SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853-3801

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POLYNOMIALITY OF THE KOJIMA-MEGIDDO-MIZUNO INFEASIBLE INTERIOR POINT ALGORITHM FOR LINEAR PROGRAMMING

by

Shinji Mizuno[1]

[1]Overseas Research Scholar of The Ministry of Science, Education and Culture, Japan.

Polynomiality of the Kojima-Megiddo-Mizuno infeasible interior-point algorithm for linear programming

Shinji Mizuno *
Cornell University, School of OR and IE
ETC Building, Ithaca, NY 14853 (April 92 - Jan. 93)
The Institute of Statistical Mathematics
4-6-7 Minami-Azabu, Minato-ku, Tokyo 106, Japan.

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Abstract

Kojima, Megiddo and Mizuno investigate an infeasible interior-point algorithm (or an exterior point algorithm) for solving a primal-dual pair of linear programming problems and they demonstrate global convergence of it. Their algorithm finds approximate optimal solutions of the pair if both problems have interior points, and they detect infeasibility when the sequence of iterates diverges. Zhang proves polynomial-time convergence of an infeasible interior-point algorithm under the assumption that both primal and dual problems have feasible points. In this paper, we show that an algorithm in the framework of the Kojima-Megiddo-Mizuno algorithm solves the pair of problems in polynomial time without this assumption. We also propose an $O(nL)$-iteration infeasible interior-point algorithm.

Key Words: Polynomial-time, Linear Programming, Primal-Dual, Interior-Point Algorithm.

Abbreviated Title: Polynomiality of an algorithm for LP.

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1 Introduction

The primal-dual infeasible interior-point algorithm for linear programming is a simple variant of the primal-dual (feasible) interior-point algorithm developed by Megiddo [7], Kojima, Mizuno, and Yoshise [2, 3], Monteiro and Adler [9, 10], and Tanabe [12]. However the theoretical behavior of the algorithm is not as well known as for the interior-point algorithm. The algorithm has been studied by Lustig [4], Lustig, Marsten, and Shanno [5], Marsten, Subramanian, Saltzman, Lustig, and Shanno [6], Tanabe [13], etc., and is known as one of the most efficient interior-point algorithms (see for example [5, 6]). Kojima, Megiddo and Mizuno [1] demonstrate global convergence of an infeasible interior-point algorithm. Their algorithm finds approximate optimal solutions of the primal-dual pair if both problems have interior points, and they detect infeasibility when the sequence of iterates diverges. Then Zhang [14] proves polynomial-time convergence of an infeasible interior-point algorithm under the assumption that both primal and dual problems have feasible points. Since the infeasible interior-point algorithm directly solves original primal and dual linear programming problems, we usually do not know whether the problems are feasible or not. In this paper, we show polynomiality of an algorithm in the framework of the Kojima-Megiddo-Mizuno algorithm without this assumption. The algorithm [1] is very flexible. We add some conditions on the initial point, the step size control and the stopping criterion to show the polynomiality.

Zhang's algorithm and ours require $O(n^2L)$ iterations. After the presentation of the original version of this paper, Potra [11] proposed an $O(n^{1.5}L)$-iteration infeasible interior-point predictor-corrector algorithm. In section 4, we construct an $O(nL)$-iteration infeasible interior-point algorithm, which is a variant of the Kojima-Megiddo-Mizuno algorithm and use the idea of the interior-point predictor-corrector algorithm proposed by Mizuno, Todd, and Ye [8].

2 A polynomial-time infeasible interior-point algorithm

Let $\boldsymbol{A}$ be an $m \times n$ matrix, $\boldsymbol{b} \in R^m$, and $\boldsymbol{c} \in R^n$. Consider the standard form linear program

$$\text{(P)} \quad \begin{array}{ll} \text{Minimize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \ \boldsymbol{x} \geq \boldsymbol{0}, \end{array}$$

and its dual

$$\text{(D)} \quad \begin{array}{ll} \text{Maximize} & \boldsymbol{b}^T\boldsymbol{y} \\ \text{subject to} & \boldsymbol{A}^T\boldsymbol{y} + \boldsymbol{z} = \boldsymbol{c}, \ \boldsymbol{z} \geq \boldsymbol{0}. \end{array}$$

Throughout the paper, we assume that the matrix $\boldsymbol{A}$ has full row rank, *i.e.*, rank $\boldsymbol{A} = m$. We call $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ an (infeasible) interior point if $\boldsymbol{x} > \boldsymbol{0}$ and $\boldsymbol{z} > \boldsymbol{0}$, and a feasible interior point if $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} > \boldsymbol{0}$, $\boldsymbol{A}^T\boldsymbol{y} + \boldsymbol{z} = \boldsymbol{c}$ and $\boldsymbol{z} > \boldsymbol{0}$.

We first describe the Kojima-Megiddo-Mizuno infeasible interior-point algorithm. Let $0 < \gamma < 1$, $\gamma_p > 0$, $\gamma_d > 0$, $\epsilon_p > 0$, and $\epsilon_d > 0$. The algorithm generates a sequence $\{(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)\}$ in the neighborhood

$$\begin{array}{rcl} \mathcal{N} = \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) & : & \boldsymbol{x} > \boldsymbol{0}, \boldsymbol{z} > \boldsymbol{0}, \\ & & x_i z_i \geq \gamma \boldsymbol{x}^T\boldsymbol{z}/n \ (i = 1, 2, \ldots, n), \\ & & \boldsymbol{x}^T\boldsymbol{z} \geq \gamma_p \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\| \ \text{ or } \ \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\| \leq \epsilon_p, \\ & & \boldsymbol{x}^T\boldsymbol{z} \geq \gamma_d \|\boldsymbol{A}^T\boldsymbol{y} + \boldsymbol{z} - \boldsymbol{c}\| \ \text{ or } \ \|\boldsymbol{A}^T\boldsymbol{y} + \boldsymbol{z} - \boldsymbol{c}\| \leq \epsilon_d\} \end{array}$$

of the path of centers, which consists of the solutions $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ to the system of equations

$$\begin{pmatrix} A\boldsymbol{x} - \boldsymbol{b} \\ A^T\boldsymbol{y} + \boldsymbol{z} - \boldsymbol{c} \\ X\boldsymbol{z} - \mu\boldsymbol{e} \end{pmatrix} = \boldsymbol{0} \tag{1}$$

for all $\mu > 0$. Here $\boldsymbol{X} = \mathrm{diag}(\boldsymbol{x})$ denotes the $n \times n$ diagonal matrix with the coordinates of a vector $\boldsymbol{x} \in R^n$ and $\boldsymbol{e} = (1, \ldots, 1)^T \in R^n$. Let $0 < \beta_1 < \beta_2 < \beta_3 < 1$. At each iteration, we assign the value $\beta_1(\boldsymbol{x}^k)^T\boldsymbol{z}^k/n$ to the parameter μ, and then compute the Newton direction $(\boldsymbol{\Delta x}, \boldsymbol{\Delta y}, \boldsymbol{\Delta z})$ at $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)$ for the system (1) of equations, that is, $(\boldsymbol{\Delta x}, \boldsymbol{\Delta y}, \boldsymbol{\Delta z})$ is the unique solution of the system of linear equations

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^T & I \\ Z^k & 0 & X^k \end{pmatrix} \begin{pmatrix} \boldsymbol{\Delta x} \\ \boldsymbol{\Delta y} \\ \boldsymbol{\Delta z} \end{pmatrix} = - \begin{pmatrix} A\boldsymbol{x}^k - \boldsymbol{b} \\ A^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c} \\ X^k\boldsymbol{z}^k - \mu\boldsymbol{e} \end{pmatrix}, \tag{2}$$

where $\boldsymbol{X}^k = \mathrm{diag}(\boldsymbol{x}^k)$ and $\boldsymbol{Z}^k = \mathrm{diag}(\boldsymbol{z}^k)$. The parameters β_2 and β_3 control the primal and dual step lengths. We can take an arbitrary initial point $(\boldsymbol{x}^1, \boldsymbol{y}^1, \boldsymbol{z}^1) \in \mathcal{N}$. Now we are ready to state the Kojima-Megiddo-Mizuno algorithm, where ϵ and ω^* are positive constants.

The Kojima-Megiddo-Mizuno infeasible interior-point algorithm

Step 1: Let $k = 1$.

Step 2: If

$$(\boldsymbol{x}^k)^T\boldsymbol{z}^k \le \epsilon, \ \|A\boldsymbol{x}^k - \boldsymbol{b}\| \le \epsilon_p \ \text{ and } \|A^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c}\| \le \epsilon_d$$

or

$$\|(\boldsymbol{x}^k, \boldsymbol{z}^k)\|_1 > \omega^* \ . \tag{3}$$

then stop.

Step 3: Let $\mu = \beta_1(\boldsymbol{x}^k)^T\boldsymbol{z}^k/n$. Compute the unique solution $(\boldsymbol{\Delta x}, \boldsymbol{\Delta y}, \boldsymbol{\Delta z})$ at $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)$ of the system (2) of equations.

Step 4: Let $\bar{\alpha}^k$ be the maximum of $\tilde{\alpha}$'s ≤ 1 such that the relations

$$\begin{aligned} &(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k) + \alpha(\boldsymbol{\Delta x}, \boldsymbol{\Delta y}, \boldsymbol{\Delta z}) \in \mathcal{N}, \\ &(\boldsymbol{x}^k + \alpha\boldsymbol{\Delta x})^T(\boldsymbol{z}^k + \alpha\boldsymbol{\Delta z}) \le (1 - \alpha(1 - \beta_2))(\boldsymbol{x}^k)^T\boldsymbol{z}^k \end{aligned}$$

hold for every $\alpha \in [0, \tilde{\alpha}]$.

Step 5: Choose a primal step length $\alpha_p^k \in (0, 1]$, a dual step length $\alpha_d^k \in (0, 1]$ and a new iterate $(\boldsymbol{x}^{k+1}, \boldsymbol{y}^{k+1}, \boldsymbol{z}^{k+1})$ such that

$$\begin{aligned} &(\boldsymbol{x}^{k+1}, \boldsymbol{y}^{k+1}, \boldsymbol{z}^{k+1}) = (\boldsymbol{x}^k + \alpha_p^k\boldsymbol{\Delta x}, \boldsymbol{y}^k + \alpha_d^k\boldsymbol{\Delta y}, \boldsymbol{z}^k + \alpha_d^k\boldsymbol{\Delta z}) \in \mathcal{N}, \\ &(\boldsymbol{x}^{k+1})^T\boldsymbol{z}^{k+1} \le (1 - \bar{\alpha}^k(1 - \beta_3))(\boldsymbol{x}^k)^T\boldsymbol{z}^k. \end{aligned}$$

Step 6: Increase k by 1. Go to Step 2.

Kojima, Megiddo and Mizuno [1] show that the algorithm terminates in a finite number of iterations, and they get a value of ω^* such that if (3) holds then there are no interior points of (P) and (D) in a wide region defined in advance.

Let

$$\rho_0 \geq \min\{\|(\boldsymbol{u}, \boldsymbol{w})\|_\infty : \boldsymbol{A}\boldsymbol{u} = \boldsymbol{b}, \boldsymbol{A}^T\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{c} \text{ for some } \boldsymbol{v}\},$$

and let $\rho \geq \rho_0$ be a constant for which we want to find the optimal solutions $\boldsymbol{x}^*$ of (P) and $(\boldsymbol{y}^*, \boldsymbol{z}^*)$ of (D), if they exist, such that

$$\|(\boldsymbol{x}^*, \boldsymbol{z}^*)\|_\infty \leq \rho.$$

Note that it is easy to compute a value of ρ_0 and we do not assume existence of the optimal solutions.

Now we state our main results which we will prove in the next section.

Theorem 2.1 *Let $(\boldsymbol{x}^1, \boldsymbol{y}^1, \boldsymbol{z}^1) = \gamma_0\rho(\boldsymbol{e}, \boldsymbol{0}, \boldsymbol{e})$ for a constant $\gamma_0 \in (0, 1]$. In the Kojima-Megiddo-Mizuno algorithm, suppose that we use*

$$\begin{aligned} \mathcal{N}' = \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \ : \ & \boldsymbol{x} > \boldsymbol{0}, \boldsymbol{z} > \boldsymbol{0}, \\ & x_i z_i \geq \gamma \boldsymbol{x}^T\boldsymbol{z}/n \ (i = 1, 2, \ldots, n), \\ & \|\boldsymbol{A}\boldsymbol{x}^1 - \boldsymbol{b}\|\boldsymbol{x}^T\boldsymbol{z} \geq \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|(\boldsymbol{x}^1)^T\boldsymbol{z}^1, \\ & \|\boldsymbol{A}^T\boldsymbol{y}^1 + \boldsymbol{z}^1 - \boldsymbol{c}\|\boldsymbol{x}^T\boldsymbol{z} \geq \|\boldsymbol{A}^T\boldsymbol{y} + \boldsymbol{z} - \boldsymbol{c}\|(\boldsymbol{x}^1)^T\boldsymbol{z}^1\} \end{aligned}$$

instead of $\mathcal{N}$, we take

$$\alpha_p^k = \alpha_d^k = \bar{\alpha}^k,$$

and we add one more stopping criterion

$$\|(\boldsymbol{x}^k, \boldsymbol{z}^k)\|_1 > \frac{1 + \gamma_0}{\gamma_0^2 \theta^k \rho}(\boldsymbol{x}^k)^T\boldsymbol{z}^k \text{ and } \theta^k > 0 \tag{4}$$

at Step 2, where $\theta^1 = 1$ and

$$\theta^{k+1} = (1 - \bar{\alpha}^k)\theta^k \text{ for } i = 1, 2, 3, \ldots .$$

When γ, γ_0, β_1, β_2 and β_3 are independent of the input data, the algorithm terminates in

$$O(n^2 L')$$

iterations, where

$$L' = \max\{\ln((\boldsymbol{x}^1)^T\boldsymbol{z}^1/\epsilon), \ln(\|\boldsymbol{A}\boldsymbol{x}^1 - \boldsymbol{b}\|/\epsilon_p), \ln(\|\boldsymbol{A}^T\boldsymbol{y}^1 + \boldsymbol{z}^1 - \boldsymbol{c}\|/\epsilon_d)\}.$$

Moreover if the algorithm stops by the condition (4), there are no optimal solutions $\boldsymbol{x}^*$ of (P) and $(\boldsymbol{y}^*, \boldsymbol{z}^*)$ of (D) such that $\|(\boldsymbol{x}^*, \boldsymbol{z}^*)\|_\infty \leq \rho$.

As shown in [1], we have

$$(\boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b}, \boldsymbol{A}^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c}) = \theta^k(\boldsymbol{A}\boldsymbol{x}^1 - \boldsymbol{b}, \boldsymbol{A}^T\boldsymbol{y}^1 + \boldsymbol{z}^1 - \boldsymbol{c}). \tag{5}$$

So $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k) \in \mathcal{N}'$ implies

$$(\boldsymbol{x}^k)^T\boldsymbol{z}^k \geq \theta^k(\boldsymbol{x}^1)^T\boldsymbol{z}^1. \tag{6}$$

This condition for $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)$ is used in Zhang [14].

3 Proof of the Theorem

In this section, we prove Theorem 2.1. The next Lemma is due to Kojima Megiddo and Mizuno [1].

Lemma 3.1 *Suppose that*

$$\mid \Delta x_i \Delta z_i - \gamma \boldsymbol{\Delta x}^T \boldsymbol{\Delta z}/n \mid \leq \eta \ \ \textit{and} \ \ \mid \boldsymbol{\Delta x}^T \boldsymbol{\Delta z} \mid \leq \eta \tag{7}$$

hold at the k*th iteration. Then* $\bar{\alpha}^k \geq \alpha^{k*}$ *for*

$$\alpha^{k*} = \min\left\{1,\ \frac{\beta_1(1-\gamma)(\boldsymbol{x}^k)^T\boldsymbol{z}^k}{n\eta},\ \frac{\beta_1(\boldsymbol{x}^k)^T\boldsymbol{z}^k}{\eta},\ \frac{(\beta_2-\beta_1)(\boldsymbol{x}^k)^T\boldsymbol{z}^k}{\eta}\right\}.$$

Proof: In Section 3 of [1], the result above is shown for a lower bound α^* instead of α^{k*}, where

$$\alpha^* = \min\left\{1,\ \frac{\beta_1(1-\gamma)\epsilon^*}{n\eta},\ \frac{\beta_1\epsilon^*}{\eta},\ \frac{(\beta_2-\beta_1)\epsilon^*}{\eta}\right\} \ \ \text{and} \ \ \epsilon^* = \min\{\epsilon,\ \gamma_p\epsilon_p,\ \gamma_d\epsilon_d\}.$$

We can prove Lemma 3.1 from the discussion in [1]. □

We will show that (7) holds for

$$\eta = O(n)(\boldsymbol{x}^k)^T\boldsymbol{z}^k.$$

Then from Lemma 3.1, we have

$$\bar{\alpha}^k \geq \delta/n^2$$

for a positive constant δ. Since we have

$$\begin{aligned}
\theta^k &= \prod_{i=1}^{k-1}(1-\bar{\alpha}^k) \leq \left(1-\frac{\delta}{n^2}\right)^{k-1}, \\
(\boldsymbol{x}^k)^T\boldsymbol{z}^k &\leq \prod_{i=1}^{k-1}(1-\bar{\alpha}^k(1-\beta_3))(\boldsymbol{x}^1)^T\boldsymbol{z}^1, \\
(\boldsymbol{A}\boldsymbol{x}^k-\boldsymbol{b}, \boldsymbol{A}^T\boldsymbol{y}^k+\boldsymbol{z}^k-\boldsymbol{c}) &= \theta^k(\boldsymbol{A}\boldsymbol{x}^1-\boldsymbol{b}, \boldsymbol{A}^T\boldsymbol{y}^1+\boldsymbol{z}^1-\boldsymbol{c}),
\end{aligned}$$

we obtain the first assertion in Theorem 2.1, that is, the algorithm terminates in $O(n^2L')$ iterations where L' is defined in Theorem 2.1.

Lemma 3.2 *At each iteration of Algorithm A, we have*

$$\begin{aligned}
\boldsymbol{D}^{-1}\boldsymbol{\Delta x} &= -\theta^k\boldsymbol{Q}\boldsymbol{D}^{-1}(\boldsymbol{x}^1-\boldsymbol{u}^1)+\theta^k(\boldsymbol{I}-\boldsymbol{Q})\boldsymbol{D}(\boldsymbol{z}^1-\boldsymbol{w}^1)-(\boldsymbol{I}-\boldsymbol{Q})(\boldsymbol{X}\boldsymbol{Z})^{-0.5}(\boldsymbol{X}\boldsymbol{z}^k-\mu\boldsymbol{e}), \\
\boldsymbol{\Delta y} &= -\theta^k(\boldsymbol{y}^1-\boldsymbol{v}^1) \\
&\quad -(\boldsymbol{A}\boldsymbol{D}^2\boldsymbol{A}^T)^{-1}\boldsymbol{A}\boldsymbol{D}\left(\theta^k\boldsymbol{D}^{-1}(\boldsymbol{x}^1-\boldsymbol{u}^1)+\theta^k\boldsymbol{D}(\boldsymbol{z}^1-\boldsymbol{w}^1)-(\boldsymbol{X}\boldsymbol{Z})^{-0.5}(\boldsymbol{X}\boldsymbol{z}^k-\mu\boldsymbol{e})\right), \\
\boldsymbol{D}\boldsymbol{\Delta z} &= \theta^k\boldsymbol{Q}\boldsymbol{D}^{-1}(\boldsymbol{x}^1-\boldsymbol{u}^1)-\theta^k(\boldsymbol{I}-\boldsymbol{Q})\boldsymbol{D}(\boldsymbol{z}^1-\boldsymbol{w}^1)-\boldsymbol{Q}(\boldsymbol{X}\boldsymbol{Z})^{-0.5}(\boldsymbol{X}\boldsymbol{z}^k-\mu\boldsymbol{e}),
\end{aligned}$$

where $\boldsymbol{X} = \mathrm{diag}(\boldsymbol{x}^k)$, $\boldsymbol{Z} = \mathrm{diag}(\boldsymbol{z}^k)$, $\boldsymbol{D} = \boldsymbol{X}^{0.5}\boldsymbol{Z}^{-0.5}$, $\boldsymbol{Q} = \boldsymbol{D}\boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{D}^2\boldsymbol{A}^T)^{-1}\boldsymbol{A}\boldsymbol{D}$ *and* $(\boldsymbol{u}^1, \boldsymbol{v}^1, \boldsymbol{w}^1)$ *is a solution of* $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{b}$, $\boldsymbol{A}^T\boldsymbol{v}+\boldsymbol{w} = \boldsymbol{c}$.

Proof: It can be directly verified that $(\boldsymbol{\Delta x}, \boldsymbol{\Delta y}, \boldsymbol{\Delta z})$ is the solution of the system (2). □

Since $\boldsymbol{Q}$ and $\boldsymbol{I}-\boldsymbol{Q}$ are orthogonal projection, we have

$$\begin{aligned}\|\boldsymbol{D}^{-1}\boldsymbol{\Delta x}\| &\leq \theta^k\|\boldsymbol{D}^{-1}(\boldsymbol{x}^1-\boldsymbol{u}^1)\| + \theta^k\|\boldsymbol{D}(\boldsymbol{z}^1-\boldsymbol{w}^1)\| + \|(\boldsymbol{XZ})^{-0.5}(\boldsymbol{X}\boldsymbol{z}^k-\mu\boldsymbol{e})\|,\\ \|\boldsymbol{D}\boldsymbol{\Delta z}\| &\leq \theta^k\|\boldsymbol{D}^{-1}(\boldsymbol{x}^1-\boldsymbol{u}^1)\| + \theta^k\|\boldsymbol{D}(\boldsymbol{z}^1-\boldsymbol{w}^1)\| + \|(\boldsymbol{XZ})^{-0.5}(\boldsymbol{X}\boldsymbol{z}^k-\mu\boldsymbol{e})\|.\end{aligned}$$

From the definition of ρ and ρ_0, we can assume that

$$\|(\boldsymbol{u}^1, \boldsymbol{w}^1)\|_\infty \leq \rho_0 \leq \rho.$$

Then $-\rho\boldsymbol{e} \leq \boldsymbol{x}^1-\boldsymbol{u}^1 \leq 2\rho\boldsymbol{e}$ and $-\rho\boldsymbol{e} \leq \boldsymbol{z}^1-\boldsymbol{w}^1 \leq 2\rho\boldsymbol{e}$. Thus we have

$$\begin{aligned}\|\boldsymbol{D}^{-1}\boldsymbol{\Delta x}\| &\leq \theta^k 2\rho\|\boldsymbol{D}^{-1}\boldsymbol{e}\| + \theta^k 2\rho\|\boldsymbol{D}\boldsymbol{e}\| + \|(\boldsymbol{XZ})^{-0.5}(\boldsymbol{X}\boldsymbol{z}^k-\mu\boldsymbol{e})\|\\ &\leq 2\theta^k\rho\|(\boldsymbol{XZ})^{-0.5}\|(\|\boldsymbol{z}^k\|+\|\boldsymbol{x}^k\|) + \sqrt{\sum_{i=1}^n((x_i^kz_i^k)^{0.5}-\mu(x_i^kz_i^k)^{-0.5})^2}\\ &\leq 4\theta^k\rho\|(\boldsymbol{XZ})^{-0.5}\|\|(\boldsymbol{x}^k,\boldsymbol{z}^k)\| + \sqrt{(\boldsymbol{x}^k)^T\boldsymbol{z}^k - 2n\mu + \sum_{i=1}^n \mu^2(x_i^kz_i^k)^{-1}}.\end{aligned}$$

Assume that $\theta^k > 0$. By using $x_i^kz_i^k \geq \gamma(\boldsymbol{x}^k)^T\boldsymbol{z}^k/n$ for each i, $\mu = \beta_1(\boldsymbol{x}^k)^T\boldsymbol{z}^k/n$, and $\|(\boldsymbol{x}^k,\boldsymbol{z}^k)\| \leq \|(\boldsymbol{x}^k,\boldsymbol{z}^k)\|_1 \leq \frac{1+\gamma_0}{\gamma_0^2\theta^k\rho}(\boldsymbol{x}^k)^T\boldsymbol{z}^k$, we see

$$\begin{aligned}\|\boldsymbol{D}^{-1}\boldsymbol{\Delta x}\| &\leq 4\theta^k\rho\sqrt{\frac{n}{\gamma(\boldsymbol{x}^k)^T\boldsymbol{z}^k}}\frac{1+\gamma_0}{\gamma_0^2\theta^k\rho}(\boldsymbol{x}^k)^T\boldsymbol{z}^k + \sqrt{(\boldsymbol{x}^k)^T\boldsymbol{z}^k - 2\beta_1(\boldsymbol{x}^k)^T\boldsymbol{z}^k + \frac{\beta_1^2}{\gamma}(\boldsymbol{x}^k)^T\boldsymbol{z}^k}\\ &= \left(\frac{4(1+\gamma_0)\sqrt{n}}{\sqrt{\gamma}\gamma_0^2} + \sqrt{1-2\beta_1+\frac{\beta_1^2}{\gamma}}\right)\sqrt{(\boldsymbol{x}^k)^T\boldsymbol{z}^k}.\end{aligned}$$

We also have this inequality without the first term in the bracket if $\theta^k = 0$. Thus we have $\|\boldsymbol{D}^{-1}\boldsymbol{\Delta x}\| = O(\sqrt{n})\sqrt{(\boldsymbol{x}^k)^T\boldsymbol{z}^k}$ and $\|\boldsymbol{D}\boldsymbol{\Delta z}\| = O(\sqrt{n})\sqrt{(\boldsymbol{x}^k)^T\boldsymbol{z}^k}$ as well, which imply that $|\boldsymbol{\Delta x}^T\boldsymbol{\Delta z}| = O(n)(\boldsymbol{x}^k)^T\boldsymbol{z}^k$ and $|\Delta x_i\Delta z_i| = O(n)(\boldsymbol{x}^k)^T\boldsymbol{z}^k$ for each i. Hence we have (7) for $\eta = O(n)(\boldsymbol{x}^k)^T\boldsymbol{z}^k$, and we have proved the former assertion of Theorem 2.1. We show the latter assertion of Theorem 2.1 in the next lemma.

Lemma 3.3 *If $\|(\boldsymbol{x}^k,\boldsymbol{z}^k)\|_1 > \frac{1+\gamma_0}{\gamma_0^2\theta^k\rho}(\boldsymbol{x}^k)^T\boldsymbol{z}^k$ holds at the kth iteration of the algorithm, then there are no optimal solutions $\boldsymbol{x}^*$ of (P) and $(\boldsymbol{y}^*,\boldsymbol{z}^*)$ of (D) such that $\|(\boldsymbol{x}^*,\boldsymbol{z}^*)\|_\infty \leq \rho$.*

Proof: Assume that we have optimal solutions $\boldsymbol{x}^*$ of (P) and $(\boldsymbol{y}^*,\boldsymbol{z}^*)$ of (D) such that $\|(\boldsymbol{x}^*,\boldsymbol{z}^*)\|_\infty \leq \rho$. From (5), we see that

$$\begin{aligned}&\boldsymbol{A}(\theta^k\boldsymbol{x}^1 + (1-\theta^k)\boldsymbol{x}^* - \boldsymbol{x}^k) = \boldsymbol{0},\\ &\boldsymbol{A}^T(\theta^k\boldsymbol{y}^1 + (1-\theta^k)\boldsymbol{y}^* - \boldsymbol{y}^k) + (\theta^k\boldsymbol{z}^1 + (1-\theta^k)\boldsymbol{z}^* - \boldsymbol{z}^k) = \boldsymbol{0}.\end{aligned}$$

So we have

$$(\theta^k\boldsymbol{x}^1 + (1-\theta^k)\boldsymbol{x}^* - \boldsymbol{x}^k)^T(\theta^k\boldsymbol{z}^1 + (1-\theta^k)\boldsymbol{z}^* - \boldsymbol{z}^k) = 0,$$

which implies

$$(\theta^k \boldsymbol{x}^1 + (1-\theta^k)\boldsymbol{x}^*)^T \boldsymbol{z}^k + (\theta^k \boldsymbol{z}^1 + (1-\theta^k)\boldsymbol{z}^*)^T \boldsymbol{x}^k = (\theta^k \boldsymbol{x}^1 + (1-\theta^k)\boldsymbol{x}^*)^T (\theta^k \boldsymbol{z}^1 + (1-\theta^k)\boldsymbol{z}^*) + (\boldsymbol{x}^k)^T \boldsymbol{z}^k.$$

By using this equality, $\boldsymbol{x}^1 = \boldsymbol{z}^1 = \gamma_0 \rho \boldsymbol{e}$, $\boldsymbol{x}^* \leq \rho \boldsymbol{e}$, $\boldsymbol{z}^* \leq \rho \boldsymbol{e}$, and $x_i^* z_i^* = 0$ for each i, we have

$$\begin{aligned}
\theta^k(\gamma_0\rho)\|(\boldsymbol{x}^k, \boldsymbol{z}^k)\|_1 &= \theta^k((\boldsymbol{z}^1)^T \boldsymbol{x}^k + (\boldsymbol{x}^1)^T \boldsymbol{z}^k) \\
&\leq (\theta^k \boldsymbol{x}^1 + (1-\theta^k)\boldsymbol{x}^*)^T \boldsymbol{z}^k + (\theta^k \boldsymbol{z}^1 + (1-\theta^k)\boldsymbol{z}^*)^T \boldsymbol{x}^k \\
&= (\theta^k \boldsymbol{x}^1 + (1-\theta^k)\boldsymbol{x}^*)^T (\theta^k \boldsymbol{z}^1 + (1-\theta^k)\boldsymbol{z}^*) + (\boldsymbol{x}^k)^T \boldsymbol{z}^k \\
&\leq n\theta^k \gamma_0 \rho^2 + (\boldsymbol{x}^k)^T \boldsymbol{z}^k.
\end{aligned}$$

From (6), $(\boldsymbol{x}^k)^T \boldsymbol{z}^k \geq \theta^k (\boldsymbol{x}^1)^T \boldsymbol{z}^1 = n\theta^k \gamma_0^2 \rho^2$. Hence we have

$$\theta^k \gamma_0 \rho \|(\boldsymbol{x}^k, \boldsymbol{z}^k)\|_1 \leq (1 + 1/\gamma_0)(\boldsymbol{x}^k)^T \boldsymbol{z}^k,$$

which contradicts the assumption of the lemma. □

4 An $O(nL)$-iteration variant

We have shown that the Kojima-Megiddo-Mizuno infeasible interior-point algorithm terminates in $O(n^2L)$ iterations by adding some conditions. The algorithm is based on the $O(nL)$-iteration primal-dual path following algorithms proposed by Kojima, Mizuno and Yoshise [2] and Mizuno, Todd and Ye [8], which use the neighborhood $\{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) : x_i z_i \geq \gamma \boldsymbol{x}^T \boldsymbol{z}/n \ (i = 1, 2, \cdots, n)$ and $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is a feasible interior point $\}$ of the path of centers. In this section, we construct an $O(nL)$-iteration variant of the infeasible interior-point algorithm. The variant is based on the $O(\sqrt{n}L)$-iteration interior-point predictor-corrector algorithm proposed by Mizuno, Todd, and Ye [8]. For $\gamma_1 \in (0, 1)$, we define a neighborhood of the path of centers:

$$\mathcal{N}_2(\gamma_1) = \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) : \boldsymbol{x} > \boldsymbol{0}, \boldsymbol{z} > \boldsymbol{0}, \|X\boldsymbol{z} - (\boldsymbol{x}^T \boldsymbol{z}/n)\boldsymbol{e}\| \leq \gamma_1 \boldsymbol{x}^T \boldsymbol{z}/n\}.$$

An $O(nL)$-iteration infeasible interior-point algorithm

Step 1: Let $k = 1$, $0 < \beta_1 < \beta_2 < 1$, $0 < \gamma_0 \leq 1$, $\gamma_1 = 1/4$, $\rho > 0$, $\theta^1 = 1$, and $(\boldsymbol{x}^1, \boldsymbol{y}^1, \boldsymbol{z}^1) = \gamma_0 \rho(\boldsymbol{e}, \boldsymbol{0}, \boldsymbol{e})$.

Step 2: If

$$(\boldsymbol{x}^k)^T \boldsymbol{z}^k \leq \epsilon, \ \|A\boldsymbol{x}^k - \boldsymbol{b}\| \leq \epsilon_p \ \text{ and } \|A^T \boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c}\| \leq \epsilon_d$$

or

$$\|(\boldsymbol{x}^k, \boldsymbol{z}^k)\|_1 > \frac{1 + \gamma_0}{\gamma_0^2 \theta^k \rho}(\boldsymbol{x}^k)^T \boldsymbol{z}^k \ \text{ and } \ \theta_k > 0 \tag{8}$$

then stop.

Step 3: Let $\mu = \beta_1 (\boldsymbol{x}^k)^T \boldsymbol{z}^k / n$. Compute the unique solution $(\Delta \boldsymbol{x}, \Delta \boldsymbol{y}, \Delta \boldsymbol{z})$ at $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)$ of the system (2) of equations.

Step 4: Let $\bar{\alpha}^k$ be the maximum of $\tilde{\alpha}$'s ≤ 1 such that the relations

$$\begin{array}{rcl} (\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k) + \alpha(\boldsymbol{\Delta x}, \boldsymbol{\Delta y}, \boldsymbol{\Delta z}) & \in & \mathcal{N}_2(2\gamma_1), \\ (\boldsymbol{x}^k + \alpha\boldsymbol{\Delta x})^T(\boldsymbol{z}^k + \alpha\boldsymbol{\Delta z}) & \leq & (1-\alpha(1-\beta_2))(\boldsymbol{x}^k)^T\boldsymbol{z}^k, \\ (\boldsymbol{x}^k + \alpha\boldsymbol{\Delta x})^T(\boldsymbol{z}^k + \alpha\boldsymbol{\Delta z}) & \geq & (1-\alpha)\theta^k(\boldsymbol{x}^1)^T\boldsymbol{z}^1 \end{array} \tag{9}$$

hold for every $\alpha \in [0, \tilde{\alpha}]$.

Step 5: Let $(\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}') = (\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k) + \bar{\alpha}^k(\boldsymbol{\Delta x}, \boldsymbol{\Delta y}, \boldsymbol{\Delta z})$ and $\theta^{k+1} = (1-\bar{\alpha}^k)\theta^k$.

Step 6: Compute the solution $(\boldsymbol{\Delta x}', \boldsymbol{\Delta y}', \boldsymbol{\Delta z}')$ of the system of equations:

$$\begin{pmatrix} A & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & A^T & I \\ Z' & \boldsymbol{0} & X' \end{pmatrix} \begin{pmatrix} \boldsymbol{\Delta x}' \\ \boldsymbol{\Delta y}' \\ \boldsymbol{\Delta z}' \end{pmatrix} = - \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ X'\boldsymbol{z}' - \frac{(\boldsymbol{x}')^T\boldsymbol{z}'}{n}\boldsymbol{e} \end{pmatrix}. \tag{10}$$

Let $(\boldsymbol{x}^{k+1}, \boldsymbol{y}^{k+1}, \boldsymbol{z}^{k+1}) = (\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}') + (\boldsymbol{\Delta x}', \boldsymbol{\Delta y}', \boldsymbol{\Delta z}')$.

Step 7: Increase k by 1. Go to Step 2.

Theorem 4.1 *When γ_0, β_1, and β_2 are independent of the input data, the algorithm above terminates in $O(nL')$ iterations, where L' is defined in Theorem 2.1. If the algorithm stops by the condition (8), there are no optimal solutions $\boldsymbol{x}^*$ of (P) and $(\boldsymbol{y}^*, \boldsymbol{z}^*)$ of (D) such that $\|(\boldsymbol{x}^*, \boldsymbol{z}^*)\|_\infty \leq \rho$.*

Note that we have (5) and (6) for each k by (2), (9), and (10). The proof of the second assertion is exactly same with Theorem 2.1. In the remainder of this section, we prove the first assertion of the theorem. At first, we show that $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k) \in \mathcal{N}_2(\gamma_1)$ for each k. It follows from $(\boldsymbol{x}^1, \boldsymbol{y}^1, \boldsymbol{z}^1) \in \mathcal{N}_2(\gamma_1)$ and the next Lemma.

Lemma 4.2 *(Mizuno, Todd, and Ye [8]) If $(\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}') \in \mathcal{N}_2(2\gamma_1)$ for $\gamma_1 = 1/4$ and $(\boldsymbol{\Delta x}', \boldsymbol{\Delta y}', \boldsymbol{\Delta z}')$ is the solution of (10) then $(\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}') + (\boldsymbol{\Delta x}', \boldsymbol{\Delta y}', \boldsymbol{\Delta z}') \in \mathcal{N}_2(\gamma_1)$.*

Lemma 4.3 *Suppose that*

$$\|\boldsymbol{\Delta X}\boldsymbol{\Delta z} - (\boldsymbol{\Delta x}^T\boldsymbol{\Delta z}/n)\boldsymbol{e}\| \leq \eta \quad \text{and} \quad |\ \boldsymbol{\Delta x}^T\boldsymbol{\Delta z}\ | \leq \eta \tag{11}$$

hold at the kth iteration. If $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k) \in \mathcal{N}_2(\gamma_1)$ then $\bar{\alpha}^k \geq \alpha^{k^*}$ for

$$\alpha^{k^*} = \min\left\{ .5,\ \sqrt{\frac{\gamma_1(\boldsymbol{x}^k)^T\boldsymbol{z}^k}{2n\eta}},\ \frac{\beta_1(\boldsymbol{x}^k)^T\boldsymbol{z}^k}{\eta},\ \frac{(\beta_2-\beta_1)(\boldsymbol{x}^k)^T\boldsymbol{z}^k}{\eta} \right\}.$$

Proof: From (2), we have

$$\begin{array}{rcl} (\boldsymbol{x}^k + \alpha\boldsymbol{\Delta x})^T(\boldsymbol{z}^k + \alpha\boldsymbol{\Delta z}) & = & (\boldsymbol{x}^k)^T\boldsymbol{z}^k - \alpha((\boldsymbol{x}^k)^T\boldsymbol{z}^k - \beta_1(\boldsymbol{x}^k)^T\boldsymbol{z}^k) + \alpha^2\boldsymbol{\Delta x}^T\boldsymbol{\Delta z} \\ & = & (1-\alpha+\beta_1\alpha)(\boldsymbol{x}^k)^T\boldsymbol{z}^k + \alpha^2\boldsymbol{\Delta x}^T\boldsymbol{\Delta z}. \end{array}$$

Since $\alpha^{k*} \le \min\{\beta_1(\boldsymbol{x}^k)^T\boldsymbol{z}^k/\eta, (\beta_2 - \beta_1)(\boldsymbol{x}^k)^T\boldsymbol{z}^k/\eta\}$ and $|\boldsymbol{\Delta x}^t\boldsymbol{\Delta z}| \le \eta$, we have

$$\begin{aligned} (\boldsymbol{x}^k + \alpha\boldsymbol{\Delta x})^T(\boldsymbol{z}^k + \alpha\boldsymbol{\Delta z}) &\le (1 - \alpha(1-\beta_2))(\boldsymbol{x}^k)^T\boldsymbol{z}^k, \\ (\boldsymbol{x}^k + \alpha\boldsymbol{\Delta x})^T(\boldsymbol{z}^k + \alpha\boldsymbol{\Delta z}) &\ge (1-\alpha)(\boldsymbol{x}^k)^T\boldsymbol{z}^k \end{aligned} \tag{12}$$

for any $0 \le \alpha \le \alpha^{k*}$. The second inequality implies (9). Hence we only need to show that

$$\|(\boldsymbol{X}^k + \alpha\boldsymbol{\Delta X})(\boldsymbol{z}^k + \alpha\boldsymbol{\Delta z}) - \mu(\alpha)\boldsymbol{e}\| \le 2\gamma_1\mu(\alpha)$$

for any $0 \le \alpha \le \alpha^{k*}$, where

$$\begin{aligned} \mu(\alpha) &= (\boldsymbol{x}^k + \alpha\boldsymbol{\Delta x})^T(\boldsymbol{z}^k + \alpha\boldsymbol{\Delta z})/n \\ &= (1 - \alpha + \beta_1\alpha)(\boldsymbol{x}^k)^T\boldsymbol{z}^k/n + \alpha^2\boldsymbol{\Delta x}^T\boldsymbol{\Delta z}/n. \end{aligned}$$

We see that

$$\begin{aligned} &\|(\boldsymbol{X}^k + \alpha\boldsymbol{\Delta X})(\boldsymbol{z}^k + \alpha\boldsymbol{\Delta z}) - \mu(\alpha)\boldsymbol{e}\| \\ &= \|\boldsymbol{X}^k\boldsymbol{z}^k - \alpha(\boldsymbol{X}^k\boldsymbol{z}^k - \beta_1((\boldsymbol{x}^k)^T\boldsymbol{z}^k/n)\boldsymbol{e}) + \alpha^2\boldsymbol{\Delta X\Delta z} \\ &\quad -((1-\alpha+\beta_1\alpha)(\boldsymbol{x}^k)^T\boldsymbol{z}^k/n + \alpha^2\boldsymbol{\Delta x}^T\boldsymbol{\Delta z}/n)\boldsymbol{e}\| \\ &\le (1-\alpha)\|\boldsymbol{X}^k\boldsymbol{z}^k - ((\boldsymbol{x}^k)^T\boldsymbol{z}^k/n)\boldsymbol{e}\| + \alpha^2\|\boldsymbol{\Delta X\Delta z} - (\boldsymbol{\Delta x}^T\boldsymbol{\Delta z}/n)\boldsymbol{e}\| \\ &\le (1-\alpha)\gamma_1(\boldsymbol{x}^k)^T\boldsymbol{z}^k/n + \frac{\gamma_1(\boldsymbol{x}^k)^T\boldsymbol{z}^k}{2n\eta}\eta \\ &\le 2\gamma_1(1-\alpha)(\boldsymbol{x}^k)^T\boldsymbol{z}^k/n \\ &\le 2\gamma_1\mu(\alpha), \end{aligned}$$

where the last inequality follows from (12). □

As shown in the previous section, we have $\|\boldsymbol{D}^{-1}\boldsymbol{\Delta x}\| = O(\sqrt{n})\sqrt{(\boldsymbol{x}^k)^T\boldsymbol{z}^k}$ and $\|\boldsymbol{D\Delta z}\| = O(\sqrt{n})\sqrt{(\boldsymbol{x}^k)^T\boldsymbol{z}^k}$. So we have (11) for $\eta = O(n)(\boldsymbol{x}^k)^T\boldsymbol{z}^k$. Hence there exists $\delta > 0$ such that $\bar{\alpha}^k \ge \delta/n$ at each iteration of the algorithm. Then we can prove Theorem 4.1 in the same way with the proof of Theorem 2.1.

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