Characterizations of Camion Trees and
Depth-first Search Trees
by Excluded Configurations\textsuperscript{1}

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Abstract

Let $G$ be a connected graph and let $T$ be a spanning tree of $G$. The tree $T$ is a Camion tree of $G$ if the edges of $G$ can be oriented such that the fundamental cycles with respect to $T$ are directed. The ordered pair $(G, T)$ is then a Camion pair. Analogously, $(G, T)$ is a dfs pair if $T$ is a depth-first search tree of the underlying graph $G$. Every dfs pair $(G, T)$ is also a Camion pair. This paper presents characterizations of Camion pairs and dfs pairs by excluded configurations.

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1 Introduction

Depth-first search [Tar] is an elementary and powerful graph search technique. Among the nice properties that derive from depth-first search trees is one on orientability: for every depth-first search tree $T$ of a connected graph $G$ (CP) there exists an orientation of $G$ such that every fundamental cycle of $G$ with respect to $T$ is directed.

We call (CP) the Camion Property, and say that a spanning tree $T$ of the connected graph $G$ is a Camion tree if CP is satisfied. That every connected graph has such a tree was first observed as a consequence of a more general result of Paul Camion [Cam], which we will discuss later.

Certainly every tree $T$ is a depth-first search tree, and therefore a Camion tree, of some graph $G$; take $G = T$. Given a graph-tree pair $(G, T)$, where $T$ is a spanning tree of the connected graph $G$, one may ask whether $T$ is a depth-first search tree or Camion tree of the specified graph $G$; the main contributions of this paper are characterizations of dfs pairs and Camion pairs, pairs $(G, T)$ for which $T$ is a depth-first search tree of $G$ or Camion tree of $G$, respectively. In particular we present characterizations of each class by exclusion of a small set of configurations $(G_i, T_i)$; any pair $(G, T)$ is a member of the class in question if none of the excluded $(G_i, T_i)$ for that class arises from $(G, T)$ by a sequence of simple operations. This approach clarifies the relationship of dfs pair and the much broader class of Camion pairs; a relationship further clarified by another result in Section 3, which shows that the Camion trees of a fixed $G$ arise by “gluing” depth-first search subtrees in a special way. Our excluded configuration characterization of Camion pairs has also been studied by Swaminathan and Wagner [SW].

Korach and Ostfeld ([KO88], [KO90]) identified those connected graphs $G$ for which every spanning tree $T$ is a depth-first search tree of $G$. They gave several characterizatoins
of such graphs, including one by excluded minors, which is discussed in more detail in Section 4 of this paper.

The Camion property of trees in graphs can be generalized to a property of column bases of real matrices. Let $G$ be a connected graph, and $T$ be a Camion tree of $G$. Give $G$ some orientation and let $A_G$ be the node-edge incidence matrix of the resulting directed graph, with any one row deleted. Then $A_G$ has full row rank and the column bases of $A_G$ correspond exactly to spanning trees of $G$. The pair $(G, T)$ satisfying CP implies that $G$ can be oriented in such a manner that if $B_T$ is the column basis corresponding to $T$, then $B_T^{-1} A_G$ has nonnegative components. In general, if $A$ is a real $m \times n$ matrix of full row rank, and $B$ is a column basis of $A$, then $B$ is a Camion basis of $A$ if there are diagonal matrices $D_m$ and $D_n$, with diagonal entries $\pm 1$, such that $D_m B^{-1} A D_n$ has nonnegative components. ($D_n$ and $D_m$ "reorient" the columns of $A$ and $B$, respectively.) It may seem surprising that every real matrix $A$ of full row rank has a Camion basis; however they abound. Camion [Cam] first showed that each real matrix has at least one such basis. Later Roudneff and Sturmfels [RS] showed that every column of every real $m \times n$ matrix of rank $m$ is in at least $m$ Camion bases of the matrix. The matrix version of the Camion property leads naturally to a geometric interpretation. If $S_A$ is the set of points in $\mathbb{R}^m$ whose coordinate vectors are given by the columns of $A$, and $S_B$ is defined similarly, then a subset of points in $S_A$ can be reflected about the origin such that after the reflection, the convex cone with extreme rays through the $m$ points in $S_B$ contains all points in $S_A$. (One can obtain a "polar" version of this observation by interpreting the columns of $A$ as the normals of hyperplanes.) The study of Camion bases of real matrices, and the still more general subject of Camion bases of oriented matroids, are interesting; however, we shall limit the scope of this paper to a study of these bases in graphs.

This paper is organized as follows. Section 1 contains introductory and background material. In Section 2 Camion pairs are defined and characterized. In Section 3 a charac-
terization of dfs pairs is developed. Section 4 discusses some results of Korach and Ostfeld, and how they relate to the results contained in this paper. Section 5 concludes the paper. Appendix A contains the lengthy proofs omitted in Section 3. For simplicity, we shall refer to depth-first search trees as dfs trees in the remainder of the paper. The notation and terminology we use for graphs are consistent with those in [BM].

2 Camion Pairs

First consider Camion trees of connected directed graphs; in a moment we shall explain why the orientation of the edges is really superfluous information. Let $D = (V, E)$ be a connected directed graph, and $T$ a spanning tree of $D$. For each edge $e$ not in $T$, the fundamental cycle of $e$ with respect to $T$ is the unique cycle in $T \cup e$ where the edge $e$ is a forward edge. The incidence vector of a cycle $C$ in $D$ is a vector in $\{1, 0, -1\}^E$ such that $C_e$ is 0 if $e$ is not in $C$, 1 if $e$ is a forward edge in $C$, and $-1$ is $e$ is a backward edge in $C$. The support of a vector $x \in \{1, 0, -1\}^E$, supp($x$), is the set of edges at which $x$ is nonzero, and two vectors $y, z \in \{0, +, -\}^E$ are consistent if $z$ and $y$ either agree or disagree in sign over all of supp($y$) ∩ supp($z$).

Definition 1. $T$ is a Camion tree of $D$ if the incidence vectors of every pair of fundamental cycles with respect to $T$ are consistent. Equivalently, $T$ is a Camion tree of $D$ if and only if there is a subset $S$ of $E$ such that by reorienting edges in $S$, every fundamental cycle with respect to $T$ is directed.

Since in Definition 1 the reorientation of edges is allowed, two directed graphs with the same underlying graph have the same Camion trees. Hence we may consider instead of $D$ the underlying graph $G$, and give $G$ an orientation only when necessary. With $G$ given some orientation and the edges of $G$ ordered such that edges not in $T$ precede edges in $T$, the fundamental cycle matrix with respect to $T$ is the matrix $[I, N(T)]$, where the rows are
the incidence vectors of fundamental cycles with respect to $T$. By definition then, $T$ is a Camion basis if and only if the edges of $G$ can be oriented such that the matrix $N(T)$ is nonnegative. If $(G, T)$ is not a Camion pair, then there must be some minimal obstructing submatrix in $N(T)$ that is not nonnegative for any orientation of $G$. Any submatrix of the following form is such an obstruction.

**Definition 2.** A $W_k$ matrix is a $k \times k$ matrix with 1's in positions $(i, i + 1)$, $1 \leq i < k$, and on the diagonal, a $-1$ in position $(k, 1)$, and zeros elsewhere.

We shall show that edges of $G$ cannot be oriented to make $N(T)$ nonnegative if and only if $G$ can be oriented such that $N(T)$ contains a submatrix $PW_kQ$ for some $k$, where $P$ and $Q$ are permutation matrices. Note that extracting a submatrix out of $N(T)$ corresponds to performing graph minor operations on $G$. In particular, removing a column from $N(T)$ corresponds to contracting an edge in $T$, and removing a row corresponds to deleting an edge not in $T$.

**Definition 3.** $(G', T')$ is a $T$-minor of $(G, T)$ if $G'$ is a subgraph of $G$ obtained by contracting a subset of edges in $T$ and deleting a subset of edges not in $T$.

We extend this definition by calling a $T$-minor of $N(T)$ any matrix $PMQ$, where $P$ and $Q$ are permutation matrices and $M$ is a submatrix of $N(T)$.

The relationship between the Camion property of $(G, T)$ and $W_k$ matrices can be made explicit by considering a coloring property of the bipartite graph $G_{N(T)} = (R, C, F)$ described below. There is a vertex in $R$ for each row in $N(T)$ and a vertex in $C$ for each column. The edge $(r, c)$ is in $F$ if and only if $N(T)_{rc} \in \{1, -1\}$. For $(r, c)$ in $F$, give it colour blue if $N(T)_{rc} = 1$, or colour red if $N(T)_{rc} = -1$.

**Lemma 4.** $(G, T)$ is a Camion pair if and only if $G_{N(T)}$ does not have any cycle with an odd number of red edges.
Proof. In each cycle of $G_{N(T)}$ containing the vertex $r$ (vertex $c$), reorienting the edge $r$ of $G$ not in $T$ (respectively the edge $c$ of $G$ in $T$) changes the colours of the two edges in the cycle that are incident with vertex $r$ (respectively vertex $c$). Thus the reorientation of any subset of edges preserves the parity of the number of red edges in any cycle in $G_{N(T)}$. If $(G,T)$ is a Camion pair, then $G$ can be reoriented such that $G_{N(T)}$ has only blue edges; i.e., $G_{N(T)}$ cannot contain a cycle with an odd number of red edges before the reorientation. Conversely, let $G_{N(T)}$ contain only cycles with an even number of red edges. It is clear that given a spanning tree $T_{N(T)}$ of $G_{N(T)}$, edges of $G$ can be reoriented such that the edges of $T_{N(T)}$ are all coloured blue, and it then follows that every fundamental cycle with respect to $T_{N(T)}$ in $G_{N(T)}$ must have all its edges coloured blue. Hence there is an orientation of $G$ such that $G_{N(T)}$ contains only blue edges, and so $(G,T)$ must be a Camion pair. \hfill \Box

Corollary 5. $(G,T)$ is a Camion pair if and only if in every orientation of $G$, there is no $W_k$ $T$-minor of $N(T)$ for any odd $k$.

Proof. First note that there never can be a $W_k$ $T$-minor of $N(T)$ if $k$ is even, since $N(T)$ is a totally unimodular matrix and the determinant of $W_k$ has absolute value 2 if $k$ is even. The only if part of the corollary is immediate, since the existence of a $W_k$ matrix in $N(T)$ implies that $G_{N(T)}$ has a cycle with one red edge. From Lemma 4, if $T$ is not a Camion tree, then $G_{N(T)}$ has at least one cycle with an odd number of red edges. Let $X$ be the smallest such cycle. $X$ must be chordless, for any chord induces a smaller cycle with an odd number of red edges. By reorientation, $X$ can be made to contain one red edge, and the submatrix corresponding to $X$ is a $W_k$ matrix. \hfill \Box

The characterization of Camion pairs given above is implicitly graphical, as there is a close correspondence between $W_k$ matrices and members of the following class of graph-tree pairs.
Definition 6. A k-wheel pair \((W,S)\) is a graph on \(k + 1\) vertices \(\{v_0, \ldots, v_k\}\), where \(v_1, \ldots, v_k\) induce a cycle, \((v_0, v_i)\) is an edge for \(1 \leq i \leq k\), and \(S\) is the spanning tree of \(W\) consisting of all edges incident with \(v_0\). An odd-wheel pair is a k-wheel pair where \(k\) is odd.

Corollary 7. \((G,T)\) is a Camion pair if and only if it does not have an odd-wheel pair \(T\)-minor.

Proof. Assign \(G\) some arbitrary orientation, and construct the fundamental cycle matrix \([I, N(T)]\). We need only show that \(G\) contains an odd-wheel \(T\)-minor if and only if \(N(T)\) contains a \(W_k\) submatrix.

\((\Rightarrow)\) Suppose \(G\) has a \(k\)-wheel \(T\)-minor for some odd integer \(k\). If we orient the edges of the odd-wheel \(T\)-minor as shown in Figure 1, then \(N(T)\) has a \(W_k\) submatrix.

\((\Leftarrow)\) Suppose by orienting and ordering the edges of \(G\) appropriately \(N(T)\) contains a \(W_k\) submatrix, \(k\) odd. Contract all tree edges and delete all non-tree edges not involved in the submatrix. Let the remaining tree edges be labeled \(t_1, \ldots, t_k\), and the remaining non-tree edges be labeled \(e_1, \ldots, e_k\). Let \(G'\) and \(T'\) denote the resulting graph and spanning
tree, respectively. The row corresponding to edge \( e_i = (x_i, y_i) \) indicates which edges in \( T' \) form the tree path \( P(x_i, y_i) \). Since \( P(x_i, y_i) \) uses only tree edges \( t_i \) and \( t_{i+1} \), they must be incident with some common vertex \( u_i \). Hence there is a path \( P \) in \( T' \), connecting edges \( t_1 \) and \( t_k \), that contains the vertices \( u_1, \ldots, u_{k-1} \). Since the unique tree path connecting the ends of \( e_1 \) contains \( t_1 \) and \( t_k \), it must include \( P \). But the row corresponding to edge \( e_1 \) has nonzero components only for edges \( t_1 \) and \( t_k \), implying \( P \) is a trivial path, and thus \( u_1 = \ldots = u_{k-1} \). It follows that \( T' \) must be a star and, consequently, \( (G', T') \) must be a \( k \)-wheel. \( \Box \)

For general \( G \), Camion trees may not necessarily be dfs trees; however, the two classes coincide if \( G \) is complete.

**Corollary 8.** In a complete graph, any Camion tree \( T \) is also a dfs tree.

**Proof.** Suppose not. Then there is a vertex \( u \) of degree greater than 2 in \( T \). Any three edges in \( T \) incident with \( u \) induce a 3-wheel \( T \)-minor. \( \Box \)

In Section 1 the Camion property of column bases of real matrices of full row rank was defined. This property extends naturally to a property of oriented matroids. Furthermore, the characterization of Camion graph-tree pairs derived in this section can be easily extended to a characterization of oriented matroid-basis pairs.

### 3 Depth-first Search Pairs

Let \( G \) be a connected graph, and let \( T \) be a dfs tree of \( G \). It is well known that by ordering the vertices by the sequence in which the vertices are visited in the depth-first search, and orienting tree edges from lower-numbered vertices to higher-numbered vertices, and non-tree edges in the opposite fashion, all fundamental cycles with respect to \( T \) are directed; therefore if \( (G, T) \) is a dfs pair, then it is a Camion pair.
In the last section we gave a characterization of Camion graph-tree pairs \((G, T)\) by excluded \(T\)-minors. In this section an analogous characterization of dfs pairs is derived.

First we establish some simple notation. Let \(T\) be a spanning tree of \(G\). Denote by \(P(s, t)\) the unique tree path between the vertices \(s\) and \(t\). If \(P\) is a tree path and \(e\) is an edge with exactly one end in \(P\), then the \(P\)-end of \(e\) is the end vertex of \(e\) that is in \(P\).

By orienting edges in \(T\) away from a vertex \(u\), in which case \(u\) is the root of \(T\), a natural partial order on the vertices can be defined; such a partial order helps to clarify the relative positions of vertices.

**Definition 9.** Two vertices \(v, w\) in \(G\) are related if there is a directed tree path from one vertex to the other. If there is no such path, then they are unrelated. We say \(w > v\), \(w\) is above \(v\), or \(w\) is an ancestor of \(v\), if there is a directed tree path from \(w\) to \(v\). In this case we also say \(v < w\), \(v\) is below \(w\), or \(v\) is a descendant of \(w\). Furthermore, if there is a directed edge from \(w\) to \(v\), then \(v\) is a child of \(w\) and \(w\) is a parent of \(v\).

Note that contrary to botanical trees, in this notation spanning trees grow downward, with the root of the tree at the top.

Clearly, if \((G, T)\) is a dfs pair and the root \(u\) is also a dfs root of \(T\), then the ends of every non-tree edge are related. It is a well known fact that the converse also holds. Denote a non-tree edge \((x, y)\) of \(G\) as a cross-edge(u) if there are edge-disjoint paths \(P(u, v), P(v, x)\) and \(P(v, y)\) where each of \(P(v, x)\) and \(P(v, y)\) must contain at least an edge. A back-edge(u) is a non-tree edge of \(G\) that is not a cross-edge(u). If \(u\) is a root, then the ends of a cross-edge(u) are unrelated, and those of a back-edge(u) are related.

**Lemma 10.** The vertex \(u\) is a dfs root of a spanning tree \(T\) of \(G\) if and only if there are no cross-edges(u).

(When there is no ambiguity, we omit the references to the vertices and paths when we use the terms cross-edge, back-edge and path.)
It follows that any vertex as defined below cannot be a dfs root.

**Definition 11.** A vertex $u$ in a tree path $P$ is a $P$-enclosed vertex if there is a non-tree edge $e = (x, y)$ such that $x, y$ are in $P$ and $u$ is an interior vertex of $P(x, y)$.

If $u$ is enclosed by $e$, then $u$ cannot be a dfs root of the dfs tree $T$, since $e$ is a cross-edge($u$).

Our goal is to find graph-tree pairs, the exclusion of which as $T$-minors characterizes dfs pairs. Described below are three classes of minimally non-dfs graph-tree pairs.

**Definition 12.** A yoke consists of two vertex-disjoint non-tree edges and a vertex that is connected by a tree edge to each of the four ends of the non-tree edges.

**Definition 13.** A 3-wheel is a cycle of three non-tree edges forming the rim, together with a vertex, the hub, that is connected to each of the three vertices on the rim by a tree edge.

**Definition 14.** A boat has a deck which is a tree path $P(s, t)$. Let $x, y$ be the vertices on the deck adjacent to $s, t$, respectively. There are two masts which are tree edges $(u, x)$ and $(v, y)$ that are not in $P(s, t)$. Vertex $u$ is joined by a line, a non-tree edge $(u, w)$, to the deck such that $x > w$ when tree-edges are directed away from $s$. Similarly there is a line $(v, z)$ such that $z < y$ if tree-edges are directed away from $t$. All other non-tree edges have both ends on $P(s, t)$. Vertices $x$ and $y$ must be $P(s, t)$-enclosed, and furthermore, $s, t$ must be biconnected.
It is clear that the spanning trees in a yoke and a 3-wheel cannot be dfs trees. In a boat, $x$ and $y$ being enclosed implies that there is a cross-edge with respect to each of $u$, $v$, $x$, and $y$. The biconnection of $s$ and $t$ ensures that every interior vertex of $P(s, t)$ is enclosed. The edge $(u, w)$ (respectively $(v, z)$) is a cross-edge of $s$ (respectively $t$). Since no vertex in a boat can be a dfs root, the spanning tree cannot be a dfs tree of the graph.

The exclusion of these graph-tree pairs as $T$-minors is sufficient to characterize dfs pairs.

**Theorem 15.** $(G, T)$ is a dfs pair if and only if $(G, T)$ does not have a yoke, a 3-wheel, or a boat $T$-minor.

An unfortunate feature of this characterization is that the number of boats is infinite. However, a more compact description of dfs pairs can be given by using a third operation (see Definition 16) that, together with $T$-minor operations, reduces every boat to one of three graph-tree pairs. We do not give the proof of Theorem 15 here since it is very similar to, but more tedious than the proof for the more compact characterization.

**Definition 16.** Let $P(s, v)$ be a tree path in $(G, T)$ with interior vertices $t$, $u$, such that $u$ is an interior vertex of $P(s, t)$ and $(s, t)$ and $(u, v)$ are non-tree edges. Then $(s, t)$ and $(u, v)$ are overlapping non-tree edges, and a $T$-cycle-reduction is the operation where the edges $(s, t)$ and $(u, v)$ are replaced by a new non-tree edge $(s, v)$.
Lemma 17. If $(G', T)$ can be obtained from $(G, T)$ by a $T$-cycle-reduction, then $(G', T)$ is a dfs pair if and only if $(G, T)$ is a dfs pair.

Proof. Let the edges $e'$ and $e''$ be replaced by $e$ in the reduction. For any vertex $u$, $e$ is a cross-edge$(u)$ if and only if $e'$ or $e''$ is a cross-edge$(u)$ before the reduction. \qed

Definition 18. $(G', T')$ is a $T$-reduction of $(G, T)$ if it can be derived from $(G, T)$ by a sequence of $T$-minor operations and $T$-cycle-reductions.

By $T$-reductions each boat can be reduced to a yoke or one of two simple boats.

Definition 19. A sail boat has a deck of five vertices, $s, x, w, y, t$, and only one non-tree edge, $(s,t)$, with both ends on the deck. Both lines are tied to the deck at $w$. A speed boat has a deck with four vertices, $s, x, y, t$, and again only one non-tree edge, $(s,t)$, with both ends on the deck. The lines of a speed boat are $(u,t)$ and $(v,x)$.

Lemma 20. Every boat has a yoke, a sail boat, or a speed boat as a $T$-reduction.

Proof. Recall the definition of a boat, and refer to non-tree edges with both ends on the deck as hulls. We first prove the lemma for boats with single hulls, and then reduce boats with multiple hulls to yokes or boats with single hulls using $T$-reductions.
Assume the boat has a single hull. Order the vertices on the deck such that \( s > \ldots > t \) and consider how the lines are tied to the deck. If \( w \geq z \), reduce the boat to a sail boat by contracting edges on the deck. If \( x \geq z > w \geq y \), by a \( T \)-cycle-reduction replace the lines with \((u, v)\), contract \( P(x, y) \) and a yoke results. If \( s = z \) and \( t = w \), then \( u, v, w, z \) induce a yoke \( T \)-minor. The remaining possibilities are the symmetric cases of \( s = z, x > w \geq y \), and \( t = w, x \geq z > y \). In either case contracting edges on the deck results in a speed boat.

Now suppose the boat has multiple hulls which do not overlap. Since there is more than one hull, each of \( x, y \) must be enclosed by a distinct hull. Again consider how the lines are tied to the deck. If \( w \geq z \) then there must be a hull, \((a, b)\), such that \( a > w \geq z > b \), for otherwise there is a cut vertex in \( P(w, z) \). But then \((a, b)\) and \((u, w)\) are overlapping non-tree edges, and may be replaced by \((u, b)\) in a \( T \)-cycle-reduction. Then \((u, b)\) and \((v, z)\) are overlapping, and may be replaced by \((u, v)\) in another \( T \)-cycle-reduction. Now flip the boat upside down, and we have a single hull boat by using \( P(u, v) \) as the deck, and \((s, x)\) and \((y, t)\) as the masts. If \( x \geq z > w \geq y \), then a \( T \)-cycle-reduction produces the non-tree edge \((u, v)\), which can then become the single hull of the boat with deck \( P(u, v) \) and masts \((s, x), (y, t)\). If \( s = z \) and \( t = w \), then there is a yoke as in the single hull case. The remaining case has one line, say \((v, z)\), tied to one end of the deck, \( s \), and the other line tied to an interior vertex of the deck. Flip the boat and use the line \((s, v)\) as the single hull of the boat with deck \( P(s, v) \) and masts \((u, x), (t, y)\).

\[ \square \]

Theorem 15 and Lemma 20 immediately imply a characterization of dfs pairs by excluded \( T \)-reductions. However, we shall give a direct proof of this characterization in Appendix A, which embodies all the ideas needed to prove Theorem 15. Here we present the two most essential ideas in the proof, and then an overview of the proof itself.

If \( (G, T) \) does not have as \( T \)-minors yokes, 3-wheels and boats, and the vertex \( u \) is the root, then all cross-edges(u) are "localized".

**Lemma 21.** If \( (G, T) \) does not contain a yoke, a 3-wheel, or a boat, as a \( T \)-minor, then
for any root $u$, there is some tree path $P(u, y)$ that contains one end of every cross-edge($u$).

The proof of this lemma is rather long and is deferred to Appendix A.

If $G$ is not 2-connected, then $T$ is a dfs tree if the subtree of $T$ in each 2-connected component is a dfs tree locally, and the subtrees in all the 2-connected components are joined together in a special way. Specifically, let $v$ be a cut vertex of $G$, $V_1'$ be the vertex set of a connected component of $G - v$, $V_1 = V_1' \cup \{v\}$, and $V_2 = V - V_1'$.

**Lemma 22.** If $T(V_1)$ is a dfs tree of $G(V_1)$ with dfs root $v$, and $T(V_2)$ is a dfs tree of $G(V_2)$, then $T$ is a dfs tree of $G$.

**Proof.** Suppose $u$ is the dfs root of $T(V_2)$. If there is a cross-edge($u$) in $(G, T)$, both ends of this edge must be in $G(V_1)$. Then this edge would also be a cross-edge($v$), which contradicts the hypothesis. Hence $T$ is a dfs tree of $G$ with dfs root $u$. \qed

Now we present the main contribution of this section.

**Theorem 23.** $(G, T)$ is a dfs pair if and only if it cannot be reduced to a yoke, a 3-wheel, a sail boat or a speed boat by $T$-reductions.

**Overview of Proof:** The only if part is trivial. For the if part, consider a non-dfs pair $(G, T)$ with minimum $|V(G)|$ subject to having none of the excluded $T$-reductions. For any vertex $u$, there is a leaf $z(u)$ of $T$ such that $P(u, z(u))$ contains an end of every cross-edge($u$). (The existence of such a path is guaranteed by Lemmas 21 and 22, and the minimality of $(G, T)$.) In the path $P(u, z(u))$, $z(u)$ is the only vertex that needs to be considered to find a contradiction; i.e., it is the only candidate to be a dfs root. The main task of the proof now lies in showing that if for some vertex $u'$ the leaf $z(u')$ is not a dfs root, then there is a vertex $u''$ in $P(u', z(u'))$ such that $z(u'')$ is a dfs root; thus a contradiction always exists, implying the non-existence of such a pair $(G, T)$. See Appendix A for the missing details of the proof.
The characterizations presented so far indicate that the class of Camion pairs is a much broader class than that of dfs pairs. Indeed dfs pairs can be seen as building blocks of non-dfs Camion pairs. Let $O$ denote an orientation of the edges of $G$. If the edges are oriented such that all fundamental cycles are directed, then $O$ is a Camion orientation. If, in addition, there is only one vertex in $G$ with in-degree 0 in $T$, then $O$ is a dfs orientation. (Note that if $(G, T)$ is a Camion and $T$ is not a path in $G$, then $(G, T)$ has a non-dfs Camion orientation.) Let $(G, T)$ be a pair with a non-dfs Camion orientation $O$, such that $r_1, \ldots, r_k$, $k > 1$, are the roots, $V_i = \{u \in V : u \leq r_i\}$, $G_i = G(V_i)$, and $T_i = T(V_i)$, $1 \leq i \leq k$.

**Proposition 24.** For each $1 \leq i \leq k$, $T_i$ is a dfs tree of $G_i$. Furthermore, $G = \cup_{i=1}^k G_i$ and $T = \cup_{i=1}^k T_i$.

**Proof.** By definition $T_i$ is a spanning tree of $G_i$, and in the orientation $O$, $T_i$ has only one root in $G_i$. Also by definition, every tree-edge is captured by some subtree $T_i$, perhaps more than one. Finally, if some non-tree edge $e$ is not captured by any of the $G_i$’s, then the ends of $e$ cannot be related, which is impossible when all fundamental cycles are directed.

Furthermore, it is easy to see that if for some $1 \leq i, j \leq k$, $G' = G_i \cap G_j$ and $T' = T_i \cap T_j$ are non-empty intersections, then the pair $(G', T')$ is also a dfs pair. In other words, non-dfs Camion pairs are just dfs pairs “glued” together at smaller dfs pairs. See Figure 5 for an example.

Analogously, one may compose dfs pairs into Camion pairs in a similar fashion. Let $D = \{(G_i, T_i) : 1 \leq i \leq k\}$ be a set of dfs pairs, each with a dfs orientation. Suppose the vertices of the pairs in $D$ are labeled and the edges oriented such that

1. $\cup_{i=1}^k T_i$ is a spanning tree of $\cup_{i=1}^k G_i$, and

2. if $U = V(G_i) \cap V(G_j) \neq \emptyset$ for some $1 \leq i, j \leq k$, then
Figure 5: Decomposition of a Camion pair into two dfs pairs.
(a) for any vertex \( u \in U \), all of \( u \)'s descendants in both \( G_i \) and \( G_j \) are also in \( U \),
(b) \( G_i(U) = G_j(U) = G_i \cap G_j \) and \( T_i(U) = T_j(U) = T_i \cap T_j \),
(c) the orientations of \( (G_i, T_i) \) and \( (G_j, T_j) \) coincide over \( (G_i \cap G_j, T_i \cap T_j) \), and
(d) \( (G_i \cap G_j, T_i \cap T_j) \) is a dfs pair.

Then \( (\bigcup_{i=1}^{k} G_i, \bigcup_{i=1}^{k} T_i) \) is a Camion pair. Moreover, all Camion pairs arise in this way. (Actually it is quite easy to produce a Camion pair from smaller Camion pairs. Let \( (G', T') \) and \( (G'', T'') \) be Camion pairs such that \( T' \cap T'' \) is a spanning tree of \( G' \cap G'' \) and \( (G' \cap G'', T' \cap T'') \) is a Camion pair, then \( (G' \cup G'', T' \cup T'') \) is also a Camion pair.)

4 Total-DFS-Graphs of Korach and Ostfeld

Korach and Ostfeld defined a connected graph \( G \) to be a Total-DFS-Graph if every spanning tree \( T \) of \( G \) is a dfs tree of \( G \). It is apparent from the definition that the class of Total-DFS-Graphs is rather restricted. This is borne out by the following characterization of Total-DFS-Graphs in terms of \( k \)-parallel-path-graphs. A connected graph \( G \) is a \( k \)-parallel-path-graph if the edges of \( G \) can be partitioned into \( k \) internally vertex disjoint paths between two fixed vertices. A \( k \)-parallel-path-graph \( G \) is determined, up to isomorphism, by a sequence of non-decreasing positive integers \( (l_1, \ldots, l_k) \), where \( l_i \) is the length (the number of edges) of the \( i \)th longest path between the two fixed vertices. Such a connected graph \( G \) is denoted by \( \text{PPG}(l_1, \ldots, l_k) \).

**Theorem 25.** ([KO88],[KO90]) A connected simple graph \( G \) is a Total-DFS-Graph if and only if \( G \) has at most one biconnected component \( C \) with at least three vertices and \( C \) is one of the following \( k \)-parallel-path graphs:

\( \text{PPG}(x, y) \), where \( x \geq 2 \) and \( y \geq 1 \);

\( \text{PPG}(x, 2, 1) \) and \( \text{PPG}(x, 2, 2) \), where \( x \geq 2 \);
$PPG(2, 2, 2, 1)$ and $PPG(2, 2, 2, 2)$.

Korach and Ostfeld [KO90] also gave an excluded minor characterization of Total-DFS-Graphs.

**Theorem 26.** [KO90] A connected simple graph $G$ is a Total-DFS-Graph if and only if none of the graphs in Figure 6 is a minor of $G$.

This result and Theorem 15 are related in that each of these excluded minors $G$ must contain some spanning tree $T$ so that $(G, T)$ contains a yoke, a 3-wheel, or a boat $T$-minor. That such is the case for the last two minors in Figure 6 is obvious; one can easily find spanning trees to obtain a 3-wheel and a yoke from these two minors. Figure 7 shows spanning trees with respect to which the first two minors in Figure 6 become boats. It is not difficult to verify the converse, that if $(G, T)$ is not a dfs pair, in particular if $(G, T)$ is a 3-wheel, a yoke, or a boat, then $G$ must contain one of the minors in Figure 6.

A third characterization of Korach and Ostfeld is one by excluded topological minors, which we omit in this paper.

![Figure 6: Graph minors not contained in any Total-DFS-Graph.](image)

**5 Concluding Remarks**

The characterization of Camion pairs in this paper extends directly in the theory of oriented matroids: $T$-minors are simply restrictive types of oriented matroid minors, and wheels
correspond closely to special oriented matroid-basis pairs. When considering the possibility of abstracting dfs trees in oriented matroids, in view of the characterizations of dfs pairs, one naturally attempts to interpret matroidally $T$-cycle-reductions and boat $T$-minors. However, one can easily construct two graph-tree pairs, for example those in Figure 8, that give rise to the same oriented matroid-basis pair, and yet a $T$-cycle-reduction is possible in one graph-tree pair but not the other. Similarly, for each boat one can find a dfs pair such that their corresponding oriented matroid-basis pairs are isomorphic; see Figure 9 for an example. Clearly Camion pairs can be recognized given only the fundamental cycles with respect to the basis in question, but dfs pairs cannot. In the case of dfs pairs, given a graphic oriented matroid $\mathcal{M}$ and a basis $B$, there may be two realizations, $(G_1, T_1)$ and $(G_2, T_2)$, of $\mathcal{M}$ and $B$ such that $(G_1, T_1)$ is a dfs pair while $(G_2, T_2)$ is not. (Note that such ambiguity does not arise if $(\mathcal{M}, B)$ has a realization $(G, T)$ such that $G$ is triconnected, in which case the realization is unique.) A plausible approach might be to define $(\mathcal{M}, B)$ to
be a matroidal dfs pair if there is any graphic realization of $\mathcal{M}$ and $B$ that is a dfs pair.

Appendix A

In this appendix we provide the technical details of the proofs of Lemma 21 and Theorem 23. The structure defined below and the following lemma are useful in the exposition of the proofs.

Recall that $T$ is a spanning tree of $G$, and the edges in $T$ are directed away from a fixed vertex $u$, the root, which determines a partial order on the vertices. Let $P$ be a path in $T$.

Definition 27. A $P$-pendant is an induced subtree of $T$ on some vertex $v$ and $v$’s descen-
dants, where v’s parent, w, is in P. The vertex w is the hinge of the P-pendant.

When there is no ambiguity, the reference to P is omitted.

**Lemma 28.** If e; and e_j, two cross-edges(u), are such that their ends are mutually unrelated, then e_i, e_j are the two non-tree edges of a yoke T-minor.

**Proof.** A yoke results from contracting all tree-edges not adjacent to the two cross-edges. □

Let a tree-homeomorph of (G’, T’) denote any graph-tree pair derived from the pair (G’, T’) by the subdivision of any subset of the set of tree edges. Clearly, if a pair (G, T) has a tree-homeomorph of (G’, T’) as a T-minor, it has (G’, T’) as a T-minor. Consequently, in the proofs below, references to yokes, 3-wheels, sail boats and speed boats may actually mean their tree-homeomorphs.

**Proof of Lemma 21:** The proof is by induction on the number of cross-edges. There is nothing to prove if there is only one cross-edge. Assume the lemma holds if there are k or fewer cross-edges. Let there be k + 1 cross-edges in (G, T), and let C denote the set of all cross-edges. Remove one of the cross-edges, e = (s, t), and by hypothesis there is a tree-path P such that every other cross-edge in C - e has one end in P. Let e’ = (x, y) be a cross-edge with P-end y, such that y is minimal among the P-ends of all cross-edges in C - e. Let T’ be the pendant containing x, and r be the hinge of T’. We shall show that either P can be extended to contain one of s or t, or there is a path Q in T’ containing r such that the path P(u, r) ∪ Q contains an end of every cross-edge(u).

If P does or can be extended to contain s or t, the lemma is proved. Therefore we may assume that each of s, t is contained in a pendant whose hinge is above y, and thus s and t are not related to y. Then one of s, t must be related to x, or there is a yoke T-minor by Lemma 28. Let s be related to x, so s is in T’. If there is no other cross-edge, (v, w), such that the P-end, v, is below r in P, then (x, y) is the only cross-edge (in (G - e, T)) with its
non-$P$-end in $T'$, in which case $P(u, x) \cup P(x, s)$ is the required path. Hence assume there is at least one such cross-edge, and consider whether the vertex $t$ is in $T'$.

Suppose $t$ is not in $T'$. We show that $t$ is not related to $v$ or $w$, and so $s$ must be related to $w$; i.e. $w$ must be in $T'$, too. If $t$ is related to $w$, by contracting $P(w, t), P(v, y), P(x, s)$, and $P(r, q)$, where $q$ is the hinge of the pendant containing $t$, we can obtain a 3-wheel with hub $r$. Hence $t$ is not related to $w$. If $t$ is related to $v$ then $y < q \leq v$, in which case we can contract $P(v, q)$ and $P(x, s)$, and we have a boat with $P(s, t)$ as the deck. Thus $t$ is not related to $v$ (and so $q > v$) and $w$, and $s$ is not related to $v$. If $w$ is not related to $s$, then there is a yoke by Lemma 28. Thus $w$ must be related to $s$. Now let $(v_1, w_1), \ldots, (v_l, w_l)$ be all the cross-edges with $P$-ends below $r$, including $e'$ and $(v, w)$, where $v_1, \ldots, v_l$ are the $P$-ends, and $w_1, \ldots, w_l$ are the non-$P$-ends, which must be in $T'$ and related to $s$. Assume there are at least two distinct minimal vertices among $w_1, \ldots, w_l$ and let $w_i$ and $w_j$ be two of them. The vertex $s$ must be an ancestor of both $w_i$ and $w_j$. Contract $P(q, r)$ and tree edges so that the vertices $v_1, \ldots, v_l$ are identified as a vertex below $q$, and there is a boat. Hence $s, w_1, \ldots, w_l$ are mutually related, and so there must be a path $Q$ in $T'$ that contains $s$ and $w_1, \ldots, w_l$. The path $P(u, r) \cup Q$ is the required path.

Now suppose $t$ is in $T'$. If there is a cross-edge, $(v, w)$, with $P$-end $v$ below $r$ and $w$ not related to $s$ or $t$, then $s$, $t$, $v$, $w$ are mutually unrelated and there is a yoke by Lemma 28. Consequently all cross-edges with $P$-ends below $r$ must have non-$P$-ends related to either $s$ or $t$ in $T'$. If there is only one such edge then there is a path $Q$ in $T'$ that contains $w$ and one of $s$, $t$, and $P(u, r) \cup Q$ is the required path. If there are $l > 1$ such cross-edges, let $(v_1, w_1), \ldots, (v_l, w_l)$ be as defined before. If there is some $w_i$ related to $s$ but not $t$, and some $w_j$ related to $t$ but not $s$, then we can find a 3-wheel by contracting $P(s, w_i), P(t, w_j)$, and $P(v_i, v_j)$. Therefore assume all of $w_1, \ldots, w_l$ are related to $s$, but not related to $t$. If there are two or more minimal vertices among $w_1, \ldots, w_l$, say $w_i$ and $w_j$, then they must be below $s$, and by contracting tree edges so that the $v_i$'s are identified, we can find a boat.
Therefore there is a path $Q$ in $T'$ that contains $s$ and $w_1, \ldots, w_l$, and $P(u, r) \cup Q$ is the required path.

\[ \Box \]

**Remark:** Let $P(u, y')$, $P(u, y'')$ be two distinct paths, each of which contains an end of every cross-edge$(u)$. Then there is a vertex $v$ such that $P(u, y') = P(u, v) \cup P(v, y')$, $P(u, y'') = P(u, v) \cup P(v, y'')$, and every cross-edge$(u)$ with neither end in $P(u, v)$ has one end in $P(v, y')$ and the other in $P(v, y'')$. It follows that there are at most two paths such that each contains an end of every cross-edge$(u)$, and they coincide near the vertex $u$.

If $(G, T)$ is a minimal counter example to Theorem 23, then more structure can be attached to the path found in Lemma 21.

**Lemma 29.** Let $(G, T)$ be a non-dfs pair with minimum $|V(G)|$ subject to having none of the excluded $T$-reductions and also no overlapping non-tree edges. If $u$ is the root, then there is a tree path $P = P(u, y) \cup P(y, z)$ such that $P(u, y)$ contains one end of every cross-edge$(u)$; every vertex on $P(y, z)$, except $z$, has degree two in $T$; and $z$ is a leaf of $T$.

**Proof.** By Lemma 21 there is a (shortest) path $P(u, y)$ that contains an end of every cross-edge. If $y$ has degree two in $T$, extend $P(u, y)$ until a vertex of degree one or greater than two in $T$ is found. Let $P$ denote the resulting path, and let $z$ be the final vertex. Let $e' = (y, y')$ be a cross-edge with $P$-end $y$, and $y'$ be in a pendant with hinge $r$. It remains to show that $z$ is a leaf of $T$.

Suppose $z$ is not a leaf. Let $S$ be a pendant with hinge $z$, such that there is no back-edge$(u)$ with one end in $S$ and another above $z$. Then $z$ is a cut vertex and there are no cross-edges$(z)$ in $G(V(S) \cup \{z\})$. By the minimality of $(G, T)$, $T(V - V(S))$ is a dfs tree for $G(V - V(S))$, and by Lemma 22 $T$ is a dfs tree of $G$, a contradiction. Hence assume each pendant with hinge $z$ contains an end of a back-edge$(u)$ whose other end is above $z$, and let $v$ be the maximal of the $P$-ends of these edges; also let $(v, v')$ and $(w, w')$ be two of these edges, where $v \geq w > z$ and $v', w'$ belong to different pendants. If $v > r$, then there
is a boat with hull \((v, v')\) and lines \((y, y')\) and \((w, w')\). If \(r \geq v > y\), then \((y, y')\) and \((v, v')\) are overlapping non-tree edges. If \(v \leq y\) and is not a cut vertex, then it must be enclosed by a non-tree edge, \(f\). If \(f\) has an end below \(z\), then its other end must not be above \(v\), and \(f\) cannot enclose \(v\). Therefore \(f\) must have an end below \(v\) but not below \(z\), and it then must overlap with \((v, v')\), a contradiction. It follows that \(v\) is a cut vertex, and again by applying Lemma 22, there is a contradiction. Consequently \(z\) must be a leaf. \(\square\)

**Proof of Theorem 23:** The only if part is trivial. First recall that none of the excluded pairs is a dfs pair. Now for any vertex \(u\), deleting a non-tree edge or contracting a tree edge cannot introduce a new cross-edge\((u)\). Hence if \((G, T)\) is a dfs pair then so is any of its \(T\)-minors; combined with Lemma 17, this implies that if \((G, T)\) has any of the excluded pairs as a \(T\)-reduction, then \((G, T)\) cannot be a dfs pair.

For the if part, let \((G, T)\) be a pair with minimum \(|V(G)|\) subject to having none of the excluded \(T\)-reductions. By Lemma 17, we may assume that there are no overlapping non-tree edges in \((G, T)\).

Pick a vertex \(u\) as the root. Let \(P\), \(y\) and \(z\) be as stipulated in Lemma 29. Let \(e' = (y, y')\) be a cross-edge with \(P\)-end \(y\), \(T'\) be the pendant containing \(y'\), and \(r\) be the hinge of \(T'\).

Note the following observations.

(O.1) There are no cross-edges\((z)\) with neither end in \(P\), since such an edge is also a cross-edge\((u)\) which must have an end in \(P\).

(O.2) If \(f\) is a cross-edge\((u)\) with non-\(P\)-end in a pendant with hinge \(q\), then there is no cross-edge\((z)\) \(f'\) with \(P\)-end above \(q\) and non-\(P\)-end unrelated to the ends of \(f\); otherwise \(f\) and \(f'\) induce a yoke.

The remainder of the proof consists of two cases. In the first case \(z\) is always shown to be a dfs root, and thus a contradiction is derived; the second case is always reducible to the first when \(z\) is not a dfs root. The two cases to consider are: Case A, where there is a
Figure 10: Path $P$ from $u$ to $y$ to $z$. 
cross-edge($u$) with $P$-end below $r$ and non-$P$-end not in $T'$; and Case B, where there is no such edge. If Case A applies, the path $P$ is said to have a T-top.

**Case A.** Let $e'' = (x, x')$ be the cross-edge($u$) with $P$-end $x$, $x < r$, and non-$P$-end $x'$ in a pendant $T''$ with hinge $q$. If there is no cross-edge($z$), then $z$ is a dfs root. So let $e = (s, t)$ be such an edge, with $s$ in $P$. The vertex $t$ must be in a pendant with hinge $p$ above $y$ but, by (O.2), not above $r$ or $q$. The vertex $s$ must not be above $r$ unless $t$ is related to $y'$, or there is a yoke; hence either $p = r$ or $r \geq s > p$. Now consider the location of $r$ relative to $p$ and $q$.

$q \geq p = r$: If $p = q$ then $t$ must be related to $x'$, and thus located in $T''$, or there is a yoke; likewise if $p = r$ then $t$ must be in $T'$. Hence $q$ cannot be the same as $p$, and $t$ is in $T'$. Find a 3-wheel with hub $r$ by contracting $P(s, x')$, $P(t, y')$ and $P(x, y)$.

$q > r > p$: If $x > p$ and $r > s$, then by contracting $P(s, x)$ there is a sail boat with deck $P(y, y')$, masts $P(r, x')$ and $P(p, t)$, hull $e'$, and lines $e''$ and $e$. If either both $x > p$ and $s = r$, or $x = p$, then $e''$ and $e$ are overlapping non-tree edges. If $x < p$, then by contracting $P(r, s)$ and $P(x, y)$ we obtain a speed boat with deck $P(y, y')$, masts $P(r, x')$ and $P(p, t)$, hull $e'$ and lines $e''$ and $e$.

$r > q \geq p$: If $p = q$ then $t$ must be related to $x'$ in $T''$, in which case we can find a 3-wheel with hub $q$ by contracting $P(s, y')$, $P(x, y)$ and $P(t, x')$. Therefore assume $p < q$. If $x \geq p$ and $s > q$, then by contracting $P(s, y')$ and $P(p, x)$ there is a speed boat with deck $P(s, y)$, masts $P(q, x')$ and $P(p, t)$, hull $e'$, and lines $e''$ and $e$. If $x \geq p$ and $x < s \leq q$, then $e$ and $e''$ are overlapping edges. If $x \geq q$ and $s \leq x$, then $(x > p$ and) by contracting $P(s, x)$ there is a sail boat with deck $P(y, y')$, masts $P(q, x')$ and $P(p, t)$, hull $e'$ and lines $e''$ and $e$. If $x < p$, then by contracting $P(q, s)$ and $P(x, y)$ there is a speed boat with deck $P(y, y')$, masts $P(q, x')$ and $P(p, t)$, hull $e'$, and lines $e''$ and $e$.  


Hence no such edge can exist and $z$ is a dfs root of $T$ in $G$, a contradiction.

**Case B.** Let $V'$ be the set of proper descendants of $r$, and $V'' = V - V' - r$.

Suppose $V''$ is not empty. If there is no non-tree edge with one end in $V'$ and the other in $V''$, then $r$ is a cut vertex. There can be no cross-edge($r$) with both ends in $V''$ since such an edge and $e'$ induce a yoke $T$-minor. By the minimality of $(G, T)$, $T(V - V'')$ is a dfs tree of $G(V - V'')$, and by Lemma 22 $T$ is a dfs tree of $G$, a contradiction. Thus there is a non-tree edge $e = (s, t)$ with $s$ in $V'$ and $t$ in $V''$. If $e$ is a cross-edge($u$), then $s \in P(u, y)$ and $t$ is in a pendant that is not $T'$, which contradicts the assumption for Case B. Therefore $e$ is a back-edge($u$) and so $t \in P$. Consider the location of $s$.

$s \in P$: If $s$ is in $P$, then both $e$ and $e'$ are cross-edges($r$). The path $P(r, z)$ contains one end of every cross-edge($r$), and clearly has a $T$-top. Thus Case A applies with root $r$.

$s \in T'$: If $s$ is in $T'$, then $e$ is a cross-edge($r$) with one end in $T'$ and one in $P$ above $r$.

By Lemma 29 there is a path $Q$ in $T'$ that contains one end of every cross-edge($r$) and ends with a leaf of $T$. Furthermore with vertex $r$ as the root, $Q$ has a $T$-top; for if not, then there must be a cross-edge($r$) with non-$Q$-end in a $Q$-pendant, but such an edge is also a cross-edge($u$) with neither end in $P$, a contradiction. Consequently Case A applies here, again with root $r$.

$s \notin P \cup T'$: The vertex $s$ must be in a pendant whose hinge is in $P(r, y)$. Then $e$ and $e'$ induce a yoke.

Hence if $V''$ is non-empty then either a contradiction derives or Case A applies with root $r$.

Now suppose $V''$ is empty, in which case $u = r$. If there is no cross-edge($z$) then there is nothing to prove. By (O.1), any cross-edge($z$) must have an end in $P$, and so its other end must be in a pendant whose hinge is below its $P$-end. Let $w$ be the maximal vertex among the hinges of pendants containing the non-$P$-ends of these cross-edges($z$). By the
maximality of $w$, cross-edges($w$) are back-edges($u$) that enclose $w$, cross-edges($u$) with non-$P$-end in $T'$ and $P$-end below $w$, or cross-edges($z$) with $P$-end above $w$. Each cross-edge($w$) has an end in $P(r, w)$ or in $T'$. With root $w$, the path given by Lemma 29, which we denote by $Q$ here, must contain $P(r, w)$ and extend into $T'$. Furthermore there is no cross-edge($w$) with non-$Q$-end in $T'$, for such an edge is a cross-edge($u$) but with neither end in $P$; thus $Q$ has a T-top, and Case A applies. \qed

References


