SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853-3801

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STABILITY CRITICAL GRAPHS
AND RANK FACETS OF THE
STABLE SET POLYTOPE

by

E.C. Sewell and L.E. Trotter, Jr.¹

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Stability Critical Graphs and Rank Facets of the Stable Set Polytope

E. C. Sewell
John M. Olin School of Business
Washington University
St. Louis, Missouri 63130-4899

L. E. Trotter, Jr.*
School of OR&IE
Cornell University
Ithaca, New York 14853

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Abstract

This paper studies the relationship between stability critical (α-critical) graphs and rank facets of the stable set polytope. It is shown that the induced subgraph corresponding to nonzero coefficients of any rank facet contains a connected, spanning, α-critical subgraph. Stability critical graphs are then used to develop a sequence of successively tighter linear relaxations of the stable set polytope. For the class of graphs that do not contain an even subdivision of $K_4$, these results are extended to show that every rank facet is due to either an edge or odd cycle, thus leading to a polynomial time algorithm to find a maximum stable set and a set of edges and odd cycles that cover the nodes of the graph (i.e., an integral optimal dual solution).

The problem of finding a maximum stable set in a graph is a well-known, difficult problem with many real-world applications. One standard approach to understanding a difficult graph-theoretic problem is to study graphs that are "minimal" or "critical" with respect to the problem. For the stable set problem, this has led to the study of α-critical graphs, i.e., graphs with the property that the deletion of each edge increases the size of a maximum stable set. Similarly, a popular approach to understanding a difficult combinatorial optimization problem is to study the polyhedron associated with an integer programming

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formulation of the problem, thus bringing to bear on the problem the powerful tools of linear programming. For the stable set problem, this has led to the study of the \textit{stable set polytope}, i.e., the convex hull of the incidence vectors of all the stable sets in the graph, which will be denoted by $P(G)$ for a graph $G$. Both of these approaches have been studied extensively, with dozens of papers written on each. This paper studies the relationship between $\alpha$-critical graphs and rank facets (every coefficient is either 0 or 1) of the stable set polytope.

In 1975 Chvátal [4] showed that there is a close relationship between $\alpha$-critical graphs and facets of the stable set polytope by proving that there is a rank facet of $P(G)$ associated with every $\alpha$-critical graph $G$. In Section 2, this relationship is shown to be even stronger by proving that there is an $\alpha$-critical graph associated with every rank facet of the stable set polytope. Section 3 uses this relationship to develop a sequence of successively tighter linear relaxations of the stable set polytope.

Chvátal [4] also gave a polynomial time algorithm to find a maximum stable set in a series-parallel graph, i.e., a graph that does not contain a subdivision of $K_4$. Boulala and Uhry [3] proved that every nontrivial facet of the stable set polytope of a series-parallel graph is due to either an edge or a chordless odd cycle. The results of Sections 2 and 3 are used in Section 4 to show that if $G$ is a graph that does not contain an even subdivision of $K_4$, then every nontrivial, rank facet of $P(G)$ is also due to either an edge or a chordless odd cycle. This leads to a polynomial time algorithm to find a maximum stable set and a set of edges and odd cycles that cover the nodes of such a graph (i.e., an integral optimal dual solution).

1 Definitions and Notation

Throughout this paper we will assume that $G = (V, E)$ is a simple, undirected graph consisting of a finite set $V$ of nodes together with a finite set $E$ of edges. An edge is an unordered pair $(u, v)$ of nodes $u, v \in V$. The nodes $u, v$ are called the \textit{endnodes} of the edge $(u, v)$, which is said to \textit{connect} $u$ and $v$. An edge $(u, v)$ is said to be \textit{incident} to $u$ and $v$. Two edges are \textit{adjacent} if they have an endnode in common. Two nodes are \textit{adjacent (neighbors)} if there is an edge that connects them. If $X \subseteq V$, then $N(X)$ is the set of neighbors of $X$ in $G$. The
degree of a node $u$, denoted by $d(G, u)$, is the number of nodes adjacent to $u$ (whenever $G$ is
clear from the context, simply $d(u)$ will be used). A node with degree equal to zero is said
to be isolated.

A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$, denoted $G' \subseteq G$, if $V' \subseteq V$ and
$E' \subseteq E$. If $V' = V$, then $G'$ is said to be a spanning subgraph of $G$. If $W \subseteq V$, then $G[W]
is the subgraph induced by $W$ (i.e., $G[W]$ has $W$ as its node set and two nodes are adjacent
in $G[W]$ if and only if they are adjacent in $G$). If $v \in V$, then $G - v$ will also be used to
denote $G[V \setminus \{v\}]$. If $(u, v) \in E$, then $G - (u, v)$ is the subgraph obtained by deleting $(u, v)$
from $E$.

A set of mutually nonadjacent nodes in a graph $G$ is called a stable (independent) set.
A maximum stable set (MSS) is a stable set of maximum cardinality. The stability number
of $G$, denoted by $\alpha(G)$, is the cardinality of any maximum stable set in $G$. The problem of
finding an MSS in a graph $G = (V, E)$ can be formulated as

$$\begin{align*}
\text{max} \quad & \mathbf{x} \\
x_v + x_w & \leq 1 \quad \forall (v, w) \in E \\
x & \geq 0 \\
x_v &= 0 \text{ or } 1 \quad \forall v \in V,
\end{align*}$$

where a stable set $S$ is associated with every feasible solution $x$ of SP by including $v$ in $S$ if
and only if $x_v = 1$. The linear program obtained by deleting the binary constraints of SP is
called the fractional stable set problem (FSP). The optimal objective function value of FSP
is denoted by $\bar{\alpha}(G)$ and the set of feasible solutions to FSP is denoted by $\bar{P}(G)$. Note that
$\bar{P}(G)$ is bounded if and only if $G$ does not contain any isolated nodes.

An edge $e \in E$ is said to be critical if $\alpha(G - e) > \alpha(G)$. $G$ is stability critical, denoted
by $\alpha$-critical, if every edge of $G$ is critical. Throughout this paper, critical will always mean
stability critical.

The number $|V| - 2\alpha(G)$, denoted by $\delta(G)$, plays an important role in the study of critical
graphs, as demonstrated by a theorem given by Hajnal [10]: If $G$ is a critical graph with
no isolated nodes, then $d(v) \leq \delta(G) + 1 \forall v \in V$. This theorem is useful in characterizing
critical graphs with small values of $\delta(\cdot)$. Let $\Gamma^{\delta}$ be the set of all critical graphs with $\delta(G) = \delta$
and let $\Gamma^{\delta}_c$ be the set of all connected graphs in $\Gamma^{\delta}$. If $G \in \Gamma^{\delta}_c$, then every node of $G$ has
degree at most one, which implies that \( G \) is \( K_2 \). If \( G \in \Gamma_1^1 \), then every node of \( G \) has degree at most two. Since \( G \) is connected, \( G \) must be either a simple path or a cycle. But \( \delta(G) < 1 \) for simple paths and even cycles, so \( G \) must be an odd cycle. A subdivision of a graph is obtained by replacing its edges by simple paths, i.e., by inserting new nodes of degree two into the edges. An even subdivision results when the number of new nodes inserted into each edge is even. Hence, \( \Gamma_1^1 \) consists of even subdivisions of \( K_3 \).

2 Rank Facets and Critical Graphs

In this section we show how rank facets of the stable set polytope and critical graphs are related.

The stable set polytope of a graph \( G = (V,E) \), denoted \( P(G) \), is full-dimensional, since it always contains the unit vectors and the origin. It is well-known that this implies that \( P(G) \) has a unique set of facet-defining inequalities (up to positive scalar multiplication). It is easy to see that the inequality \( x_v \geq 0 \) is a facet of \( P(G) \) for all \( v \in V \). These facets are called the nonnegativity facets of \( P(G) \).

A valid inequality (facet) \( bx \leq b_0 \) of the stable set polytope is called a rank inequality (facet) if \( b_v \) equals either 0 or 1 for all \( v \in V \). It is evident that if \( \sum_{v \in W} x_v \leq b_0 \) is a facet of \( P(G) \), then \( b_0 = \alpha(G[W]) \). One class of rank inequalities are the clique inequalities. The inequality \( \sum_{v \in V(K)} x_v \leq 1 \) is a facet of \( P(G) \) if and only if \( K \) is a maximal clique in \( G \) (e.g., see Padberg [16]). This implies that \( x_v \leq 1 \) is a facet of \( P(G) \) if and only if \( v \) is an isolated node in \( G \). Thus, the only facets of \( P(G) \) that have exactly one nonzero coefficient are the nonnegativity facets and the clique facets for isolated nodes. These facets are called the trivial facets. In the special case when \( K \) is an edge, say \((v,w) \in E\), the valid inequality \( x_v + x_w \leq 1 \) is called an edge inequality.

Another class of rank inequalities are the odd cycle inequalities. Suppose \( C \) is an odd cycle in \( G \). Then every stable set in \( G \) can contain at most \((|C| - 1)/2\) nodes in \( C \), so

\[
\sum_{v \in C} x_v \leq (|C| - 1)/2
\]

is a valid inequality for \( P(G) \). It is not difficult to show that Equation 1 is a facet of \( P(C) \)
if and only if $C$ is chordless, but, in general, it need not be a facet of $P(G)$.

A cutset $D \subseteq E$ in a graph $G = (V, E)$ is the set of edges joining a nonempty, proper subset $W \subseteq V$ to $N(W)$ and is denoted by $D = (W, V \setminus W)$. A cutset $D = (W, V \setminus W)$ is said to be critical if $\alpha(G[W]) + \alpha(G[V \setminus W]) > \alpha(G)$. Critical cutsets were introduced by Balas and Zemel [1], who showed that if the rank inequality associated with $G$ is a facet of $P(G)$, then every cutset of $G$ is critical.

**Lemma 1 (Balas and Zemel)** If $\sum_{v \in V} x_v \leq \alpha(G)$ is a nontrivial facet of $P(G)$, then every cutset of $G$ is critical.

As pointed out by Balas and Zemel [1], when the rank inequality associated with $G$ is a nontrivial facet of $P(G)$, then $G$ must be connected, since the empty cutset is not critical.

In 1975 Chvátal [4] showed that if the set of critical edges in $G$ forms a connected subgraph, then the rank inequality associated with $G$ is a facet of $P(G)$.

**Theorem 2 (Chvátal)** Let $G = (V, E)$ be a graph and let $E'$ be the set of its critical edges. If $G' = G(V, E')$ is connected, then the rank inequality associated with $G$, $\sum_{v \in V} x_v \leq \alpha(G)$, is a facet of $P(G)$.

An immediate consequence of this theorem is that the rank inequality associated with any connected, critical graph is a facet. The converse of this is not true. The rank inequality associated with the graph in Figure 1 is a facet, but the graph is not critical, since $e$ can be removed without increasing the stability number. But it is easy to see that once $e$ is removed, the resulting graph is critical. This is not a coincidence, as demonstrated by Theorem 4, which provides a partial converse of Chvátal's theorem.

We require the following lemma (see Sewell [18] or Lovász and Plummer [14]) in the proof of Theorem 4. We include for completeness a sketch of the proof.

**Lemma 3** For $G = (V, E)$, if $\sum_{v \in V} x_v \leq \alpha(G)$ is a nontrivial facet of $P(G)$, then $|X| \leq |N(X)|$ for every stable set $X$ in $G$.

**Proof.** (Sketch) Suppose $X \subseteq V$ is a minimal node set for which $|X| > |N(X)|$. Then it can be shown that $X$ is contained in every MSS in $G$. Now $G$ must have $|V|$ linearly
independent MSSs, since $\sum_{v \in V} x_v \leq \alpha(G)$ is a facet of $P(G)$. But this contradicts that $X$ is contained in every MSS (unless $\sum_{v \in V} x_v \leq \alpha(G)$ is a trivial facet of $P(G)$).

\begin{tikzpicture}
\node (s) at (0,0) [circle, draw] {};
\node (t) at (2,2) [circle, draw] {};
\node (u) at (4,0) [circle, draw] {};
\node (v) at (2, -2) [circle, draw] {};
\node (w) at (0, -4) [circle, draw] {};
\node (x) at (4, -4) [circle, draw] {};
\node (y) at (-2, 2) [circle, draw] {};
\node (z) at (-2, -2) [circle, draw] {};
\draw (s) -- (t) -- (u) -- (v) -- (w) -- (x) -- (s);
\draw (s) -- (y) -- (z) -- (s);
\end{tikzpicture}

Figure 1: A facet-producing graph that is not critical.

**Theorem 4** If $\sum_{v \in W} x_v \leq \alpha(G[W])$ is a nontrivial facet of $P(G)$, then $G[W]$ contains a spanning subgraph $G'$ such that $G' \in \Gamma^\circ$, where $\delta = \delta(G[W]) = 2(\bar{\alpha}(G[W]) - \alpha(G[W]))$.

**Proof.** First, note that $\sum_{v \in W} x_v \leq \alpha(G[W])$ must also be a facet of $P(G[W])$. Next, remove noncritical edges from $G[W]$ recursively until all remaining edges are critical. Call the resulting graph $G'$. By construction $G'$ is critical and $\alpha(G') = \alpha(G[W])$. Since the removal of an edge never destroys a stable set, there must exist $|W|$ linearly independent stable sets of size $\alpha(G')$ in $G'$, so $\sum_{v \in W} x_v \leq \alpha(G')$ is a facet of $P(G')$. Hence, $G'$ must be connected, by Lemma 1.

All that remains is to calculate $\delta(G')$. Lemma 3 implies that $|N(X)| - |X| \geq 0$ for every stable set $X$ in $G'$. Nemhauser and Trotter [15] (see also Hammer, Hansen and Simeone [11]) have shown that this implies that $x_v = 1/2 \forall v \in V$ is an optimal solution to FSP for $G'$, so $\bar{\alpha}(G[W]) = |W|/2$. It follows that

$$\delta(G') = |W| - 2\alpha(G') = |W| - 2\alpha(G[W]) = 2\bar{\alpha}(G[W]) - 2\alpha(G[W]).$$

Loosely stated, Chvátal's theorem says that there is a rank facet associated with every critical graph, whereas Theorem 4 says that there is a critical graph associated with every rank facet.
3 Relaxations of the Stable Set Polytope

In this section we show how "critical subgraphs" of $G$ can be used to generate valid inequalities for $P(G)$, where $G' \subseteq G$ is a critical subgraph of $G$ if $G' \in \Gamma_{c}^{\delta}$ for some $\delta \geq 0$. We then show how these valid inequalities can be used to define linear relaxations of the stable set polytope.

As already mentioned in Section 1, a standard linear relaxation of the stable set polytope is

$$\tilde{P}(G) = \{ x \in R_{+}^{n} : x_{u} + x_{w} \leq 1 \ \forall (u, w) \in E \},$$

where $R_{+}^{n}$ is the set of nonnegative vectors in $R^{n}$. One way to view the edge inequalities used in $\tilde{P}(G)$ is as rank inequalities associated with subgraphs of $G$ that are in $\Gamma_{c}^{0}$. Furthermore, if $G' = (V', E')$ is any subgraph of $G$, then the rank inequality associated with $G'$, $\sum_{v \in V'} x_{v} \leq \alpha(G')$, is a valid inequality for $P(G)$. So a natural generalization of $\tilde{P}(G)$ is to use the rank inequalities associated with subgraphs of $G$ that are in $\Gamma_{c}^{\delta}$ as valid inequalities. For $\delta \geq 0$ define

$$P^{\delta}(G) = \{ x \in R_{+}^{n} : \sum_{v \in V'} x_{v} \leq \alpha(G') \ \forall G' \subseteq G \text{ with } G' \in \Gamma_{c}^{\delta}, \delta \leq \delta \}.$$ 

Clearly, $P^{0}(G) = \tilde{P}(G)$. Now, $P^{1}(G)$ uses subgraphs that are in $\Gamma_{c}^{0}$ and $\Gamma_{c}^{1}$, i.e., edges and odd cycles. Thus, we have

$$P^{1}(G) = \{ x \in R_{+}^{n} : x_{u} + x_{w} \leq 1 \ \forall (u, w) \in E ; \sum_{v \in C} x_{v} \leq (|C| - 1)/2 \ \forall \text{ odd cycles } C \subseteq G \}.$$

Since every inequality in $P^{\delta}(G)$ is valid for $P(G)$, we must have $P(G) \subseteq P^{\delta}(G) \ \forall \delta \geq 0$; i.e., $P^{\delta}(G)$ is a linear relaxation of $P(G)$. Furthermore, if $\delta > \delta'$, then the linear description of $P^{\delta}$ contains all of the inequalities present in the definition of $P^{\delta'}$, so we have

$$P^{0}(G) \supseteq P^{1}(G) \supseteq \cdots \supseteq P^{\delta}(G) \supseteq P(G).$$

It is not true that $\delta$ can always be chosen large enough so that $P^{\delta}(G) = P(G)$, since there may be some nonrank inequalities that are essential for $P(G)$. But, in view of Theorem 4, it is true that for a fixed graph, $G$, $\delta$ can be chosen large enough so that $P^{\delta}(G)$ contains all of the rank facets of $P(G)$. In this case, $P^{\delta}(G)$ approximates $P(G)$ "locally", i.e., in the
neighborhood of the MSSs in $G$. Key questions are: How well does it approximate $P(G)$ and how large must $\delta$ be?

In consideration of these questions, let $T_\delta^c$ be the set of graphs which do not contain a subgraph in $\Gamma_c^\delta$ with $\delta' > \delta$. So $T_0^c$ is the set of graphs which do not contain a subgraph in $\Gamma_c^\delta$ for $\delta > 0$. Odd cycles comprise $\Gamma_1^1$, so $T_0^0$ is exactly the set of bipartite graphs. $T^1_1$ is the set of graphs which do not contain a subgraph in $\Gamma_c^\delta$ for $\delta > 1$, which, by using the following theorem from Sewell and Trotter [19], is exactly the set of graphs which do not contain an even subdivision of $K_4$.

**Theorem 5 (Sewell and Trotter)** If $G \in \Gamma_c^\delta$ with $\delta \geq 2$, then $G$ contains an even subdivision of $K_4$.

One way to measure how well $P^\delta(G)$ approximates $P(G)$ in the neighborhood of the MSSs of $G$ is to consider the “gap” between the optimal solution to $\max 1x$ over $P^\delta(G)$ and the size of an MSS in $G$. Let us define

$$\alpha^\delta(G) = \max\{1x : x \in P^\delta(G)\}.$$

Then this “gap” is $\alpha^\delta(G) - \alpha(G)$.

The following theorem shows that $\delta$ can indeed be chosen large enough so that the “gap” is zero and gives the necessary value of $\delta$ in terms of critical subgraphs of $G$.

**Theorem 6** If $G \in T^\delta$ and has no isolated nodes, then

(i) $\exists$ a partition $V_0, V_1, \ldots, V_r$ of $V$ ($V_0$ may be empty) such that $r\delta \geq \delta(G)$ and

$$\alpha(G) = \alpha(G[V_0]) + \alpha(G[V_1]) + \ldots + \alpha(G[V_r]),$$

(ii) $\alpha^\delta(G) - \alpha(G) = 0$,

(iii) there is an integer-valued optimal solution to the linear programming dual of $\max \{1x : x \in P^\delta(G)\}$.

**Proof.** We begin by building an edge cover for nodes that are contained in every MSS in $G$. Let $M = \emptyset$, $S = \emptyset$ and $G' = G$. Choose any nonisolated node, say $u$, that is contained in every MSS in $G'$. Let $w$ be a node that is adjacent to $u$ in $G'$. Let $M = M \cup \{(u, w)\}$, $S =$
$S \cup \{u\}$ and $G' = G' - u - w$. Repeat this procedure until every node that is contained in every MSS in $G'$ is isolated. Arbitrarily choose a distinct edge from $E$ incident to each isolated node in $G'$ and add it to $M$ (such edges must exist, since there are no isolated nodes in $G$). Next, remove all the isolated nodes from $G'$ and add them to $S$. Note that $\alpha(G') = \alpha(G) - |S|$, since every time a node was removed from $G'$ and placed into $S$ it was contained in every MSS in $G'$. In addition, $|M| = |S|$. Let $V_0 = \{v \in V : v$ is an endnode of an edge in $M\}$ and $H_0$ be the graph with node set $V_0$ and edge set $M$. It is easy to see that $\alpha(H_0) = |S|$ and $\delta(H_0) \leq 0$. Note that $V_0$ may be empty, in which case $G' = G$.

Now we build a dual solution for $G'$. Remove noncritical edges from $G'$ recursively until all remaining edges are critical. Call the resulting graph $H$. No node is contained in every MSS in $H$, since no node is contained in every MSS in $G'$. Hence $H$ does not contain any isolated nodes. Furthermore, $\alpha(H) = \alpha(G')$ by construction. Let $H_1, \ldots, H_r$ be the components of $H$ and denote the nodes of $H_i$ by $V_i$ for $1 \leq i \leq r$. Since $\alpha(H) = \alpha(G')$ we have

$$\alpha(G) = \alpha(H_0) + \alpha(H_1) + \ldots + \alpha(H_r).$$

Furthermore, $G \in T^\delta$ implies $H_1, \ldots, H_r \in T^\delta$, so $\delta(H_i) \leq \delta$. It then follows that

$$\delta(G) = |V| - 2\alpha(G)$$

$$= \sum_{i=0}^r |V_i| - 2\sum_{i=0}^r \alpha(H_i)$$

$$= \sum_{i=0}^r \delta(H_i)$$

$$\leq \sum_{i=1}^r \delta(H_i)$$

$$= \sum_{i=1}^r \delta$$

$$= r\delta.$$

Thus, $r\delta \geq \delta(G)$.

Now, $H_i \in T^\delta'$ with $\delta' \leq \delta$ implies the inequality $\sum_{v \in V_i} x_v \leq \alpha(H_i)$ is included in the system of linear inequalities for $P^\delta(G)$, for $i = 1, \ldots, r$. Assign the value 1 to the dual variables corresponding to each of these rank inequalities and to the dual variables corresponding to the edges in $M$. Set all of the remaining dual variables to 0. The resulting dual solution
is feasible, integral and has value $\alpha(G)$. Since any MSS provides a feasible solution to the primal problem with value $\alpha(G)$, we conclude that this dual solution is optimal and that $\alpha^\delta(G) - \alpha(G) = 0$.

A few remarks are in order. First, the inequalities included in defining $P^\delta(G)$ are not all necessarily facets of $P(G)$. In fact, Theorem 6 is not true if $P^\delta(G)$ is restricted to using only rank facets of $P(G)$. Consider the graph in Figure 2, where $x_v$ is indicated next to each node. Then $x$ satisfies all of the rank facets, but $1x - \alpha(G) = .2 > 0$. Of course, $x$ violates the valid inequality $x(V) \leq 2$, but this inequality is not a facet of $P(G)$. Second, the number of inequalities included in the definition of $P^\delta(G)$ may grow exponentially with $\delta$, although, for fixed value of $\delta$, Lovász[13] has shown that there is a finite basis for $\Gamma^\delta_e$ (see [14] for details regarding basis graphs for $\alpha$-critical graphs), which implies that there is a finite basis for these valid inequalities. Third, the system for $P^\delta(G)$ is not necessarily totally dual integral — the dual integral solution is guaranteed only for the objective function $1x$, not all objective functions (see [17] or [5] for more details concerning total dual integrality). Finally, even though $\alpha^\delta(G) - \alpha(G) = 0$, $P^\delta(G)$ may contain nonintegral extreme points that are optimal for $\alpha^\delta(G)$, see Figure 5 in Section 4 for an example.

![Figure 2: A point satisfying all rank facets, but not in $P^\delta(G)$.](image)

4 Graphs without Even Subdivisions of $K_4$

In this section we apply and extend the results of the previous two sections to the case when $G$ does not contain an even subdivision of $K_4$. Recall that $T^1$ is precisely the class of graphs
that do not contain an even subdivision of $K_4$.

It is well-known that $P(G) = P^0(G)$ if and only if $G$ is a bipartite graph without isolated nodes. Since $T^0$ is precisely the set of bipartite graphs, it is natural to ask if $P(G) = P^1(G)$ whenever $G \in T^1$. The answer is no. The graph in Figure 3(a) is in $T^1$, but the inequality $\sum_{i=1}^6 x_i + 2x_7 \leq 3$ is a facet of the stable set polytope of this graph. The converse is also false. The graph in Figure 3(b) contains an even subdivision of $K_4$, so it is not in $T^1$, but the only nontrivial facets of $P(G)$ are due to edges and odd cycles. Nonetheless, it follows from Theorem 4 that edges and odd cycles are the only rank facets of $P(G)$ whenever $G \in T^1$; we elaborate on this in Theorem 8 below.

![Figure 3: A nonrank facet for a graph in $T^1$ and a graph not in $T^1$ with $P(G) = P^1(G)$.](image)

Before proving Theorem 8, we need the following theorem due to Padberg [16] regarding lifting the odd cycle inequality associated with any chordless odd cycle in $G$ to a facet-defining inequality of $P(G)$.

**Theorem 7 (Padberg)** Suppose $G$ is a graph, $G[C]$ is a chordless odd cycle in $G$ and $V \setminus C = \{v_1, \ldots, v_r\}$. Then nonnegative, integers $b_1, \ldots, b_r$ can be determined such that

$$
\sum_{v \in C} x_v + \sum_{v \in V \setminus C} b_v x_v \leq (|C| - 1)/2
$$

is a facet of $P(G)$.
Theorem 8 If a graph $G$ without isolated nodes contains no even subdivision of $K_4$, then every rank facet of $P(G)$ corresponds to either an edge or a chordless odd cycle. Furthermore, if $G[C]$ is a chordless odd cycle in $G$, then $\sum_{v \in C} x_v \leq (|C| - 1)/2$ is a facet of $P(G)$.

Proof. Suppose $\sum_{v \in W} x_v \leq a_0$ is a nontrivial facet of $P(G)$. Theorem 4 implies that $G[W]$ contains a spanning, connected, critical subgraph $G'$. Now, $G$ does not contain an even subdivision of $K_4$, so $G'$ is in either $\Gamma^0_2$ or $\Gamma^1_2$ by Theorem 5, hence $G'$ is either an edge or a chordless odd cycle. It is not difficult to see that $G[W]$ must in fact be the same as $G'$.

Now let $G[C]$ be a chordless odd cycle in $G$. It is always the case that $\sum_{v \in C} x_v \leq (|C| - 1)/2$ is a facet of $P(G[C])$. What needs to be shown here is that when the odd cycle inequality is lifted to a facet of $P(G)$, as in Theorem 7, every node in $V \setminus C$ receives a coefficient equal to 0. So suppose to the contrary that when the odd cycle inequality is lifted some node $u \in V \setminus C$ receives a positive coefficient, say $b_u \geq 1$ (Theorem 7 insures that $b_u$ is integral). Let $G' = G[C \cup \{u\}]$. Obviously, $\alpha(G') \geq \alpha(G[C]) = (|C| - 1)/2$. On the other hand, we must have $\alpha(G') \leq \alpha(G[C])$, since any $x \in P(G)$ satisfies

$$\sum_{v \in C} x_v + x_u \leq \sum_{v \in C} x_v + \sum_{v \in V \setminus C} b_u x_u \leq (|C| - 1)/2.$$ 

Remove noncritical edges from $G'$ recursively until all remaining edges are critical. Call the resulting graph $G''$. Every cutset in $G[C]$ is critical, so every cutset in $G'$ must also be critical. Therefore, $G''$ must be connected. But $\delta(G'') = (|C| + 1) - 2\alpha(G'') = (|C| + 1) - 2(|C| - 1)/2 = 2$. Hence, $G'' \in \Gamma^2_2$. But this contradicts $G$ does not contain an even subdivision of $K_4$, by Theorem 5. \qed

An edge cover of $G$ is a set of edges, $E' \subseteq E$, such that every node in $G$ is incident to at least one edge in $E'$. The minimum cardinality of an edge cover of $G$ is denoted by $\rho(G)$. An odd cycle cover of $G$ is a set of edges and odd cycles such that every node in $G$ is either incident to one of the edges or contained in one of the odd cycles. The cost of an edge is 1 and the cost of an odd cycle with $2k + 1$ edges is $k$. The cost of an odd cycle cover is the sum of the costs of its elements. The minimum cost of an odd cycle cover of $G$ is denoted by $\rho(G)$. The inequalities in $P^1(G)$ correspond to edges and odd cycles of $G$, so for any graph
we have
\[ \alpha(G) \leq \alpha^1(G) \leq \bar{\rho}(G) \leq \rho(G). \]

Every odd cycle cover of \( G \) corresponds in a natural way to an integer-valued solution to the linear programming dual of max \( \{ 1x : x \in P^1(G) \} \), by assigning the value 1 to the dual variables corresponding to the edges and odd cycles in the cover and assigning the value 0 to the remaining dual variables. Note that the value of such a dual solution equals the cost of the corresponding odd cycle cover.

In the literature, a graph has been called a Kőnig graph when \( \alpha(G) = \rho(G) \). A well-known theorem of Kőnig states that bipartite graphs without isolated nodes are Kőnig graphs (see, e.g., [2]). Thus, it is natural to ask whether \( \alpha(G) = \bar{\rho}(G) \) for graphs in \( T^1 \). It is easy to see that the next theorem implies the answer to this question is yes.

**Theorem 9** If a graph \( G \) without isolated nodes contains no even subdivision of \( K_4 \), then \( \alpha^1(G) - \alpha(G) = 0 \) and max \( \{ 1x : x \in P^1(G) \} \) has an integer-valued optimal dual solution which assigns positive value only to variables corresponding to edge and odd cycle inequalities that are facets of \( P(G) \).

**Proof.** Since graphs containing no even subdivisions of \( K_4 \) are in \( T^1 \), Theorem 6 implies that \( \alpha^1(G) - \alpha(G) = 0 \) and that the dual of max \( \{ 1x : x \in P^1(G) \} \) has an integral optimal solution. This dual solution assigns positive value to variables corresponding to subgraphs in \( \Gamma^0_\epsilon \) and \( \Gamma^1_\epsilon \), i.e., edges and odd cycles. All that remains to be shown is that these edges and odd cycles can be chosen in such a manner that they correspond to facets of \( P(G) \). Consider an odd cycle \( C \) whose variable has received positive weight in the optimal dual solution. If \( C \) is chordless, then it is a facet of \( P(G) \) by Theorem 8. If it is not chordless, then any chord divides \( C \) into an even cycle and an odd cycle, say \( C' \). Then \( C' \) together with the alternate edges from \( C \setminus C' \) can be used in place of \( C \) in the dual solution. Repeated application of this procedure reduces \( C \) to a chordless odd cycle. Now consider an edge \( e = (u, w) \) whose variable has positive weight in the optimal dual solution. If \( x_u + x_w \leq 1 \) is not a facet of \( P(G) \), then it must be contained in a larger clique. Since \( G \) does not contain an even subdivision of \( K_4 \), this clique must be a triangle, i.e., an odd cycle of length three. This odd cycle can be used in place of \( e \) in the optimal dual solution. It is straightforward
to verify that both of the replacements discussed here leave the value of the dual solution unchanged.

Gerards and Schrijver [6] called a subdivision of $K_4$ in which the four triangles have become odd cycles an odd-$K_4$. They proved that if $G$ does not contain an odd-$K_4$, then $P(G) = P^1(G)$. Gerards [7] extended this result by showing that $\alpha(G) = \bar{\rho}(G)$ for such graphs. (He actually showed that this equality holds in the case of weighted stable sets with an appropriate modification of the definition of an odd cycle cover.) Even subdivisions of $K_4$ are odd-$K_4$'s, so the class of graphs without odd-$K_4$'s is contained in the class of graphs without even subdivisions of $K_4$. This containment is proper; Figures 3(a) and 4 provide examples of graphs that contain an odd-$K_4$, but do not contain an even subdivision of $K_4$. Thus Theorem 9 generalizes Gerards' result in the unweighted case. (Unfortunately, Theorem 9 cannot be directly generalized to the weighted case as in [7] for odd-$K_4$'s. The graph in Figure 3(a), which does not contain an even subdivision of $K_4$, with node weights of $b_1 = \cdots = b_6 = 1$ and $b_7 = 2$, has maximum weight stable set equal to 3, but the minimum cost of an odd cycle cover (where each $v \in V$ must be covered at least $b_v$ times) is 4.)

![Graph](image)

Figure 4: A graph containing an odd-$K_4$, but not containing an even subdivision of $K_4$.

Let $\tau(G)$ denote the cardinality of any smallest node cover in $G$, i.e., a set of nodes such that every edge in $G$ has at least one endnode in the set. Let $\nu(G)$ denote the cardinality of any largest matching in $G$, i.e., a set of edges that do not have an endnode in common. Jeurissen [12] conjectured that the maximum number of edge-disjoint odd cycles in $G$ is greater than or equal to $\tau(G) - \nu(G)$, for any graph. Although Seymour [20] provided a counterexample to this conjecture, a slightly stronger result actually does hold for graphs

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in $T^1$; namely, the maximum number of node-disjoint odd cycles is greater than or equal to $\tau(G) - \nu(G)$. To see this, suppose $G$ is a graph in $T^1$ without isolated nodes, $H$ is a minimum cost odd cycle cover of $G$, and $r$ is the number of odd cycles in $H$. Theorem 9 shows that we may assume that the odd cycles in $H$ are node-disjoint. An edge cover of $G$ can be obtained from $H$ by using all of the edges in $H$ and $k + 1$ edges from each odd cycle of length $2k + 1$ in $H$. The cost of this edge cover is $\rho(G) + r$. Thus we have

$$r \geq \rho(G) - \bar{\rho}(G) = \rho(G) - \alpha(G) = \tau(G) - \nu(G),$$

where the final equality is obtained from the following well-known identity due to Gallai (see, e.g., [14]): $\alpha(G) + \tau(G) = |V| = \rho(G) + \nu(G)$. The addition of isolated nodes to $G$ does not alter the quantity $\tau(G) - \nu(G)$, so the inequality holds for all graphs in $T^1$, not just those without isolated nodes. As a final note on disjoint odd cycles, we remark that the proof of Theorem 6 implies that if $G \in T^1$, then there exists a partition $V_0, V_1, \ldots, V_s$ of $V$ such that

$$\delta(G) \leq \sum_{i=1}^{s} \delta(G[V_i])$$

and $\delta(G[V_i]) \leq 1$, for $i = 1, \ldots, s$. If a graph is bipartite, then it must have $\delta(\cdot) \leq 0$, so at least $\delta(G)$ of the graphs $G[V_1], \ldots, G[V_s]$ must contain an odd cycle. Thus $G$ contains at least $\delta(G)$ node-disjoint odd cycles.

As mentioned in Section 3, $P^1(G)$ may contain nonintegral extreme points that are optimal for $\alpha^1(G)$. Consider the graph in Figure 5, where $x_v$ is indicated next to each node. One can easily verify that $x$ satisfies all of the edge and odd cycle inequalities, but $1x - \alpha(G) = 4 - 4 = 0$. It is a straightforward task to show that $x$ is an extreme point of $P^1(G)$.

Now we turn our attention to actually finding an MSS in a graph that does not contain an even subdivision of $K_4$. We will need the following lemma which was proved by Grötschel, Lovász and Schrijver [9].

**Lemma 10 (Grötschel, Lovász and Schrijver)** The linear programming problem $\max \{1x : x \in P^1(G)\}$ can be solved in polynomial time for any graph $G$. 

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Figure 5: A nonintegral, optimal, extreme point in $P^1(G)$.

With the aid of this lemma, we are able to find an MSS in polynomial time for graphs in $T^1$.

**Theorem 11** If $G$ does not contain an even subdivision of $K_4$, then a maximum stable set can be found in polynomial time. Furthermore, if $G$ does not contain any isolated nodes, then a corresponding odd cycle cover can be found in polynomial time.

**Proof.** Isolated nodes belong to every MSS and can be removed immediately. First, calculate $\alpha^1(G)$. Theorem 9 implies $\alpha^1(G) = \alpha(G)$. Next, choose a node $v \in V$ and let $G' = G[V \setminus \{v\} \cup N(v)]$. Clearly, $v$ is in an MSS in $G$ if and only if $\alpha(G') = \alpha(G) - 1$. So, calculate $\alpha^1(G') = \alpha(G')$. If $\alpha(G') = \alpha(G) - 1$, then replace $G$ by $G'$, otherwise replace $G$ by $G - v$. By repeating this procedure at most $|V|$ times, we obtain an MSS in $G$. The overall running time is polynomial, since the running time of each iteration is polynomial by Lemma 10.

To find a corresponding odd cycle cover, consider any edge $e \in E$. Now, $e$ is critical if and only if $\alpha^1(G - e) > \alpha^1(G)$. Thus, whether $e$ is critical or not can be determined in polynomial time. Consider each edge $e \in E$, consecutively. Determine if $e$ is critical; if it is not, then replace $G$ by $G - e$. The resulting graph is critical. Therefore, it must consist of isolated nodes, edges and odd cycles. By arbitrarily choosing an edge incident to each
isolated node and adding it to this graph, we obtain a cover of weight $\alpha(G)$. As in the proof of Theorem 9, this cover can be transformed into a cover that uses edges and odd cycles corresponding to facets of $P(G)$.

We do not know a polynomial time algorithm to decide whether a graph contains an even subdivision of $K_4$. But the algorithm given in the above proof can be modified to take an arbitrary graph as input and, in polynomial time, either produce an MSS or prove that the graph is not in $T^1$ (without actually finding an even subdivision of $K_4$). All that needs to be added is a check at the end of the algorithm. If the stable set found by the algorithm has cardinality $\alpha^1(G)$, then it is an MSS in $G$, otherwise the graph is not in $T^1$.

Finally, we observe that a polynomial time algorithm for finding an MSS in a graph in $T^d$ can be designed for any value of $\delta$, provided the separation problem for the inequalities included in the linear system defining $P^d(G)$ can be solved in polynomial time.

5 Final Remarks

Recall that in Section 4 we showed that every nontrivial, rank facet of $P(G)$, where $G$ is a graph that does not contain an even subdivision of $K_4$ (i.e., $G \in T^1$), is either an edge or odd cycle facet. Furthermore, we also proved that $\alpha^1(G) = \alpha(G)$. These results together with the fact that $\alpha^1(H)$ can be determined in polynomial time for an arbitrary graph $H$ yielded a polynomial time maximum stable set algorithm for graphs in $T^1$ (Theorem 11). Since the determination of $\alpha^1(G)$ depends implicitly on the ellipsoid method for solving linear programming problems, so does our MSS algorithm. It would be of interest to find a "combinatorial" MSS algorithm for graphs in $T^1$, i.e., an algorithm that does not depend implicitly on the ellipsoid method.

Another interesting question is whether there is a polynomial time algorithm to find a maximum weight stable set for graphs in $T^1$. Of course, a graph in $T^1$ may have nontrivial facets other than edges and odd cycles, as illustrated by the graph in Figure 3(a), so any such algorithm could not depend solely upon edge and odd cycle inequalities. On the other hand, a polynomial time maximum weight stable set algorithm has been developed for the class of graphs which do not contain odd-$K_4$'s (see [6, 7, 8]). Perhaps this algorithm can be
modified to handle graphs in $T^1$.

A final question concerns variable fixing. Nemhauser and Trotter [15] have shown that if $x$ is an optimal solution to FSP for a graph $G = (V, E)$, then the set $V_1 = \{v \in V : x_v = 1\}$ is contained in an MSS. This implies that after solving FSP the variables corresponding to $V_1$ can be “fixed” at value one and $V \cup N(V_1)$ can be removed from further consideration in finding an MSS. Is this also a property of $P^\delta(G)$? First, consider the case when $G$ is in $T^6$ and suppose $\bar{x}$ is an optimal solution to max $\{1x : x \in P^\delta(G)\}$. Let $S = \{v \in V : \bar{x}_v = 1\}$, $V' = V \setminus (S \cup N(S))$ and $G' = G[V']$. Theorem 6 implies that $1\bar{x} = \alpha^\delta(G) = \alpha(G)$.

Furthermore, the restriction of $\bar{x}$ to $V'$ must be an optimal solution to max $\{1x : x \in P^\delta(G')\}$. Now $G \in T^6$ implies $G' \in T^6$, hence $\alpha(G') = \alpha^\delta(G') = \sum_{v \in V'} \bar{x}_v$. Thus, there is a stable set $S'$ in $G'$ such that $|S'| = \sum_{v \in V'} \bar{x}_v = \alpha(G')$. Then $S \cup S'$ is stable in $G$ and $|S \cup S'| = \sum_{v \in V} \bar{x}_v = \alpha(G)$.

Hence, $S$ is contained in an MSS in $G$. Therefore, variable fixing remains valid for $P^\delta(G)$ in the special case when $G \in T^6$. The question thus reduces to: Does $P^\delta'(G)$ have the variable fixing property for $\delta' = 1, \ldots, \delta - 1$ whenever $G \in T^6$? We point out that $\delta' = 0$ has been omitted from the question; this is because the result of [15] is an affirmative answer in this case. As a final note, we give an example that shows that it is necessary to include all of the inequalities that define $P^\delta(G)$. No MSS in the graph depicted in Figure 6 contains node 10, but if we maximize $1x$ over the edge inequalities together with the odd cycle inequalities for the three triangles $\{1, 2, 3\}$, $\{4, 5, 6\}$ and $\{7, 8, 9\}$, then $x_{10} = 1$ in every optimal solution. If all of the odd cycle inequalities are included, however, then every optimal solution is entirely fractional.

References


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Figure 6: A counterexample to variable fixing for arbitrary relaxations of the stable set polytope.


