Obtaining exact solution from a near optimum solution for convex QP

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Abstract

Consider a convex quadratic program of size L involving m inequality constraints and possibly some equality constraints in n variables. Given a feasible solution whose objective value is within 2^{-2L} of the optimum objective value, we discuss a technique that finds a true optimum solution in at most m iterations, where each iteration involves solving at most three systems of linear equations.

Key Words: Convex quadratic programming, near optimum solution, optimum solution.

1 Introduction

Given a near optimum solution for a linear program (LP), techniques for finding a true optimum solution from it using a small number of Gaussian elimination steps have been well known for a long time, and are discussed in most mathematical programming books, see for example [1, 2]. For an LP of size L involving m inequality constraints and possibly some equality constraints in n variables, if a feasible solution whose objective value is within 2^{-L} of the optimum objective value is given, these techniques find a true optimum solution for the problem in at most m iterations, where each iteration requires the solution of a system of linear equations.

With the development of polynomial time interior point methods for LP in recent years, these techniques for moving from a near optimum to a true optimum solution have become very important theoretically. Interior point methods do not find a true optimum solution directly, they find a near optimum solution in polynomial time. And hence at the end they

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need efficient techniques for moving from a near optimum to a true optimum solution in order to claim that they can solve LPs in polynomial time.

In recent years, several polynomial time interior point methods have been developed for finding a near optimum solution for convex quadratic programs (CQP) too. If we have both primal and dual solutions with duality gap small, then it is easy to get exact solutions. But so far, if only a primal solution is available, there has been no technique to move from a near optimum solution for a CQP to a true optimum solution. In this note we discuss a simple and efficient technique for doing just that. For a CQP of size L involving m inequality constraints and possibly some equality constraints in n variables, given a feasible solution whose objective value is within 2^{-2L} of the optimum objective value, our technique finds a true optimum solution in at most m iterations, where each iteration involves solving at most three systems of linear equations. This technique is theoretically important because it closes the gap and makes it possible to claim that interior point methods find a true optimum solution for CQPs in polynomial time.

2 The algorithm

Consider the CQP

$$\begin{array}{lll} P & \text{Minimize} & q(x) = \frac{1}{2}x^T Q x + c^T x \\ & \text{subject to} & a_i^T x = b_i, i \in I, \\ & & a_i^T x \geq b_i, i \in J, \end{array}$$

where Q is a positive semidefinite matrix. Assume that all the data are integral. Let L denote the size of the data. The optimality conditions (Karush-Kuhn-Tucker conditions) for the problem are a system of linear constraints; and complementarity conditions which list several pairs of variables (each pair consists of a slack variable for an inequality constraint and its Lagrange Multiplier) and require that at least one quantity in each pair must be zero. From this we conclude that if P has an optimum solution, then it has a rational optimum solution in which each variable and the the optimum objective value are all ratios whose denominator and numerator are integers between -2^L and 2^L. Suppose that P has a finite but unknown optimum objective value of v^* and that we have a feasible solution x^1 such that

$$q(x^1) - v^* < 2^{-2L}. \tag{1}$$

We show an efficient algorithm which computes the exact solution of P from such a feasible solution x^1 using only a small number of pivot steps.

Let J_1 and J_2 be a partition of the index set J: $J_1 \cup J_2 = J$ and $J_1 \cap J_2 = \emptyset$. Consider the CQP

$$\begin{array}{lll} P(J_1, J_2) & \text{Minimize} & q(x) = \frac{1}{2}x^T Q x + c^T x \\ & \text{subject to} & a_i^T x = b_i, i \in I \cup J_1, \\ & & a_i^T x \geq b_i, i \in J_2. \end{array}$$

Let $v(J_1, J_2)$ denote the optimum objective value in this problem. $v(J_1, J_2)$ is defined to be ∞ if this problem is infeasible. Since the feasible region of $P(J_1, J_2)$ is a subset (a face) of that of P,

$$v(J_1, J_2) \geq v^*.$$

Since the size of the data of $P(J_1, J_2)$ is L, if $v(J_1, J_2)$ is finite, it is also a ratio whose denominator and numerator are integers between -2^L and 2^L. Thus we have the following result.

Proposition 1

$$v(J_1, J_2) - v^* \geq 2^{-2L} \quad \textit{if} \quad v(J_1, J_2) \neq v^*.$$

Now define $\bar{J}_1 = \{i : a_i^T x^1 = b_i, i \in J\}$ and $\bar{J}_2 = \{i : i \in J, i \notin \bar{J}_1\}$. From the properties satisfied by x^1, Proposition 1 implies that any optimum solution of $P(\bar{J}_1, \bar{J}_2)$ must be an optimum solution of P.

We now solve the following quadratic program involving equality constraints only:

$$\begin{array}{lll} \bar{P}(\bar{J}_1) & \text{Minimize} & q(x) = \frac{1}{2}x^T Q x + c^T x \\ & \text{subject to} & a_i^T x = b_i, i \in I \cup \bar{J}_1. \end{array}$$

It is well known that the problem of minimizing a convex quadratic function in n variables, subject only to a feasible system of linear equality constraints in the variables, can be solved by solving at most three systems of linear equations [3]. Actually $\bar{P}(\bar{J}_1)$ may not have a finite optimum solution, but we can find a point x' which is either an optimum solution for it, or a point x' feasible to $\bar{P}(\bar{J}_1)$ satisfying $q(x') \leq q(x^1) - 2^{-2L}$, using the procedure described below.

Procedure for Minimizing a Convex Quadratic Function Subject to Linear equality Constraints

Consider the problem

$$\begin{array}{lllll} A & \text{Minimize} & f(x) & = & \frac{1}{2}x^T D x + e^T x \\ & \text{subject to} & Bx & = & d \end{array} \tag{2}$$

where we assume that the system of equations (2) is feasible. If B is of order $r \times n$, using at most r Gauss-Jordan pivot steps on (2), we can select a basic vector for it. Let y denote the basic vector obtained, and z the vector of nonbasic variables. So, these pivot steps

partition x into basic, nonbasic parts as (y, z); and transform (2) into a system of the form $y = Fz + g$. Substituting this expression for y in $f(x)$, it becomes purely a function of the nonbasic variables z, say, $h(z) = \frac{1}{2}z^T Ez + \alpha^T z + \beta$, where E is a symmetric positive semidefinite matrix; and A gets transformed into the following unconstrained minimization problem.

$$\text{Minimize} \quad h(z) = \frac{1}{2}z^T Ez + \alpha^T z + \beta$$

Now solve the following system of linear equations

$$z^T E + \alpha^T = 0 \tag{3}$$

If (3) has a solution, let z^* be a solution for it. Let $y^* = Fz^* + g$, and $x^* = (y^*, z^*)$. In this case x^* is an optimum solution for A, terminate the procedure.

Suppose (3) has no solution. In this case $f(x)$ is unbounded below in A [3]. Since (3) has no solution, α^T is not in the row space of E, and hence the system

$$\begin{aligned} z^T E &= 0 \\ z^T \alpha^T &= -1 \end{aligned} \tag{4}$$

has a solution. Find a solution $\hat{z}$ for (4). Let $\hat{y} = F\hat{z}$. Let $\bar{x}$ be a feasible solution for (2), for example $\bar{x} = (\bar{y} = g, \bar{z} = 0)$. It can be verified that $\bar{x} + \lambda(\hat{y}, \hat{z})$ is feasible to (2) for all λ and $f(\bar{x} + \lambda(\hat{y}, \hat{z})) \to -\infty$ as $\lambda \to +\infty$, and so a positive value for λ can be found such that $f(\bar{x} + \lambda(\hat{y}, \hat{z})) < \gamma$ for any specified γ. Terminate the procedure with such a point.

Strategy for Computing an Exact Optimum Solution for P Beginning with the Near Optimum Solution x^1 Satisfying (1)

Using the Procedure described above, find either an optimum solution for $\bar{P}(\bar{J}_1)$, or a feasible solution x' for it satisfying $q(x') \leq q(x^1) - 2^{-2L}$. If x' is optimal to $\bar{P}(\bar{J}_1)$ and is feasible for $P(\bar{J}_1, \bar{J}_2)$ it is an optimum solution for P, terminate. Otherwise x' must be infeasible to $P(\bar{J}_1, \bar{J}_2)$. In this case as you move from x^1 to x' on the line segment joining them, let $\{x : a_{i_1}^T x = b_{i_1}\}$ be one of the first hyperplanes that intersects this line segment among all the hyperplanes $\{x : a_i^T x = b_i\}$ for $i \in \bar{J}_2$. Let x^2 be the point of intersection of the line segment joining x^1 to x' with the hyperplane $\{x : a_{i_1}^T x = b_{i_1}\}$. Clearly x^2 is feasible to P and satisfies $q(x^2) < q(x^1)$. x^2 satisfies at least one more inequality constraint among those in P as an equation, than x^1. Now we repeat the same process with x^2 as the current near optimum solution instead of x^1.

Since the number of inequalities constraints in P satisfied by the current near optimum solution strictly increases after each iteration, we can obtain a solution of P after at most $\#J$ iterations of this process.

If $Q = 0$ in P, it reduces to an LP. In this case it can be verified that the procedure discussed here reduces to the well known technique for moving from a near optimum solution to a true optimum solution in LP. Hence, this process can be viewed as a generalization of the technique for moving from a near optimum to a true optimum solution in LP, to the case of CQP.

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