STOCHASTIC STORAGE PROCESSES
WITH FINITE BOUNDARIES

By

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Stochastic storage processes with finite boundaries arise naturally in a variety of settings. Some of the more important models are the single server queue with a finite capacity, (either due to a finite waiting room or due to a work-load regulating policy), inventory models with a constraint on the size of the warehouse, insurance-risk models with an upper limit on the risk-reserve, and dams or storage reservoirs with a finite capacity.

We investigate a class of stochastic models that gives rise to stochastic storage processes with finite boundaries. On a suitable probability space, a construction is given, by piecing together independent copies of the underlying netput process. In discrete time this underlying process would be a random walk, whereas in continuous time a Levy process would be appropriate. The properties of the constructed storage process are studied. We obtain the transient as well as limiting distribution of the storage level. Comparisons are made with the case of a single boundary (the origin).
The class of models studied here generally covers the situations described above. Special cases of the problem of the finite dam have been studied under restrictive assumptions by several authors. For the single server queue our formulation is relevant to a system where the server regulates the total virtual workload in the system and denies access to any customer whose arrival would cause the workload to exceed an allowable maximum.

Models with a single finite boundary have been studied extensively in the literature. An important property of such processes is their regenerative nature, the principal tool of analysis being renewal theory. Our approach to the two-boundary case exploits the semiregenerative nature of the process and uses results from Markov-renewal theory.
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Chapter 1

Introduction and Summary

1.1 Problem formulation

Problems in the theory of storage have been studied in the past three or four decades by several authors. Classical models have mostly concentrated on problems with a single (lower) finite boundary. These provide a general framework for dealing with problems of queues, dams, insurance-risk, inventory and many other uncapacitated stochastic flow systems. For a few of these models see Arrow, Karlin & Scarf [1], Moran [10], and Prabhu [11].

It is practically always the case that queues have only a finite waiting room, dams have only a finite capacity and finished goods inventories have finite buffer spaces. Therefore it is more realistic to study stochastic storage models with a finite capacity. Due to the presence of a lower as well as upper boundary, such models will be referred to as stochastic storage processes with (two) finite boundaries.
The common features of the storage systems mentioned above are an input process (production, customer arrivals, rainfall), an output process (demand, customer departures, released water), and a finite intermediate storage buffer (consisting of finished goods, customers in queue, stored water), which in some sense decouples the input from the output.

We first give an abstract description of the class of problems to be studied here, in terms of the input process, the output process, and the finite storage buffer. Two different models are described, one in discrete time and the other with time as a continuous parameter.

**Storage Model 1** (Discrete time): Consider a store of finite capacity $c < \infty$. Let $A_0 = 0$ and let $A_n$, $(n \geq 1)$ be the amount (of goods, water, etc.) entering the store during the time interval $(n-1, n]$ provided there is no overflow. Thus, $\sum_{i=1}^{n} A_i$ is the amount that enters the store in time $(0, n]$ provided the store never becomes full in that period. We shall call the stochastic process $\{\sum_{i=1}^{n} A_i : n = 0, 1, 2, \ldots\}$, the *potential input* process. Clearly, $\{\sum_{i=1}^{n} A_i : n = 0, 1, 2, \ldots\}$ is an increasing process.

Let $B_0 = 0$ and let $B_n$ be the demand at time $n \geq 1$. Thus, $\sum_{i=1}^{n} B_i$ is the total output from the store in time $(0, n]$ provided that the store is never empty during that period. The stochastic process $\{\sum_{i=1}^{n} B_i : n = 0, 1, 2, \ldots\}$, which will be referred to as the *potential output* process, is once again an increasing process. If the level in the store rises above $c$ at any time, then the excess input is lost instantaneously due to overflow. This overflow does not contribute towards the demand, which is drawn out from the store after any overflow has taken place.
If at any time the store cannot fully satisfy the demand, then the demand is satisfied to the extent possible and that portion of it which cannot be fulfilled is lost. There is no backlogging of demand and lost demand does not affect future supply or demand. Thus $A_n$ and $B_n$ represent potential rather than actual input and output respectively. Let $S_n = \sum_{i=1}^{n}(A_i - B_i) \ (n \geq 1)$. We shall require that the sequence of random variables $\{X_n = A_n - B_n\}$ be independent and identically distributed. The assumption sometimes made in the literature that the sequences $\{A_n : n = 0,1,2,\ldots\}$ and $\{B_n : n = 0,1,2,\ldots\}$ be independent is unnecessary; indeed it is often the case that the demand in a period is correlated with the supply. We shall use the terminology of the economic theory of production and call the process $\{S_n : n = 0,1,2,\ldots\}$, the underlying netput process.

Let us denote by $Z_n$, the storage level at time $n \geq 0$. We may describe the above model mathematically, as follows.

Let $\eta_0 = 0$, and let $\eta_n \ (n \geq 1)$ be the actual input into the store in the time interval $(n, n+1]$. Due to the possibility of overflow, we may express the actual input after overflow (if any) as

$$\eta_n = \min(A_n, c - Z_{n-1}) \ (n \geq 1) \quad (1.1)$$

Let $\xi_0 = 0$, and let $\xi_n$ be the actual output at time $n$. Since there is a possibility that the demand exceeds the available storage, the actual output may be described by the release rule

$$\xi_n = \min(Z_{n-1} + \eta_n, B_n), \ (n \geq 1) \quad (1.2)$$
We may now express the storage level at time \( n \) as

\[
Z_n = Z_{n-1} + \eta_n - \xi_n \quad (n \geq 1),
\]

where \( Z_0 \), the initial storage level lies in \([0, c]\).

In this thesis, we are interested in providing a general framework in which to study the process \( \{Z_n : n = 0, 1, 2, \ldots\} \).

**Storage Model 2 (Continuous time):** Let us denote by \( A_t \), the total amount (of commodities, workload, etc.) flowing into a store of finite capacity \( c < \infty \), during the time interval \((0, t]\). Unless the store is empty, there is a continuous release from the store, at a unit rate. During those intervals in which the store is empty, no release occurs. If the store becomes full, the excess amount above \( c \) is lost instantaneously. Perhaps, it is most suitable here to think of a dam of finite capacity into which water flows according to an input process \( \{A_t : t \geq 0\} \). To satisfy demand downstream water is released continuously and at a unit rate as long as the dam is not dry. Any water that causes overtopping is lost instantaneously and does not contribute towards satisfying the demand. We assume that \( \{A_t : t \geq 0\} \) is an increasing stochastic process. Of primary interest is \( Z_t \), storage level at time \( t \geq 0 \). In this thesis, we discuss a class of models that enables us to understand the process \( \{Z_t : t \geq 0\} \).

### 1.2 Survey of the literature

Some of the earliest analytical work on the problem of the finite dam is by Moran [10]. In the discrete-time problem, if one assumes the input, output and capacity
to be discrete valued, then the dam content process \( \{Z_n : n = 0, 1, 2, \ldots\} \) may be modelled as a Markov chain. A detailed analysis of such a model involving a study of the transient behaviour of \( \{Z_n : n = 0, 1, 2, \ldots\} \) can be found in the book by Prabhu [11]. The exact form of the distribution of \( Z_n \) even in the seemingly simple case of geometric inputs is extremely complicated. Transient solutions for other input distributions have not been found.

The problem of finding the limiting distribution of the dam content is somewhat more tractable. The book by Prabhu [11] contains a description of the methods developed for obtaining the stationary distribution of \( Z_n \), which have been successfully applied for a number of input distributions.

For inputs with a continuous distribution, the stationary distribution of the dam content satisfies a certain integral equation written in terms of the distribution function of the inputs. This integral equation may, in some sense, be viewed as an appropriate limit of the system of equations describing the stationary distribution of the Markov chain \( \{Z_n : n = 0, 1, 2, \ldots\} \) for the case of discrete inputs. The solution of this integral equation is known for the case of exponential as well as gamma inputs and can be found in the book by Prabhu [11].

In continuous time, one possible formulation is to define the dam content \( Z_t \) by the equation

\[
Z_{t+dt} = \min\{Z_t + dA_t, c\} - \min\{Z_t, dt\}
\]  

(1.4)

Equation 1.4 shows that \( \{Z_t : t \geq 0\} \) is a time-homogeneous Markov process. Letting \( F(z, t) \) denote its transition function, one can establish the forward Kol-
mogorov differential equation

\[
\frac{\partial F}{\partial t} - \frac{\partial F}{\partial z} = -\lambda(F(z, t) - F(z - h, t)), \quad (0 \leq t \leq \infty, 0 \leq z \leq c)
\]

along with the boundary conditions

\[
F(z, t) = 0, \quad z < 0 \tag{1.6}
\]

\[
F(c, t) = 0, \quad t \geq 0 \tag{1.7}
\]

However, this representation is not generally valid. See, for example, Kingman [7].

Weesakul and Yeo [18] have treated a special case of the problem, when the inputs form a compound Poisson process with exponentially distributed jumps. However, they are unable to solve the equation 1.5 directly. Instead, they obtain a solution for the discrete time problem with geometric inputs and define the solution to the continuous time problem by taking the limit of the discrete time problem in an appropriate manner.

Srinivasan [16] treats the problem of the finite dam whose inputs constitute a stationary renewal point process, the interval between two renewals having a known density. The magnitudes of the successive inputs are assumed to form a sequence of mutually independent, identically distributed random variables. However, the emphasis in this work is on the distribution of the first wet period and the distribution of the storage level during the first wet period.

In view of the inherent difficult nature of the problem caused by the finite boundaries, other indirect methods have been employed by several authors of which we mention only the following. Takacs [17] uses a generalization of the classical ruin theorem, to treat the continuous time problem when the inputs
form a compound Poisson process. His methods are combinatorial in nature and quite involved.

Roes [14], [15] employs another method to make the problem more tractable. The dam content process \( \{Z_t : t \geq 0\} \) is enclosed between two processes \( \{U_t : t \geq 0\} \) and \( \{V_t : t \geq 0\} \) which are allowed to have jumps only at the epochs \( t = n\Delta, \quad n = 1, 2, \ldots, \quad \Delta > 0 \). The enclosing processes are dealt with by a method developed by Cohen [4] which involves the solution of a certain integro-differential equation. However, as the authors remark, this approach is not quite satisfactory because of the inherent problem of uniqueness of the integro-differential equation.

In the queueing context, the class of models studied here can be applied to finite maximum waiting time situations. Here the server has a policy of regulating the workload in the system and rejecting any arrival whose service time would cause an 'overflow' in the virtual workload. See Kleinrock [9] for a discussion of such problems.

Stochastic models of buffered flow as applied to production planning have been discussed by Harrison [5]. He considers a firm that produces a single durable commodity on a make-to-stock basis. Production flows into a finished goods inventory, and demand that cannot be met is simply lost, with no adverse effect on future demand. Such models have particular relevance to systems where individual inventory items are physically and economically insignificant, and the volume of flow is high enough to be represented by a continuous stochastic process.
1.3 Models with a single finite boundary

In order to gain a better understanding of the problem and the nature of the difficulties involved in treating the problem of a storage process with two finite boundaries, we first give a brief description of a storage process with a single finite boundary (the origin).

Let \( \{ \sum_{i=1}^{n} A_i : n = 0, 1, 2, \ldots \} \), \( \{ \sum_{i=1}^{n} B_i : n = 0, 1, 2, \ldots \} \) and \( \{ \sum_{i=1}^{n} S_i : n = 0, 1, 2, \ldots \} \) be the potential input, output and netput processes respectively, as described in section 1.1.

We shall assume that the store has an infinite capacity. If the storage level is not adequate to meet a demand, then the demand is satisfied to the extent possible. Unsatisfied demand is lost and does not affect future demand or supply. Denote by \( Z_n \), the storage level at time \( n \geq 0 \).

\( Z_n \) may be expressed in terms of the maximum and minimum functionals of netput process as shown below. Define

\[
m_n = \min(0, S_1, S_2, \ldots, S_n) \tag{1.8}
\]

Then, we have the following

**Theorem 1.3.1** For the infinite capacity store with initial level \( Z_0 \), the storage level may be expressed as

\[
Z_n = \min(Z_0 + S_n, S_n - m_n). \tag{1.9}
\]

**Proof:**

Let \( X_n = A_n - B_n \) \( (n \geq 1) \)
Then,

\[ Z_1 = \max(0, Z_0 + X_1) \]
\[ Z_2 = \max(0, Z_1 + X_2) = \max(0, X_2, Z_0 + X_1 + X_2). \]

Proceeding inductively,

\[ Z_n = \max(0, X_n, X_n + X_{n-1}, \ldots, X_2, Z_0 + X_n + \ldots X_1) \]
\[ = \max(S_n - S_n-r, (0 \leq r \leq n), Z_0 + S_n). \]

Since

\[ \max(-S_{n-r} (0 \leq r \leq n)) = -\min(S_r (0 \leq r \leq n)) = -m_n \quad (1.10) \]

we get

\[ Z_n = \min(Z_0 + S_n, S_n - m_n) \quad (1.11) \]

Further, the process \( \{Z_n : n = 0, 1, 2, \ldots \} \) has embedded in it, two regenerative phenomena, namely the ladder processes defined by

\[ N_1 = \min\{n : S_n > 0\} \quad (1.12) \]
\[ N_r = \min\{n : S_n > S_{N_{r-1}} \} \quad (r \geq 2) \quad (1.13) \]

and

\[ \bar{N}_1 = \min\{n > 0 : S_n \leq 0\} \quad (1.14) \]
\[ \bar{N}_r = \min\{n > 0 : S_n \leq S_{\bar{N}_{r-1}} \} \quad (r \geq 2) \quad (1.15) \]
A graph of $Z_n$ and $S_n$ versus $n$ shows that between the successive epochs $N_1, N_2, \ldots$, the two processes are congruent. Using the regenerative property of the process \{$Z_n : n = 0, 1, 2, \ldots$\}, it is possible to find the distribution of $Z_n$. We first need to define two renewal functions associated with the ladder processes \{$N_r : r = 0, 1, 2, \ldots$\}, and \{$ar{N}_r : r = 0, 1, 2, \ldots$\}.

Define

\[ v_n = P \{ S_n \leq S_m (0 \leq m \leq n - 1), S_n \geq x \} (n \geq 1, x \leq 0) \quad (1.16) \]

\[ v_0 = \begin{cases} 1 & \text{if } x \leq 0 \\ 0 & \text{if } x > 0 \end{cases} \quad (1.17) \]

and

\[ u_n = P \{ S_n > S_m (0 \leq m \leq n - 1), S_n \leq x \} (n \geq 1, x > 0) \quad (1.18) \]

\[ u_0 = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} \quad (1.19) \]

We then have the following (See Prabhu [11]).

**Theorem 1.3.2** For the infinite capacity store with initial level $Z_0$, the distribution of the storage level is given by

\[ P\{Z_n \leq x\} = \sum_{i=1}^{n} u_m(x - Z_0)v_{n-m}(-Z_0) + \sum_{i=1}^{n} u_m(x)v_{n-m}(-\infty) \]

\[ -\sum_{i=1}^{n} u_m(x)v_{n-m}(-Z_0) \quad (1.20) \]
To illustrate the preceeding ideas, let us consider an M/M/1 queue in which the interarrival times \( u_1, u_2, \ldots \) have the exponential density \( \lambda e^{-\lambda z} \) and the service times \( v_1, v_2, \ldots \) have the exponential density \( \mu e^{-\mu z} \), where \( 0 < \lambda < \infty \), \( 0 < \mu < \infty \). We shall assume that the traffic intensity \( \rho = \frac{\lambda}{\mu} < 1 \). Let us denote by \( W_n \), the waiting time of the \( n \)th customer. Let \( X_n = u_n - v_n \), \( (n \geq 1) \), and \( S_n = \sum_{i=1}^{n} X_i \) \( (n \geq 1) \). It can be shown that the waiting time process \( \{W_n : n = 0, 1, 2, \ldots\} \) is the storage process induced by the netput process \( \{S_n : n = 0, 1, 2, \ldots\} \). In order to find the distribution of \( W_n \), we first need the following

**Lemma 1.3.1** For the netput process \( \{S_n : n = 0, 1, 2, \ldots\} \) of the M/M/1 queue, we have

\[
E(z^{N_1} e^{i\omega S_{N_1}}) = \frac{\lambda \xi(z)}{\mu - i\omega} \tag{1.21}
\]

where \( 0 < z < 1 \), \( \omega \in \mathcal{R} \), and \( \xi(z) = \frac{(\lambda+\mu)-\sqrt{(\lambda+\mu)^2-4\lambda\mu z}}{2\lambda} \)

This lemma follows from the Weiner-Hopf factorization for the random walk \( \{S_n : n = 0, 1, 2, \ldots\} \). For a proof, we refer to Prabhu [12]. Using this result, we can obtain the distribution of \( W_n \).

**Theorem 1.3.3** In the M/M/1 queue, the limit distribution function of the \( n \)th customer's waiting time is given by

\[
\lim_{n \to \infty} P\{W_n \leq x\} = 1 - \rho e^{-(\mu-\lambda)z}, \quad x \geq 0 \quad (\rho > 1) \tag{1.22}
\]

This concludes our discussion of infinite capacity storage processes in discrete time. In continuous time, the input process may be allowed to belong to a general class of processes namely the class of Lévy processes. For example, in the case
of the M/G/1 queue, if we consider the virtual workload process as the storage process, then the input process is of the compound Poisson type. For the case of a dam, the input may be of a more general member of the class of infinitely divisible processes. The class of Lévy processes is rich enough to contain both the compound Poisson process as well as Brownian motion as special members. To give rigour to our treatment of continuous time storage processes, we need the following definitions.

Let \((W, \mathcal{A}, P)\) be a probability space on which \(\{A_t : t \geq 0\}\) is a stochastic process taking values in \((\mathbb{R}_+, \mathbb{R}_+)\).

**Definition 1.3.1** The stochastic process \(\{A_t : t \geq 0\}\) is said to have stationary and independent increments if

(i) \(A_0 = 0\) \(\text{ w.p. } 1\)

(ii) for every \((t_0, t_1, \ldots, t_n)\), the family of random variables \(\{A_{t_i} - A_{t_{i-1}}\}_i\) is independent.

(iii) for every \(s < t\), the distribution of \(A_t - A_s\) depends only on \(t - s\).

**Definition 1.3.2** The stochastic process \(\{A_t : t \geq 0\}\) is said to be continuous in probability, if for every \(\epsilon > 0\),

\[
P(|A_t| > \epsilon) \to 0 \quad \text{as } t \to 0^+.
\]

**Definition 1.3.3** The stochastic process \(\{A_t : t \geq 0\}\) is said to be RCLL (right continuous with left limits) if

(i) the limits \(A_{t-}\) and \(A_{t+}\) exist, and
(ii) $A_{t+} = A_t$.

**Definition 1.3.4** The stochastic process $\{A_t : t \geq 0\}$ is said to be a Lévy process if it satisfies definitions 1.3.1, 1.3.2, and 1.3.3, with probability 1.

It can be shown that the characteristic function of a Lévy process may be written as

$$
E[\exp(i\omega A_t)] = \exp[-t(i\omega + \frac{\omega^2 \sigma^2}{2} + \int_{-\infty}^{\infty} (1 - e^{i\omega y} + \frac{i\omega y}{1 + y^2})]M(dy)
$$

$$(i = \sqrt{-1}, \omega \in \mathbb{R}) \quad (1.23)$$

where $a$ is a real constant and $M(dy)$ is a measure on the Borel sets of $\mathbb{R}$ called the Lévy measure of the process $\{A_t : t \geq 0\}$. The above is called the Lévy-Khintchine representation of the characteristic function of a Lévy process. The constant $a$ is called the drift of the process and the interpretation given to the measure $M$ and the constant $\sigma^2$ is the following. The process $\{A_t : t \geq 0\}$ has a pure jump component as well as a Brownian motion component. The number of positive (negative) jumps of magnitude $\geq x > 0 \ (\leq x < 0)$ is a Poisson process with parameter $M((x, \infty)) \ (M((\infty, x))$. If $\sigma^2 = 0$ then there is no Brownian component.

**Definition 1.3.5** The stochastic process $\{A_t : t \geq 0\}$ is said to be a subordinator if it is a Lévy process with non-negative sample paths, i.e., if $\{A_t : t \geq 0\}$ is a Lévy process on $(0, \infty)$.

**Definition 1.3.6** A Lévy process $\{A_t : t \geq 0\}$ is of bounded variation if in the
representation 1.23 above, \( \sigma^2 = 0 \) and

\[
\int_0^\infty \min(y,1)M(dy) < \infty.
\]  

(1.24)

We can now define an infinite capacity, continuous time storage model as follows.

Let \((W, \mathcal{A}, P)\) be a probability space. Let the input process be \(\{A_t : t \geq 0\}\), a subordinator of bounded variation on \((W, \mathcal{A}, P)\), taking values in \((\mathbb{R}_+, \mathbb{R}_+)\). Define \(\Omega = \mathbb{R}_+ \times W\), and \(\mathcal{F}\) be the product \(\sigma\)-algebra \(\mathbb{R}_+ \times \mathcal{A}\). For every finite probability measure \(\nu\) on \(\mathbb{R}_+\), define \(P^\nu\) to be the product measure \(\nu \times P\) on \(\mathcal{F}\). For every \(\omega = (x, w) \in \Omega\), define \(A_t(\omega) = A_t(w)\) for \(t \geq 0\) and \(Z_0(\omega) = x\). Then \(\{A_t : t \geq 0\}\) is a subordinator of bounded variation on \((\Omega, \mathcal{F}, P)\). Define the storage level \(Z_t\) in an infinite capacity store to be the solution to the integral equation

\[
Z_t = Z_0 + A_t - \int_0^t \zeta(s)ds, \quad (t \geq 0)
\]  

(1.25)

where \(Z_0 = x \geq 0\) is the initial storage level and,

\[
\zeta(s) = \begin{cases} 
0 & \text{if } Z_t = 0 \\
1 & \text{otherwise}
\end{cases}
\]  

(1.26)

For example, in an \(M/G/1\) queue, we would identify \(Z_t\) with the virtual workload in the system at time \(t\) and then, the input process would be described by \(A_t = v_1 + v_2 + \ldots + v_{N(t)}\), where \(v_1, v_2, \ldots\) are the service times of the successive customers and \(N(t)\) is the number of customer arrivals until time \(t\).

The next theorem shows the similarity between the representation of the storage level in the discrete time model and the solution to the integral equation
above. Letting $Y_t = A_t - t$ be the netput until time $t$, with minimum functional $m_t = \inf\{Y_s : 0 \leq s \leq t\}$, we have the following result (for proof see Prabhu [11], p.75).

**Theorem 1.3.4** The integral equation 1.25 has the unique solution

$$Z_t = \max\{Z_0 + Y_t, Y_t - m_t\}, \quad (t \geq 0). \quad (1.27)$$

Using the above theorem, it is possible to derive the transform of the generating function of $Z_t$, as was done by Prabhu ([11], p.80)

**Theorem 1.3.5**

$$\int_0^\infty E[\exp(-\theta Z_t)|Z_0 = x]dt = \frac{\exp(-\theta x - \theta \exp(-x\eta)\eta^{-1})}{s - \theta + \phi(\theta)}$$

where

$$\eta = \eta(s) \quad \text{satisfies} \quad \eta = s + \phi(\eta)$$

Thus, the problem of the infinite capacity store is completely solved. However, there seems to be no straightforward modification of this integral equation to represent the storage level when the capacity of the store is finite. In particular, it is not possible to represent the storage level in terms of the supremum and infimum of the netput process, and consequently a different approach is required.

Since the analysis of the class of single boundary models relies on the identification of a regenerative phenomenon (namely one of the ladder processes), a natural question to ask is whether there exists any analogous method of analysis in the two boundary case. The answer to that question is indeed in the affirmative. In the next section we give a synopsis of this method and also a summary of this work.
1.4 Summary of thesis

In section 1.3 we defined the ascending and descending ladder variables and noted that the descending ladder epochs correspond to the successive points in time at which the store empties itself. The proof of theorem 1.3.2 relies on the fact that the random vectors \((\bar{N}_{k+1} - \bar{N}_k, S_n - S_{\bar{N}_k})\) and \((N_k, S_{\bar{N}_k})\) are independent and the former has the same distribution as \((N_1, S_{n-\bar{N}_k})\). In the two boundary case, we may similarly define the sequence of time points at which the storage level hits either boundary. However, the future evolution of the process at any of these points in time is not exactly a probabilistic replica of its evolution from the beginning; instead of a single regenerative phenomenon, we have a set of two linked regenerative phenomena, one for each of the two boundaries.

In section 2.1, we propose a Markovian model which generalizes the above idea to a class of countably many linked recurrent phenomena. We begin by explaining this idea using a Markov chain on a countable state space, which is suitably modified.

In section 2.2, we characterize the modified chain and give an example to illustrate the construction. Section 2.3 extends this construction to a Markov chain on an arbitrary measurable space (the real line, for example). In section 2.4, we study the distribution of the modified chain and give conditions under which a limiting distribution exists. The problem of the finite dam in discrete time is treated as a special case of the theory. For a similar model, see [2]. The treatment there is quite different from ours.
In chapter 3, we first prove that it is possible to construct a finite capacity storage process using the idea of piecing together stochastic processes. Such a construction is possible only when the successive epochs at which the netput process crosses the boundaries are well defined. If, however, the intervals between jumps cannot be properly defined, as in the case of an input process with an infinite jump rate, we cannot directly construct the process by this method. In section 3.1, we give a few preliminary results. Section 3.2 contains the construction and distribution of the storage process when the input process is of the compound Poisson type. In section 3.3, we show how this construction can be extended for other input processes, by defining the storage process to be the limit of a sequence of storage processes, each of which has a compound Poisson input process.
Chapter 2

Discrete Time Storage Models

In this chapter, we begin to develop the main theme of this work. We discuss a discrete-time Markovian model which underlies our approach to the problem of the finite store. The exposition in this chapter is quite intuitive and may be considered preparatory to the more rigorous and formal development of the theory for continuous time which we present in the next chapter.

In the last chapter we saw that the analysis of storage models with a single finite boundary proceeds by identifying regenerative phenomena embedded within the storage level process and using renewal arguments to establish a renewal-type equation whose solution is then obtained from the general theory of regenerative phenomena. In the case of two finite boundaries, a method that suggests itself is to try and identify epochs at which the process regenerates itself. However, it is not possible to find a single recurrent event such that, beyond the epoch of its occurrence, the future of the process is a probabilistic replica of the process
starting at time 0.

However, if we consider the successive epochs at which the store either becomes empty or becomes full, we can identify two different types of 'recurrent' probabilistic copies of the process. We call such events semirecurrent, an elegant theory for which exists (see for example, Prabhu [13], Kingman [8]). We construct the finite capacity storage process by, roughly speaking, piecing together several independent copies of unconstrained processes, each piece representing the evolution of the capacititated process between the epochs at which it crosses either of its two boundaries.

To illustrate these ideas we begin with the following simple model which describes a ‘modified’ Markov chain.

2.1 A Modified Markov Chain Model

Consider a Markov chain on a countable state space. The chain is allowed to evolve freely according to its transition probability matrix until its first passage into a specified subset of the state space, which for reasons that will become clear presently, is called the taboo set. We assume that the taboo set is the union of countably many disjoint component subsets of the state space. We do not allow the chain to make direct transitions into the taboo set. Consequently, when the chain enters the taboo set, its value is replaced according to a probability distribution function which is allowed to depend on the particular component of the subset that was entered. Now, the whole process is repeated until the next passage through the taboo set and so on.
The stochastic process that results in the above described manner is also a Markov chain. In addition, it has other interesting properties which we shall explore later.

Let $X = \{X_n : n = 0, 1, 2, \ldots \}$ be a Markov chain on a countable state space $S$, with transition probability matrix $(P_{ij})_{i,j \in S}$. Suppose that we are given a sequence of Markov chains, $\{X^{(r)} : r = 0, 1, 2, \ldots \}$, which are independent copies of $X$, in the following sense.

**Assumptions 2.1.1** On some suitable probability space, for $r = 1, 2, 3, \ldots$, let $X^{(r)} = \{X_n^{(r)} : n = 0, 1, 2, \ldots \}$ be a Markov chain on the state space $S$ with transition probability matrix $(P_{ij})_{i,j \in S}$, and the additional property that, given its initial value $X_0^{(r)}$, the finite dimensional distribution of the chain $X^{(r)}$ is conditionally independent of the 'previous' chains $X^{(1)}, X^{(2)}, \ldots, X^{(r-1)}$.

Let $H$ be a subset of $S$ such that

$$H = \bigcup_{i \in I} H_i,$$  \hspace{1cm} (2.1)

for some $I \subseteq \{0, 1, 2, \ldots \}$

where

$$H_i \in S, \ i \in I, \text{ and } H_i \cap H_j = \emptyset, \ i, j \in I, i \neq j.$$  

We shall refer to the set $H$ as the *taboo set*.

For the $r$th chain $X^{(r)}$, define its first passage time into the taboo set as

$$\tilde{N}^{(r)} = \min\{n > 0 : X_n^{(r)} \in H\}$$  \hspace{1cm} (2.2)

and also, define

$$J^{(r)} = j, \text{ if } X_{\tilde{N}^{(r)}}^{(r)} \in H_j \quad (r \geq 1).$$  \hspace{1cm} (2.3)
Let
\[ N_r = \sum_{i=1}^{r} \tilde{N}^{(r)} \quad \text{and} \quad J_r = J^{(r)} \quad (r \geq 1). \tag{2.4} \]

Let \( \theta_j : j \in I \) be a family of probability distributions (p.m.f.'s) on \( S \). These are the distributions according to which the value of the chain is replaced in the taboo set, and we shall refer to them as the replacement distributions. Thus, for \( r \geq 2 \), given \( J_{r-1} = j \), the conditional p.m.f. of \( X_0^{(r)} \) is
\[ \theta_j(y) = P\{X_0^{(r)} = y | J_{r-1} = j \}, \quad (y \in S, r \geq 2). \tag{2.5} \]

Note that we allow \( \theta_j(y) > 0 \) for \( y \in H \), and thus, there is no restriction on replacing the value of the modified Markov chain with a value in the taboo set.

We construct a modified chain \( Z = \{Z_n : n = 0, 1, 2, \ldots \} \) by piecing together the chains \( X^{(1)}, X^{(2)}, \ldots, X^{(r-1)} \) between the times \( 0, N_1, N_2, \ldots \), in the following manner.

Let \( Z_0 \) be the initial state and define
\[ Z_n = X_n^{(1)}, \quad \text{for } 0 < n < N_1 \tag{2.6} \]

and
\[ Z_n = X_{n-N}^{(r)}, \quad \text{for } N_r \leq n < N_{r+1} \quad (r \geq 1). \tag{2.7} \]

Thus, the process \( Z \) takes on the values \( (Z_0, Z_1, Z_2, \ldots Z_{N_1}, Z_{N_1-1}, \ldots) \equiv (X^{(0)}, X^{(1)}, \ldots, X^{(1)}_{N_1-1}, X^{(2)}, \ldots) \). It can be readily seen that \( Z \) is a Markov chain on the state space \( S \) with transition probability matrix \( (\tilde{P}_{ij})_{i,j \in S} \) given by
\[ \tilde{P}_{ij} = P\{Z_n = j, \text{ and permissible} | Z_0 = i\} \tag{2.8} \]
\[ + P\{Z_n = j, \text{ and not permissible} | Z_0 = i\} \tag{2.9} \]
\[ P_{ij} = \begin{cases} P_{ij}, & j \notin \mathcal{H} \\ \sum_{l \in I} \sum_{k \in \mathcal{H}_l} P_{ik} \theta_l(j), & j \in \mathcal{H}. \end{cases} \quad (2.10) \]

Defining
\[ Q_{ij} = \begin{cases} P_{ij}, & j \notin \mathcal{H} \\ 0, & j \in \mathcal{H} \end{cases} \quad (2.11) \]

we may write
\[ \hat{P}_{ij} = Q_{ij} + \sum_{l \in I} \sum_{k \in \mathcal{H}_l} (P_{ik} - Q_{ik}) \theta_l(j). \quad (2.12) \]

More interestingly however, the process \( Z \) has embedded in it, a semirecurrent phenomenon, as we shall see in the following section. Arjas and Speed [2] have discussed a similar model. However, they do not discuss the connections with storage theory which we explore in this work.

### 2.2 An Embedded Semirecurrent Phenomenon

We first give the following definition, following Prabhu [13].

Let \( I \subseteq \{0, 1, 2, \ldots \} \) be a countable set.

**Definition 2.2.1** A semirecurrent phenomenon \( Z = \{Z_{nj} : n \in \{0, 1, 2, \ldots \}, j \in I\} \) is a stochastic process on the state space \( \{0, 1\} \), such that for \((n_r, j_r) \in \{0, 1, 2, \ldots \} \times I, (r \geq 0, 0 = n_0 < n_1 < \ldots < n_r),\)

\[ P\{Z_{n_1j_1} = \ldots Z_{n_rj_r} = 1|Z_{n_0j_0}\} = \prod_{i=1}^{r} P\{Z_{n_i-n_{i-1}j_i} = 1|Z_0 = 1\} \quad (2.13) \]
Associated with a semirecurrent phenomenon, we may define a semirecurrent set

\[ \zeta = \{(n, l) : \mathcal{Z}_{nl} = 1\} \quad (2.14) \]

Prabhu [12] has studied such phenomena in detail. In particular, he has established the correspondence between the set \( \zeta \) and the range of a Markov renewal process. We refer to his work for the details. Here, we shall only prove the following

**Theorem 2.2.1** The process \((N, J) = \{(N_r, J_r) : r = 0, 1, 2, \ldots\}\) defined in equations 2.2 - 2.4 is a Markov renewal process on the state space \(\{0, 1, 2, \ldots\} \times \mathcal{I}\) with the semi-Markov transition function

\[ q_{ij}(n) = \sum_{l \in \mathcal{I}} P\{X_k \not\in \mathcal{H} (1 \leq k \leq n-1), X_n \in \mathcal{H}_j|X_0 = l\} \theta_i(l) \quad (n \geq 1, i, j \in \mathcal{I}). \quad (2.15) \]

**Proof:** Let

\[ \tilde{q}_{ij}(n) = \sum_{x \in \mathcal{I}} P\{X_k \not\in \mathcal{H} (1 \leq k \leq n-1), X_n \in \mathcal{H}_j|X_0 = x\} \theta_i(x) \quad (n \geq 1, i, j \in \mathcal{I}). \quad (2.16) \]

Then, we have

\[
P\{N_r = n_r, J_r = j_r (1 \leq r \leq m)|N_0 = n_0, J_0 = j_0\}
\]

\[
= P\{\tilde{N}_r = \tilde{n}_r, J_r = j_r (1 \leq r \leq m)|\tilde{N}_0 = \tilde{n}_0, J_0 = j_0\}\sum_{i=1}^{n} \tilde{n}_i, \ r \geq 1)
\]

\[
= \sum_{x_1 \in \mathcal{I}} P\{X_1^{(r)} \not\in \mathcal{H}, \ldots, X_{\tilde{n}_r-1}^{(r)} \not\in \mathcal{H}, X_{\tilde{n}_r}^{(r)} \in \mathcal{H}_{j_1} (1 \leq r \leq m)|X_0^{(1)} = x_1\} \theta_{j_0}(x_1)
\]

\[
= \prod_{r=2}^{m} \sum_{x_r \in \mathcal{I}} P\{X_1^{(r)} \not\in \mathcal{H}, \ldots, X_{\tilde{n}_r-1}^{(r)} \not\in \mathcal{H}, X_{\tilde{n}_r}^{(r)} \in \mathcal{H}_{j_r}|X_0^{(r)} = x_r\} \theta_{j_{r-1}}(x_r)
\]
by assumption 2.1.1
\[
= \prod_{r=2}^{m} \tilde{q}_{jr-1,jr}(n_r)
\]
since the Markov chains \(X^{(1)}, \ldots, X^{(r)}\) have the same transition matrix.

Thus,
\[
P\{N_{m+1} = n_{m+1}, J_{m+1} = j_{m+1} | N_r = n_r, J_r = j_r (1 \leq r \leq m)\}
= \frac{P\{N_r = n_r, J_r = j_r (1 \leq r \leq m+1) | N_0 = n_0, J_0 = j_0\}}{P\{N_r = n_r, J_r = j_r (1 \leq r \leq m) | N_0 = n_0, J_0 = j_0\}}
\]
\[
= \frac{\prod_{r=2}^{m+1} \sum_{x_r \in \mathcal{I}} P\{X_1^{(r)} \notin \mathcal{H}, \ldots, X_{n_{r}-1}^{(r)} \notin \mathcal{H}, X_{n_{r}-1}^{(r)} \in \mathcal{H}_{jr} | X_0^{(r)} = x_r \} \theta_{jr-1}(x_r)}{\prod_{r=2}^{m} \sum_{x_r \in \mathcal{I}} P\{X_1^{(r)} \notin \mathcal{H}, \ldots, X_{n_{r}-1}^{(r)} \notin \mathcal{H}, X_{n_{r}-1}^{(r)} \in \mathcal{H}_{jr} | X_0^{(r)} = x_r \} \theta_{jr-1}(x_r)}
\]
\[
= \frac{\prod_{r=2}^{m+1} \tilde{q}_{jr-1,jr}(n_r)}{\prod_{r=2}^{m} \tilde{q}_{jr-1,jr}(n_r)}
= \tilde{q}_{jm-1,jm}(n_m)
\]

which proves the Markov renewal property of \((N, J)\). Moreover, the function
\[
\tilde{q}_{ij}(n) = P\{N_1 = n, J_1 = j | J_0 = i\} \quad (2.17)
\]
is its Markov renewal transition function. \(\Box\)

### 2.3 Construction of the Process on a general state space

The extension of the above Markov chain model to the case when the underlying Markov chains take values in a general state space, is quite straightforward. For completeness, we provide the details below.

We first need the following.
Definition 2.3.1 A map \( P : \mathbb{R} \times \mathcal{R} \rightarrow [0,1] \) is said to be a Markov transition kernel on \( \mathbb{R} \times \mathcal{R} \) if it satisfies the following conditions.

(i) for every \( x \in \mathbb{R} \), \( P(x, \cdot) \) is a probability measure on \( \mathcal{R} \)

(ii) for every \( A \in \mathcal{R} \), \( P(\cdot, A) \) is a Borel-measurable function on \( \mathcal{R} \)

(iii) \( P(x, \mathcal{R}) = 1 \).

Suppose we are given a sequence of independent (in the sense of assumption 2.1.1) Markov chains \( X^{(1)}, X^{(2)}, \ldots; \ X^{(r)} = \{X_n^{(r)} : n = 0, 1, 2, \ldots\} \), with the same Markov transition kernel \( P(x, A), \ x \in \mathbb{R}, \ A \in \mathcal{R} \). Also let \( \mathcal{H} \) be any Borel subset of \( \mathbb{R} \) such that for some \( \mathcal{I} \subseteq \{0, 1, 2, \ldots\} \), and a collection \( \{\mathcal{H}_i\}_{i \in \mathcal{I}} \) of disjoint subsets of \( \mathcal{R} \) we have \( \mathcal{H} = \bigcup_{i \in \mathcal{I}} \mathcal{H}_i \). For the \( r \)th chain, \( X^{(r)} \), define \( (N_r, J_r) \) as in equations 2.2 to 2.4. Further, let \( \mu(x, A), \ x \in \mathcal{H}, \ A \in \mathcal{R} \) be another Markov transition kernel on \( \mathcal{H} \times \mathcal{R} \) (the replacement kernel). We shall assume that \( \mu(x, A) \) is constant over each of the sets \( \mathcal{H}_j \), i.e.,

\[
\mu(x, A) = \mu(y, A) \quad \text{if} \ \exists j \in \mathcal{I}, \ \text{s.t. both} \ x, y \in \mathcal{H}_j.
\]  

(2.18)

We may then write \( \mu_j(A) = \mu(x, A) \) for all \( x \in \mathcal{H}_j \). Let the initial value of the \( r \)th chain depend on the \( (r - 1) \)th chain in the following manner.

\[
P\{X_0^{(r)} \in A | X_{N_{r-1}}^{(r-1)} = x\} = \mu(x, A), \quad (r \geq 2).
\]  

(2.19)

We define a new process \( Z = \{Z_n : n = 0, 1, 2, \ldots\} \) exactly as in equations 2.6 to 2.7. Then, the following holds.
Theorem 2.3.1 The process \((N, J) = \{(N_r, J_r) : r = 0, 1, 2, \ldots\}\) defined above is a Markov renewal process on the state space \(\{0, 1, 2, \ldots\} \times \mathcal{I}\) with semi-Markov transition function

\[
q_{ij}(n) = \int_{x \in \mathbb{R}} P\{X_k \notin \mathcal{H} (1 \leq k \leq n - 1), X_n \in \mathcal{H}_j | X_0 = x\} \mu_i(dx)
\]

\((n \geq 1, i, j \in \mathcal{I}). \tag{2.20}\)

Proof: Let

\[
\tilde{q}_{ij}(n) = \int_{x \in \mathbb{R}} P\{X_k \notin \mathcal{H} (1 \leq k \leq n - 1), X_n \in \mathcal{H}_j | X_0 = x\} \mu_i(dx)
\]

\((n \geq 1, i, j \in \mathcal{I}) \tag{2.21}\)

Then, we have

\[
P\{N_r = n_r, J_r = j_r (1 \leq r \leq m) | N_0 = n_0, J_0 = j_0\}
\]

\[
= P\{\tilde{N}_r = \tilde{n}_r, J_r = j_r (1 \leq r \leq m) | \tilde{N}_0 = \tilde{n}_0, J_0 = j_0\}(n_r = \sum_{i=1}^{r} \tilde{n}_i, \ r \geq 1)
\]

\[
= \int_{x_1 \in \mathbb{R}} P\{X^{(1)}_1 \notin \mathcal{H}, \ldots, X^{(r)}_{n_r-1} \notin \mathcal{H}, X^{(r)}_{n_r} \in \mathcal{H}_{j_r} (1 \leq r \leq m) | X^{(1)}_0 = x_1\} \mu_j(\mu_0(dx_1)
\]

\[
= \prod_{r=2}^{m} \int_{x_r \in \mathbb{R}} P\{X^{(1)}_1 \notin \mathcal{H}, \ldots, X^{(r)}_{n_r-1} \notin \mathcal{H}, X^{(r)}_{n_r} \in \mathcal{H}_{j_r} | X^{(r)}_0 = x_r\} \mu_{j_r-1}(dx_r)
\]

by assumption 2.1.1

\[
= \prod_{r=2}^{m} \tilde{q}_{j_{r-1}j_r}(n_r)
\]

since the Markov chains \(X^{(1)}, \ldots, X^{(r)}\) have the same transition matrix.

Thus,

\[
P\{N_{m+1} = n_{m+1}, J_{m+1} = j_{m+1} | N_r = n_r, J_r = j_r (1 \leq r \leq m)\}
\]

\[
= \frac{P\{N_r = n_r, J_r = j_r (1 \leq r \leq m+1) | N_0 = n_0, J_0 = j_0\}}{P\{N_r = n_r, J_r = j_r (1 \leq r \leq m) | N_0 = n_0, J_0 = j_0\}}
\]
\[
\begin{align*}
\prod_{r=2}^{m+1} \int_{x_r \in \mathbb{R}} & \mathbb{P}\{X_1^{(r)} \notin \mathcal{H}, \ldots, X_{n_{r-1}}^{(r)} \notin \mathcal{H}, X_{n_{r-1}}^{(r)} \in \mathcal{H}_{j_{r-1}} | X_0^{(r)} = x_r\} \mu_{j_{r-1}}(dx_r) \\
\prod_{r=2}^{m} \int_{x_r \in \mathbb{R}} & \mathbb{P}\{X_1^{(r)} \notin \mathcal{H}, \ldots, X_{n_{r-1}}^{(r)} \notin \mathcal{H}, X_{n_{r-1}}^{(r)} \in \mathcal{H}_{j_{r-1}} | X_0^{(r)} = x_r\} \mu_{j_{r-1}}(dx_r) \\
= & \frac{\prod_{r=2}^{m+1} q_{j_{r-1}j_{r}}(n_r)}{\prod_{r=2}^{m} q_{j_{r-1}j_{r}}(n_r)} \\
= & q_{j_{m}j_{m+1}}(n_{m+1})
\end{align*}
\]

which proves the Markov renewal property of \((N, J)\). Moreover, the function

\[
\tilde{q}_{ij}(n) = \mathbb{P}\{N_1 = n, J_1 = j | J_0 = i\} \tag{2.22}
\]

is its Markov renewal transition function.

\[\square\]

## 2.4 Distribution of the Process

In order to derive the distribution of the process \(Z\) we first introduce the following definitions and notation.

We know from general Markov renewal theory that the process \(J = \{J_n : n = 0, 1, 2, \ldots\}\) is a Markov chain on the state space \(\mathcal{I}\). For a fixed \(j \in \mathcal{I}\), define \(M_0^{(j)} = 0\) a.s. and for \(k = 1, 2, 3, \ldots\), let \(M_k^{(j)}\) be the epoch of the \(k\)th visit of the Markov chain \(J\) to the state \(j\). Thus, define

\[
\begin{align*}
M_0^{(j)} & = 0 \text{ a.s.}, \\
M_k^{(j)} & = \min\{n > M_{k-1}^{(j)} : J_n = j\} \quad (k \geq 1). \tag{2.23}
\end{align*}
\]

Denote

\[
T_k^{(j)} = N_{M_k^{(j)}} \quad (k \geq 0). \tag{2.24}
\]

The process \(T^{(j)} = \{T_n^{(j)} : n = 0, 1, 2, \ldots\}\) is renewal process embedded in \((N, J)\).
Let
\[ F_{ij} = P\{T_1^{(j)} < \infty | J_0 = i\} \leq 1, \]  
(2.25)
\[ \mu_{ij} = E[T_1^{(j)} | J_0 = i] \leq \infty \]  
(2.26)

Denote by \( V_j(m) \) the total number of visits of the Markov chain \( J \) to the state \( j \) upto time \( m \) and let \( V(m) \) be the total number of transitions which occur in \( J \), upto time \( m \).

Thus
\[ V_j(m) = \max\{k : T_k^{(i)} \leq m\} \]  
(2.27)

and
\[ V(m) = \sum_{i \in I} V_i(m). \]  
(2.28)

Let \( (P_{ij}^{(n)})_{i,j \in I} \) denote the \( n \)-step transition probability matrix of \( J \), i.e.,
\[ P_{ij}^{(n)} = P\{J_n = j | J_0 = i\} \quad (n \geq 0, i, j \in I) \]  
(2.29)

and let \( q_{ij}^{(r)}(n) \) be the \( r \)-step Markov renewal transition function of \( (N, J) \), i.e.,
\[ q_{ij}^{(1)}(n) = q_{ij}(n) = P\{N_1 = n, J_1 = j\} \]  
(2.30)
\[ q_{ij}^{(r)}(n) = q_{ij}^{(r*)}(n) = P\{N_r = n, J_r = j\} \quad (r \geq 2, n \geq 0). \]  
(2.31)

Let \( u_{ij}(n) \) be the Markov renewal distribution measure of \( (N, J) \),
\[ u_{ij}(n) = \sum_{r=0}^{\infty} q_{ij}^{(r)}(n) = \sum_{r=0}^{n} q_{ij}^{(r)}(n) \quad (n \geq 0, i, j \in I), \]  
(2.32)

the sum being effectively finite, since \( N_r \geq r \) for all \( r \geq 0 \).

Finally, we shall denote by \( P_j\{\cdot\} \) the conditional probability \( P\{\cdot | J_0 = j\} \). We then have the following
Theorem 2.4.1 The transient distribution of the process $Z$ defined in section 2.3 is given by

$$P_j\{Z_n \in A\} = \sum_{l \in I} \sum_{m=0}^{n} u_{jl}(n-m)P_{l}\{Z_m \in A, N_1 > m\}, \quad (n \geq 0, j \in I, A \in \mathcal{R}). \quad (2.33)$$

Proof: For every $j \in I$, we have

$$P_j\{Z_n \in A\}$$

$$= P_j\{Z_n \in A, N_1 > n\} + \sum_{m=1}^{n} P\{Z_n \in A, N_1 = m\}$$

$$= P_j\{Z_n \in A, N_1 > n\} + \sum_{m=1}^{n} \sum_{l \in I} P\{Z_n \in A, N_1 = m, J_1 = l\}$$

$$= P_j\{Z_n \in A, N_1 > n\} + \sum_{m=1}^{n} \sum_{l \in I} P_j\{N_1 = m, J_1 = l\} \times P_l\{Z_{n-m} \in A\}$$

since $(N_1, J_1)$ is a point of semi-regeneration for the process $Z$.

Thus we obtain

$$P_j\{Z_n \in A\} = P_j\{Z_n \in A, N_1 > n\} + \sum_{m=1}^{n} \sum_{l \in I} q_{jl}^{(m)} \times P_l\{Z_{n-m} \in A\}$$

for every $j \in I \quad (2.34)$

which is a Markov renewal (matrix) equation.

Since the function $g_j(n) = P_j\{Z_n \in A\}$ is non-negative and bounded for every $A \in \mathcal{R}$ and $n \geq 0$, the above equation has a particular solution given by

$$P_j\{Z_n \in A\} = \sum_{l \in I} \sum_{m=1}^{n} u_{jl}(n-m) \times P_l\{Z_{n-m} \in A\}, \quad j \in I \quad (2.35)$$
Further, since $V(m)$ is finite for every $m \geq 0$, the solution is unique. \qed

In the case when the state space of the Markov chain $J$ is finite, i.e., when the number of component subsets $\mathcal{H}_i$, $i \in \mathcal{I}$ is finite, we obtain the following result for the limiting distribution of $Z$, under the assumption that $N_1$ has finite expectation.

**Corollary 2.4.1** For the process $Z$ defined in section 2.3, if the following conditions hold

(i) the set $I$ has finite cardinality; and

(ii) $E[N_1|J_0 = j] < \infty$ for every $j \in \mathcal{I}$.

Then the limiting distribution exists, and is given by

$$
\lim_{n \to \infty} P_j\{Z_n \in A\} = \sum_{l \in \mathcal{I}} \frac{F_{jl}}{\mu_l} \sum_{m=1}^{\infty} xP_l\{Z_m \in A, N_1 > m\}, \quad \text{for every } j \in \mathcal{I}
$$

(2.36)

where $F_{jl}$ and $\mu_l$ are as defined in equations 2.25 - 2.26.

**Proof:** From standard renewal theory, we know that

$$
\lim_{n \to \infty} u_{jl}(n) = \frac{F_{jl}}{\mu_l} \quad \text{for all } j, l \in \mathcal{I}
$$

(2.37)

Thus, from theorem 2.4.1, we have

$$
\lim_{n \to \infty} P_j\{Z_n \in A\}
= \lim_{n \to \infty} \sum_{l \in \mathcal{I}} \sum_{m=1}^{n} u_{jl}(n-m) \times P_l\{Z_m \in A, N_1 > m\}
= \sum_{l \in \mathcal{I}} \lim_{n \to \infty} \sum_{m=1}^{n} u_{jl}(n-m) \times P_l\{Z_m \in A, N_1 > m\},
$$
since \( \mathcal{I} \) is finite

\[
= \sum_{i \in \mathcal{I}} \frac{F_{ii}}{\mu_{ii}} \times P_i \{ Z_m \in A, N_1 > m \},
\]

where the inner sum on the r.h.s. is finite, since

\[
\sum_{m=1}^{\infty} P_i \{ Z_m \in A, N_1 > m \} \leq \sum_{m=1}^{\infty} P_i \{ N_1 > m \} = E[N_1 | J_0 = i] < \infty \quad \text{by assumption.}
\]

The above construction provides a natural framework for treating discrete time stochastic storage processes with finite boundaries: we piece together copies of the netput process between epochs of overflow or emptiness. This is illustrated in the next section.

### 2.5 The finite dam

To illustrate the application of the stochastic process constructed in section 2.3 let us consider the problem of the finite dam. For a description of the problem, we refer to the discrete time model (Model 1) of chapter 1. Let the distribution measure of the potential inputs be

\[
G\{dx\} = P\{A_i \in dx\} \quad (i \geq 1, x \in \mathbb{R}). \tag{2.38}
\]

We shall assume that the support of the distribution function of the potential outputs lies in \((0, c)\). Denote

\[
H\{dx\} = P\{B_i \in dx\} \quad (i \geq 1, x \in (0, c)). \tag{2.39}
\]
Then, the netput process is the random walk \( \{ S_n : n = 0, 1, 2, \ldots \} \), where

\[
S_n = S_0 + A_n - B_n (n \geq 1)
\]  \hspace{1cm} (2.40)

where \( 0 \leq S_0 \leq c \) is the initial content.

To construct the dam content process \( \{ Z_n : n = 0, 1, 2, \ldots \} \) we start with independent copies \( S^{(1)}, S^{(2)}, \ldots \) of the netput process. First, for \( r \geq 1 \), define

\[
T_r^1 = \min \{ n > 0 : S_n^{(r)} \leq 0 \},
\]

\[
T_r^2 = \min \{ n > 0 : S_{n-1}^{(r)} + A_n \geq c \}
\]  \hspace{1cm} (2.41) \hspace{1cm} (2.42)

and

\[
\bar{N}_r = \min \{ T_r^1, T_r^2 \}.
\]  \hspace{1cm} (2.43)

Set

\[
N_0 = 0, \text{ a.s.}
\]

\[
N_r = \sum_{i=1}^{r} N_i (r \geq 1)
\]  \hspace{1cm} (2.44)

Also, define

\[
J_r = \begin{cases} 
1 & \text{if } N_r = T_r^1 \\
0 & \text{if } N_r = T_r^2 
\end{cases}
\]  \hspace{1cm} (2.45)

The appropriate replacement kernel is

\[
\mu_0(dy) = P\{ S_0^{(r)} \in dy | J_{r-1} = 0 \} = \epsilon_0(dy)
\]  \hspace{1cm} (2.46)

\[
\mu_1(dy) = P\{ S_0^{(r)} \in dy | J_{r-1} = 1 \} = H\{c - dy\} \quad (r \geq 2)
\]  \hspace{1cm} (2.47)

and then, the storage level is defined to be

\[
Z_n = S_{n-N_{r-1}}^{(r)}, \text{ for } N_{r-1} \leq n < N_r \quad (n \geq 1, r \geq 1)
\]  \hspace{1cm} (2.48)
where $Z_0$ is the initial level. This is perhaps best illustrated in Figure 1 which depicts a typical sample path of the storage process.

From theorem 2.3.1, we know that $(N, J)$ is a Markov renewal process on the state space $\{0, 1, 2, \ldots\} \times \{0, 1\}$. Its semi-Markov transition function can be expressed in terms of the netput process as follows:

\[ q_{00}(n) = \]
\[ P\{0 < S - i < c, 0 < S_{i-1} + A_i < c, (1 \leq i \leq n - 1), S_n \leq 0|S_0 = 0\} \quad (2.49) \]

\[ q_{01}(n) = \]
\[ P\{0 < S - i < c, 0 < S_{i-1} + A_i < c, (1 \leq i \leq n - 1), S_{n-1} + A_n \geq c|S_0 = y\} \quad (2.50) \]

\[ q_{10}(n) = \]
\[ \int_{y \in (0, c)} P\{0 < S_i < c, 0 < S_{i-1} + A_i < c, (1 \leq i \leq n - 1), S_n \leq 0|S_0 = c - y\} \times H\{dy\} \quad (2.51) \]

and

\[ q_{11}(n) = \]
\[ \int_{y \in (0, c)} P\{0 < S_i < c, 0 < S_{i-1} + A_i < c, (1 \leq i \leq n - 1),
S_{n-1} + A_n \geq c|S_0 = c - y\} \times H\{dy\} \quad (2.52) \]

Let

\[ q_{ij}^{(r)}(n) = P\{N_r = n, J_r = j|J_0 = i\}, \quad (2.53) \]
Figure 2.1: A sample path of the dam content process

\[ u_{ij}(n) = \sum_{r=0}^{\infty} q_{ij}^{(r)}(n) \quad (n \geq 1, r \geq 1) \quad (2.54) \]

The (one-step) transition probability matrix of the Markov chain \( J \) is

\[ P_{ij} = \sum_{n=1}^{\infty} q_{ij}(n), \quad (i, j \in \{0, 1\}) \quad (2.55) \]

It will be convenient to define

\[ g_{in}(x) = \mathbb{P}\{N_1 > n, S_n \leq x\} \]

\[ = \mathbb{P}\{0 < S_i < c, 0 < S_{i-1} + A_i < c, (1 \leq i \leq n-1) 0 < S_n < c, S_n \leq x\} \]

\[ (0 < x < c) \quad (2.56) \]

Let us also introduce the generating functions

\[ \tilde{g}_i(s, x) = \sum_{n=0}^{\infty} s^n g_{in}(x) \quad (2.57) \]
\[ G_j(s, x) = \sum_{n=0}^{\infty} s^n P_j \{ Z_n \leq x \} \]  
\[ \bar{q}_{ij}(s) = \sum_{n=0}^{\infty} s^n q_{ij}(n) \]  
\[ \bar{u}_{ij}(s) = \sum_{n=0}^{\infty} s^n u_{ij}(n) \]

\((i, j, l \in \{0, 1\}, 0 < x < c, |s| < 1)\)

Then, for the transient distribution of the dam content, we have

**Theorem 2.5.1**

(a) \( \sum_{n=0}^{\infty} s^n P_j \{ 0 < Z_n \leq x \} = \sum_{l=0,1} q_l(x, s) \int_{q_{ji}(s)}^{1} \) \( (0 < x < c) \)

(b) \( \sum_{n=0}^{\infty} s^n P_j \{ Z_n = 0 \} = \frac{1}{1 - q_{j0}(s)} \)

**Proof:** From theorem 2.4.1, we have

\[ P_j \{ 0 < Z_n \leq x \} = \sum_{l=0,1} \sum_{m=0}^{\infty} u_{jl}(n - m) P_l \{ N_l > m, Z_n \leq x \}. \]  
\[ (2.61) \]

Since

\[ u_{jl}(n - m) = \sum_{r=0}^{\infty} q^{(r)}_{jl}(n - m), \]  
\[ (2.62) \]

it follows that

\[ \bar{u}_{jl}(s) = \sum_{r=0}^{\infty} (q_{jl}(s))^r = (1 - q_{jl}(s))^{-1}. \]

Therefore

\[ \sum_{n=0}^{\infty} s^n P_j \{ 0 < Z_n \leq x \} = \sum_{l=0,1} (1 - \bar{q}_{jl}(s))^{-1} \bar{q}_l(x, s). \]

Also,

\[ P_j \{ Z_n = 0 \} = u_{j0}(s) \]
which implies that
\[
\sum_{n=0}^{\infty} s^n P_j \{ Z_n = 0 \} = \frac{1}{1 - \tilde{q}_j(s)}.
\] (2.63)

\[ \square \]

We remark that the distribution has an atom at zero. If the distribution of the inputs is absolutely continuous, then so is the distribution of the dam content, in the interval \((0, c)\). For the limiting distribution of the dam content we have

**Theorem 2.5.2**  
(a) \( \lim_{n \to \infty} P_j \{ 0 < Z_n \leq x \} = \sum l = 0, 1 \frac{F_{il}}{\mu_i} \sum_{m=1}^{\infty} g_{lm}(x) \)

(b) \( \lim_{n \to \infty} P_j \{ Z_n = 0 \} = \frac{F_{j0}}{\mu_{00}} \quad (j \in \{0, 1\}, 0 < x < c) \)

**Proof:**  We know from Wald's theorem that the first passage time \(N_1\) of the random walk \(\{S_n : n = 0, 1, 2, \ldots\}\) has finite moments of all order. The above result then follows directly from corollary 2.4.1.

\[ \square \]

We remark that unlike the infinite capacity case where the system is in 'equilibrium' only if \(E[A_n - B_n] < 0\), the limiting distribution in the finite capacity case exists even if this is not true.
Chapter 3

Continuous Time Storage Process

In this chapter we extend the discrete time storage model of the previous chapter to continuous time. Referring to Storage Model 2 of Section 1.1, we allow the potential input process to be a Lévy process of bounded variation with nondecreasing sample functions and finite or infinite jump rate. The output is assumed to occur at a constant unit rate when the store is not empty. In the case when the jump rate is finite, the input is essentially a compound Poisson process. In that case the epochs at which the netput process crosses the two finite boundaries are well defined and we may carry out a natural extension of the construction of the storage process in discrete time. However, if the jump rate of the potential input process is infinite, then the epochs at which the boundaries are crossed are not well defined and consequently a different approach is required. Before we present
these results, we need the following preliminaries.

### 3.1 Preliminaries

In order to construct the finite boundary storage level process formally, we will need the following definitions and results.

**Lemma 3.1.1** Let \( \{ (\Omega_j, \mathcal{F}_j, P_j) : j = 0, 1, 2, \ldots \} \) be a sequence of probability spaces. Let \( \tilde{\Omega} = \prod_{j=1}^{\infty} \Omega_j \) denote the infinite Cartesian product \( \Omega_1 \times \Omega_2 \times \ldots \) and let \( \tilde{\mathcal{F}} = \prod_{j=1}^{\infty} \mathcal{F}_j \) be the minimal \( \sigma \)-algebra over the measurable rectangles in \( \tilde{\Omega} \).

There exists a unique probability measure \( \tilde{P} \) on \( (\tilde{\Omega}, \tilde{\mathcal{F}}) \) such that

\[
\tilde{P}(d\omega_1, d\omega_2, \ldots, d\omega_n) = P_1(d\omega_1) \times P_2(d\omega_2) \times \ldots \times P_n(d\omega_n)
\]  

for every \( n \geq 1 \) and \( \omega_i \in \Omega_i \) (\( i \geq 1 \))

See Ash [3], p.111.

**Corollary 3.1.1** Let \( (\Omega, \mathcal{F}, P^x) \) be a probability space for every fixed \( x \in \mathbb{R} \). Let \( \mu(w, dy) \) be a probability kernel on \( \Omega \times \mathbb{R} \). (the replacement kernel) Let \( \Omega_j = \Omega \times \mathbb{R} \), \( \mathcal{F}_j = \mathcal{F} \times \mathbb{R} \) and \( P_j^x(d\omega, dy) = P^x(d\omega) \times \mu(\omega, dy) \), for \( \omega \in \Omega, y \in \mathbb{R} \) and for every \( j = 1, 2, 3, \ldots \). Let \( \tilde{\Omega} = \prod_{j=1}^{\infty} \Omega_j, \tilde{\mathcal{F}} = \prod_{j=1}^{\infty} \mathcal{F}_j \). Then, for every \( x \in \mathbb{R} \), there exists a unique probability measure \( \tilde{P}^x \) on \( (\tilde{\Omega}, \tilde{\mathcal{F}}) \) satisfying

\[
\tilde{P}^x(d\tilde{\omega}) = \prod_{j=1}^{\infty} P_j^x(d\omega_j, dy_j) = P^x(d\omega_1)\mu(\omega_1, dy_1)P^x(d\omega_2)\mu(\omega_2, dy_2)\ldots
\]

\[
\tilde{\omega}_j = (\omega_j, y_j) \in \Omega_j
\]

**Proof:** Since \( P^x \times \mu \) is a probability measure on \( \Omega_j \), this result follows directly from the infinite dimensional product measure theorem (Lemma 3.1.1). \( \square \)
3.2 Compound Poisson Input

Let \((W, \mathcal{A}, \mathcal{P})\) be a probability space. Let the potential input into a finite store, 
\(\{A_t : t \geq 0\}\) be a compound Poisson process on \((W, \mathcal{A}, \mathcal{P})\) taking values in 
\((\mathbb{R}_+, \mathcal{R}_+), \) with \(P\{A_0 = 0\} = 1.\) Thus in the terminology of section 1.3, each 
\(\{A_t : t \geq 0\}\) is a Lévy process with nonnegative increments, bounded variation 
and having characteristic function

\[
E[e^{i\theta A_t}] = e^{-t(-i\theta\alpha + \int_0^\infty (1-e^{i\theta y})M(dy))}
\] (3.3)

where

\[
\lambda = M((0, \infty)) < \infty \text{ is the jump intensity of } A_t
\]

and 
\[
F(x) = \frac{M((0, x))}{\lambda} \text{ is the distribution function of the jump sizes.}
\]

Without loss of generality, we may assume that the successive jumps occur at 
times \(\tau_1, \tau_2, \ldots\) and have magnitudes \(\gamma_1, \gamma_2, \ldots.\) Define \(W^{(r)} = \mathbb{R}_+ \times W\) and \(\mathcal{A}^{(r)} = \mathcal{R}_+ \times \mathcal{A}\). If \(\nu\) is a probability measure on \(\mathcal{R}_+,\) then define \(P_\nu\) to be the product 
measure \(\nu \times P\) on \(\mathcal{A}^{(r)}\). In particular, when \(\nu = \epsilon_x\) is Dirac measure concentrated 
at \(x,\) we write \(P_x\) for \(\epsilon_x \times P.\) For each \(t \geq 0,\) let \(B_t = \mathcal{R}_+ \times \sigma(A_s : s \leq t).\) 
For every \(\omega^{(r)} = (z_0^{(r)}, w^{(r)}) \in W^{(r)},\) set \(A_t(\omega^{(r)}) = A_t(w^{(r)})\) for all \(t \geq 0.\) Then, 
\(\{A_t(\omega^{(r)}) : t \geq 0\}\) is a standard Markov process over \((W^{(r)}, \mathcal{A}^{(r)}, P_x)\) which is 
adapted to the filtration \(\{B_t\}_{t \geq 0}.\)

For \(r = 1, 2, 3, \ldots\) let

\[
\Omega^{(r)} = W^{(r)} \times \mathbb{R}_+ \quad (3.4)
\]

\[
\mathcal{F}^{(r)} = \mathcal{A}^{(r)} \times \mathcal{R}_+ \quad (3.5)
\]
and let $\mu(\omega, dy)$ be a probability kernel on $\Omega^r$. Further, let

$$Q_\omega(\omega, da) = P_\omega(\omega) \mu(\omega, dy)$$

(3.6)

For $\omega^r = (z_0^r, w^r) \in \Omega^r$, define

$$Y_t(\omega^r) = \max \{z_0^r + A_t(\omega^r) - t, \sup_{0 \leq s \leq t} [(A_t - t) - (A_s - s)]\}$$

(3.7)

Thus, $Y_t(\omega^r)$ is (a sample path of) the storage level in an infinite capacity store whose potential input process is $\{A_t(\omega^r) : t \geq 0\}$ and whose initial content is $z_0^r$. Define the first epoch at which $Y_t(\omega^r)$ crosses the upper and lower boundaries respectively as

$$U(\omega^r) = \inf \{t > 0 : Y_t(\omega^r) \geq c\}$$

(3.9)

and

$$V(\omega^r) = \inf \{t > 0 : Y_t(\omega^r) \leq 0, Y_t(\omega^r) > 0\}$$

(3.10)

and let

$$T(\omega^r) = \min \{U(\omega^r), V(\omega^r)\}.$$  

(3.11)

Also, define

$$J(\omega^r) = \begin{cases} 1 & \text{if } Y_{T(\omega^r)} \geq c \\ 0 & \text{if } Y_{T(\omega^r)} \leq 0. \end{cases}$$

(3.12)

Now, define a new process $\{\tilde{Z}_t : t \geq 0\}$ that is identical with $\{Y_t : t \geq 0\}$ until time $T(\omega^r)$ and thereafter takes the value $y$, i.e., for $\tilde{\omega}^r = (\omega^r, y)$, we let

$$Z_t(\tilde{\omega}^r) = \begin{cases} Y_t(\omega^r), & \text{for } t < T(\omega^r) \\ y, & \text{for } t \geq T(\omega^r) \end{cases}$$

(3.13)
Define
\[
\tilde{\Omega} = \prod_{r=1}^{\infty} \Omega^{(r)} \quad (3.14)
\]
\[
\tilde{\mathcal{F}} = \prod_{r=1}^{\infty} \mathcal{F}^{(r)} \quad (3.15)
\]

From corollary 3.1.1, we know that there exists a unique probability measure \( \tilde{P}_x \) on \((\tilde{\Omega}, \tilde{\mathcal{F}})\) satisfying
\[
\tilde{P}_x(\tilde{\omega}) = Q_x(\tilde{\omega}^{(1)}, \tilde{\omega}^{(2)}, \ldots). \quad (3.16)
\]

On \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}_x)\), define the storage level of the finite capacity store, to be
\[
\tilde{Z}_t(\tilde{\omega}) = \begin{cases} 
Z_t(\tilde{\omega}^{(1)}), \\
\text{for } t \leq T(\omega^{(1)}) Z_{t- T(\omega^{(1)})}(\tilde{\omega}^{(2)}), \\
\text{for } T(\omega^{(1)}) < t \leq T(\omega^{(1)}) + T(\omega^{(2)}) \\
\ldots \\
Z_{t- (T(\omega^{(1)}) + T(\omega^{(2)}) + \ldots + T(\omega^{(r)}) + T(\omega^{(r+1)}))}, \\
\text{for } \sum_{i=1}^{r} T(\omega^{(i)}) < t \leq \sum_{i=1}^{r+1} T(\omega^{(i)}) 
\end{cases} \quad (3.17)
\]

for \(\tilde{\omega} = (\omega^{(1)}, \omega^{(1)}, \ldots) \in \tilde{\Omega}\). In order that all random variables of interest be defined on the same probability space, set
\[
T_r(\tilde{\omega}) = \sum_{i=1}^{r} T(\omega^{(i)}) \quad (3.18)
\]
\[
J_r(\tilde{\omega}) = J(\omega^{(r)}) \quad (r \geq 1) \quad (3.19)
\]

For the remainder of this chapter, we shall suppress the argument \(\tilde{\omega}\) and write \(T_r, J_r, \text{ etc. instead of } T_r(\tilde{\omega}), J_r(\tilde{\omega}), \text{ etc.}\).

The construction above may be thought of as identifying \(\{\tilde{Z}_t : t \geq 0\}\) with a sample path of the netput process until the first time at which the netput process
enters the taboo set \((-\infty, 0] \cup [c, \infty)\). It is then instantaneously replaced according to the replacement kernel \(\mu\). Then the cycle is repeated.

We remark that since

\[ T_r = \sum_{i=1}^{r} T(\omega(i)) \to \infty, \text{ as } r \to \infty \]

the process is well defined.

It can be shown that the above constructed process is a standard Markov process with sample paths in \(D[0, \infty)\). See for example [6]. However, that by itself does not enable us to find its distribution. Suppose now that the replacement kernel is such that it only depends on \(\omega\) only through \(Y_{T(\omega)}(\omega)\), i.e., \(\mu\) satisfies the following property.

\[ \mu(\omega, dy) = \begin{cases} 
\mu_0(dy), & \text{if } Y_{T(\omega)}(\omega) \leq 0 \\
\mu_1(dy), & \text{if } Y_{T(\omega)}(\omega) \geq c 
\end{cases} \]

(3.20)

Then, we have the following result, which we state without proof, since it is essentially similar to theorem 2.3.1

**Theorem 3.2.1** The process \((T, J) = \{(T_r, J_r) : r = 0, 1, 2, \ldots \}\) defined above is a Markov renewal process on \((\tilde{\Omega}, \tilde{\mathcal{F}})\) taking values in \(\mathbb{R} \times \{0, 1\}\).

In view of the above, we shall refer to the process \(\tilde{Z}\) as a semiregenerative process. Let us denote the semi-Markov transition distribution measure of \((T, J)\) by

\[ Q_{ij}^{(1)} \{dt\} = Q_{ij} \{dt\} = \tilde{P}_i \{T_1 \in dt, J_1 = j\} \]

\[ Q_{ij}^{(r)} \{dt\} = \tilde{P}_i \{T_r \in dt, J_r = j\} \quad (i, j \in \{0, 1\}, \ t \in \mathbb{R}) \quad (r \geq 2) \]  

(3.21)
and the corresponding renewal function by
\[ U_{ij}(dt) = \sum_{r=1}^{\infty} Q^{(r)}_{ij} dt \quad (i, j \in \{0, 1\}, \quad t \in \mathbb{R}). \quad (3.22) \]

The marginal process \( J = \{ J_n : n = 0, 1, 2 \ldots \} \) is a Markov chain. For \( j \in \{0, 1\} \) denote
\[ N_0^{(j)} = 0 \quad (3.23) \]
\[ N_r^{(j)} = \min \{ n > N_{r-1}^{(j)} : J_n = j \}; \quad (3.24) \]

Then, \( \{(N_r^{(j)}, T_{N_r^{(j)}}) : r = 0, 1, 2, \ldots \} \) is a renewal process on \( \{0, 1, 2, \ldots \} \times \mathbb{R} \). Let us denote
\[ N_j(t) = \sum_{n=1}^{\infty} 1\{T_n \leq t, J_n = j\}, \quad (3.25) \]
\[ N(t) = \sum_{j=0,1} N_j(t) \quad (3.26) \]
so that \( N_j(t) \) is the number of visits to the state \( j \) up to time \( t \), and \( N(t) \) is the total number of such visits.

We then have the following.

**Theorem 3.2.2** The transition probabilities of the finite capacity storage process \( \{ \tilde{Z}_t : t \geq 0 \} \) defined in equation 3.17 are given by
\[ P_i(t, A) = \tilde{P}\{ \tilde{Z}_t \in A | J_0 = i \} \]
\[ = \sum_{j=0,1} \int_{s \in (0,t)} U_{ij}\{ds\} \int \mu_j(dx) \tilde{P}^x \{ Y_{i-s}^x \in A, T_1 > t - s \} \quad (3.27) \]
where
\[ Y_u^x = \max \{ x + A_u - u, \sup_{0 \leq \tau \leq u}[ (A_u - u) - (A_{\tau} - \tau) ] \} \quad (3.28) \]
is the storage level at time $u$, beginning with an initial, level of $x$, provided that there has been no overflow or emptying (re-emptying, if $x = 0$) of the store.

**Proof:** For each $i = 0, 1$, we have

$$\tilde{\mathbb{P}}_i\{\tilde{Z}_t \in A\} =$$

$$\tilde{\mathbb{P}}_i\{\tilde{Z}_t \in A, T_1 > t\} + \int_{s \in (0,t)} \tilde{\mathbb{P}}_i\{\tilde{Z}_t \in A, T_1 > ds\}$$

$$\tilde{\mathbb{P}}_i\{\tilde{Z}_t \in A, T_1 > t\} + \int_{s \in (0,t)} \tilde{\mathbb{P}}_i\{\tilde{Z}_t \in A, T_1 > ds, J_1 = j\}$$

$$\tilde{\mathbb{P}}_i\{\tilde{Z}_t \in A, T_1 > t\} + \int_{s \in (0,t)} \sum_{j=0,1} Q_{ij} ds \tilde{\mathbb{P}}_j\{\tilde{Z}_{t-s} \in A\}$$

since $(T_1, J_1)$ is a point of semi-regeneration for $\tilde{Z}$.

The above is a Markov renewal equation. Since $\tilde{\mathbb{P}}_i\{\tilde{Z}_t \in A, T_1 > t\}$ is bounded for all $t \in \mathbb{R}$ and $j = 0, 1$, it has a particular solution given by

$$\tilde{\mathbb{P}}_i\{\tilde{Z}_t \in A\} = \sum_{j=0,1} U_{ij}(ds) \tilde{\mathbb{P}}_j\{\tilde{Z}_{t-s} \in A, T_1 > t-s\}$$

where

$$\tilde{\mathbb{P}}_j\{\tilde{Z}_{t-s} \in A, T_1 > t-s\} = \int_x \mu_j(dx) \tilde{\mathbb{P}}_x\{Y_{t-s}^z \in A, T_1 > t-s\}$$

Further, since the state space of $J$ is finite, this solution is unique. \qed

The following limit result is a direct consequence of the previous theorem.

**Corollary 3.2.1** For the finite capacity process $\{\tilde{Z}_t : t \geq 0\}$, we have

$$\lim_{t \to \infty} P_i(t, A) = \sum_{j=0,1} \frac{F_{ij}}{\mu_{jj}} \int_{s \in (0, \infty)} \tilde{\mathbb{P}}_j\{\tilde{Z}_s \in A, T_1 > s\}. \quad (3.29)$$
3.3 General Input

The assumption made in the previous section that the input process has a finite jump rate was necessary in order for the construction of the storage process (equation 3.17) to be meaningful. However, as in the case of a finite dam, it has been suggested that such an assumption may not be realistic. For example, if the Lévy measure of the process \( \{A_t(\omega(r)) : t \geq 0\} \) in equation 3.3 is taken to be

\[
M(dy) = \frac{\beta}{y} \exp^{-\gamma y} dy \quad (\beta > 0, \gamma > 0)
\]  

(3.30)

then the jump rate is

\[
M((0, \infty)) = \int_{0,\infty} \frac{\beta}{y} \exp^{-\gamma y} dy = \infty.
\]  

(3.31)

In that case our original definition is no longer meaningful. However, it is well known that \( \{A_t(\omega(r)) : t \geq 0\} \) is the uniform limit of an increasing sequence of compound Poisson processes. Let us introduce on the probability space \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}) \) a sequence of potential input processes, \( \{A^n_{t}\}_{n\geq 1} \), where for \( n = 1, 2, \ldots \),

\[
A^n_{t} = \sum_{s \leq t} (A_{s} - A_{s-}) \mathbf{1}_{\{A_{s} - A_{s-} \geq \frac{1}{n}\}} \quad (t \geq 0)
\]  

(3.32)

Then, the Lévy measure of \( \{A^n_{t} : t \geq 0\} \) is \( M((\frac{1}{n}, \infty)) \), which is finite for every \( n \geq 1 \). and therefore the construction of the previous section can be used to define a finite capacity storage process \( \{\tilde{Z}^n_{t} : t \geq 0\} \) associated with \( \{A^n_{t} : t \geq 0\} \).

The following lemma is then easily verified

**Lemma 3.3.1** For all \( n \geq 1 \),

\[
0 \leq \tilde{Z}^{n+1}_{t} - \tilde{Z}^{n}_{t} \leq A^{n+1}_{t} - A^{n}_{t}
\]  

(3.33)
Proof: This follows from the fact that the jumps (epochs and magnitudes) of the process \( \{A^n_t : t \geq 0\} \) are a subset of those for the process \( \{A^{n+1}_t : t \geq 0\} \). \( \square \)

We then have the following

**Theorem 3.3.1** The sequence \( \tilde{Z}^n_t \) converges a.s. \( \tilde{P}_x \) to a limit. Further, the convergence is uniform on any compact interval \( [0, t] \).

Proof: Since

\[
\tilde{Z}^1_t \leq \tilde{Z}^2_t \leq \ldots
\]

and

\[
\tilde{Z}^n_t \leq x + A^n_t \leq x + A_t, \quad \text{for all } n \geq 1,
\]

therefore \( \lim_{n \to \infty} \tilde{Z}^n_t(\tilde{\omega}) = \tilde{Z}_t \) exists, for all \( \tilde{\omega} \in \tilde{\Omega}, t \geq 0 \).

By lemma 3.3.1, we have

\[
|\tilde{Z}^n_t - \tilde{Z}^m_t| \leq |A^n_t - A^m_t| \text{ for all } n, m \geq 1, t \geq 0
\]

which implies

\[
\tilde{P}_x\{ \lim_{N \to \infty} \sup_{n, m \geq N} \sup_{0 \leq s \leq t} |\tilde{Z}^n_t - \tilde{Z}^m_t| > 0 \} \\
\leq \tilde{P}_x\{ \lim_{N \to \infty} \sup_{n, m \geq N} \sup_{0 \leq s \leq t} |A^n_t - A^m_t| > 0 \}
\]

But this last probability is zero since \( A^n_t \) converges uniformly to \( A_t \) as \( n \to \infty \), and so \( \tilde{Z}^n_t \) converges uniformly, a.s. \( \tilde{P}_x \) on \( \tilde{\Omega} \) to \( \tilde{Z}_t \). \( \square \)

We may then define the storage level in a finite capacity store with input process \( \{A_t : t \geq 0\} \) to be equal to this limit \( \tilde{Z}_t \).
Bibliography


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