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TECHNICAL REPORT NO. 924

September 1990

MOVING AVERAGES OF RANDOM
SERIES WITH RANDOM COEFFICIENTS
AND RANDOM COEFFICIENT
AUTOREGRESSIVE MODELS

by

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¹Partially supported by the Mathematical Sciences Institute of Cornell
University and the Cornell Applied Mathematics Center as part of the Special
Focus on Extremes, Stable Processes and Heavy Tailed Phenomena.

²Research was partially supported by NSF Grant MCS-8801034 at Cornell
University.
Moving Averages of Random Series with Random Coefficients
and Random Coefficient Autoregressive Models

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September 1990

Abstract. Consider the series \( \sum_{n} C_n Z_n \) where \( \{Z_n\} \) are iid \( R^d \)-valued random vectors and \( \{C_n\} \) are random matrices independent of the \( \{Z_n\} \). Under suitable summability conditions on the \( \{C_n\} \), if the distribution of \( Z_1 \) is multivariate regularly varying at \( \infty \) then so is the distribution of the sum. Application is made to stationary solutions of the first order random difference equation in \( R^d \)

\[
X_{n+1} = M_{n+1} X_n + Q_{n+1}
\]

and to the \( p \)th order random difference equation

\[
X_t = \sum_{i=1}^{p} M_i(t) X_{t-i} + \epsilon_t.
\]

Under circumstances where explicit solution of the difference equations is impossible, this provides some information about the form of the solution.

Introduction.

The purpose of this paper is to show how methods for analyzing tail behavior of moving averages of iid heavy tailed random variables extend naturally to analyzing the tail behavior of moving averages of iid random \( d \)-dimensional random vectors with random matrix coefficients. The methods used for moving averages have been discussed in Cline (1983), Davis and Resnick (1985a,b; 1986), Davis, Marengo and Resnick (1985), Rootzen (1978, 1986), Resnick (1987).

In section 2 we consider tail behavior of an infinite series of the form

\[
\sum_{n} C_n Z_n
\]

where \( \{Z_n\} \) are iid \( d \)-dimensional \( R^d \) valued random vectors whose common distribution satisfies a multivariate regular variation condition at \( \infty \) and where the \( \{C_n\} \) are random matrices independent of the \( \{Z_n\} \) and whose norms satisfy suitable summability conditions. We are able to show that the distribution of \( \sum_{n} C_n Z_n \) is also regularly varying at \( \infty \) and that a situation of tail equivalence prevails between the distributions of \( Z_1 \) and \( \sum_{n} C_n Z_n \).

A particular case of these results arises when we consider stationary solutions of a random difference equation of the form

\[
X_{n+1} = M_{n+1} X_n + Q_{n+1}
\]

where we assume \( \{Q_n\} \) are iid random \( R^d \) valued vectors independent of the iid \( d \times d \) random matrices \( \{M_n\} \) which have non-negative entries. Such equations have been widely
studied in a variety of contexts. See for example Vervaat (1979), Nicholls and Quinn (1982), Grincevicius (1975, 1981), Goldie (1989), Kesten (1973), Furstenberg and Kesten (1960). It is rare to be able to explicitly exhibit the stationary solution of (1.1) (cf. Vervaat (1979)) so any qualitative information about the solution is very welcome. Assuming $Q_n$ has a distribution which is regularly varying at $\infty$ we are able to obtain under suitable conditions that the stationary solution has marginal distributions which are also regularly varying.

The vector setting of our results allows treatment of the $p$th order random coefficient autoregressive models of the form

$$X_t = \sum_{i=1}^{p} M_t(t)X_{t-i} + \epsilon_t$$

discussed for example in Nicholls and Quinn (1982). See also Grenander (1968). Here we assume that $\{\epsilon_t\}$ are iid non-negative random variables independent of the iid iid $R^d$-valued random vectors $\{(M_t(t), 1 \leq i \leq p), t \in N\}$. With modest extra effort, the vector version of (1.2) could be analyzed as well.

2. Regularly Varying Tails.

Suppose $\{Z_n\}$ are iid column vectors in $R^d_+ = [0, \infty)^d, d \geq 1$, with distributions which have tails which are regularly varying with index $\alpha, 0 < \alpha < \infty$. One convenient way to describe the property of multivariate regular variation is that there exists a Radon measure $\nu$ on $E := [0, \infty)^d \setminus \{0\}$ which is not identically 0, and a univariate regularly varying function $a(\cdot)$ of index $1/\alpha$ such that

$$nP[a_{n-1} Z_1 \in \cdot] \xrightarrow{\nu}$$

on $E$, where $a_n = a(n)$ and "\xrightarrow{\nu}" denotes vague convergence. (Cf. Resnick, 1987; O'Mey, 1989.)

Let $\{C_n\}$ be a sequence of $m \times d$ random matrices with positive components and suppose $\{C_n\}$ is independent of $\{Z_n\}$. We investigate the tail behavior of the series

$$\sum_{n=1}^{\infty} C_n Z_n.$$ 

First some comments on the convergence of the series.

For $z \in R^d$ or $R^m$ let $|z|$ be a norm, making $R^d$ or $R^m$ a Banach space and let $\|C\|$ be the usual Banach norm of the linear transformation $C$ defined for instance by

$$\|C\| = \sup_{|x| \leq 1} |Cx|,$$

so that $|Cx| \leq \|C\||x|$.

For analyzing tail behavior in the infinite series (2.1), the following assumptions are made. (These assumptions are stronger than what would be required for convergence in (2.2).) If $\alpha < 1$, we assume there exists $0 < \eta < \alpha$ such that $\alpha + \eta < 1$ and

$$\sum_{n} E\|C_n\|^{\alpha + \eta} < \infty.$$ 

If $\alpha \geq 1$, assume there exists $\eta > 0$ such that

$$\sum_{n} (E\|C_n\|^{\alpha + \eta})^{\frac{1}{\alpha + \eta}} < \infty, \quad \sum_{n} (E\|C_n\|^{\alpha - \eta})^{\frac{1}{\alpha - \eta}} < \infty.$$ 

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With these conditions one readily verifies that the series (2.2) converges.

If the series (2.2) were replaced by a sum of two terms, then by arguments almost identical to what is given in Davis, Marengo and Resnick (1986) or Resnick (1987, section 4.5) we would get

\[
(2.4) \quad nP[a_n^{-1}(C_1Z_1 + C_2Z_2) \in \cdot] \xrightarrow{\nu} E(\nu\{x : C_1x \in \cdot\} + \nu\{x : C_2x \in \cdot\})
\]

With the assumptions (2.1) and (2.3) or (2.3') in force we now show how to extend (2.4) to

\[
(2.5) \quad nP[a_n^{-1}\sum_{k=1}^{\infty} C_kZ_k \in \cdot] \xrightarrow{\nu} \sum_{k=1}^{\infty} E\nu\{x : C_kx \in \cdot\}
\]

The methods of analysis follow Cline (1983); see also the summary in Resnick (1987), section 4.5.

Temporarily, we suppose \( d = 1 \). It is convenient to collect some needed results for reference.

**Lemma 2.1.** Let \( Z \) be a non-negative random variable with distribution \( F \) such that \( 1 - F \) is regularly varying with index \( -\alpha \), \( \alpha > 0 \). If \( \epsilon > 0, c > 0 \) are given, there exist constants \( x_0, K > 0 \) such that

\[
(1) \quad \frac{1 - F(x/c)}{1 - F(x)} \leq \begin{cases} 
(1 + \epsilon)c^{\alpha - \epsilon}, & \text{if } c \geq 1, x/c \geq x_0 \\
(1 + \epsilon)c^{\alpha + \epsilon}, & \text{if } c < 1, x \geq x_0.
\end{cases}
\]

\[
(2) \quad E(cZ \wedge x)^{\alpha + \epsilon} \leq \begin{cases} 
Kc^{\alpha - \epsilon}x^{\alpha + \epsilon}(1 - F(x)), & \text{if } c \geq 1, x/c \geq x_0 \\
Kc^{\alpha + \epsilon}x^{\alpha + \epsilon}(1 - F(x)), & \text{if } c < 1, x \geq x_0.
\end{cases}
\]

**Proof:** (1) is basically the Potter inequalities (Bingham, Goldie, Teugels, 1987, page 25; Resnick, 1987, page 23). For (2), write

\[
E(cZ \wedge x)^{\alpha + \epsilon} = c^{\alpha + \epsilon} \int_0^{x/c} (\alpha + \epsilon)s^{\alpha + \epsilon - 1}P[Z > s]ds.
\]

From Karamata's theorem we get for large \( x \)

\[
E(cZ \wedge x)^{\alpha + \epsilon} \leq 2P[Z > x/c]x^{\alpha + \epsilon}.
\]

Now apply (1) to get (2). \( \blacksquare \)

Continue to suppose \( d = 1 \) and set \( \mathcal{C} = \sigma\{C_n, n \geq 1\} \). We observe that by Cline (1983) (see also Resnick, 1987, p. 227)

\[
(2.6) \quad \lim_{\epsilon \to \infty} \frac{P[\sum_{j=1}^{\infty} C_jz_j > x|\mathcal{C}]}{P[Z_1 > x]} = \sum_{j=1}^{\infty} C_j^z
\]

almost surely so that if the expectation of this limit is the limit of the expectations we will get the desired result. The taking of expectations will be justified by the following variant of Fatou's lemma attributed to Johns (1957) or Pratt (1960): If \( 0 \leq X_n \leq Y_n \) for \( n \geq 0 \) and \( X_n \to X_0 \) and \( Y_n \to Y_0 \) and \( EY_n \to EY_0 \), then \( EX_n \to EX_0 \).
To get a bound for the ratio in (2.6) write

\[ P(\sum_{j=1}^{\infty} C_j Z_j > x | \mathcal{C}) \leq P(\sum_{j=1}^{\infty} C_j Z_j > x, \bigvee_j C_j Z_j > x | \mathcal{C}) + P(\sum_{j=1}^{\infty} C_j Z_j > x, \bigvee_j C_j Z_j \leq x | \mathcal{C}) \]

\[ \leq P(\bigvee_j C_j Z_j > x | \mathcal{C}) + P(\sum_{j=1}^{\infty} C_j Z_j > x, \bigvee_j C_j Z_j \leq x | \mathcal{C}) \]

\[ = A + B. \]

Now we further decompose \( A \) as (\( x_0 \) comes from Lemma 2.1)

\[ A \leq P(\bigvee_j C_j Z_j > x, \bigvee_j C_j \leq x/x_0 | \mathcal{C}) + P(\bigvee_j C_j Z_j > x, \bigvee_j C_j > x/x_0 | \mathcal{C}) \]

\[ = A_1 + A_2. \]

Now

\[ \frac{E(A_2)}{P[Z_1 > x]} \leq \frac{\sum_{j=1}^{\infty} P(C_j > x/x_0)}{P[Z_1 > x]} \]

\[ \leq \frac{\sum_{j=1}^{\infty} E[C_j^\alpha + \eta (x/x_0)^{(\alpha + \eta)}}{P[Z_1 > x]} \]

where \( \eta \) comes from (2.3) and because of (2.3) and the fact that \( P[Z_1 > x] \) is \(-\alpha\)-varying, we see that \( E(A_2)/P[Z_1 > x] \to 0 \) as \( x \to \infty \). For \( A_1 \) we have the bound (set \( P[Z_1 \leq x] = F(x) \))

\[ \sum_j (1 - F(x/C_j)) I_{[C_j \leq x/x_0]} \]

so that applying Lemma 2.1 we get (\( K' > 0 \) is a constant)

\[ \frac{A_1}{1 - F(x)} \leq \sum_{j; C_j > 1} K' C_j^\alpha - \eta + \sum_{j; C_j \leq 1} K' C_j^\alpha + \eta \]

\[ \leq K' \left( \sum_j C_j^{\alpha - \eta} + \sum_j C_j^{\alpha + \eta} \right) \]

which by (2.3) is integrable.

Now we must deal with \( B \). We have

\[ B \leq P(\sum_j (C_j Z_j \wedge x) > x | \mathcal{C}) \]

\[ = P(\sum_j (C_j Z_j \wedge x) > x, \bigvee_j C_j \leq x/x_0 | \mathcal{C}) + P(\sum_j (C_j Z_j \wedge x) > x, \bigvee_j C_j > x/x_0 | \mathcal{C}) \]

\[ \leq P(\sum_j (C_j Z_j \wedge x) > x, \bigvee_j C_j \leq x/x_0 | \mathcal{C}) + P(\bigvee_j C_j > x/x_0 | \mathcal{C}) \]

\[ = B_1 + B_2. \]
We handle $B2/(1 - F(x))$ as we handled $A2/(1 - F(x))$. If $\alpha < 1$, choose $\eta$ as in (2.3) and then

$$B1 \leq 1_{\{V_j, C_j \leq x/\sigma_0\}}P[\sum (C_j Z_j \wedge x) > x|C]$$

$$\leq x^{-(\alpha+\eta)}1_{\{V_j, C_j \leq x/\sigma_0\}}E \left( \sum_j (C_j Z_j \wedge x)^{\alpha+\eta}|C \right)$$

and applying the second part of Lemma 2.1 we get the bound

$$\leq K x^{-(\alpha+\eta)} 1_{\{V_j, C_j \leq x/\sigma_0\}} \left( \sum_{j:C_j > 1} C_j^{\alpha-\eta} x^{\alpha+\eta}(1 - F(x)) + \sum_{j:C_j \leq 1} C_j^{\alpha+\eta} x^{\alpha+\eta}(1 - F(x)) \right)$$

from which it is abundantly clear that $B1/(1 - F(x))$ has the required bound when $\alpha < 1$.

When $\alpha \geq 1$ we have

$$B1 \leq x^{-(\alpha+\eta)} E \left( \sum_j (C_j Z_j \wedge x)^{\alpha+\eta}|C \right)$$

and applying Minkowski we get the bound

$$\leq x^{-(\alpha+\eta)} \left( \sum_j [E( (C_j Z_j \wedge x)^{\alpha+\eta}|C)]^{\frac{1}{\alpha+\eta}} \right)^{\alpha+\eta}$$

and applying Lemma 2.1 we get the bound

$$\leq x^{-(\alpha+\eta)} \left( \sum_{j:C_j > 1} [K x^{\alpha+\eta}(1 - F(x))C_j^{\alpha-\eta}]^{\frac{1}{\alpha+\eta}} + \sum_{j:C_j \leq 1} [K x^{\alpha+\eta}(1 - F(x))C_j^{\alpha+\eta}]^{\frac{1}{\alpha+\eta}} \right)^{\alpha+\eta}$$

$$\leq K'(1 - F(x)) \left\{ \left( \sum_j C_j^{\frac{\alpha+\eta}{\alpha+\eta}} \right)^{\alpha+\eta} + \left( \sum_j C_j \right)^{\alpha+\eta} \right\}.$$  

Since

$$E(\sum_j C_j)^{\alpha+\eta} \leq \left( \sum_j (EC_j^{\alpha+\eta})^{\frac{1}{\alpha+\eta}} \right)^{\alpha+\eta}$$

with a similar Minkowski bound for

$$E \left( \sum_j C_j^{\frac{\alpha+\eta}{\alpha+\eta}} \right)^{\alpha+\eta}$$

we get the desired bound in the $\alpha \geq 1$ case via (2.3')

This proves (2.5) when $d = 1$. To generalize this to $d > 1$ under conditions (2.1) and (2.3) or (2.3') we may suppose that

$$nP(|Z_1| > a_n) \to 1.$$
Note that $a_n \to \infty$. Observe that for compact $K \subset E$ we have for any integer $L$

$$nP[a_n^{-1} \sum_{j=1}^{\infty} C_j Z_j \in K] = nP[a_n^{-1} \left( \sum_{j=1}^{L} C_j Z_j + \sum_{j=L+1}^{\infty} C_j Z_j \right) \in K]$$

$$\leq nP[a_n^{-1} \sum_{j=1}^{L} C_j Z_j \in K^\delta] + nP[\sum_{j=L+1}^{\infty} \|C_j\|\|Z_j\| > \delta]$$

where $K^\delta$ is the closed $\delta$-neighborhood of $K$ in the norm topology. Thus from (2.4) and the result for $d = 1$ we get

$$\limsup_{n \to \infty} nP[a_n^{-1} \sum_{j=1}^{\infty} C_j Z_j \in K]$$

$$\leq \sum_{j=1}^{L} E\nu\{x : C_j x \in K^\delta\} + \sum_{j=L+1}^{\infty} E\|C_j\|^2 \delta^{-\alpha}$$

so letting $L \to \infty$ and then $\delta \to 0$ yields

(2.7) $$\limsup_{n \to \infty} nP[a_n^{-1} \sum_{j=1}^{\infty} C_j Z_j \in K] \leq \sum_{k=1}^{\infty} E\nu\{x : C_k x \in K\}.$$ 

On the other hand if $G$ is open and relatively compact, there exist open, relatively compact $\{G_m\}$ such that

$$G_m \subset \overline{G_m} \subset G_{m+1} \uparrow G.$$ 

The distance from $G_m$ to $G^c$ must be strictly positive so for $L, m$ given there must exist $\epsilon > 0$ such that

$$[a_n^{-1} \sum_{n=1}^{L} C_n Z_n \in G_m] \bigcap \bigcap_{n=L+1}^{\infty} C_n Z_n \leq a_n \epsilon \subset [a_n^{-1} \sum_{n=1}^{\infty} C_n Z_n \in G],$$

whence

$$\liminf_{n \to \infty} nP[a_n^{-1} \sum_{n=1}^{\infty} C_n Z_n \in G] \geq \liminf_{n \to \infty} P[a_n^{-1} \sum_{n=1}^{L} C_n Z_n \in G_m] P[\bigcap_{n=L+1}^{\infty} C_n Z_n \leq a_n \epsilon]$$

and from (2.3) we get the lower bound

$$\geq \sum_{n=1}^{L} E\nu\{x : C_n x \in G_m\}.$$ 

Letting first $m \to \infty$ and then $L \to \infty$ yields

(2.8) $$\liminf_{n \to \infty} nP[a_n^{-1} \sum_{n=1}^{\infty} C_n Z_n \in G] \geq \sum_{n=1}^{\infty} E\nu\{x : C_n x \in G\}.$$ 

Combining (2.8) and (2.9) yields (2.6).

We now summarize our findings.
THEOREM 2.2. Suppose \( \{Z_n\} \) is iid independent of the random matrices \( \{C_n\} \) and that the multivariate regular variation condition

\[
(2.1) \quad nP[a_n^{-1}Z_1 \in \cdot] \xrightarrow{\nu} \nu
\]

holds and that the summability conditions (2.3), (2.3') on the random coefficients \( \{C_n\} \) hold. Then the tail behavior of the series

\[
(2.2) \quad \sum_{n=1}^{\infty} C_n Z_n
\]

is given by

\[
(2.6) \quad nP\left[a_n^{-1} \sum_{k=1}^{\infty} C_k Z_k \in \cdot\right] \xrightarrow{\nu} \sum_{k=1}^{\infty} E\nu\{x : C_k x \in \cdot\}
\]

If \( d = 1 \) this result says

\[
(2.9) \quad \lim_{x \to \infty} \frac{P[\sum_{k=1}^{\infty} C_k Z_k > x]}{P[Z_1 > x]} = \sum_{k=1}^{\infty} EC_k^{\nu}.
\]


Consider a stochastic difference equation of the form \( n \in \mathbb{N} \)

\[
(3.1) \quad X_{n+1} = M_{n+1} X_n + Q_{n+1},
\]

where \( \{Q_n\} \) are iid random \( [0, \infty)^d \)-valued vectors independent of the iid \( d \times d \) random matrices \( \{M_n\} \). We assume the components of \( M_n \) are non-negative almost surely. Such equations have been widely studied in a variety of contexts. See for example Vervaat (1979), Nickols and Quinn (1982), Grincevicius (1975,1981), Goldie (1989), Kesten (1973), Furstenberg and Kesten (1960). We are interested in the implications of Theorem 2.2 for such equations.

COROLLARY 3.1. Suppose \( Q_1 \) has a distribution with a regularly varying tail of index \( \alpha > 0 \) so that for a regularly varying function \( a(\cdot) \) of index \( 1/\alpha \) we have with \( a_n = a(n) \) and some limit measure \( \nu \):

\[
(3.2) \quad nP[a_n^{-1}Q_1 \in \cdot] \xrightarrow{\nu} \nu(\cdot)
\]

on \( [0, \infty)^d \setminus \{0\} \) with \( \nu \not= 0 \). Assume that \( \{M_n\} \) satisfies for some \( \beta > 0 \)

\[
E\|M_1\|^{\alpha+\beta} < \infty
\]

and that for some \( p \geq 1 \)

\[
(3.3) \quad E\left[ \prod_{i=1}^{p} M_i \right]^{\alpha} < 1.
\]

The stationary solution \( \{X_n\} \) of (3.1) exists with marginal distributions satisfying

\[
(3.4) \quad X_1 \overset{d}{=} \sum_{n=1}^{\infty} \left( \prod_{i=1}^{n-1} M_i \right) Q_n
\]
where an empty product is set equal to 1. The tail behavior of the marginal distribution is given by

\[(3.5) \quad nP[a_n^{-1}X_1 \in \cdot] \to \sum_{n=1}^{\infty} E\nu\{x : \prod_{i=1}^{n-1} M_i x \in \cdot\}.\]

If \(d = 1\), (3.5) becomes

\[nP[a_n^{-1}X_n > x] \to \sum_{n=1}^{\infty} E \left( \prod_{i=1}^{n-1} M_i^\alpha \right) x^{-\alpha} = \frac{1}{1 - EM_1^\alpha} x^{-\alpha}.\]

The result for \(d = 1\) is given in Grincevicus (1975) under slightly weaker conditions.

**Proof:** The distributional equivalence in (3.4) is clear after iterating the relation (3.1) several times; cf. Vervaat, 1979, page 753. If we set

\[C_n = \prod_{i=1}^{n-1} M_i\]

then we must analyze tail behavior of \(\sum_{n=1}^{\infty} C_n \eta_n\), and Theorem 2.2 is applicable provided (2.3) or (2.3') is applicable. Because of (3.3), there exists \(\eta < \beta\) such that

\[E\left\| \prod_{i=1}^{p} M_i \right\|^{\alpha \pm \eta} < 1.\]

For (2.3) we have

\[\sum_{n=p}^{\infty} E\|C_{n+1}\|^{\alpha \pm \eta} = \sum_{n=p}^{\infty} E\left\| \prod_{i=1}^{n} M_i \right\|^{\alpha \pm \eta}\]

\[= \sum_{m=1}^{\infty} \sum_{l=0}^{m+p} E\| \prod_{i=1}^{m} M_i \|^{\alpha \pm \eta} \left( E\left\| \prod_{i=1}^{p} M_i \right\|^{\alpha \pm \eta} \right)^m < \infty.\]

Verification of (2.3') is almost the same since if

\[E\left\| \prod_{i=1}^{p} M_i \right\|^{\alpha \pm \eta} < 1\]

then

\[\left( E\left\| \prod_{i=1}^{p} M_i \right\|^{\alpha \pm \eta} \right)^{\frac{1}{\alpha \pm \eta}} < 1. \]
Now consider the process of one dimensional random variables defined by the $p$th order equation $(t \in N)$

$$X_t = \sum_{i=1}^{p} M_i(t)X_{t-i} + \epsilon_t$$

where $\{\epsilon_t\}$ are iid non-negative random variables independent from the iid $[0, \infty)^p$-valued random vectors $\{(M_i(t), 1 \leq i \leq p), t \in N\}$. (Obviously, in what follows we could have taken the $\epsilon$'s vector valued; needed changes would be slight.)

The standard method of analyzing (3.6) is to convert it into a vector valued first order equation. This is accomplished by setting

$$Q_t = (0, \ldots, 0, \epsilon_t)' , \quad Y_t = (X_{t-p+1}, \ldots, X_t)'$$

and

$$M_t = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
M_p(t) & M_{p-1}(t) & \cdots & M_1(t)
\end{pmatrix}$$

so that from (3.6) we get the first order equation $(t \in N)$

$$Y_t = M_t Y_{t-1} + Q_t.$$

If $\epsilon_1$ has a distribution with a regularly varying tail

$$\lim_{t \to \infty} \frac{P[\epsilon_1 > tx]}{P[\epsilon_1 > t]} = x^{-\alpha}, \quad x > 0$$

then the same is true for the multivariate distribution of $Q_1$:

$$nP[a_n^{-1}Q_1 \in dx] \sim \delta_0(dx_1) \times \cdots \times \delta_0(dx_{p-1}) \times (x_1^{p-\alpha-1}dx_p) =: \nu(dx)$$

where $\delta_0$ is the measure concentrating all mass on 0. We may now apply Corollary 3.1.

**Corollary 3.2.** Suppose (3.6), (3.8) hold and that

$$E\|\prod_{j=1}^{p} M_j\|^\alpha < 1, \quad EM_i(t)^{\alpha+\beta} < \infty$$

for some $\beta > 0$ and $i = 1, \ldots, p$. Then $X_1$ is tail equivalent to $\epsilon_1$:

$$nP[a_n^{-1}X_1 > x] \sim x^{-\alpha} \sum_{n=1}^{\infty} \chi_n$$

where

$$\chi_n = E\left\{\left(\prod_{j=1}^{n-1} M_j(0, \ldots, 0, 1)\right)^\alpha\right\}.$$
(The notation is \((x)_p\) is the pth component of the p-dimensional vector \(x\).)

**Proof:** From Corollary 3.1 we conclude

\[
nP[a_n^{-1}Y_1 \in \cdot] \rightarrow \sum_{n=1}^{\infty} E \nu \{y : \prod_{i=1}^{n-1} M_i \in \cdot\}.
\]

Thus we have

\[
nP[a_n^{-1}X_1 > x] = nP[a_n^{-1}Y_1 \in [0, \infty) \times \cdots \times [0, \infty) \times (x, \infty)]
\]

\[
\rightarrow \sum_{n=1}^{\infty} E \nu \{(0, \ldots, 0, y)' : \prod_{i=1}^{n-1} M_i(0, \ldots, 0, y)' \in [0, \infty) \times \cdots \times [0, \infty) \times (x, \infty)\}
\]

and because of the special form of \(\nu\) given in (3.9) this is

\[
= \sum_{n=1}^{\infty} E \int_{y > 0, y' > x} \alpha y^{-\alpha - 1} dy
\]

\[
= x^{-\alpha} \sum_{n=1}^{\infty} \chi_n.
\]

For \(p = 2\) suppose for instance that \(\alpha < 1\) and

\[
E|M_1(t)|^\alpha + E|M_2(t)|^\alpha =: c
\]

satisfies

\[
c(c + 1) < 1
\]

and that in addition for \(\beta > 0\) we have \(E|M_i(t)|^{\alpha + \beta} < \infty\) for \(i = 1, 2\) and any \(t\). Then

\[
M_1M_2 = \begin{pmatrix} 0 & 1 \\ M_2(1) & M_1(1) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ M_2(2) & M_1(2) \end{pmatrix}
\]

so that

\[
\|M_1M_2\| \leq M_2(2) + M_1(2) + |M_1(1)M_2(2) + M_2(1) + M_1(1)M_1(2)|
\]

and from here it is easy to see that (3.10) holds.

**References**


*Keywords*. multivariate regular variation, infinite sums with random coefficients, random coefficient autoregressive models.

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