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ASSOCIATION OF
INFINITELY DIVISIBLE
RANDOM VECTORS

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Association of infinitely divisible random vectors \(*\ddagger\)

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Abstract

We show that the Lévy measure of an associated infinitely divisible random vector in \(R^d\) may charge those quadrants of the space where the coordinates have different signs. We describe further certain families of infinitely divisible random vectors for which association does require the Lévy measure to be concentrated on \(R^d_+ \cup R^d_-\).

Running title: Association of infinitely divisible random vectors

1 Introduction

A random vector \(X \in R^d\) is called associated if \(cov(f(X), g(X)) \geq 0\) for any two measurable functions \(f, g : R^d \rightarrow R\), nondecreasing in each coordinate, and such that the covariance exists. The notion of association is a very important one in probability theory and its applications; we refer the reader to [EPW67] for more information on associated random variables and their properties. It is well known that normal random vectors are associated if and only if their correlations are all nonnegative. The necessity of this condition is trivial, but its sufficiency is not (unless the dimension \(d\) is very small), and this fact is due to [Pit82]. The situation is quite different for infinitely divisible random vectors without a Gaussian component, in which case

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sufficient conditions for association have been described first. Specifically, let $X$ be an infinitely divisible random vector with characteristic function

$$\phi_X(\theta) = \exp\left\{ \int_{R^d \setminus \{0\}} \left( e^{i\langle \theta, x \rangle} - 1 - i1(\|x\| \leq 1)(\theta, x) \right) \nu(dx) + i\langle \theta, b \rangle \right\}. \quad (1.1)$$

We refer to $\nu$ as the Lévy measure of $X$, and $b$ as its shift vector. It has been shown by [Res88] that a sufficient condition for association of the components of $X$ is

$$\nu\left\{ x = (x_1, \ldots, x_d) : x_ix_j < 0 \text{ for some } i \neq j \right\} = 0. \quad (1.2)$$

In other words, the Lévy measure $\nu$ is concentrated on the positive ($R^d_+$) and the negative ($R^d_-$) quadrants of $R^d$. Later, [LRS90] proved that for $\alpha$-stable random vectors (1.2) is a necessary and sufficient condition for association, but the question of necessity of (1.2) in general remained open. In fact, this question was posed to the author by Professor A. Jakubowski, and the present paper is the result. We give an answer to the above question in Section 2, and the answer is negative.

This negative result notwithstanding, (1.2) is necessary for association of certain sub-families of infinitely divisible random vectors, more general than that of stable random vectors. These positive results are discussed in Section 3.

2 An associated infinitely divisible random vector with Lévy measure that is not supported by $R^d_+ \cup R^d_-$

The main purpose of this section is to prove the following theorem.

**Theorem 2.1** There is an associated infinitely divisible random vector $X \in R^2$ with Lévy measure $\nu$ such that

$$\nu\left\{ x = (x_1, x_2) \in R^2 : x_1x_2 < 0 \right\} > 0. \quad (2.1)$$

We prove the theorem through a sequence of steps. We start with the following simple proposition.

**Proposition 2.1** Suppose that $Y$ is an infinitely divisible random variable such that $P(Y < 0) > 0$ and, moreover,

$$Y_1 + Y_2 \geq Y_1, \quad (2.2)$$
where $Y_1, Y_2$ are independent copies of $Y$. Let $N$ be a Poisson random variable with mean 1, independent of two independent sequences \( \{Y_j^{(i)}, j = 1, 2, \ldots \}, i = 1, 2 \) of independent copies of $Y$. Then the random vector \( X = (X_1, X_2) \) defined by

\[
X_i = \sum_{j=1}^{N+1} Y_j^{(i)}, \ i = 1, 2
\]  

(2.3)

is infinitely divisible, associated, and its Lévy measure satisfies (2.1).

**Proof:** Write

\[
(X_1, X_2) = \left( \sum_{j=1}^{N} Y_j^{(1)}, \sum_{j=1}^{N} Y_j^{(2)} \right) + (Y_1, Y_2) =: (Z_1, Z_2) + (Y_1, Y_2),
\]

where the vector \((Z_1, Z_2)\) is independent of the two independent copies $Y_1, Y_2$ of $Y$.

Obviously, the random vector \((Z_1, Z_2)\) is infinitely divisible (it is compound Poisson), and its Lévy measure is equal to $P_Y \times P_Y$, where $P_Y$ is the law of $Y$. By the assumption we have $P(Y < 0) > 0$, and then (2.2) implies further that $P(Y > 0) > 0$ as well. Therefore, we conclude that the Lévy measure of \((Z_1, Z_2)\) satisfies (2.1). Since \((Y_1, Y_2)\) is infinitely divisible by assumption, it follows that \((X_1, X_2)\) is infinitely divisible as well, and that its Lévy measure satisfies (2.1).

Finally, the assumption (2.2) implies that

\[
Y_1 + \ldots + Y_n \overset{st}{\ge} Y_1 + \ldots + Y_m
\]

(2.4)

for all $n \geq m \geq 1$. Now association of \((X_1, X_2)\) follows, e.g., from Theorem 4.1 of [MO90]; see also [ALP78].

It is clear now that Theorem 2.1 will follow from Proposition 2.1 once we construct an infinitely divisible random variable $Y$ satisfying its assumptions. This is done in the following two lemmas.

**Lemma 2.1** Let $U$ be a random variable which is absolutely continuous, apart from a possible atom at 0. Let $f_U$ be the density of the absolutely continuous part of $U$. Assume that there is a $p > 0$ and finite positive constants $C_i, i = 1, 2, 3, 4$ such that the following conditions hold.

\[
f_U(x) \leq C_1 \text{ for almost every } x \in \mathbb{R},
\]  

(2.5)

\[
f_U(x) \leq C_2(x - 1)^{-2}e^x \text{ for almost every } x < 0,
\]  

(2.6)

\[
P(U \leq x) \geq C_3(x - 1)^{-2}e^x \text{ for every } x < 0,
\]  

(2.7)

\[
P(U > x) \sim C_4x^{-p} \text{ as } x \to \infty.
\]  

(2.8)

Then there is an $h > 0$ such that the random variable $Y = U + h$ satisfies (2.2).
PROOF: We start with showing that there is an $h > 0$ such that
\[
\lim_{x \to -\infty} \frac{P(Y_1 + Y_2 \leq x)}{P(Y_1 \leq x)} < 1. \quad (2.9)
\]
Here, as usually, $Y_1$ and $Y_2$ are independent copies of $Y = U + h$.
First of all, it follows from (2.7) that
\[
\lim_{x \to -\infty} \frac{P(Y_1 \leq x)}{x^{-2}e^{\varepsilon x}} \geq C_3 e^{-h}. \quad (2.10)
\]
Further, let $p = P(U = 0)$. Then for every $x < 0$ we have
\[
P(Y_1 + Y_2 \leq x) = 2pP(U_1 \leq x - 2h) + P(U_1 + U_2 \leq x - 2h, U_1 \neq 0, U_2 \neq 0), \quad (2.11)
\]
where $U_1$ and $U_2$ are independent copies of $U$. It follows from (2.6) that for any $x < 0$
\[
2pP(U_1 \leq x - 2h) \leq 2pC_2e^{-2h}x^{-2}e^{\varepsilon x}. \quad (2.12)
\]
Further, for any $x < 0,$
\[
P(U_1 + U_2 \leq x - 2h, U_1 \neq 0, U_2 \neq 0) = \int_{-\infty}^{\infty} f_U(y - h)P(U \leq x - y - h, U \neq 0)dy \quad (2.13)
\]
\[
= \int_{-\infty}^{x-M} + \int_{x-M}^{x+1} + \int_{x+1}^{\infty} + \int_{M}^{\infty} =: I_1(x) + I_2(x) + I_3(x) + I_4(x),
\]
where $M$ is a (large) positive constant to be chosen later. We have, as in (2.12) that
\[
I_1(x) \leq P(U \leq x - M - h) \leq C_2 e^{-(M+h)x^{-2}e^{\varepsilon x}}. \quad (2.14)
\]
Further, if $h > \max(1, M)$, a repeated use of (2.6) shows that for every $x < 0$,
\[
I_2(x) \leq \int_{x-M}^{x+1} f_U(y - h)C_2(x - y - h - 1)^{-2}e^{\varepsilon y - h}dy \quad (2.15)
\]
\[
\leq C_2(h - M)^{-2}e^{M-h}P(U \leq x + 1 - h) \leq C_2^2(h - M)^{-2}e^{M+1-2h}x^{-2}e^{\varepsilon x}.
\]
A similar repeated use of (2.6) and a few lines of algebra show that, if $h$ is as above, then for every $x < 0$
\[
I_3(x) \leq \int_{x+1}^{M} (y - h - 1)^{-2}e^{\varepsilon y - h}(x - y - h - 1)^{-2}e^{\varepsilon y - h}dy \leq 4C_2^2 e^{-2h}x^{-2}e^{\varepsilon x}. \quad (2.16)
\]
Finally, by (2.5) and (2.6) we have
\[
I_4(x) \leq C_1 \int_{M}^{\infty} P(U \leq x - y - h)dy \quad (2.17)
\]
\[ \leq C_1 C_2 \int_M^\infty (x - y - h - 1)^{-2} e^{x - y - h} \, dy \leq C_1 C_2 e^{-(h + M)} x^{-2} e^x. \]

Combining the estimates (2.14)-(2.17) we conclude by (2.10), (2.12) and (2.13) that

\[ \lim_{x \to -\infty} \frac{P(Y_1 + Y_2 \leq x)}{P(Y_1 \leq x)} \leq \frac{C_2}{C_3} (2pe^{-h} + e^{-M} + C_2 h - M - 2e^{M+1-h} + 4C_2 e^{-h} + C_1 e^{-M}), \]

and the right hand side above can be made as small as we wish by first choosing \( M \) large, and then choosing \( h \) large comparatively to \( M \). Therefore, there is an \( h > 0 \) such that (2.9) holds.

It is a simple consequence of (2.8) that for any \( h > 0 \)

\[ \lim_{x \to -\infty} \frac{P(Y_1 + Y_2 > x)}{P(Y_1 > x)} = 2. \]  

Fix now any \( h_0 > 0 \) such that (2.9) holds for \( h = h_0 \). It follows then from (2.9) and (2.18) that there is a \( T \in (0, \infty) \) such that for every \( x \in R \) with \( |x| \geq T \) we have

\[ P(Y_1 + Y_2 > x) > P(Y_1 > x). \]  

We claim that \( h = h_0 + 2T \) is as required in the lemma. To this end we need to show that for every \( x \in R \)

\[ P(W_1 + W_2 > x) \geq P(W_1 > x), \]  

where \( W_i = Y_i + 2T = U_i + h_0 + 2T, \ i = 1, 2. \) Consider two cases. If \( x \leq 3T \), then \( x - 4T \leq -T \), and so by (2.19)

\[ P(W_1 + W_2 > x) = P(Y_1 + Y_2 > x - 4T) > P(Y_1 > x - 4T) \geq P(W_1 > x), \]

as required. If, on the other hand, \( x > 3T \), then \( x - 2T > T \), and so by (2.19) we have

\[ P(W_1 > x) = P(Y_1 > x - 2T) < P(Y_1 + Y_2 > x - 2T) \leq P(W_1 + W_2 > x), \]

thus proving (2.20) in all cases and completing the proof of the lemma.

Obviously, the random variable \( Y \) of Lemma 2.1 satisfies the assumptions of Proposition 2.1, except, perhaps, the requirement of infinite divisibility. Therefore, Theorem 2.1 will follow once we construct an infinitely divisible random variable \( U \) satisfying the assumptions of Lemma 2.1. This is done in the following lemma.

**Lemma 2.2** Let \( Z \) be an absolutely continuous random variable with the following density:

\[ f_Z(z) = \begin{cases} 
(z - 1)^{-2} e^z & \text{if } z \leq 0, \\
0 & \text{if } 0 < z < 1, \\
bz^{-(1+p)} & \text{if } z \geq 1,
\end{cases} \]  

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where $b$ is a normalizing constant. Then the random variable

$$U = \sum_{i=1}^{N} Z_i$$

is infinitely divisible and satisfies the assumptions of Lemma 2.1, where $N$ is a Poisson random variable with mean 1, independent of a sequence $Z_1, Z_2, \ldots$ of independent copies of $Z$.

Proof: Clearly $U$ is infinitely divisible. Since $Z$ is absolutely continuous, so is $U$, apart from an atom at 0 (corresponding to $N = 0$). The density of the absolutely continuous part of $U$ can be written in the form

$$f_U(x) = \sum_{n=1}^{\infty} e^{-1} \frac{1}{n!} f_{Z_1 + \ldots + Z_n}(x). \quad (2.22)$$

Since $f_Z(z) \leq \hat{b} = \max(b, 1)$ for every $z$, it follows that $f_{Z_1 + \ldots + Z_n}(x) \leq \hat{b}$ for every $x \in R$ and $n \geq 1$ as well. Therefore, (2.5) holds with $C_1 = b(1 - e^{-1})$.

A straightforward induction argument together with some algebra shows that that for every $n \geq 1$ and $x < 0$

$$f_{Z_1 + \ldots + Z_n}(x) \leq 6^{n-1}(x - 1)^{-2}e^x, \quad (2.23)$$

and it follows from (2.22) and (2.23) that $U$ satisfies (2.6) with $C_2 = (e^6 - 1)/6e$. Furthermore, another piece of simple calculations shows that for every $x < 0$

$$P(U \leq x) \geq e^{-1}P(Z_1 \leq x) \geq (e^{-1}(1 - e^{-1})/4)(x - 1)^{-2}e^x,$$

and so $U$ satisfies (2.7) with $C_3 = e^{-1}(1 - e^{-1})/4$. It remains to check (2.8). But it follows from, say, Theorem 1 of [EGV79] that $P(U > x) \sim P(Z > x)$ as $x \to \infty$, and so $U$ does satisfy (2.8) with $C_4 = bp^{-1}$. This completes the proof of the lemma, and so Theorem 2.1 has been established as well. \qed

3 Some situations where (1.2) is necessary for association

Two results are presented in this section. The first one sheds additional light on the phenomenon discovered in the previous section: (1.2) is not, in general, necessary for the infinitely divisible random vector $X$ to be associated. Namely, it turns out that (1.2) implies more than just association of the components of $X$, and it is this point that we are going to explore now.
Let then $X$ be an infinitely divisible random vector with characteristic function 
$\phi_X(\theta)$ given by (1.1). For every $\gamma > 0$, $(\phi_X(\theta))^{\gamma}$ is also a characteristic function in $R^d$, and we let $X^{*\gamma}$ be an infinitely divisible random vector with this characteristic function. Observe that $X \overset{d}{=} X^{*1}$.

**Theorem 3.1** The condition (1.2) is equivalent to the following statement: for every $\gamma > 0$ the vector $X^{*\gamma}$ is associated.

**Proof:** Of course, if (1.2) holds, then $X^{*\gamma}$ must be associated for every $\gamma > 0$ by the result of [Res88] mentioned in the Introduction, and so we concentrate on the necessity of (1.2). Observe first of all that we may assume without loss of generality that $X$ is a symmetric random vector. Indeed, let $Y$ be an independent copy of $X$, and $Y^{*\gamma}$ an independent copy of $X^{*\gamma}$, $\gamma > 0$. Since

$$(X - Y)^{*\gamma} \overset{d}{=} X^{*\gamma} - Y^{*\gamma},$$

we conclude that the symmetric infinitely divisible random vector $Z = X - Y$ satisfies the assumptions of the theorem, and if (1.2) fails for $X$, it does so for $Z$ as well.

Further, we may assume that $d = 2$.

For a $\gamma > 0$ let $P^\gamma$ be the law of $X^{*\gamma}$ (the family $\{P^\gamma, \gamma > 0\}$ is referred to as the convolution semigroup generated by the infinitely divisible random vector $X$). Then the generator $G$ of this convolution semigroup has the form (remember that we have assumed $X$ to be symmetric, and so, in particular, the shift vector $b = 0$)

$$Gg(y) = \int_{R^2 - \{0\}} (g(x + y) - g(y) - 1(||x|| \leq 1)(x, \Delta g(y))\nu(dx),$$

for $y \in R^2$, where $g : R^2 \to R$ is in the domain $D$ of the generator $G$. Further, any $g \in C^\infty_b$ (the space of all infinitely differentiable functions $R^2 \to R$ with bounded derivatives) is in the domain $D$.

Let $f, g$ be two $C^\infty_b$ functions, nondecreasing in each coordinate. Then the assumption of the theorem implies that for every $\gamma > 0$,

$$P^\gamma f g(0) \geq P^\gamma f(0) P^\gamma g(0).$$

It follows that there is a sequence $\gamma_n \downarrow 0$ such that for every $n \geq 1$

$$\left( P^\gamma f(0) \right)_{\gamma = \gamma_n} \geq \left( P^\gamma f(0) P^\gamma g(0) \right)_{\gamma = \gamma_n},$$

which is the same as

$$P^{m} G f g(0) \geq P^{m} G f(0) P^{m} g(0) + P^{m} f(0) P^{m} G g(0).$$
Letting $n \to \infty$ we obtain

$$G_f g(0) \geq g(0)G_f(0) + f(0)G_g(0),$$

that is,

$$\int_{R^d - \{0\}} (f(x)g(x) - f(0)g(0) - 1(||x|| \leq 1)(x, \triangle f(0))\nu(dx) \geq g(0)\int_{R^d - \{0\}} f(x) - f(0) - 1(||x|| \leq 1)(x, \triangle f(0))\nu(dx)$$

$$+ f(0)\int_{R^d - \{0\}} g(x) - g(0) - 1(||x|| \leq 1)(x, \triangle g(0))\nu(dx).$$

Suppose now that (1.2) fails. Then

$$\nu\left\{(0, \infty) \times (-\infty, 0)\right\} > 0,$$

and so there is an $a > 0$ such that

$$\nu\left\{(a, \infty) \times (-\infty, -a)\right\} > 0.$$  \hspace{1cm} (3.3)

Choose now an $0 < \epsilon < 1$, and take two arbitrary $C^\infty_0$ functions $f$ and $g$, which are nondecreasing in each variable, and such that

$$f(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 \leq \epsilon a, \\ 1 & \text{if } x_1 \geq a, \end{cases}$$

$$g(x_1, x_2) = \begin{cases} 0 & \text{if } x_2 \leq -\epsilon a, \\ -1 & \text{if } x_2 \leq -a, \end{cases}$$

Since $\triangle f(0) = \triangle g(0) = 0$, we can rewrite (3.2) as

$$\int_{R^d - \{0\}} f(x)g(x)\nu(dx) \geq 0.$$  

We conclude that

$$0 \leq -\nu\{(a, \infty) \times (-\infty, -a)\} + \int_{[a, \infty) \times [-a, -\epsilon a]} g(x)\nu(dx)$$

$$- \int_{[\epsilon a, a] \times (\infty, -a]} f(x)\nu(dx) + \int_{[\epsilon a, a] \times [-a, -\epsilon a]} f(x)g(x)\nu(dx)$$

$$\leq -\nu\{(a, \infty) \times (-\infty, -a)\}$$

because $f \geq 0$ and $g \leq 0$. Clearly, this contradicts (3.3), and so (1.2) must hold.

This completes the proof of the theorem. □
Remark The assumption that $X^{*\gamma}$ is associated for all $\gamma > 0$ is equivalent to the following, apparently weaker assumption.

There is a sequence of positive numbers $\gamma_n \downarrow 0$ such that

$$X^{*\gamma_n} \text{ is associated for every } n \geq 1. \tag{3.5}$$

Indeed, (3.5) implies that $X^{*\gamma_n}$ is associated for all $m \geq 1, n \geq 1$, and since the set \{m\gamma_n, m \geq 1, n \geq 1\} is dense in $(0, \infty)$ and association is preserved under weak convergence, it follows that $X^{*\gamma}$ is associated for every $\gamma > 0$.

Example 3.1 Semistable Random Vectors

An infinitely divisible random vector $X$ is called $r$-semistable index $\alpha$ (or $r$-SS($\alpha$)), $0 < r < 1, 0 < \alpha < 2$, if for every $n \geq 1$ there is a non-random vector $D_n \in R^d$ such that

$$X^{*r^n} \overset{d}{=} r^{n/\alpha}X + D_n. \tag{3.6}$$

If $X$ is $r$-SS($\alpha$) for all $0 < r < 1$, then it is, in fact, $\alpha$-stable. We refer the reader to [CRT82] for more information on semistable random vectors.

The following corollary extends the result of [LRS90] from stable to semistable random vectors.

Corollary 3.1 An $r$-SS($\alpha$) random vector $X$ is associated if and only if its Lévy measure is concentrated on $R^d_+ \cup R^d_-$.

Proof: Only the “only if” part requires an argument. It follows from (3.6) that the association of $X$ implies the association of $X^{*r^n}$ for all $n \geq 1$. Therefore, (3.5) holds with $\gamma_n = r^n, n \geq 1$, and our statement follows from Theorem 3.1. \qed

We now present another family of infinitely divisible random vectors in which association does imply (1.2).

Let $\rho$ be a Lévy measure on $(-\infty, \infty)$. We denote by $S_\rho$ the family of infinitely divisible random vectors in $R^d$ whose Lévy measure is given by

$$\nu(A) = \int_0^\infty m(t^{-1}A)\rho(dt), \tag{3.7}$$

for a Borel set $A \in R^d$, where $m$ ranges over finite Borel measures on the unit sphere $S_d$ of $R^d$. Observe that the family of $\alpha$-stable random vectors, $0 < \alpha < 2$, is $S_\rho$ with $\rho(dt) = t^{-(1+\alpha)}dt$.

Theorem 3.2 Assume that there is an $r \geq 0$ such that for every $k > 0$

$$\lim_{x \to \infty} \frac{\rho(\{kx, \infty\})}{\rho(\{x, \infty\})} = k^{-r} \tag{3.8}$$

(“$\rho$ has a regularly or slowly varying tail at infinity”). Then a random vector $X \in S_\rho$ is associated if and only if (1.2) holds (i.e. if and only if the measure $m$ in (3.7) is concentrated on $(R^d_+ \cup R^d_-) \cap S_d$).
PROOF: Again only the "only if" part requires an argument, and again we may assume without loss of generality that the infinitely divisible random vector $X$ is symmetric (that is, the measure $m$ is symmetric).

Let $I_+$ and $I_-$ be two disjoint subsets of $\{1, \ldots, d\}$, and let $\{k_i\}$ be a collection of real numbers such that $k_i > 0$ for $i \in I_+$ and $k_i < 0$ for $i \in I_-$. It follows from Theorem 4.1 of [RS93] with $\phi(X) = \phi(X_1, \ldots, X_d) = \max_{i \in I_+ \cup I_-} \frac{X_i}{k_i}$ that

$$P\left( \max_{i \in I_+ \cup I_-} \frac{X_i}{k_i} > \lambda \right) \sim \int_{S_d} \rho\left( \frac{\lambda}{\max_{i \in I_+ \cup I_-} \left( \frac{k_i}{\lambda} \right)_+} \right) m(ds) \quad (3.9)$$

as $\lambda \to \infty$, and the regular (slow) variation assumption (3.8) together with (3.9) shows that, as $\lambda \to \infty$,

$$P\left( \max_{i \in I_+ \cup I_-} \frac{X_i}{k_i} > \lambda \right) \sim \rho\left( (\lambda, \infty) \right) \int_{S_d} \left( \max_{i \in I_+ \cup I_-} \left( \frac{s_i}{k_i} \right)_+ \right)^r m(ds) \quad (3.10)$$

Here $a_+ = a \vee 0$.

Suppose now that (1.2) fails and, say,

$$\nu \{ x = (x_1, \ldots, x_d) \in R^d : x_1 > 0, x_2 < 0 \} > 0.$$

Then, of course,

$$m \{ s = (s_1, \ldots, s_d) \in S_d : s_1 > 0, s_2 < 0 \} > 0 \quad (3.11)$$

We have by (3.10)

$$P(X_1 > \lambda \text{ or } X_2 < -\lambda) \sim \rho\left( (\lambda, \infty) \right) \int_{S_d} \left( (s_1)_+ \lor (-s_2)_+ \right)^r m(ds) \quad (3.12)$$

as $\lambda \to \infty$, while

$$P(X_1 > \lambda) \sim \rho\left( (\lambda, \infty) \right) \int_{S_d} (s_1)_+^r m(ds) \quad (3.13)$$

and

$$P(X_2 < -\lambda) \sim \rho\left( (\lambda, \infty) \right) \int_{S_d} (-s_2)_+^r m(ds) \quad (3.14)$$

as $\lambda \to \infty$. It follows from (3.12), (3.13) and (3.14) that

$$P(X_1 > \lambda, X_2 < -\lambda) \sim \rho\left( (\lambda, \infty) \right) \int_{S_d} \left( (s_1)_+ \land (-s_2)_+ \right)^r m(ds) \quad (3.15)$$

as $\lambda \to \infty$. Therefore, we conclude by (3.13) and (3.15) that

$$\lim_{\lambda \to \infty} P(X_2 < -\lambda | X_1 > \lambda) = \frac{\int_{S_d} \left( (s_1)_+ \land (-s_2)_+ \right)^r m(ds)}{\int_{S_d} (s_1)_+^r m(ds)} > 0. \quad (3.16)$$
However, as $X$ is associated, so is $(X_1, X_2)$. In particular,

$$P(X_2 < -\lambda | X_1 > \lambda) \leq P(X_2 < -\lambda). \quad (3.17)$$

It is obvious that the right hand side of (3.17) goes to zero as $\lambda$ goes to infinity, thus contradicting (3.16) which says that the left hand side of (3.17) converges to a positive limit. This contradiction proves (1.2). □

**Remark** Theorem 3.2 represents still another generalization of the result of [LRS90]. We leave to the interested reader to explore the various ways in which the assumptions of Theorem 3.2 can be relaxed. The simple formulation which we have chosen here is designed to demonstrate the main ingredients which make the argument work: (i) Lévy measure is “balanced in all directions”, and (ii) certain regularity of the “tail” of Lévy measure.

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