\]
To summarize one has
\[
\lim_{t \to \infty} \frac{P(c_1 Z_1 + c_2 Z_2 > t) - (c_1^a + c_2^a) \bar{F}(t)}{\bar{F}(t)A(t)} = (H(c_1^{-1}) + H(c_2^{-1})) \lim_{t \to \infty} \frac{A(t)}{A(t)}
\]
\[
+ \alpha \mu(c_1 c_1^a + c_2 c_2^a) \lim_{t \to \infty} \frac{1}{tA(t)} + 2^{a+1} c_1 c_2 (2^a - 1) \frac{\bar{F}(t)}{A(t)}.
\]
Making specific choices of $\bar{A}(t)$ one recovers the different limit functions specified in items II and III of the theorem. □
4. Central limit theorem.

The first order regular variation of distribution tails has an exact probabilistic equivalent in the weak convergence of associated point processes to a Poisson process limit. This has been a very useful tool in studying heavy tailed phenomena which are quite complicated functionals of iid random variables. See Resnick (1986, 1987). We present a probabilistic equivalent of second order regular variation which is then applied to discuss the equivalence of second order regular variation and asymptotic normality of Hill’s estimator.

The connection between second order regular variation and the central limit theorem stems from the following invariance principle.

**Proposition 4.1.** Suppose \( \{Z_n, n \geq 1\} \) are iid non-negative random variables with common distribution \( F \) whose tail is regularly varying so that (1.1) holds. Let \( b(t) \) be the quantile function defined by

\[
b(t) = \left( \frac{1}{1 - F} \right)^{-}(t).
\]

Let the tail empirical measure be

\[
\nu_n(\cdot) = \frac{1}{k} \sum_{i=1}^{n} \epsilon_{Z_i/b(n/k)}(\cdot)
\]

so that \( k \nu_n(A) \) is the cardinality of \( \{i : Z_i/b(n/k) \in A\} \). Then if \( k = k(n) \) satisfies \( k \to \infty \) and \( k/n \to 0 \), we have

\[
\sqrt{k} \left[ \nu_n((x, \infty]) - E \nu_n((x, \infty]) \right] \Rightarrow W(x^{-\alpha})
\]

in \( D((0, \infty]) \), where \( \{W(t), t \geq 0\} \) is a standard Brownian motion.

Note that

\[
E \nu_n((x, \infty]]) = \frac{n}{k}(1 - F(b_n(x))).
\]


Here is a characterization of second order regular variation based on the central limit theorem. The setup in Proposition 4.1 is still in force.

**Theorem 4.2.** Suppose \( 1 - F \in RV_{-\alpha} \). We have that \( 1 - F \) is second order regularly varying iff for some \( \theta \in [0, 1) \) there exists a function \( U(t) \in RV_{\theta} \) such that \( U(t) \to \infty \) as \( t \to \infty \) and there exists a function \( g(x), x \geq 1 \) not identically zero such that with \( k = \lfloor U(n) \rfloor \) we have for each \( x \geq 0 \)

\[
\sqrt{k} \left[ \frac{1}{k} \sum_{i=1}^{n} \epsilon_{Z_i/b(n/k)}(x, \infty] - x^{-\alpha} \right] \Rightarrow W(x^{-\alpha}) + g(x)
\]

in \( (0, \infty) \). In this case, \( 1 - F \in 2RV(-\alpha, \rho) \) with

\[
\theta = \frac{2|\rho|}{\alpha + 2|\rho|}, \quad \rho = \frac{-a \theta / 2}{1 - \theta},
\]

and

\[
\frac{1 - F(tx)}{A(t)} - x^{-\alpha} \to g(x)
\]
where the function $A$ is specified as follows. Define

\begin{equation}
(4.4) \quad h(t) = \frac{t}{U(t)} \in RV_{1-\theta},
\end{equation}

where $0 < 1 - \theta \leq 1$,

\begin{equation}
(4.5) \quad \chi(t) = U \circ h^+(t) \in RV_{\frac{\rho}{\alpha}}
\end{equation}

and

\begin{equation}
(4.6) \quad A(b(t)) = \frac{1}{\sqrt{\chi(t)}} \in RV_{\frac{\rho}{\alpha+1}}.
\end{equation}

**Proof.** Suppose first that $1 - F \in 2RV(-\alpha, \rho)$ and (1.2) holds. Then

\begin{equation}
A(t) \in RV_\rho, \quad b(t) \in RV_{1/\alpha}
\end{equation}

so that

\begin{equation}
A(b(t)) \in RV_{\rho/\alpha}.
\end{equation}

Define

\begin{equation}
V(x) = \frac{\sqrt{x}}{A(b(x))} \in RV_{\frac{\rho}{\alpha+1}}
\end{equation}

so that

\begin{equation}
V^+ \in RV_{\frac{\rho}{\alpha+1}}
\end{equation}

and set

\begin{equation}
U(t) = \frac{t}{V^-(\sqrt{t})} \in RV_{\frac{\rho}{\alpha+1}}.
\end{equation}

We may set $\theta = \frac{2|\rho|}{\alpha+|\rho|}$ and then $0 \leq \theta < 1$. Thus $U(t)/t \to 0$ as $t \to \infty$. Furthermore we claim that $U(t) \to \infty$. If $|\rho| > 0$, this claim is obvious. If not, note $U(t) \to \infty$ iff $t^2/V^-(t) \to \infty$ iff $V^2(t)/t \to \infty$ iff $1/A^2(b(t)) \to \infty$ which follows from the fact that $A(t) \to 0$.

Now we may set $k = \lceil U(n) \rceil$ confident that $k \to \infty$ and $k/n \to 0$. Also we observe that

\begin{equation}
\sqrt{k}A(b(n/k)) = \sqrt{n}\left(\frac{\sqrt{k/n}A(b(n/k))}{\sqrt{n/[U(n)]}}\right) \sim \sqrt{n}\left(V\left(\frac{n}{U(n)}\right)\right)^{-1} \sim 1.
\end{equation}

as $n \to \infty$. So it follows that

\begin{equation}
\sqrt{k}\left(\frac{1}{k} \sum_{i=1}^{n} \epsilon Z_i/b(n/k)(x, \infty) - x^{-\alpha}\right) = \sqrt{k}\left(\frac{1}{k} \sum_{i=1}^{n} \epsilon Z_i/b(n/k)(x, \infty) - \frac{n}{k}(1 - F(b(n/k)x))\right)
\end{equation}

\begin{equation}
+ \sqrt{k}\left(\frac{n}{k}(1 - F(b(n/k)x)) - x^{-\alpha}\right) = W(x^{-\alpha}) + o_p(1) + \sqrt{k}A(b(n/k)) \left(\frac{\int (1 - F(b(n/k)x)) - x^{-\alpha}}{A(b(n/k))}\right) \Rightarrow W(x^{-\alpha}) + 1H(x),
\end{equation}
so the desired result holds with \( g = H \).
Conversely, suppose
\[
\sqrt{\epsilon} \sum_{i=1}^{n} \epsilon z_{i/b(n/k)}(x, \infty) - x^{-\alpha} \Rightarrow W(x^{-\alpha}) + g(x).
\]
Referring to Proposition 4.1, we conclude that
\[
\frac{\sqrt{\epsilon} (1 - F(b(n/k)x)) - x^{-\alpha}}{1/\sqrt{\epsilon}} \to g(x).
\]
Define
\[
A(b(n/k)) = \frac{1}{\sqrt{U(n)}}, \quad \chi(t) = \frac{1}{A^{2}(b(t))},
\]
So
\[
\chi \left( \frac{n}{U(n)} \right) \sim U(n),
\]
where \( U \in RV_{\theta}, \ 1 > \theta \geq 0 \) and
\[
h(t) := \frac{t}{U(t)} \in RV_{1-\theta}, \quad h^{-}(t) \in RV_{1/(1-\theta)}.
\]
It follows that
\[
\chi(t) \sim U(h^{+}(t)) \in RV_{1-\theta}
\]
so
\[
A(b(t)) \sim \frac{1}{\sqrt{\chi(t)}} \in RV_{1-\theta/2}.
\]
Therefore
\[
\frac{\sqrt{\epsilon} (1 - F(b(n/k)x)) - x^{-\alpha}}{A(b(n/k))} \to g(x)
\]
and a standard argument (Geluk and de Haan, 1987) allows the conclusion that
\[
\frac{1-F(t)}{1-\epsilon F(t)} - x^{-\alpha} \to g(x)
\]
and with
\[
\rho = \frac{-\alpha \theta / 2}{1-\theta},
\]
we get \( 1 - F \in 2RV(-\alpha, \rho) \) as claimed. \( \square \)

**Remark:** Examining Theorem 4.2, one sees that (4.2) in fact holds in \( D(0, \infty) \) and with \( g = H \).
We now discuss the relationship between asymptotic normality of Hill’s estimator and second order regular variation.

**Theorem 4.3.** Suppose \( 1-F \in RV_{-\alpha} \) and that the von Mises condition holds: \( F \) has a density \( F' \) satisfying
\[
\lim_{x \to \infty} \frac{x F'(x)}{1 - F(x)} = \alpha.
\]
Then \(1 - F\) is second order regularly varying \(\text{iff for some } \theta \in [0, 1) \text{ there exists a function } U \in RV_\theta \text{ such that}\)
\(U(t) \to \infty \text{ as } t \to \infty \) and there exist non-zero constants \(c\) and \(\sigma > 0\) such that with \(k = [U(n)]\) we have
\[
(4.7) \quad \sqrt{k}(H_{k, n} - \alpha^{-1}) \Rightarrow N(c, \sigma^2).
\]

**Proof.** Suppose \(1 - F \in 2RV(-\alpha, \rho)\) so that (1.2) holds. From Theorem 4.2, there exists \(U \in RV_\theta, U(t) \to \infty\)
and with \(k = [U(n)]\) we have
\[
(4.8) \quad \sqrt{k}\left(\frac{1}{k} \sum_{i=1}^{n} \epsilon Z_{i/b(n/k)}(x, \infty) - x^{-\alpha}\right) \Rightarrow W(x^{-\alpha}) + H(x)
\]
in \(D(0, \infty)\). Applying Vervaat’s lemma (Vervaat, 1972) to the convergence
\[
\sqrt{k}\left(\frac{1}{k} \sum_{i=1}^{n} \epsilon Z_{i/b(n/k)}(x^{-1/\alpha}, \infty) - x\right) \Rightarrow W(x) + H(x^{-1/\alpha})
\]
in \(D[0, \infty)\), we get on taking inverses
\[
(4.9) \quad \sqrt{k}\left(\left(\frac{1}{k} \sum_{i=1}^{n} \epsilon Z_{i/b(n/k)}(x^{-1/\alpha}, \infty)\right)^{-\alpha} - x\right) \Rightarrow -(W(x) + H(x^{-1/\alpha}))
\]
and thus
\[
(4.10) \quad \sqrt{k}\left(\frac{Z_{[k/x]}}{b(n/k)}\right)^{-\alpha} - x \Rightarrow -(W(x) + H(x^{-1/\alpha}))
\]
in \(D[0, \infty)\) and
\[
(4.11) \quad \frac{Z_{[k/x]}}{b(n/k)} \Rightarrow 1
\]
in \(\mathbb{R}\). In fact, (4.8), (4.10) and (4.11) hold jointly in \(D(0, \infty) \times D[0, \infty) \times \mathbb{R}\). Applying composition of the third and the first components of this joint convergence yields
\[
\sqrt{k}\left(\frac{1}{k} \sum_{i=1}^{n} \epsilon Z_{i/Z_{i+k+1}}(x, \infty) - \left(\frac{Z_{[k+1]x}}{b(n/k)}\right)^{-\alpha}, \left(\frac{Z_{[k+1]x}}{b(n/k)}\right)^{-\alpha} - x^{-\alpha}\right)
\]
\[
\Rightarrow (W(x^{-\alpha}) + H(x), -(W(1) + H(1))x^{-\alpha})
\]
in \(D(0, \infty) \times \mathbb{R}\). Remembering that \(H(1) = 0\), we get by addition
\[
\sqrt{k}\left(\frac{1}{k} \sum_{i=1}^{n} \epsilon Z_{i/Z_{i+k+1}}(x, \infty) - x^{-\alpha}\right)
\]
\[
= \sqrt{k}\left(\frac{1}{k} \sum_{i=1}^{n} \epsilon Z_{i/Z_{i+k+1}}(x, \infty) - \left(\frac{Z_{[k+1]x}}{b(n/k)}\right)^{-\alpha}\right)
\]
\[
+ \sqrt{k}\left(\frac{Z_{[k+1]x}}{b(n/k)} - x^{-\alpha}\right)
\]
\[
\Rightarrow W(x) + H(x^{-\alpha}) - x^{-\alpha} W(1)
\]
and so a continuous mapping argument (map \( x(\cdot) \mapsto \int_1^\infty x(s)/s \, ds \)) yields (cf. Resnick and Stărică, 1995b)

\[
\sqrt{k}(H_{k,n} - \frac{1}{\alpha}) \Rightarrow \int_1^\infty W(x^{-\alpha}) \frac{dx}{x} + \int_1^\infty H(x) \frac{dx}{x} - \frac{W(1)}{\alpha}.
\]

Note

\[
\int_1^\infty H(x) \frac{dx}{x} \neq 0
\]

and so the limit is normal with non-zero mean as required.

Conversely, suppose (4.7) holds. From Davis and Resnick (1984) or Csorgo and Mason (1985) we have

\[
\sqrt{k} \left( H_{k,n} - \frac{n}{k} \int_{b(n/k)}^\infty (1 - F(s)) \frac{ds}{s} \right) \Rightarrow N(0, \frac{1}{\alpha^2})
\]

and the convergence to types theorem yields

\[
\sqrt{k} \left( \frac{n}{k} \int_{b(n/k)}^\infty (1 - F(s)) \frac{ds}{s} - \frac{1}{\alpha} \right) \to c \neq 0
\]

from which it follows that

\[
\frac{\int_t^\infty \frac{1 - F(s)}{1 - F(t)} \frac{ds}{s} - \frac{1}{\alpha}}{A(t)} \to c
\]

where \( A(t) \) is defined as in (4.6). Second order regular variation then follows from the following Proposition and the proof of Theorem 4.3 is complete. □

The following result is the second order version of Karamata’s theorem. It is similar to the second remark following de Haan’s (1995) Theorem 1.

**Proposition 4.4.** Suppose \( F \) is a distribution concentrating on \([0, \infty)\). Then

\[ 1 - F \in 2RV(-\alpha, \rho) \]

iff there exists a function \( A(t) \) satisfying \( A > 0, \ A(t) \to 0 \) and \( A \in RV_\rho \) for some \( \rho \leq 0 \) such that

\[
\lim_{t \to \infty} \frac{\int_t^\infty \frac{1 - F(s)}{1 - F(t)} \frac{ds}{s} - \frac{1}{\alpha}}{A(t)} \to c \neq 0,
\]

where \( c \) is a non-zero constant,

**Proof.** Begin by assuming (4.12) and for specificity suppose that \( c > 0 \). Then there exists a function \( V \in RV_\rho \) such that

\[
\frac{\int_1^{t^\alpha} (1 - F(x)) \frac{dx}{x}}{1 - F(t)} = \frac{1}{\alpha} + V(t).
\]

Thus

\[
- \left( \log \int_t^{t^\alpha} (1 - F(x)) \frac{dx}{x} \right)' = \frac{1 - F(t)/t}{\int_t^{t^\alpha} (1 - F(x)) \frac{dx}{x}} = \frac{t^{-1}}{\alpha^{-1} + V(t)}.
\]

So integrating from 1 to \( x \) gives for some \( k > 0 \)

\[
\int_1^x (1 - F(s)) \frac{ds}{s} = k \exp \left\{ - \int_1^x \frac{1}{\alpha^{-1} + V(s)} \frac{ds}{s} \right\}
\]
and therefore we get a representation for $1 - F$, namely

$$1 - F(x) = \frac{k}{\alpha - x + V(x)} \exp\left\{ - \int_1^x \frac{1}{\alpha - x + V(s)} \frac{ds}{s} \right\}.$$

We may now use this representation to prove the second order regular variation. We have for $x > 1$

$$\frac{1 - F(tx)}{1 - F(t)} - x^{-\alpha} = \frac{a^{-1} + V(t)}{a^{-1} + V(tx)} \exp\left\{ - \int_1^x \frac{1}{a^{-1} + V(tx)} \frac{ds}{s} \right\} - x^{-\alpha}$$

(4.13)

and writing

$$\frac{a^{-1} + V(t)}{a^{-1} + V(tx)} = 1 + \frac{V(t) - V(tx)}{a^{-1} + V(tx)}$$

we get the second order ratio in (4.13) equal to

$$= x^{-\alpha} \left[ e^{\alpha \int_1^x \frac{V(t)}{V(tx)} \frac{ds}{s} - 1} \frac{V(t) - V(tx)}{V(t)(a^{-1} + V(tx))} + \alpha \int_1^x \frac{V(t)}{V(tx)} \frac{ds}{s} \right]$$

$$= x^{-\alpha} [I + II].$$

Since $V(t) \to 0$,

$$\int_1^x \frac{\alpha V(ts)}{1 + \alpha V(ts) s} \frac{ds}{s} \to 0$$

and therefore as $t \to \infty$

$$I \sim \frac{\alpha \int_1^x \frac{\alpha V(ts)}{1 + \alpha V(ts) s} \frac{ds}{s}}{V(t)} \to \alpha^2 \int_1^x s^{\alpha - 1} ds = \alpha^2 \left( \frac{x^{\rho - 1}}{\rho} \right),$$

and

$$II \sim \alpha \left( 1 - \frac{V(tx)}{V(t)} \right) \to \alpha (1 - x^\rho).$$

So the limit of the second order ratio is of the form

$$k' x^{-\alpha} \left( \frac{x^\rho - 1}{\rho} \right)$$

as desired.

Conversely, suppose $1 - F \in 2RV(-\alpha, \rho)$ so that (1.2) holds. Write

$$\frac{\int_t^x \frac{1 - F(x)}{1 - F(t)} \frac{dx}{x} - \frac{1}{\alpha}}{A(t)} = \frac{\int_1^x \left( \frac{1 - F(tx)}{1 - F(t)} - s^{-\alpha} \right) ds}{A(t)}.$$

The result follows by applying dominated convergence to the integral on the right. For $\rho = 0$, this step is justified by Theorem 1.20(ii) of Geluk and de Haan (1987) and for $\rho < 0$ the justification is Theorem 1.8 of Geluk and de Haan.
References


