THE WILD CARD OPTION IN TREASURY BOND FUTURES IS ALMOST WORTHLESS

by

Hugh I. Cohen*
Abstract

This paper clarifies misconceptions concerning the wild card option in treasury bond futures contracts. In an economy with no arbitrage and two additional futures contracts, a theoretical upper bound for the value of the wild card option is established. Through a simulation it is shown that this upper bound is small.

1 Introduction

The wild card option in the treasury bond futures contract gives the short trader, during the delivery month, the option to initiate the delivery process at the 2 P.M. settlement price until 8 P.M. of the same day. Valuation of the wild card option has been complicated because it is difficult to isolate the value to the futures contract of the wild card option. There are two methods of valuing the wild card option in the finance literature; both of which are flawed. The first method, which is found in textbooks by Duffie and Kolb and which is implied by the literature of the Chicago Board of Trade, is to value a wild card option which is not part of a futures contract. That is, suppose there exists an option which permits it's holder to receive the cash settlement of delivering a futures contract at it's 2 P.M. settlement price until 8 P.M. of the same day. Gay & Manaster, and Kane & Marcus both show that the optimal exercise strategy for the wild card option may be to exercise the wild card option when treasury bond prices rise (when conversion factor of the cheapest bond to deliver is less than one). This exercise strategy is inconsistent with an independent put option which would never be optimally exercised if treasury bond prices rise because it would be out of the money.

A second method of valuing the futures contract was proposed by Kane & Marcus which overcomes this first problem. However, they are unable to isolate the wild card option and instead value the timing option. The timing option includes the wild card option; but, it also includes the option of the short trader to deliver on any day of the delivery period. This paper illustrates a technique which permits us to isolate just the wild card option and establishes a theoretical upper bound on its value. Since, under the assumptions established in section 2, the upper bound of the value of the wild option is shown to be small, it is concluded that the majority of the value found in the timing option is due to the short trader’s choice of when to initiate delivery, not from the wild card play.

The order of the rest of the paper is as follows: section 2 establishes the economy which is being considered. In section 3, a theoretical expression for the upper bound of the value of the wild card option, both in terms of cash value and futures price, is given. Section 4 gives a numerical example of the magnitude of these bounds. Section 5 states the conclusions of the paper. Proofs of more than a few lines are given in separate appendices.

---

1[5]
2[8]
3[1]
4[3]
5[4]

Kane and Marcus [4] value V(t) where $F(t) = \frac{B(t)}{q} - v(t)$. This is the entire timing option, not just the wild card option.
2 The Economy

Consider an economy $(\Omega, F, Q)$ where information evolves over time according to a right continuous filtration $\mathcal{F}_t$. In this economy three treasury bond futures contracts trade. The first futures contract is identical to the treasury bond futures contract which trades on the CBT. Thus the first futures contract contains the wild card option. $F(t)$ will denote the futures price of this contract at time $t$ and we will call this the original futures contract. The second treasury bond futures contract does not permit the wild card play. That is, while a short trader in the original futures contract may initiate delivery until 8 P.M. at the 2 P.M. settlement price, a short trader in the second futures contract may only initiate delivery until 2 P.M.. With this one exception, the terms of these two futures contracts are identical. We denote the futures price of the second futures contract at time $t$ as $F_B(t)$ and we will call this futures contract B. Finally, we need to consider a third "futures contract" which is fundamentally different from a standard futures contract. One term of this contract, which we denote futures contract C, is that its futures price is equal to the futures price of the original futures contract. That is $F_C(t) = F(t)$. However as in futures contract B, the short trader in futures contract C is not permitted the wild card play. In order to compensate the short trader for losing the wild card option, there is a one-time cash flow from the long trader to the short trader when the position is taken. The amount of this cash flow is determined by the market. It’s amount at time $t$ will be denoted WC$(t)$. Observe that after the position is taken and WC$(t)$ is transferred from the long to the short trader, marking to market cash flows and delivery cash flows in contract C are identical to these cash flows in the original futures contract.

To summarize the differences in these contracts, the original futures contract is the only futures contract which gives the short trader the wild card option. Futures contract B does not permit the wild card option and trades at a different futures price. Futures contract C does not permit the wild card play. However since its futures price is $F(t)$, the price of the futures contract with the wild card option, it compensates the short trader with a one-time cash flow when the position is taken, WC$(t)$. In addition to the above assumptions, the following are added:

A1) Frictionless markets.
A2) The delivery process of all three futures contracts is instantaneous.
A3) There is no end of the month option in any of the futures contracts.\(^7\)
A4) There exists an equivalent martingale measure.
A5) The market is complete.
A6) The settlement price of the futures contracts is the closing market price.
A7) Deliverable treasury bonds trade and have no coupon payments during the delivery month.\(^8\)
A8) For time changes less than one day, changes in the forward rate curve are according to the continuous Ho & Lee model.\(^9\)

\(^7\)Instead there is a final settlement time when delivery must occur.
\(^8\)This assumption is for convenience only, it is not necessary.
\(^9\)The technique used to derive the upper bound requires some model of the forward rate curve. The continuous Ho & Lee model was chosen because it is simple. It models changes in the forward rate curve by parallel shifts. For those who prefer a different model, the results given should remain valid since the forward rate model is only used to show that $\epsilon$ (defined later) is small. Other models will get different $\epsilon$’s, but they too should be small. This model is described in Heath, Jarrow, Morton [6].
3 An Upper Bound on the Wild Card Option

We define $S_n$ during the delivery month to be the 2 P.M. closing time with $n$ wild card plays available to the short trader. Thus closing time the next day will be denoted $S_{n-1}$. $B_i(T)$ is the market price of bond $i$ at time $t$. This price does not include accrued interest, $AI_i(t)$. Also, $CF_i$ is the conversion factor of bond $i$. The following lemmas are a result of no arbitrage, a consequence of the assumptions.

**Lemma 1** To avoid arbitrage at any closing time during the delivery month, $S_n$,

$$F(S_n) \leq \min_{i \in I} \left( \frac{B_i(S_n)}{CF_i} \right)$$

Furthermore,

$$F(S_0) = \min_{i \in I} \left( \frac{B_i(S_0)}{CF_i} \right)$$

where $I$ is the set of possible deliverable treasury bonds.

**Proof.** To avoid arbitrage at $S_n$, it must not be possible to purchase a treasury bond in the bond market, short the futures contract and deliver the bond at a profit. This implies,

$$F(S_n) \cdot CF_i - B_i(S_n) \leq 0 \quad \forall i \in I$$

$$F(S_n) \leq \frac{B_i(S_n)}{CF_i} \quad \forall i \in I$$

$$F(S_n) \leq \min_{i \in I} \left( \frac{B_i(S_n)}{CF_i} \right)$$

For the second part, note that without the end of the month option, the short trader must deliver the treasury bond at $S_0$. Thus at $S_0$ it must not be possible to long the futures contract and sell the immediately delivered bond at a profit. So,

$$B_i(S_0) - F(S_0) \cdot CF_i \leq 0 \text{ for some deliverable bond } i$$

$$F(S_0) \geq \left( \frac{B_i(S_0)}{CF_i} \right) \text{ for some deliverable bond } i$$

$$F(S_0) \geq \min_{i \in I} \left( \frac{B_i(S_0)}{CF_i} \right)$$

Combining this with the previous result yields equality and proves the lemma. ■
Lemma 2 To avoid arbitrage,
\[ WC(t) \geq 0 \quad \forall \ t \]

Proof. Suppose \( WC(t) < 0 \). Then the following is an arbitrage portfolio. At time \( t \), long futures contract \( C \) and short the original futures contract. Initially at time \( t \), we receive a cash flow of \( WC(t) \) since the initial payment from the long trader to the short trader is negative for contract \( C \). At each marking to market we break even since \( F(t) = F_C(t) \). Holding both contracts we wait until our long position is delivered, and deliver our short position at the same time using the same bond. Since the futures prices are the same, the invoice amounts are the same and we break even at delivery. Thus we earn \( WC(t) \) at time \( t \) without risk. This implies arbitrage. ■

Lemma 3 For \((S_{t-1} - t)\) less than one day,
\[ E[\text{Value}(S_{t-1}) \mid \mathcal{F}_t] \leq \frac{Value(t)}{b(t, S_{t-1})} + \varepsilon \]

where \( \text{value}(t) \) is the value of a treasury bond at time \( t \) (price plus accrued interest), \( b(t, T) \) is the price of a pure discount bond at time \( t \) which pays \$1 at time \( T \), \( S_{t-1} \) is the closing time immediately following \( t \), and \( \varepsilon \) is small.

Proof. See appendix 1. ■

Lemma 4 For \((S_{t-1} - t)\) less than one day, and no coupon payment between the two times,
\[ E[B(S_{t-1}) \mid \mathcal{F}_t] \leq \frac{B(t) + AI(t)}{b(t, S_{t-1})} - AI(S_{t-1}) + \varepsilon \]

where \( \varepsilon \) is small.

Proof. The proof comes from combining the following equation with lemma 3,
\[ E[\text{Value}_j(T) \mid \mathcal{F}_t] = E[B_j(T) \mid \mathcal{F}_t] + AI_j(T) \]

Lemma 5 The max expected value at stopping time \( \tau \) of the cash flows of Trader I, a short trader in the original futures contract who exercises the wild card option at stopping time \( \tau \), over the expected cash flow of Trader II, a short trader who does not exercise the wild card option at time \( \tau \), is given by the following relationship:
\[ E[\text{cash flow}_I(\tau) - \text{cash flow}_{II}(\tau)] \leq [F(S_\tau) - \frac{B_j(\tau)}{CF_j}] \times [CF_j - 1] + \varepsilon' \]

where \( S_\tau \) is the closing time immediately preceding \( \tau \) and \( \varepsilon' \) is small.

Proof. see appendix 2. ■
Theorem 6

\[ WC(t) \leq \sup_{\tau} \left[ E[\text{Max}[F(S_{\tau}) - \frac{B_j(\tau)}{CF_j}] \cdot [CF_j - 1] + \epsilon', 0] \mid \mathcal{F}_t] \right] \]

Where \( \tau \) is the set of times that the wild card option may be exercised.

Proof. To avoid arbitrage, the market price of the futures contract C can be no larger than the max expected value under an equivalent martingale measure of the additional cash flows generated by the optimal exercising of the wild card option. Thus the theorem follows directly from lemma 5. \( \blacksquare \)

Now that we have established an upper bound on the cash value of the wild card option, we want to understand how the option affects the futures price of the contract. That is we wish to compare the futures price of the original futures contract to the futures price of contract B.

Lemma 7

\[ F(t) \leq F_B(t) \quad \forall \ t \]

Proof. Suppose \( F(t) > F_B(t) \). This would mean that a short trader in the original futures contract would, in addition to receiving a greater invoice amount as a short trader in futures contract B, receive the wild card option. Clearly no trader would long the original futures contract or short futures contract B until the prices were adjusted. \( \blacksquare \)

We wish to show that \( F_B(t) - F(t) \) is small and thus the wild card influences the futures price by a small amount.

Theorem 8 To avoid arbitrage if \( t \) is any closing time during the delivery month \( S_N \),

\[ F_B(S_N) \leq F(S_N) + \text{Max} \left[ \frac{WC(S_N)}{CF_i} \cdot E(F_B(S_{N-1}) - F(S_{S-1})) \cdot (1 - CF_i) + \frac{WC(S_N)}{b(S_N, S_{N-1})} \right] \]

if \( t \) is during the delivery month, but not a closing time,

\[ F_B(t) \leq F(t) + \text{Max} \left[ E(F_B(S_{N-1}) - F(S_{S-1})) \cdot (1 - CF_i) + \frac{WC(S_N)}{b(t, S_{N-1})} \right] \]

Finally if we are prior to the delivery month, and delivery is not allowed at the next settlement time, we get

\[ F_B(t) \leq F(t) + E(F_B(S_{Next}) - F(S_{Next}) - E(WC(S_{Next})) + \frac{WC(t)}{b(t, S_{Next})} \]

where \( i \) is the cheapest bond to deliver, and \( S_{Next} \) is the next settlement time.

Proof. See appendix 3 \( \blacksquare \).

Theorem 6 is important because it relates an upper bound for the cash value of the wild card option to an upper bound for the discount of the futures price because of the wild card option. The next section will give an example which illustrates that \( WC(t) \) is small.
4 Numerical Evaluation

A computer simulation was performed to observe the magnitude of max WC(t). The simulation started at the settlement time with one wild card play remaining and simulated the change in forward rates from two to eight P.M.. At eight P.M., the value of the bonds was computed from the forward curve and the max value of the wild card option was computed using theorem 6 with \( \epsilon' = 0 \). Once the max value of WC(t) was known for one day, we went back to the settlement time with two wild plays available, and only exercised if the payoff was greater than the expected payoff with 1 wild card play. Working backwards in this way the max value of a European wild card option was calculated over the delivery month.

An analog of the continuous Ho and Lee model was chosen to simulate the change in the forward rate curve.\(^{10}\) Briefly, the model uses a single parameter to change the forward curve by parallel shifts. Since the time period over which the fluctuation occurs is small, six hours, parallel shifts in the forward curve seem reasonable. For the simulation, a standard deviation of .02 was used as the annual standard deviation in the forward rates.

The bonds used and their conversion factors were taken from Kolb \(^{11}\) as the bonds available for delivery on 6/1/1987. In addition an initial stepwise forward rate curve was specified each day as:

\[
\text{forward rate}(0.0, 0.25, 0.5, 1.0, 5.0, 10.0, 35.0) = (0.076, 0.079, 0.082, 0.087, 0.089, 0.09, 0.09)
\]

The max value of the WC(t) is given below. For comparison, the futures price in this example is approximately $88.

<table>
<thead>
<tr>
<th>Wild Card Plays Remaining</th>
<th>Max Wild Card Option</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.017</td>
</tr>
<tr>
<td>2</td>
<td>0.026</td>
</tr>
<tr>
<td>3</td>
<td>0.033</td>
</tr>
<tr>
<td>4</td>
<td>0.037</td>
</tr>
<tr>
<td>5</td>
<td>0.043</td>
</tr>
<tr>
<td>6</td>
<td>0.045</td>
</tr>
<tr>
<td>7</td>
<td>0.050</td>
</tr>
<tr>
<td>8</td>
<td>0.054</td>
</tr>
<tr>
<td>9</td>
<td>0.056</td>
</tr>
<tr>
<td>10</td>
<td>0.058</td>
</tr>
<tr>
<td>11</td>
<td>0.060</td>
</tr>
<tr>
<td>12</td>
<td>0.062</td>
</tr>
<tr>
<td>13</td>
<td>0.063</td>
</tr>
<tr>
<td>14</td>
<td>0.065</td>
</tr>
<tr>
<td>15</td>
<td>0.067</td>
</tr>
<tr>
<td>16</td>
<td>0.068</td>
</tr>
</tbody>
</table>

All values are in dollars with a standard error of 1 cent.

\(^{10}\)This method is presented in Heath, Jarrow, and Morton, [6].

\(^{11}\)[8]
5 Conclusion

In this paper a theoretical upper bound for the wild card option is presented.

\[ WC(t) \leq \sup_{\tau} \left[ E[\max(F(S_\tau) - \frac{B_j(\tau)}{CF_j}) \ast (CF_j - 1) + \epsilon', 0] \mid F_t] \right] \]

If the conversion factor of the cheapest bond to deliver is less than one, then the value of the wild card option is realized when it is exercised after bond prices increase. Further observe that the value of exercising the wild card option is at most a percentage \((CF_j - 1)\) of what it would be if it could be separated from the futures contract. Thus valuing the option by separating it from the contract is clearly incorrect.

Past studies have concluded that the value of the timing option in the treasury bond futures contract to be approximately 20 cents. \(^{12}\) However, the contribution of the wild card option to that value has not been explored until now, due to the difficulty of isolating the wild card option. We conclude that the majority of this value stems from permitting the short trader to choose which settlement time at which to deliver, not from the wild card play and that the importance of the wild card option in past literature is exaggerated. This model suggests that ignoring the wild card option when pricing the treasury bond futures contract will result with only a small loss of accuracy, provided the short trader is permitted which settlement time to deliver.

\(^{12}\)The futures price in this study was approximately $72. See Kane and Marcus [4].
6 Appendix 1

The purpose of this appendix is to give a formal proof that when \((S_{t-1} - t)\) is less than one day, the following inequality holds.

\[
E[Value(S_{t-1}) \mid \mathcal{F}_t] \leq \frac{Value(t)}{b(t, S_{t-1})} + \epsilon
\]

where \(Value(t)\) is the value of a treasury bond at time \(t\) (denoted \(V(t)\) from now on), and \(\epsilon\) is small. Recall, that \(b(t, T)\) is the price at time \(t\) of a $1$ discount bond which matures at time \(T\).

\[
b(t, T) = \exp[-\int_t^T f(t, v)dv]
\]

Define a money market account \(M(t)\) such that, \(M(t)\) is the price at time \(t\) of a money market account which started at time \(0\) with $1$.

\[
M(t) = \exp[\int_0^t f(v, v)dv]
\]

Observe that for \(b(t, T)\) and \(V(t)\), the value of a treasury bond at time \(t\), \(\frac{b(t, T)}{M(t)}\) and \(\frac{V(t)}{M(t)}\) are martingales.

Thus,

\[
E\left[ \frac{V(S_{t-1})}{M(S_{t-1})} \mid \mathcal{F}_t \right] = \frac{V(t)}{M(t)} \tag{1}
\]

From the definition of the money market account,

\[
M(S_{t-1}) = \exp[\int_0^{S_{t-1}} f(v, v)dv]
\]

\[
= M(t) * \exp[\int_t^{S_{t-1}} f(v, v)dv]
\]

Substituting back into equation 1,

\[
E\left[ \frac{V(S_{t-1})}{M(t) * \exp[\int_t^{S_{t-1}} f(v, v)dv]} \mid \mathcal{F}_t \right] = \frac{V(t)}{M(t)}
\]

Since \(M(t)\) is \(\mathcal{F}_t\) measurable,

\[
E\left[ \frac{V(S_{t-1})}{\exp[\int_t^{S_{t-1}} f(v, v)dv]} \mid \mathcal{F}_t \right] = V(t) \tag{2}
\]

Now,

\[
\exp[\int_t^{S_{t-1}} f(v, v)dv] = \exp[\int_t^{S_{t-1}} f(t, v)dv] * \exp[\int_t^{S_{t-1}} [f(v, v) - f(t, v)]dv]
\]

\[
= \frac{\exp[\int_t^{S_{t-1}} f(v, v)dv] - \exp[\int_t^{S_{t-1}} f(t, v)dv]}{\exp[\int_t^{S_{t-1}} f(t, v)dv]}
\]
\[
E\left[\frac{V(S_{t-1})}{\exp\int_{t}^{S_{t-1}}[f(v,v) - f(t,v)]dv}\right] | \mathcal{F}_t = V(t)
\]

Since \(b(t,S_{t-1})\) is \(\mathcal{F}_t\) measurable,

\[
E\left[\frac{V(S_{t-1})}{\exp\int_{t}^{S_{t-1}}[f(v,v) - f(t,v)]dv}\right] | \mathcal{F}_t = \frac{V(t)}{b(t,S_{t-1})}
\]

(3)

For convenience, define

\[
u \equiv \int_{t}^{S_{t-1}}[f(v,v) - f(t,v)]dv
\]

In the context of the problem, \((S_{t-1} - t)\) is less than one day. If we assume that the change in the forward curve can be modeled by the continuous Ho and Lee model, then \(f(v,v) - f(t,v)\) can be modeled by a random variable with a normal distribution, say \(N(\alpha_1, \sigma_1)\). An annual standard deviation of \(.02/\text{year}\) is an historically high estimate of \(\sigma_1\). This would imply a daily standard deviation of \(.001/\text{day}\) and using the continuous Ho and Lee model with \((S_{t-1} - t)\) less than a day, \(\alpha_1 < \sigma_1^2\). Observe \(u\) is the integral from \(t\) to \(S_{t-1}\) of the integrand just approximated. So \(u\) may be approximated by \(\frac{1}{365} * \text{Normal}(\alpha_1, \sigma_1^2)\) distribution, which we will write \(\text{Normal}(\alpha, \sigma^2)\) with \(\sigma = 3 \times 10^{-6}\). Observe also the the distribution of \(u\) is stationary as \(t\) varies.

Now we establish an upper bound on,

\[
E[V(S_{t-1}) | \mathcal{F}_t] - E\left[\frac{V(S_{t-1})}{\exp(u)} | \mathcal{F}_t\right] =
\]

\[
E[V(S_{t-1})\left[1 - \frac{1}{\exp(u)}\right] | \mathcal{F}_t]
\]

Recall that any correlation coefficient is \(\leq 1\). This implies that,

\[
\frac{E(XY) - E(X)E(Y)}{\sigma(X)\sigma(Y)} \leq 1
\]

So,

\[
E[V(S_{t-1})\left[1 - \frac{1}{\exp(u)}\right] | \mathcal{F}_t] \leq E[V(S_{t-1}) | \mathcal{F}_t] * E[1 - \exp(-u) | \mathcal{F}_t] +
\]

\[+ \sigma[V(S_{t-1}) | \mathcal{F}_t] * \sigma[1 - \exp(-u) | \mathcal{F}_t]\]

There are four terms which need evaluating in this equation. The first term, the conditional expected value of the bond is in the order of magnitude of \$100\) (par value). The third term, the conditional standard deviation is less than the expected value. The following will show that for \((S_{t-1} - t)\) equal to one day, the second and fourth terms are so small that when you multiply through by them the value for the entire right hand side of the inequality is very small.
Using the normal moment generating function and recalling $\sigma^2 \approx 9 \times 10^{-12}$ and $\alpha < \sigma^2$,

$$E[1 - \exp(-u) \mid \mathcal{F}_t] = 1 - \exp[-\alpha + \frac{\sigma^2}{2}] \approx 0 \equiv \epsilon_1$$

Now we calculate $\sigma(1 - \exp(-u))$. Recall,

$$\text{Var}(x) = E(x^2) - [E(x)]^2$$

So,

$$E[(1 - \exp(-u))^2] = E[1 - 2\exp(-u) + \exp(-2u)]$$

$$= 1 - 2\exp[-\alpha + \frac{\sigma^2}{2}] + \exp[-2\alpha + 2\sigma^2] \approx 0$$

This implies,

$$\sigma(1 - \exp(-u)) \approx 0 \equiv \epsilon_2$$

So,

$$E[V(S_{t-1}) - \frac{V(S_{t-1})}{\exp(u)} \mid \mathcal{F}_t] \leq \epsilon_1 \cdot E[V(S_{t-1}) \mid \mathcal{F}_t] + \epsilon_2 \cdot \sigma[V(S_{t-1}) \mid \mathcal{F}_t]$$

We have established an upper bound, which is very small. We will refer to the right hand side as $\epsilon$.

$$E[V(S_{t-1}) \mid \mathcal{F}_t] \leq E[\frac{V(S_{t-1})}{\exp(u)} \mid \mathcal{F}_t] + \epsilon$$

By equation 3,

$$V(S_{t-1}) \mid \mathcal{F}_t \leq \frac{V(t)}{b(t, S_{t-1})} + \epsilon$$
7 Appendix 2

In proving lemma 3, we will assume that at time \( t \), it is optimal to exercise the wild card option. We are interested in comparing the cashflow of Trader I, a short trader who exercises the wild card option, with the max expected cashflow of Trader II who is also a short trader in the original futures contract, but does not exercise the wild card option. Since it is optimal to exercise the wild card option at time \( t \), under the equivalent martingale measure, the expected cashflow of Trader II \( \leq \) the cashflow generated by the wild card play. What we are interested in is finding an upper bound for the expected loss at time \( t \) of Trader II by not exercising the wild card option. We will define \( L(t) \) to be the expected loss of Trader II if s/he follows the best strategy other than executing the wild card option at time \( t \). Note that if it were not optimal to exercise the wild card option at time \( t \), \( L(t) < 0 \). Letting \( S_t \) be the closing time immediately preceding \( t \), \( S_{t-1} \) be the closing time immediately after \( t \), and letting bond \( j \) be the cheapest to deliver,

\[
CASHFLOW_I(t) = F(S_t) * CF_j - B_j(t)
\]

One strategy available to Trader II is the following: at time \( t \), wait until immediately before the market closes the following day, at time \( S_{t-1} \), and execute a reversing trade (long the futures contract). The expected cash flow of Trader II at time \( t \) following an optimal strategy is \( \geq \) the expected cashflow of this strategy. So,

\[
E[CASHFLOW_{II}(S_{t-1}) | \mathcal{F}_t] \geq E[F(S_t) - F(S_{t-1}) | \mathcal{F}_t]
\]

\[
E[CASHFLOW_{II}(S_{t-1}) | \mathcal{F}_t] \geq F(S_t) - E[F(S_{t-1}) | \mathcal{F}_t]
\]

At time \( t \), we could put the proceeds from \( CASHFLOW_I \) into a default free bond until time \( S_{t-1} \), denoted \( b(t, S_{t-1}) \), which yields,

\[
E[L(S_{t-1}) | \mathcal{F}_t] \leq \frac{F(S_t) * CF_j - B_j(t)}{b(t, S_{t-1})} - (F(S_t) - E[F(S_{t-1}) | \mathcal{F}_t])
\]

Recall Lemma 1,

\[
F(S_{t-1}) \leq \min_i \left( \frac{B_i(S_{t-1})}{CF_i} \right)
\]

So,

\[
E[L(S_{t-1}) | \mathcal{F}_t] \leq \frac{F(S_t) * CF_j - B_j(t)}{b(t, S_{t-1})} - (F(S_t) - E[\min_i \left( \frac{B_i(S_{t-1})}{CF_i} \right) | \mathcal{F}_t])
\]

So for bond \( j \),

\[
E[L(S_{t-1}) | \mathcal{F}_t] \leq \frac{F(S_t) * CF_j - B_j(t)}{b(t, S_{t-1})} - (F(S_t) - \frac{1}{CF_j} * E[B_j(S_{t-1}) | \mathcal{F}_t])
\]

Using Lemma 4,

\[
E[L(S_{t-1}) | \mathcal{F}_t] \leq \frac{F(S_t) * CF_j - B_j(t)}{b(t, S_{t-1})} - F(S_t) + \left( \frac{B_j(t) + AI_j(t)}{b(t, S_{t-1})} - AI_j(S_{T-1}) + \epsilon \right) \frac{1}{CF_j}
\]
Rearranging we have,

\[ E[L(S_{t-1}) \mid \mathcal{F}_t] \leq \frac{1}{b(t, S_{t-1})} \ast [F(S_t) - \frac{B_j(t)}{CF_j}] \ast [CF_j - 1] + F(S_t) \ast \left[ \frac{1}{b(t, S_{t-1})} - 1 \right] + \]

\[ + \frac{1}{CF_j} \ast \left[ \frac{AI_j(t)}{b(t, S_{t-1})} - AI_j(S_{t-1}) + \epsilon \right] \]

Thus we have bound the expected difference in cash flows of a trader who exercises the wild card option to one who does not exercise it as a sum of three terms. Observe that in our problem \( S_{t-1} - t \) is less than 24 hours. The second term is only the interest of receiving the settlement price a day earlier and is thus small. The third term may be either positive or negative. Ignoring epsilon, it is positive if the 24 hour interest rate is greater than the coupon rate of the delivered treasury bond. This also will be small since the excess overnight accrued interest would be minimal. Thus the majority of the value of the wild card option stems from the first term. Discounting by \( b(t, S_{t-1}) \) leads to the approximation that,

\[ L(t) \leq [F(S_t) - \frac{B_j(t)}{CF_j}] \ast [CF_j - 1] + \epsilon' \]

where \( \epsilon' \) is the discounted sum of the second and third terms.
8 Appendix 3

The purpose of this appendix is to show that if $F_B(t) - F(t)$ exceeds a certain bound then the portfolio of shorting contract B, longing contract C, and borrowing WC(t), which has no initial cash flow, has a strategy with positive expected value under the equivalent martingale measure, implying arbitrage. The strategy has three possibilities. The first is that you entered the portfolio at a settlement time, and the long position was immediately delivered. In this case, you would instantaneously deliver your short position and repay the loan. The second possibility is that the long position is delivered at the next settlement time. In this case you would deliver your short position at the same time and repay the loan. The final possibility is that the long position is not delivered at either time. In this case we would execute a reverse trade in both markets and repay the loan at the next closing time. Note that if time t is not a closing time, immediate delivery is not a possibility. Furthermore if we are prior to the delivery month, then delivery at the next closing time is not possible either.

Recall for all $t$ in the future we know from lemma 7,

$$E(F(t)) \leq E(F_B(t))$$

We wish to show that the cash flows of these strategies are positive. Therefore if we are at a closing time $S_N$ we want the cashflow of immediate delivery,

$$F_B(S_N) \cdot CF_i - F(S_N) \cdot CF_i - WC(S_N) > 0$$

For all time $t$, the cashflow of delivery at the next settlement time,

$$E(F_B(S_{N-1})) \cdot CF_i + F_B(t) - E(F(S_{N-1})) \cdot CF_i - F(t) + E(F(S_{N-1}) - \frac{WC(t)}{b(t, S_{N-1})} > 0$$

The cashflow of a reverse trade at the next settlement time,

$$F_B(t) - E(F_B(S_{N-1})) - F(t) + E(F(S_{N-1}) - \frac{WC(t)}{b(t, S_{N-1})} > 0$$

If we are at a closing time $S_N$ and all three cash flows are positive, it implies $F_B(S_N) - F(S_N)$ is greater than

$$\max\left[ \frac{WC(S_N)}{CF_i}, E(F_B(S_{N-1}) - F(S_{N-1})) \cdot (1 - CF_i) + \frac{WC(S_N)}{b(S_N, S_{N-1})}, \right.$$

$$\left. E(F_B(S_{N-1}) - F(S_{N-1}) - WC(S_{N-1})) + \frac{WC(S_N)}{b(S_N, S_{N-1})} \right]$$

If we look at what occurs as at $N=1$, the bound becomes,

$$\max\left( \frac{WC(S_1)}{CF_i}, \frac{WC(S_1)}{b(S_1, S_0)} \right)$$

14
If we are not at a settlement time, then only the second and third constraints need to hold because we do not have to worry about immediate delivery. In this case we get $F_B(t) - F(t)$ is greater than

$$\text{Max}[E(F_B(S_{N-1}) - F(S_{S-1})) \ast (1 - CF_i) + \frac{WC(t)}{b(t, S_{N-1}}),$$

$$E(F_B(S_{N-1}) - F(S_{N-1}) - E(WC(S_{N-1}))), + \frac{WC(t)}{b(t, S_{N-1})}]$$

If $S_{t-1} = S_0$ then the bound becomes simply,

$$\frac{WC(t)}{b(t, S_0)}$$

Finally, if we are not in the delivery month and we do not have to worry about delivery at the next settlement time, then only the third constraint is appropriate. In this case we get $F_B(t) - F(t)$ is greater than

$$E(F_B(S_{N_{ext}}) - F(S_{N_{ext}}) - E(WC(S_{N_{ext}}))), + \frac{WC(t)}{b(t, S_{N_{ext}}})]$$

We conclude that if $F_B(t) - F(t)$ is greater than the appropriate above bound then the strategy of shorting contract B, longing contract C, and borrowing WC(t) is an arbitrage portfolio. Thus to avoid arbitrage $F_B(S_N) - F(S_N)$ must fall below these bounds.

References


