APPROXIMATION ALGORITHMS FOR THE
GEOMETRIC COVERING SALESMAN PROBLEM

By
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Abstract

We introduce a geometric version of the Covering Salesman Problem: Each of the \( n \) salesman's clients specifies a \textit{neighborhood} in which they are willing to meet the salesman. Identifying a tour of minimum length that visits all neighborhoods is an NP-hard problem, since it is a generalization of the Traveling Salesman Problem. We present simple heuristic procedures for constructing tours, for a variety of neighborhood types, whose length is guaranteed to be within a constant factor of the length of an optimal tour. The neighborhoods we consider include, parallel unit segments, translates of a polygonal region, and circles.

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1 Introduction

A salesman wants to meet a set of potential buyers. Each buyer specifies a compact set in the plane, his neighborhood, within which he is willing to meet. For example, the neighborhoods may be disks centered at the buyers locations, and each disk's radius specifies the distance that a buyer is willing to travel to the meeting place. The salesman wants to compute a tour of shortest length that will intersect all of the buyers neighborhoods and finally returns to his initial departure point. (Note that the neighborhoods may overlap partially.) The problem generalizes the Euclidean Traveling Salesman Problem (TSP) in which the areas specified by the buyers are single points, and consequently it is NP-hard [Pa,GGJ].

On the other hand, it is known that the optimal tour of a Euclidean Traveling salesman (and in fact any symmetric TSP obeying the triangle inequality) can be approximated by a tour of length at most one and a half times the optimal tour [Ch]. Such approximation algorithms are available also for some generalizations of the TSP (see [BCCM,BGSF,Fr1,FHK,Fri1,Fri2,JP,RS]). In this paper we construct algorithms with a bounded error ratio for some important cases of the Traveling Salesman with Neighborhoods Problem. Our motivation is mainly to demonstrate the existence of such algorithms and therefore we do not try to obtain the lowest possible ratios, avoiding reductions in the bounds, even when such reductions are obvious, if this complicates the exposition.

The general method we use is to "represent" each neighborhood by a carefully chosen point in the neighborhood, and then apply a known approximation algorithm to these points in the plane. However, some naive choices for such representing points fail to deliver an approximation algorithm with a bounded error ratio. In Section 4, we discuss such examples. In Section 2, we give a method for choosing representative points for neighborhoods that are parallel unit segments. We show that this method does produce a constant approximation to the optimal tour. In Section 3, we discuss some extensions of this method to neighborhoods that are unequal length parallel segments, and to translates of a connected region. We also give a Combination Lemma that allows us to approximate a problem with regions of several different types, by combining approximations of each type. Thus for instance, we can approximate regions that are unit segments parallel to one of k different directions, or of several different lengths.

We will assume (unless otherwise stated) that the initial location of the salesman can be viewed as a region of the same type as the customers regions. An alternative is to consider the salesman's initial location as a point region and combine this region with an approximate tour on all other regions using the Combination Lemma.

It is interesting to compare our methods to those used by Current and Schilling [CS]. The problem considered in their paper is a graph version of ours: Given a directed graph, non-negative costs associated with each arc, and a constant S, find a tour of minimum length such that all nodes not in the tour are at distance at most S from some node in the tour. Their heuristic proceeds by first finding a minimum vertex cover of the nodes and then approximating the shortest tour on the covering nodes. Unfortunately the first step of this procedure requires a solution of another NP-hard problem, and even if somehow this solution is obtained, there is no guarantee on how well this heuristic will perform. Our heuristic (designed independently) also starts with a covering problem which can be solved optimally in linear time after sorting, and results in a bounded performance ratio.

2 Parallel Unit-Segments

In this section, we assume throughout that the regions are unit segments parallel to the x-axis. Our result is the following:
Let $p$ denote the constant factor by which we can approximate an optimal tour on a set of points in the plane. (Currently $p = 1.5$, [Ch].)

**Theorem 1** Given parallel equal length segments in the plane, we can find, in polynomial time, a tour visiting all segments, of length at most $4\sqrt{2}p$ times the length of an optimal such tour.

**Proof:** Our approximation algorithm is simple: We first cover the unit segments by a minimum number of vertical lines. (A set of lines is said to cover a set of segments if each segment is intersected by at least one line from the covering set. We refer to the lines as stubbers. We use the terms stabber or covering line interchangeably.) We do this in a greedy fashion. Our leftmost line is as far right as possible, namely at the leftmost right endpoint of a segment. Removing all segments covered by previous lines, we repeat this procedure, until all segments are covered. If one or two covering lines suffice, our approximation is trivial, and is described below. Otherwise, three or more covering lines are necessary, and the second step of our algorithm is to represent each unit segment by the point in which it intersects the covering lines. Note that by our construction of covering lines, each unit segment has a unique representative point. Finally we use these points as input to a bounded error TSP algorithm for points (such as Christofides’ algorithm). Clearly, the resulting tour is a tour on the original segments. We will show that its length is within a constant factor of the optimal tour, but first we complete the discussion of the one or two covering lines cases.

It is interesting to note that we do not use the fact that the segments are of equal length, in the special case that one or two covering lines suffice. Indeed, as long as arbitrary length segments parallel to the $x$-axis can be stabbed by at most two lines parallel to the $y$-axis, an approximation algorithm is trivial. We use the following notation: $y_1$ is the minimum $y$-value of a segment, and $y_2$ the maximum value.

**Case (1)** All segments stabbed by a single line: It is easy to construct an optimal tour: Double the segment on a single covering line from $y_1$ to $y_2$.

**Case (2)** Two covering lines are necessary and sufficient: We construct a tour as follows: Let $x_1$ be the smallest $x$-value of a right endpoint of a segment. Let $x_2$ be the greatest $x$-value of a left endpoint of a segment. Clearly, $x_1 < x_2$, otherwise one line could have covered all segments. Let $y_1$ and $y_2$ be as before. The tour constructed is a rectangle whose sides are parallel to the axes, cornered at $(x_1, y_1)$, $(x_2, y_1)$, $(x_2, y_2)$, and $(x_1, y_2)$. Clearly, this tour visits all segments, and its length is $2(x_2 - x_1 + y_2 - y_1)$. In Section 4 we see that $LB_1 = 2\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, is a lower bound on the length of an optimal tour. Hence we produced a tour of length at most $\sqrt{2}$ times the length of an optimal tour.

**Case (3)** Three or more lines are needed to cover the segments: We will show that an optimal tour using the representative points is of length at most $4\sqrt{2}$ times the length of the optimal tour on the unit segments. We need some notation: Let $l_j$, $j = 1, \ldots, k$ be the collection of vertical covering lines. Let $\{d_i\}$ be the sequence of intersection points of the optimal tour (on the unit segments) with the lines, in cyclic order around the tour. We consider the corresponding sequence of intersected lines, and see that there may be multiple consecutive crossings of each particular line. We pick a subsequence of $\{d_i\}$ corresponding to the first crossing point in each such consecutive sequence. To avoid double subscripts we denote this subsequence of the intersection points, in cyclic order around the tour, by $\{b_i\}$, $i = 1, \ldots, m$. Now, the optimal tour can be partitioned by the points $\{b_i\}$ into blocks, $B_i$, which are the parts of the tour between $b_i$ and $b_{i+1\bmod m}$ (To simplify notation, from now on we drop the $(\bmod m)$ term.) Figure 1 illustrates the definition of a block in which three stabbing lines are involved: $l_{j-1}$, $l_j$ and $l_{j+1}$. The optimal tour (shown in part by the dotted polygonal line) crosses line $l_j$ three consecutive times, the first such crossing is
defined to be $b_i$. Then $OPT$ crosses $l_{j+1}$, and this crossing point defines $b_{i+1}$. The block $B_i$ is the part of $OPT$ from $b_i$ to $b_{i+1}$. We define the height of a block $B_i$, 

$$h(B_i) \equiv \max_{(x,y) \in B_i} y - \min_{(x,y) \in B_i} y$$

where $(x, y)$ are the coordinates of points in the block. Similarly we define the width of a block, 

$$w(B_i) \equiv \max_{(x,y) \in B_i} x - \min_{(x,y) \in B_i} x.$$ 

See Figure 1. From these definitions it is easy to see that the length of a block $B_i$ is at least $\sqrt{h^2(B_i) + w^2(B_i)}$. (We use this fact to justify inequality (4) below.) We drop the $B_i$ when it is clear which is the block in question. Note that the width of a block is at most one less than the distance between the covering lines in the block. See Figure 1 where the distance between $l_{j-1}$ and $l_{j+1}$ is at most one more than $w(B_i)$. (This is true because a block starts and ends at two different covering lines, hence, either the rightmost or leftmost point of the block is on a covering line. Thus we underestimate only on one side, and this under-estimation is by at most the length of a segment.)

To complete the proof we exhibit a tour, $T$, which is a union of paths $\{T_i\}_{i=1}^m$, where $T_i$ starts at $b_i$ and ends at $b_{i+1}$. These paths satisfy two properties: First, each $T_i$ "visits" all unit segments that $B_i$ "visits", but does so at the representative points of the segments (i.e., their crossing points with the covering lines). Second, the length of $T_i$ is at most a constant multiple of the length of $B_i$. Since the length of the optimal tour on the representative points is no longer than the length of $T$, and the length of the optimal tour on the unit segments is the sum of the lengths of $B_i$, we get the desired result.

Denote $I(B_i)$ the set of unit segments intersected by block $B_i$. The construction of $T_i$ depends on $I(B_i)$, and on the lines covering these segments. We note that at most three stabbing lines cover the segments $I(B_i)$: the two (different) lines on which $b_i$ and $b_{i+1}$ lie (i.e., where the block begins and ends) and possibly the line on $b_i$'s opposite side. Without loss of generality, let $b_i$ be on line $l_j$ and $b_{i+1}$ be on line $l_{j+1}$. The unit segments in $I(B_i)$ are of three types: Segments covered by $l_{j+1}$,
segments covered by \( l_j \), and segments covered by \( l_{j-1} \). By our construction, no other unit segment can be intersected by \( B_i \). There are four cases of interest:

(3.0) \( I(B_i) \) is empty.

(3.1) \( I(B_i) \) contains only unit segments covered by one of the three lines.

(3.2) \( I(B_i) \) contains unit segments covered by two of the three lines.

(3.3) \( I(B_i) \) contains unit segments covered by all three lines.

Call the top (resp. bottom) of a line with respect to a block, the point on the line intersected by the highest (resp. lowest) \( y \) valued unit segment in the block. We are now ready to describe \( T_i \) in each of these cases:

Case (3.0): \( T_i \) is the straight line segment connecting \( b_i \) to \( b_{i+1} \). Clearly, \( T_i \) covers all unit segments covered by \( B_i \) and has the same length.

Case (3.1): \( T_i \) is comprised of the following line segments: From \( b_i \) to the top of the line in question, (length bounded by \( w + 1 + h \)), to the bottom of the same line (length bounded by \( h \)), to \( b_{i+1} \) (length bounded by \( w + 1 + h \)). Clearly, the length of \( T_i \) is bounded by \( 2w + 2 + 3h \).

Case (3.2): \( T_i \) is comprised of the following line segments: From \( b_i \) to the top of one of the lines, to the bottom of the same line, to the bottom of the other line, to the top of that other line, to \( b_{i+1} \). It can be seen that the length of \( T_i \) is bounded by \( 2w + 2 + 4h \) (by choosing the second line to be the one closer to \( b_{i+1} \) and thus not doing unnecessary backtracking).

Case (3.3): Assume without loss of generality that \( b_i \) is higher than \( b_{i+1} \) (i.e., has a larger \( y \) coordinate). \( T_i \) is comprised of the following line segments: From \( b_i \) to the top of line \( l_{j-1} \), to the bottom of the same line, to the bottom of the line \( l_j \), to the top of that line, to the top of \( l_{j+1} \), to the bottom of that line, to \( b_{i+1} \). It can be seen that the length of \( T_i \) is bounded by \( 2w + 2 + 4h \).

Notice that in all four cases we have the vertical portions of \( T_i \) traverse each of the covering lines, each line between its top and bottom, and thus \( T_i \) visits all unit segments visited by \( B_i \). Hence \( T \) is a tour meeting all segments, in an order possibly different from the order they are visited by the optimal tour.

To bound the length of \( T_i \) we note that: (Here \( | \cdot | \) denotes the Euclidean length.)

\[
|T_i| \leq 2w(B_i) + 2 + 4h(B_i) \quad (1)
\]
\[
\leq 4w(B_i) + 4h(B_i) \quad (2)
\]
\[
\leq 4\sqrt{2}\sqrt{h^2(B_i) + w^2(B_i)} \quad (3)
\]
\[
\leq 4\sqrt{2}|B_i| \quad (4)
\]

Inequality (1) follows directly in all four cases. (2) relies on the fact that \( w \geq 1 \). This is true because the width of any block is at least the distance between two consecutive covering lines, which is at least one, the length of the segments. (3) is simple calculus, and (4) was previously obtained. This concludes our proof for parallel unit line segments.

3 Extensions

3.1 Unequal segments

Our first extension relaxes the condition that all segments are of equal length, although point regions are not allowed. (Point regions can most easily be incorporated using the Combination
Lemma below.) Without loss of generality, let the shortest region be a segment of unit length, and the longest segment be of length \( r \). To simplify matters, we further assume that \( r \) is an integer, otherwise we can replace \( r \) by its ceiling. The algorithm and analysis are straightforward generalizations of the previous ones: Cover the segments by a minimum number of vertical lines. If one or two lines suffice to cover the segments, the analysis is identical to the equal length segment case. Otherwise, choose as a representative point of each segment, the rightmost intersection with a covering line. Define blocks, their heights and widths as before. Note that whereas a block visited segments covered by at most three different lines when all segments were of the same length, now a block may visit segments whose representative point is on one of at most \( r + 2 \) different covering lines. This follows from our choice of the representative point as the rightmost crossing by a stabbing line. (Leftmost crossing would work equally well.) If we were to choose any crossing point as the representative point, a block could possibly visit segments stabbed by up to \( 2r + 1 \) different stabbing lines. Furthermore, the width of a block is at most \( 2r \) less than the distance between any two covering lines in the block (because we may be "off" by \( r \) on each side of the block). We construct \( T_i \) to go between the top and bottom of each line in the block (if the line has a representative point of any visited segments on it). Between covering lines, \( T_i \) simply travels from the top (bottom) of one line to the top (bottom) of another, as necessary. The length of this vertical and horizontal traveling of \( T_i \) is bounded by \( (r + 3)h(B_i) + 2(w(B_i) + 2r) \). We conclude the proof as before:

\[
|T_i| \leq (r + 3)h(B_i) + 4r + 2w(B_i) \\
\leq (r + 3)h(B_i) + (4r + 2)w(B_i) \\
\leq (3r + 3)\sqrt{2}h(B_i) + w(B_i) \\
\leq (3r + 3)\sqrt{2}|B_i|
\]

Recall that \( p \) denotes the constant factor by which we can approximate an optimal tour on a set of points in the plane. We have shown the following:

**Theorem 2** Given parallel segments in the plane, of lengths between 1 and \( r \), we can find, in polynomial time, a tour visiting all segments, of length at most \((3r + 3)\sqrt{2}p\) times the length of an optimal such tour.

Note that the bound is not quite as good as the one obtained when \( r = 1 \), namely all segments are of equal length.

### 3.2 Translate regions

Our next generalization is to regions that are translates of the same convex body, e.g., a unit circle or rectangle. (Simple modifications that allow the regions to be non-convex are discussed below.) Our idea is to imitate our algorithm for segments. Define the diameter, \( \delta \), of a region to be the distance between the two points in the region farthest apart. Without loss of generality we assume that the diameter is between two points whose \( y \)-value is the same (i.e., the two points in the region determining the diameter lie parallel to the \( x \) axis). Now treating these diameters as equal parallel segments of length \( \delta \), we find a minimum cover by vertical lines (covering these diameters). Next, we pick a representing point from each region to be the point of intersection of the diameter and the covering line. By the convexity assumption, this point is in the region. Let \( \eta \) be the height of a region, namely the vertical distance between the points in a region with highest and lowest \( y \)-value.
Note that $\eta \leq \delta$. We further define $\eta_1 (\eta_2)$ to be the vertical distance between the representing point and the highest (resp. lowest) point in the region. By definition $\eta = \eta_1 + \eta_2$. Again, we separate our discussion to the cases in which one, two, or three or more covering lines are necessary. However, unlike the unit segment case, in which the one and two covering lines cases were trivial, here, these are the more difficult to extend. The intuitive reason is that very short optimal tours are possible in these cases, and a constant factor approximation is harder to obtain.

A first attempt to extend the treatment of the segment regions to convex (or general) regions if one line suffices to stab all regions, is to pick again a vertical segment that is part of the stabbing line and double it. The following examples illustrate the failure of this straightforward generalization:

**Example 1**: The regions are parallelograms, one "sitting" on the $x$-axis with its right side going from $(0,0)$ to $(\epsilon,1)$. A second parallelogram is translated to the right so that its left side goes from $(\epsilon,0)$ to $(2\epsilon,1)$. The one line cover is at $x = \epsilon$ which gives an approximation of length 2, whereas the optimum is of length less than $2\epsilon$.

**Example 2**: The regions are disks. Consider two unit disks separated by and tangent to a vertical line. Let $2x$ be the length of the vertical segment between the tangent points, and hence the approximation is $4x$. Let $2y$ be the distance between the disks (implying that $4y$ is the optimal tour length). Then $(1 + y)^2 = 1 + x^2$, so the ratio of the approximated tour to the optimal tour is equal to $\frac{x}{y} = \sqrt{1 + \frac{x^2}{y^2}}$, which tends to infinity as $y$ tends to zero.

Next we describe an approximation method for translate convex regions which does produce a constant performance ratio.

Recall that $p$ denotes the constant factor by which we can approximate an optimal tour on a set of points in the plane.

**Theorem 3** *Given translates of a convex region in the plane we can find, in polynomial time, a tour visiting all regions, of length at most $4\sqrt{5}p$ times the length of an optimal such tour.*

**Proof**: We separate our discussion into three cases, depending on whether one two or more lines are necessary to cover the regions.

**Case (1)** One stabbing line suffices: Find the smallest perimeter rectangle, whose sides are aligned with the axes, that touches all regions. (Here, and whenever discussing minimum perimeter rectangles touching all regions, we consider a rectangle to be the two dimensional region enclosed by its perimeter.) Note that some regions may lie completely inside this rectangle. Denote the width of the rectangle by $W$ and its height by $H$. Clearly, the length of an optimal tour is at least $2\sqrt{W^2 + H^2}$. Note that the perimeter of the rectangle may not be a “legal” tour for the regions, because it may not visit all regions (namely the regions completely inside the rectangle). We add (twice) the vertical segment from the bottom of the rectangle to its top. This doubled segment is placed at the middle of the horizontal sides. With this addition we get a tour that is guaranteed to visit all regions. This crucially relies on the fact that all regions can be stabbed by a single vertical line, and thus all lie in a vertical strip of width $2\delta$. Hence $W \leq 2\delta$, implying that the regions are at least as wide as half of the rectangle. The length of this tour is $2W + 4H$ which is at most $2\sqrt{5W^2 + H^2}$ which is at most $\sqrt{5}$ times the optimal tour length.

We discuss briefly how to find (in polynomial time) a minimum-perimeter rectangle touching all regions whose sides are parallel to the coordinate axes. If the regions in question are simple polygons, then a minimum-perimeter rectangle is determined by four contact points, which will be vertices of the regions touching edges of the rectangle. A naive algorithm follows immediately: Examine all rectangles determined by quadruples of region vertices, check each to see if it touches all regions, and select a minimum-perimeter such rectangle. If the regions are circles, then it is easy
to show that the only contact points between circle boundaries and the boundary of a minimum-perimeter rectangle are points tangent to lines parallel to the axes and to 45° lines (in order to accommodate corner solutions). The naive algorithm can then be applied to this case as well. Using techniques similar to [AS], a faster algorithm can be designed.

**Case (2)** Two stabbing lines: Let us assume that we pick the two stabbing lines to be as close to each other as possible, and denote by $D$ the (horizontal) distance between these two lines. There are two cases to consider: (2.1) $D \geq \delta$, and (2.2) $D < \delta$.

Case (2.1) $D \geq \delta$: This case is similar to the unit segment case. Define $x_1$ to be the smallest $x$-value of a right endpoint of a diameter, and $x_2$ to be the largest $x$-value of a left endpoint of a diameter (as we did for the unit segment case). Then by definition $x_2 - x_1 = D$. Next, let $y_1$ be the minimum $y$-value of a diameter, and $y_1(R)$ be the maximum $y$-value of that region. Similarly, let $y_2$ be the maximum $y$-value of a diameter, and $y_2(R)$ be the minimum $y$-value of that region. Again these correspond to the unit segment case, where $y_1 \leq y_2$, but we may have $y_1(R) \geq y_2(R)$.

However, $y_2 - y_1 \leq (y_2(R) - y_1(R))^+ + \eta$, where $\eta$ is the height of the regions and $a^+ \equiv \max(a, 0)$.

A lower bound on the length of the optimal tour is $2\sqrt{(x_2 - x_1)^2 + ((y_2(R) - y_1(R))^+)^2}$. (See Section 4.) The approximation tour we construct is a rectangle whose sides are parallel to the axes, and cornered at $(x_1, y_1)$, $(x_2, y_1)$, $(x_2, y_2)$, and $(x_1, y_2)$. Let the length of this tour be denoted by $APX$, and the length of the optimal tour be denoted by $OPT$.

\[
APX = 2(x_2 - x_1) + 2(y_2 - y_1) \\
\leq 2D + 2(y_2(R) - y_1(R))^+ + 2\eta \\
\leq 4D + 2(y_2(R) - y_1(R))^+ \\
\leq 2\sqrt{5D^2 + ((y_2(R) - y_1(R))^+)^2} \\
\leq \sqrt{5OPT}
\]

Here (5) follows from our assumption of Case (1): $\eta \leq \delta \leq D$.

Case (2.2) $D < \delta$: Here we find a rectangle of minimum perimeter, whose sides are parallel to the axes, that touches all regions. Denote the width of the rectangle by $W$ and its height by $H$. Clearly, the length of an optimal tour is at least $2\sqrt{W^2 + H^2}$. Note that the perimeter of the rectangle may not be a tour, because it may not visit all regions, (namely the regions completely inside the rectangle). However, adding (twice) the vertical segments from the bottom of the rectangle to its top, at its one third and two third points width-wise, does produce a tour guaranteed to visit all regions. The reason is similar to the one stabbing line case: All regions are stabbed by two vertical lines which are separated by at most $\delta$ (by the assumption $D < \delta$), and thus all lie in a vertical strip of width $3\delta$. Hence $W \leq 3\delta$, implying that the regions are at least as wide as one third of the rectangle. The length of this tour is $2W + 6H$, which is at most $2\sqrt{10W^2 + H^2}$, which is at most $\sqrt{10}$ times the optimal tour length.

**Case (3)** Three or more covering lines: This case is very similar to the segment case. We define blocks, $B_i$, and their heights and widths as before. The definition of the top (and bottom) of a line with respect to a block is modified slightly to reflect the fact that regions have a height. The top of the line in a block is the higher of the highest point visited by the block and the highest representing point in the block. Similarly define the bottom of a line with respect to a block. Noting that the top (bottom) of a line with respect to block $B_i$ is at most $\eta_i$ higher ($\eta_i$ lower) than the highest (lowest) crossing point of this line by block $B_i$, we get that the distance between the top and bottom of a line in block $B_i$ is at most $\eta + h(B_i)$. $T_i$ is defined as before noting the modifications of the top and bottom definitions. Clearly, $T_i$ visits all regions that are visited by $B_i$. To bound the length of $T_i$ we have:
\[ |T_i| \leq 2w(B_i) + 2\delta + 4(h(B_i) + \eta) \]
\[ \leq 2w(B_i) + 2\delta + 4h(B_i) + 4\delta \]  
(6)
\[ \leq 8w(B_i) + 4h(B_i) \]  
(7)
\[ \leq 4\sqrt{2}(B_i) + w^2(B_i) \]  
(8)
\[ \leq 4\sqrt{5}|B_i|. \]  
(9)

(6) relies on the fact that \( \eta \leq \delta \). (7) follows because \( w(B_i) \geq \delta \). (8) is simple calculus, and (9) was previously obtained. This concludes our proof for translates of a convex region.

The analysis for the case of three or more covering lines did not require the full description of the body, of which the regions were translates, only its diameter. In fact we do not require that all regions be translates of one body; it suffices that the diameters of all the regions are parallel equal length segments, and that the regions are convex. However, the seemingly simpler cases in which one or two covering lines suffice require us to find a rectangle as described. This can be done in polynomial time for regions such as polygons or splines, but may present a problem for more general regions.

Translates of connected non-convex regions can also be approximated, assuming their representation is such that the computation of the minimum perimeter rectangle is easy. In the following theorem, the diameter of a region is the maximum Euclidean distance between any pair of its points.

**Theorem 4** Given translates of a connected (not necessarily convex) region in the plane we can find, in polynomial time, a tour visiting all regions, of length at most \( 4\sqrt{10}p \) times the length of an optimal such tour.

**Proof:** We begin, as in other cases, by covering the regions greedily by vertical lines. Since the regions are connected, we have, as before, that the distance between covering lines is at least the diameter of the regions. (Note that if the regions are not connected, a greedy cover might use very close stabbers. As a result, we are not able to use an inequality similar to (2) and (7) to prove a bound in this case.)

If one or two covering lines suffice to cover the regions, then our approximation scheme is identical to the convex case.

If three or more vertical lines are necessary to cover the diameters, only a slight modification to the definitions and analysis is needed to obtain a constant error ratio. The ratio obtained is only somewhat \( (\sqrt{2} \times \text{times}) \) worse than the convex case. We must modify our definition of a representing point, since the intersection between the covering lines and the diameter of a region may be outside a region. Instead, we choose as a representing point (arbitrarily) any point in the intersection of the region with the covering line. The top (bottom) of a line with respect to a block is the point on the line with highest (resp. lowest) y-value in a region visited by this block. Here we bound the top (and bottom) of a line in a block to be at most \( \eta \) away from the highest (lowest) point visited by the block, and so the distance between the top and bottom of a block is at most \( 2\eta + h(B_i) \).

We proceed as before:

\[ |T_i| \leq 2w(B_i) + 2\delta + 4(h(B_i) + 2\eta) \]
\[ \leq 2w(B_i) + 2\delta + 4h(B_i) + 8\delta \]
\[ \leq 12w(B_i) + 4h(B_i) \]
\[ \leq 4\sqrt{10}h^2(B_i) + w^2(B_i) \]
\[ \leq 4\sqrt{10}|B_i| \]

3.3 Combining approximations

The lemma we describe next, which we refer to as the **Combination Lemma**, allows us to approximate a problem with regions of several different types, by combining approximations for each type. This lemma can be applied for instance, to the case in which the regions are unit segments parallel to one of \(k\) different directions, (e.g., \(k = 2\) and segments are parallel to either the \(x\)-axis or the \(y\)-axis). The error ratio obtained is \(k(c+2) - 2\), where \(c\) denotes the error ratio of the single direction problem. Another application is to the case in which the segments are parallel, but may be of one of \(k\) different lengths, including zero length segments, namely points.

**Lemma 5 (Combination Lemma)** Given regions that can be partitioned into two types, and constants \(c_1, c_2\) bounding the error ratios with which we can approximate the optimal tours on regions of types 1 and 2, then we can approximate the optimal tour on all regions with an error ratio bounded by \(c_1 + c_2 + 2\).

**Proof:** Let \(OPT_1\) and \(OPT_2\) be the optimal tour lengths for regions of type 1 and 2. Let \(OPT\) be the overall optimal tour length. By the triangle inequality, each subproblem’s optimal value is bounded by the optimal value to the original problem (\(OPT_1 \leq OPT\)). Denote by \(APX_1\), \(APX_2\) and \(APX\) the approximate tour lengths for regions of type 1,2, and all regions, obtained by methods described below. Let \(\delta_1\) and \(\delta_2\) be the diameters of the two region types. Our proof consists of two cases: Case (1): \(2\delta_1 + 2\delta_2 \leq OPT\), Case (2): \(2\delta_1 + 2\delta_2 > OPT\).

Case (1): Obtain \(APX_1\) and \(APX_2\) by the hypothesis of the theorem, with corresponding bounds \(c_1\) and \(c_2\). Let \(D\) be the minimum distance between a point in a type 1 region and a point in a type 2 region. Clearly \(2D \leq OPT\). We obtain \(APX\) by combining the two approximate solutions into a tour visiting all regions by “gluing” the tours together at the place in which the two region types are closest to each other. This “glue” has length bounded by \(2(D + \delta_1 + \delta_2)\). Thus we have

\[
APX \leq APX_1 + APX_2 + 2D + 2\delta_1 + 2\delta_2 \\
\leq c_1OPT_1 + c_2OPT_2 + OPT + OPT \\
\leq (c_1 + c_2 + 2)OPT
\]

Case (2): We begin by constructing a minimum perimeter rectangle (whose sides are parallel or perpendicular to a fixed direction of our choice) that touches all regions (of both types). Denote the lengths of the sides of the rectangle by \(W\) and \(H\). We know that \(OPT \geq 2\sqrt{W^2 + H^2}\). Without loss of generality we assume that \(\delta_1 \leq \delta_2\). We further partition Case (2) into two subcases: Case (2a): \(\delta_1 \geq OPT/4\) (and hence \(\delta_2 > OPT/4\)), Case (2b): \(\delta_1 < OPT/4\) (and hence \(\delta_2 > OPT/4\)).

Case (2a): Build \(APX\) by going around the perimeter of the rectangle combined with two (doubled) stabbing segments, one for each region type, that visit all regions not visited by the perimeter of the rectangle. Finding such stabbers is an easy task: We ignore all regions stabbed by the boundary of the rectangle and find a line cover for regions of each type, using lines perpendicular to the direction of the diameter. We claim that one line suffices, since the length of each diameter is
at least half the length of the rectangle's diagonal in Case (2a). In fact, only the part of the covering line inside the rectangle is sufficient to stab all regions of one type completely in the rectangle. We use this segment of the covering line, as the stabbing segment needed by $APX$. The length of the segment is bounded by the length of the diagonal of the rectangle, and since we are adding two segments, each doubled, our approximation length is bounded by the perimeter of the rectangle $(2W + 2H)$ plus four diagonals $(4\sqrt{W^2 + H^2})$.

$$
APX \leq 2W + 2H + 4\sqrt{W^2 + H^2} \\
\leq 8\sqrt{W^2 + H^2} \\
\leq 4OPT
$$

Case(2b): Obtain $APX_1$ by the hypothesis of the theorem, with bound $c_1$. Build $APX_2$ as in Case (2a), this time using only one stabber, to visit only regions of type 2 not visited by the rectangle perimeter. Next we glue together the two approximations. The length of this glue depends on the distance between the rectangle of $APX_2$ and $APX_1$. If $APX_1$ crosses the rectangle, clearly, the glue length is zero. If $APX_1$ lies completely inside the rectangle then the length of glue is at most the minimum of $W$ and $H$. The remaining possibility is that $APX_1$ lies completely outside of the rectangle, but recall this rectangle touches all regions, and hence $APX_1$ can be at most $\delta_1$ away from the rectangle. In this case glue of length $2\delta_1 < OPT/2$ suffices. In summary, in all three cases the glue length is bounded by $OPT/2$.

$$
APX \leq APX_1 + 2W + 2H + 2\sqrt{W^2 + H^2} + \text{glue} \\
\leq c_1OPT_1 + 2(1 + \sqrt{2})\sqrt{W^2 + H^2} + OPT/2 \\
\leq c_1OPT + (1 + \sqrt{2})OPT + OPT/2 \\
\leq (c_1 + 1.5 + \sqrt{2})OPT
$$

To complete the proof we recall that $c_1 \geq 1$ in all cases, and thus the bounds of Cases (2a) and (2b) are at least as good as the bound claimed in the lemma.

We can use this lemma repeatedly to obtain approximations to more than two region types. The bound we obtain for combining $k$ different regions types with individual approximation bounds of $c_1, c_2, \ldots, c_k$ is $c_1 + c_2 + \cdots + c_k + 2(k - 1)$. A better bound may be achievable by proving a Combination Lemma for this more general case.

4 Lower Bounds and Counterexamples

Our first lower bound, $LB_1$, is derived by considering a rectangle for which we know that the optimal tour (and in fact any tour visiting all regions) must "touch" all of its four sides. We begin with regions that are segments parallel to the $x$-axis.

Let $x_1$ be the smallest $x$-value of a right endpoint of a segment. Let $x_2$ be the greatest $x$-value of a left endpoint of a segment. Assume $x_1 \leq x_2$. (If this assumption is not satisfied then the segments have a common $x$ coordinate and an optimal solution is obvious.) Let $y_2$ be the maximum $y$-value of a segment, $y_1$ the minimum value. Set $LB_1 = 2\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Any tour visiting all segments must go as far to the left as $x_1$, as far to the right as $x_2$, as far down as $y_1$ and as far up as $y_2$. Thus a tour must visit all four sides of the rectangle whose corners are $(x_1, y_1), (x_1, y_2), (x_2, y_2)$, and $(x_2, y_1)$. An easy calculus exercise shows that a tour touching all four sides of a rectangle must
have length at least twice the diagonal, implying that $LB_1$ is a lower bound for $OPT$. If we wish to include the special instances for which $x_1 > x_2$ in this lower bound we can write more generally $LB_1 = 2\sqrt{((x_2 - x_1)^+)^2 + (y_2 - y_1)^2}$.

We can state a corresponding lower bound for regions whose diameters are parallel segments. Here, we define $x_1$, $x_2$, $y_1$ and $y_2$ as above, using the diameters of the regions as the segments. Next, let $y_1(R)$ be the maximum $y$-value of the region by which $y_1$ was defined, and let $y_2(R)$ be the minimum $y$-value of the region by which $y_2$ was defined. We may have $y_1(R) \geq y_2(R)$. A lower bound on the length of the optimal tour is $LB_1 = 2\sqrt{((x_2 - x_1)^+)^2 + ((y_2(R) - y_1(R))^+)^2}$.

In Section 2 we used another rectangle to generate such a lower bound, when we found the smallest perimeter rectangle touching all regions. The fact that this rectangle is minimal implies that there are contact-critical points, one on each side of the rectangle, where a region “barely touches” the rectangle. Thus again we can lower bound the length of an optimal tour by twice the length of the diagonal of this rectangle. Note that we don’t have to restrict such a rectangle to have sides parallel or perpendicular to the diameters, any direction will do. The important property is that an optimal tour must visit all four sides of the rectangle and thus have length bounded below by twice the length its diagonal.

A second lower bound can be obtained by considering distances between pairs of regions. Let $d_{ij}$ be the distance between regions $i$ and $j$, measured as the distance between the nearest pair of points on these two regions. Consider a complete graph $G$ where each node corresponds to a region and the length of the arc connecting nodes $i$ and $j$ is $d_{ij}$. Let $LB_2$ be the length of a shortest tour on $G$. Clearly, $LB_2$ is a lower bound on $OPT$.

![Figure 2: $LB/OPT$ approaches zero](image)

Let $LB = \max\{LB_1, LB_2\}$. Figure 2 demonstrates that $LB/OPT$ may be arbitrarily close to zero, even when the regions are parallel unit segments. In this case we see that $LB_1$ is determined by the dotted rectangle, $LB_2$ is arbitrarily close to zero, as we can make the vertical distance between the segments very small. On the other hand, if the segments are very short compared to the sides of the rectangle, we see that the optimal tour is very long, compared to $LB_1$ and $LB_2$. 

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It is interesting to see that, in the above example, $LB_2$ (and thus possibly $LB$) may be increased if we delete regions, resulting in a tighter lower bound. This idea can be formalized as follows: Let $S$ be the set of regions. For each subset $S' \subseteq S$ let $LB_2(S')$ be the resulting lower bound when all regions in $S \setminus S'$ are deleted. Then set $LB'_2 = \max_{S' \subseteq S} \{ LB_2(S') \}$, and $LB' = \max \{ LB_1, LB'_2 \}$. It is an open question whether the ratio $LB'/OPT$ can be made arbitrarily close to zero or it is bounded below by some positive constant. If the latter is true, it is interesting to ask whether the optimizing $S'$ can be computed efficiently (i.e., in polynomial time).

We mentioned in Section 2 that the approximate tour we obtain may visit the segments in a different order than the optimal tour. Figure 3 shows an example in which the order in which the approximate tour visits the segments is very different from the order used by the optimal tour. The optimal tour (shown partially by a dotted line) is all a single block. The order APX uses is to first visit all segments stabbed by $l_1$, then all stabbed by $l_2$ and last the segments stabbed by $l_3$.

One might ask whether a better bound in the case of unequal length parallel segments is possible. In particular, is it possible to get a bound in which $r$, the ratio of the longest to shortest segment, does not appear. This does not seem possible using our algorithm as the example in Figure 4 shows. In this example covering lines are determined by the short segments on top, and every longer segment is intersected by covering lines many times. As long as we restrict our choice of which such an intersection point we select to represent a segment to the rightmost, or leftmost, or “middle” intersection, the resulting tour is “long”. (However, a better approximation may be possible here using the Combination Lemma, by separating the problem into subproblems.)

The remainder of this section contains examples that show that some other (more naïve) methods for picking a representative point in each region may not yield a constant error ratio. The first such example is the most naïve: Pick an arbitrary point in each region. It is easy to generate examples that show that even when all regions are unit segments parallel to the $x$-axis, and the representing point is chosen to be the middle point of the segment, the result can be arbitrarily
bad. Start with an arbitrary simple polygon in the plane, whose perimeter is much longer than its height. We make this polygon an optimal tour on the midpoints of some segments, by placing many (equal-length horizontal) segment midpoints along it. However, if the segments are long enough, such that one vertical line suffices to stab all segments, then an optimal tour will be a vertical line segment of length twice the height of the polygon, which is small relative to its perimeter.

Other possible choices for representing points are based on trying to extend the tree heuristic which works so well for point regions (i.e., the classical Euclidean TSP). In this heuristic we build a minimum spanning tree on the points, complete it to an Eulerian tour which is then shortcut to a TSP tour. There are several ways in which we can think of building a minimum spanning tree on a set of parallel unit segments (which are all equivalent for points in the plane). Suppose we have somehow connected a subset of the segments into a forest, and picked representing points on them. We can pick the next segment to be connected by a “Prim” type algorithm: pick a point on some segment not yet visited, that is closest to points already selected. Alternatively we can think of a “Kruskal” type algorithm that decides next to connect any two segments whose distance between them is minimized, as long as no cycles are closed. Of course for parallel segments, the points on the segments minimizing the distance may not be unique, so this algorithm is not fully specified. There are two alternatives: The simple alternative is to choose one such minimizing pair arbitrarily, the second, and more complex is to try to optimize over all choices of minimizing points. Unfortunately, we don’t know how to accomplish this in polynomial time, so we don’t analyze the possible success of this second method (and we refer to the first alternative as our “Kruskal” type algorithm).

Intuitively it is not surprising that both algorithms fail to produce the desired result, as the examples below show. The Prim type algorithm picks a representative point in a myopic fashion, a choice that may prove to be disastrous later. The Kruskal-type algorithm fails for the opposite
Figure 5: Failure of Prim-type algorithm

Figure 6: Failure of Kruskal-type algorithm
reason of allowing too much flexibility in the choice of which point(s) will be used to connect the
segments and thus allows the salesman to travel within the regions “for free”.

The approximation in Figure 5 (shown by a dotted line) picks the nearest point on a segment
not yet connected, and is thus made to zigzag, while the optimal tour is a rectangle of much shorter
length. Notice that all the segments in this example can be made to be of equal length by extending
them to the right or left appropriately.

In the example of Figure 6, each of the middle segments has two points chosen by a “Kruskal”
type algorithm. If we build our approximation by connecting points chosen in the same region by
the segment they share a very poor approximation results. Alternatively, we can choose to include
all points selected by the Kruskal-type algorithm as input to a point approximation TSP algorithm,
but this too (although successful in the example given) may yield bad approximations in general.

5 Concluding Remarks

We conclude this paper by mentioning some open problems. These correspond to cases for which
we have not yet been able to find a polynomial time algorithm to approximate, with a bounded
performance guarantee, an optimal TSP tour (or prove that no such approximation exists unless
P=NP).

The first open problem is for regions that are nonuniform parallel segments. One would
like a bound independent of the ratio r between longest and shortest segment. We assume that the
number of distinct sizes is not fixed, and more strongly, that the segments cannot be divided into
classes, such that within each class the ratio of the longest to shortest segment is small. Otherwise
an approximation can be found by combining approximations for the individual classes.

The second open problem concerns non-parallel unit segments (where the number of directions
is not fixed, and the ratio of their projections on a given direction is not bounded).

The third open problem concerns regions (convex or non convex) which can be quite general, as
long as their diameter is known. Here we can not apply our minimum perimeter rectangle approach,
because we may not be able to efficiently compute such a rectangle.

A related question is whether we can approximate non-connected regions, such as regions each
of which is comprised of of two disjoint unit segments.

Other generalizations may be quite straightforward, such as regions in higher dimensions.

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