The Cumulative Flow Plot: Understanding Basic Concepts in Material Flow

by

Peter L. Jackson

This research was supported in part by NSF Grant DDM-8819542 and by grants from the General Foods Corporation and the International Business Machines Corporation.
ACKNOWLEDGEMENTS

It is a pleasure to acknowledge the helpful suggestions and additional references provided by John Jenner, John McClain, Bill Maxwell, and Jack Muckstadt that have been freely used in the development and presentation of these ideas.
ABSTRACT

The cumulative flow plot is a plot over time of the cumulative input to and the cumulative output from a process. Like the Gantt chart, it has been in use in both academic and industrial circles for many years. The advantage of the plot is that it presents in graphical form three fundamental aspects of material flow: rate of flow, quantity in process, and time in process. The purpose of this paper is to review and extend the utility of the plot and to demonstrate its power by illustrating several basic concepts in the study of material flow. The paper is intended to complement textbook treatments of production planning and inventory control.
The Cumulative Flow Plot: Understanding Basic Concepts in Material Flow

The cumulative flow plot is a plot over time of the cumulative input to and the cumulative output from a process. The advantage of the plot, as will be shown, is that it presents in graphical form three fundamental aspects of material flow: rate of flow, quantity in process, and time in process. Like the Gantt chart, which visually presents the allocation of time to activities, the cumulative flow plot has been in use in both academic and industrial circles for many years. Newell [1971] appears to be the first to use the plot to illustrate all three aspects of material flow. Both student and practitioner can gain valuable insight into the behavior of material flow in a system by studying cumulative flow plots of the system. With the advent of graphically-oriented computer-aided logistics systems, such plots will be readily available as tools for visual analysis of simulation and production data.

To the uninitiated, however, the cumulative flow plot requires some explanation. The purpose of this paper is to provide that explanation and to motivate its use. To demonstrate the didactic and analytical power of the plot, we will use and extend the plot to illustrate several basic concepts in the study of material flow. The paper is intended to complement textbook treatments of production planning and inventory control. For this reason, it includes many problems and discussion questions and it emphasizes intuition and visualization over mathematical formulation.

The paper is organized as follows. Sections 1 - 13 present a variety of ideas drawn from the study of material flow. They are illustrated in Figures 1-27 primarily with cumulative flow plots. Section 14 reviews the limitations of the plot. Section 15 provides references for more advanced study of the topics mentioned in the earlier sections. It also points to related research in the visual display of material logistics data. The plots used throughout the paper were generated using a spreadsheet model of material flow. This model is described in detail in the Appendix.

1. Stocks and Flows: Little’s Formula

We consider any process that has a single input and a single output of material and carries inventory (stocks) of material from one period of time to the next. Figure 1 illustrates. Input, output, and inventory are all measured in the same units and are non-negative quantities. Conservation of material is observed. The inventory at the beginning of a period plus the input during the period and less the output during the period
equals the inventory at the end of the period. The inventory at the end of a period equals the inventory at the start of the next period. Input and output during a period are assumed to occur continuously in time: we are specifying input and output rates of flow.

Inventory is of interest because we must commit resources to have it exist. We may have invested money to create it and we must provide space to store it. Rates of flow are important because we are interested in the ability of the system to provide a service: processing the input (eg. waste treatment) or, more commonly, producing an output.

Figure 2(a) plots the inventory of a process over an inventory cycle: an interval of time that begins and ends with zero inventory. The slope of the plot during a period is the difference between the input and output rates for that period. The average inventory over the inventory cycle is the area under the inventory curve (shaded in Fig. 2(a)) divided by the length of the cycle. The average inventory in Fig. 2(a) is 32.6 units.

Because the inventory plot focuses on the difference between the input and output rates, it fails to provide any information on the average rate of flow either into or out of the process. Since the beginning and ending inventories are zero we know only
Figure 2. Relating Stocks, Flow Rates, and Flow Times
that the average input rate must equal the average output rate over the length of the cycle.

Figure 2(b) is the cumulative flow plot for the same process. It plots the cumulative input to and the cumulative output from the process over the same inventory cycle. At the end of the cycle, cumulative input is equal to cumulative output. The average rate of flow is the ratio of this total flow to the length of the cycle: the slope of the line joining the origin to the endpoint of either cumulative curve. In Figure 2(b) this average rate of flow is 9.4 units per period.

The cumulative flow plot also presents the complete pattern of inventory over time, but in a different form from the inventory plot. For example, at the end of period 11 the cumulative input to the process was 115 units and the cumulative output was 85 units. The inventory still in the system, therefore, was 30 units. This amount is given graphically by the vertical distance between the cumulative input plot and the cumulative output plot at time 11.

The bonus value of the cumulative flow plot is that it displays a quantity that was not even mentioned in Figure 1: the time that material spends in the system. Observe in Figure 2(b) that time 11 marks the arrival of the 115th unit into the system. Not until time 16.33 have 115 units left the system (cumulative output). If we assume that units leave the system in the same order they enter (the FIFO assumption: First-In First-Out) then the 115th unit was in the system for 5.33 time periods. Material flow time of the 115th unit is indicated by the horizontal distance between the cumulative input plot and the cumulative output plot at unit 115.

Material flow time is of special interest when material is perishable: the longer a unit is in the system the more likely it will acquire defects or go sour. For example, the longer a semiconductor wafer spends in a photolithography process before receiving a protective coating of glass, the more likely it will become contaminated by dust in the air. Stated philosophically, material flow time is of general interest because the longer a unit spends in the system the less likely it will be what the customer wants.

Based on Figure 2(b), we can trace a plot of the material flow time of each unit that entered the system (Figure 2(c)) from the first unit to the last under the FIFO assumption. The axes in Figure 2(c) have been reversed from their normal orientation to permit direct comparison of flow times with Figure 2(b). The average material flow
time is the area under this curve divided by the total amount of flow. In Figure 2(c) the
average flow time is 3.47 periods.

The shaded area in each plot is measured in terms of unit-periods: one unit-period
being equivalent to one unit of inventory existing in the system for one full period.

Q1. How are the area under the inventory plot and the area under the flow time
plot related to each other and to the area between the cumulative input and output
curves? Use that relationship to derive the following formula:

\[
\text{Average inventory} = \text{average flow rate} \times \text{average flow time}.
\]

The above formula is known by a variety of names in different disciplines: in
queueing theory, it is known as Little's formula. It has the status of a law in much
more general situations than we have considered here.

Q2. It is not necessary to restrict the analysis to inventory cycles. How can one
represent initial and terminal inventories in a cumulative flow plot? Draw an example.

In summary, the simple procedure of plotting cumulative input and output for a
process permits a graphical study of material flow rates, inventory, and flow time.

2. The Role of Inventory

How much inventory is needed in a system? To begin to answer this question, we
must first understand why inventory exists. In the next few sections, we will review a
number of the causes of inventory. Once the need for inventory is understood, it is
possible, at least in rough terms, to establish the amount of inventory that is sufficient
to meet that need.

On the other hand, for the past two decades, beginning in Japan, many industrial
engineering theorists and practitioners have been promoting the concept that all inven-
tory represents wasted resources. To recognize the need for inventory and to establish
the appropriate level of inventory for that need is not enough. Indeed, it may be the
wrong approach. The philosophy of Just-in-Time asserts that we must identify the
need for inventory and work to eliminate that need. Therefore, as each cause of inven-
tory is introduced in these next sections, consider what engineering or management
parameters give rise to that cause. These are the parameters that must change if the need for inventory is to be eliminated.

3. Pipeline Stock/Process Inventory

Inventory will exist whenever there is a positive process time. If it takes a fixed amount of time to process a unit and there are no restrictions on process or storage capacity, then the output rate of a process is completely determined by the input rate, lagged by the process time. Figure 3 illustrates. The horizontal distance between the input curve and output curve is constant and equal to the process time of two periods. The vertical distance, however, varies with the input rate. The amount of inventory that is appropriate is given by Little's formula: the product of the input rate and the process time. This is referred to as pipeline stock: the process time is the length of the pipeline and the quantity of inventory in the pipeline is determined by the rate of flow. When the input rate changes, it may be some time before the actual inventory matches this amount: it takes the fixed process time to fill or to empty the pipeline to its new level.

![Figure 3. Pipeline Stock](image-url)
The terminology of pipeline stock arose in distribution systems as planners realized that they needed to keep supply channels ("pipelines") full of inventory if they were to keep up with demand. We will use the phrase process inventory to refer to the same concept in manufacturing settings. To reduce the need for pipeline inventory, the only option is to reduce the process time.

Q3. Once every 5 days, a Japanese auto manufacturer sends a ship loaded with cars to the United States. Each ship carries 750 cars and takes 25 days to cross the Pacific Ocean. On an average day, how many cars are at sea? (Answer: 3750.) Draw a cumulative flow plot of this process with initial inventory equal to this average value.

4. Behavioral Stock

Another cause for inventory that on the surface sounds very different, turns out to behave very much like the pipeline model. For some processes, the processing rate varies directly with the amount of inventory. This phenomenon is most evident at small levels of inventory in processes involving human workers: as the amount of work in the queue decreases, the workers relax and work slower. Some professors, for example, are never able to clear their desks of work even when summer frees them from teaching duties. In such systems, there is always stock in the system: behavioral stock.

To illustrate this phenomenon, suppose that no matter how little inventory is in the system, the system will output only a fixed fraction of the remaining stock in any period. Figure 4 illustrates for an initial inventory and no further inputs. The stock is progressively reduced each period but as the number of units awaiting processing decreases, so too does the output rate.

Q4. A clerical department appears to work under a 20% rule: every day the inventory of transactions to process is reduced by 20%. There are currently 60 transactions waiting to be processed and no further transactions will come in for quite some time (Figure 4). What will be the average processing time of the remaining transactions? (Answer: 4.5 days.) How does the average processing time relate to the number of transactions currently waiting? (Answer: it is independent of the initial number of transactions.)
Figure 4. Behavioral Stock: Slowing Output Rate with Diminishing Inventory

Figure 5. Behavioral Stock: Tracking the Input Rate
In general, this particular behavioral model behaves very much like the pipeline model. Figure 5 is the behavioral counterpart to Figure 3 using a 33.33% rule (chosen to yield the same average processing time). The cumulative output plot appears to be a smoother version of a lagged output process than the fixed process time process.

Q5. If the input rate is 5 units per period and a 40% behavioral rule is operating (i.e., in every period, the output is equal to 40% of the sum of the input and the initial inventory for that period), what will be the long run, stable level of inventory? (Answer: 7.5 units.) Figure 6 illustrates with an initial inventory of 30 units.

For the model of behavioral stock in Q5, let $e$ denote the behavioral efficiency: $e = 0.40$. It is easily shown that the average material flow time is given by $(1-e)/e$. Call this ratio the behavioral flow time.

Combinations of behavioral models and lagged, or pipeline, models of inventory are common in economic and financial studies. The following question is an example.
Q6. Uncollected credit sales are referred to as accounts receivable. On average, given current credit policies, a certain company finds that 10% of its monthly sales are collected in cash in the month of the sale; 60% of the sales are collected during the next month; 5% in the second month after the sale; 10% in the third month; and the remaining 15% are collected during the fourth through sixth months after the sale. All sales are credit sales. Sales and collections occur continuously in time (the daily sales and collection rates are 1/30 of the corresponding monthly rates). The company averages $180,000 in sales per month. Estimate the average balance in accounts receivable and the average collection period (time until the sales are collected in cash). (Answer: $315,000 and 1.75 months, respectively.) Prepare a cumulative flow plot that illustrates both of these quantities.

5. Capacity

The maximum output rates of both the pipeline and the behavioral models are limited only by the input rate. Viewing the pipeline model slightly differently, there is no upper limit to the number of units in process. Manufacturing processes typically have not only a process time per unit but a limit to the number of units that can be processed at one time. For example, an oven takes two hours to bake a product and
can hold up to 40 trays of product at once. The maximum output rate is determined by the ratio of the maximum number of units that can be in process to the processing time per unit (20 trays per hour).

Figure 7 is similar to Figure 3 except that the maximum number of units that can be in process is set to 20. If the input rate exceeds the maximum output rate then units simply queue up and wait for entrance to the process. Figure 7 distinguishes between queue time and process time and between queue inventory and process inventory (pipeline stock). A measure of material flow efficiency is the *queue factor*:

\[
\text{queue factor} = (\text{average queue time} + \text{process time}) / \text{process time}.
\]

In general, small queue factors are desirable. Later sections will explore the reasons for queues.

In some cases, the process time and the number of units in process are negligible relative to the maximum production rate. For example, a screw machine can turn a screw one at a time in a few seconds. In these cases, we typically specify only the production rate. Figure 8 illustrates for the same maximum production rate as Figure 7
Figure 9. Stocks and Flows: A Four Stage Process

but with a zero process time. (The queue factor is undefined for such a situation.) Capacity is illustrated in a cumulative flow plot as a limit on the maximum slope of the cumulative output curve.

Q7. An assembly department builds units at a maximum rate of 500 units per hour. The build time per unit is 0.25 hours. What is the maximum number of units in
process? (Answer: 125 units.) Refer to this as the maximum process inventory: any inventory above this amount is simply queue inventory, as was illustrated in Figure 7.

6. Multi-Stage Systems and WIP Waves

It is obvious that inputting material at a rate faster than the maximum output rate of the process results in a queue. Queue time is wasted time as far as the material being processed is concerned and queue inventories are expensive: they consume management time and facility resources. However, some reasons for such queueing phenomena may become clear if we extend the discussion to multi-stage systems.

Figure 9 shows four stages of a process in series: the output of one stage forms the input for the next. Figure 10 plots the cumulative flows into and out of each stage. The output curve for one stage coincides with the input curve for the next stage. At any point in time, the vertical distance between the input and output curves for a stage is equal to amount of inventory at that stage. Figures 11 (a), (b), (c), and (d) show the inventory levels in the four stages at times 8, 9, 10, and 11, respectively. The information in Figure 11 can also be read directly from the vertical slices taken in Figure 12.
Figure 11. A WIP Wave
Figures 10 and 11 illustrate the phenomenon of a wave of work in process (WIP) inventory moving through a system: a WIP wave. In period 8, the input rate to the system is suddenly and temporarily increased. In period 9, the output rate in stage 1 increases in response to the inventory buildup. Stage 2 responds in period 10, and so on. It is time 15 before the system returns to normal inventory levels. For this example, we have assumed a process time of 1 period in each stage and a normal input rate of 5 units per period. In addition to the process time, we have also assumed a behavioral efficiency of 80% in each stage to smooth the output rates and to make the transitions more wave-like. To begin with, there is no inventory in the system but the system converges to near normal inventory levels within 7 periods.

In Figure 12, we continue the example by imposing production rate limits in each of the stages. Stages 1 through 4 have maximum production rates of 30, 8.5, 7, and 6 units per period, respectively. The production capacities are progressively smaller. The last and smallest production rate makes stage 4 the bottleneck of the system. The impact of progressively restricting capacity through the system is to slow the progress of the WIP wave through the stages. Note how inventory moves quickly out of stage 1 but piles up in stage 4. The system does not return to normal until period 21. The maximum slopes of the output curves in stages 2, 3, and 4 are determined by the
maximum production rates in each of these stages. The capacity of stage 1 is so large that the maximum observed slope of its output curve is determined by the maximum rate of input.

Q8. The effective process time in stage 4 is given by the sum of the fixed process time (one period) and the behavioral flow time \((1-0.8)/0.8 = 0.25\) periods. Given the maximum production rate of 6 units per period, what is the maximum process inventory for stage 4? That is, what level of inventory in stage 4 is needed to sustain the maximum production rate. (Answer: 7.5 units.) In the situation depicted by Figure 12, the maximum inventory in stage 4 occurred at time 15 and equalled 11.82 units. What is the maximum queue (non-process inventory) that was experienced in stage 4. (Answer: 4.32 units.) Also, estimate the maximum queue factor that was experienced in stage 4. (Answer: 1.576.)

In Figure 13, the temporary increase of input shown in Figure 12 is spread over four periods rather than just one. Note that the output of the system is basically the same as in Figure 12 and the system still takes until period 21 to return to normal. Compared to Figure 12, however, the average amount of inventory in the system (measured as the average vertical distance between the input curve for stage 1 and the
output curve for stage 4) is smaller. The average flow time is also smaller. The obvious lesson is that there is no point in putting material into a system faster than the bottleneck stage can process it. One of the reasons why real systems sometimes suffer unnecessarily from WIP waves is that the multi-stage nature of the system hides the problem from those responsible for its cause.

Q9. How can the manager for stage 1 (Figure 12) be made aware that the input rate needs to be moderated? After all, stage 1 had no difficulty in coping with the sudden increase in the input rate.

Q10. What can you say about the production controller for a system who, after seeing jobs come out of the system too slowly, releases jobs into the system at a faster rate?

One of the answers to question Q9 is to impose limits on the maximum inventory allowed in each of the stages. Such limits can be physical (storage capacity) or managerial (refusal to authorize material transfer from one stage to the next). Figure 14 uses the same input process as in Figure 12 but restricts the inventory in each of stages 2, 3,
and 4 to a maximum of 7.5 units. Note how the output curve is the same as in Figures 12 and 13 but the WIP bulge is concentrated in stage 1, the first stage to experience the impact of the changing input rate. With these limits on inventory, any discrepancy between the input rate and the maximum output rate of the system will show up in stage 1 (near the source of the problem) almost immediately.

Q11. Suppose the process times in the four sectors are 1, 2, 2, and 1 periods, respectively, and the production capacities are 30, 9, 8, and 6 units per period, respectively. Establish limits on inventory in stages 2, 3, and 4 that will not constrain the overall output rate of the system but will have the effect that WIP bulges are concentrated in stage 1. That is, compute the maximum process inventory in each of stages 2, 3, and 4, as determined by the maximum output rate of the system. (Answer: 13.5, 13.5, and 7.5 units, respectively.)

Observe in Figure 14 that from periods 12 to 17, stage 4 is operating at maximum (bottleneck) capacity and each of stages 2, 3, and 4 are filled with inventory. During this interval, no unit can enter stage 2 until a unit leaves stage 4 and every unit that leaves stage 4 is replaced by a unit entering at stage 2. It is as though every unit of output authorizes its own replenishment within the subsystem of stages 2 through 4. In this situation, the bottleneck is said to be pulling inventory to itself: the flow of material in the subsystem is completely determined by the bottleneck stage. Before period 12 and after period 17 this is not the case: the input rate in stage 1 is constraining the throughput of material in the system.

Another answer to question Q9 is to impose a limit not on the inventory of each stage but on the total inventory in stages 2 through 4. One way of accomplishing this is to put the material into standard size containers and restrict the total number of containers available in stages 2 through 4. When all the containers are full no material can enter stage 2 until a container leaves stage 4 and becomes empty again.

In Figure 14, the average total inventory in the system and the average total material flow time through the system are the same as in Figure 12. To achieve the reduced inventory and flow time benefits of Figure 13, it is necessary to restrict the input rate of material to the system. This can be done by extending the inventory restrictions of Figure 14 back to include stage 1. Think of stage 0 as being the stage where production orders queue up. The idea is that it is better to have production orders queue up in stage 0 than to have inventory of material pile up in any of the actual production stages 1 through 4.
In the next sections we explore situations in which it is necessary to have queue inventory at different stages.

7. Cycle Stock

In the example developed in the previous section, the production capacity of the system is determined by the capacity of the bottleneck stage. The excess capacities of the other stages are wasted: at most, these excess capacities serve to move inventory quickly through the stages in front of the bottleneck. In general, non-bottleneck stages must operate at rates less than their capacities.

Figure 15 illustrates a two stage system in which stage 2 is the bottleneck (production capacities are 10 and 5 units per period, respectively). Stage 1 is said to be upstream of the bottleneck. Stage 2 has a fixed processing time of 2 periods so a certain amount of process inventory is unavoidable. Stage 1, however, has a negligible processing time and the input rate is chosen so that the stage 1 input and output curves coincide throughout the plot. There is no inventory in stage 1. The overall input rate is chosen to keep stage 2 running at maximum (bottleneck) capacity. However, rather
than running stage 1 continuously at the same rate as stage 2, Figure 15 shows that it is run at maximum capacity for periodic intervals of time and is idle (both input and output curves are flat) over the remaining intervals. The result is a cyclic buildup and depletion of inventory in stage 2 in addition to the process inventory of stage 2. This queue inventory is referred to as cycle stock. The previous sections make clear that cycle stock has a negative impact on the queue factor of the system. On the other hand, if the resources in stage 1 (machines and labor) can be used for other purposes during the idle periods of Figure 15, such as for producing other products, then it may be better to carry additional inventory at stage 2 than to waste those stage 1 resources by devoting them to continuous production. This tradeoff is considered further in the next section when we look at multi-item problems.

There are at least two ways of specifying a cyclic production schedule at stage 1. One is to specify all the times at which production in stage 1 is to start and to stop. This is the scheduling approach and is the subject of the next section. Care must be taken under the scheduling approach to ensure that a production run is initiated soon enough to prevent stage 2 from running out of stock. The stage 2 pipeline must be kept full or stage 2 will not achieve its maximum throughput (‘‘Don’t starve the bottleneck!’’). The other is to specify MIN and MAX levels: initiate a production run at stage 1 whenever stage 2 inventory drops below this MIN level and to keep producing until stage 2 inventory reaches the MAX level.

Q12. What is the minimum appropriate value for the MIN level? (Answer: the maximum stage 2 process inventory.) What value of the MAX level corresponds to a one-for-one replenishment policy? (Answer: the maximum stage 2 process inventory.)

Q13. Suppose the bottleneck stage is located several stages downstream from the stage to be controlled by a cyclic production schedule. On what measure of inventory should the MIN and MAX levels be based? (Answer: the total inventory in the stages downstream of the controlled stage, down to and including the bottleneck stage.)

8. Scheduling Issues

If, as suggested in the previous section, the resources (machines and labor) for stage 1 are actually used to produce other products, then a simple order policy may not be sufficient to control the system. When stage 2 inventory hits its MIN level, the resources at stage 1 may be committed already to a production run for a different product. Only in extreme cases, such as infinite production capacity or costless

20
switchover from one product to another, does the scheduling problem disappear. In general, one must schedule the production runs for stage 1 and coordinate these runs among all the products that share stage 1 resources.

To illustrate these scheduling issues and their impact on inventories, consider three different products, A, B, and C, each of which is produced in a two-stage process. For each product, the second stage is the bottleneck. This bottleneck stage is used to produce the product continuously at maximum capacity and has nothing to do with the bottleneck stages for the other products. For each product, the first stage resource is a single machine that is shared among the three products. Figure 16 illustrates. The production rate of the single machine is fast enough to keep all three bottleneck stages busy but it takes time to change over from producing one product to producing another: long production runs are necessary. The scheduling problem is to schedule
Figure 17. Inventory Consequences of Production Schedules
production runs of the three products on the stage 1 machine in such a way as to minimize the total cycle stock across the three products.

Q14. The production rates for products A, B, and C at their respective second stages are 6, 9, and 6 units per period, respectively. The first stage machine can produce any of the products at a maximum rate of 30 units per period. It takes one full period to switch the machine from producing one product to producing another. During a switchover period, the machine can produce nothing. What is the minimum production run length for each product that ensures that the stage 1 machine can keep each of the bottleneck stages busy? (Answer: 2, 3, and 2 periods, respectively, for a total cycle length of 10 periods.)

Figure 17 displays cumulative flow plots for each of products A, B, and C. The stage 1 input and output curves for each product coincide, as in Figure 15: the focus is on stage 2 inventories. The stage 1 input/output curves are determined by the production schedule on the stage 1 common machine. This production schedule is illustrated in Gantt chart form at the foot of each cumulative flow plot. Shaded areas on the production schedule indicate periods of product switchover.

The production schedule in stage 1 is optimal in the sense that it minimizes the total cycle stock across the three products. Note that this optimal schedule is possible only because the initial inventories of products B and C are exactly enough to last until the first production runs of each of these products. Note how the production schedule and inventories are perfectly coordinated: a production run for a product begins precisely when inventory for that product drops to its process inventory level. It would be possible to generate this same production schedule using MIN and MAX levels for the three products but only if the initial inventories match those in the example.

In many cases, production schedules are not tightly coordinated with inventory requirements. Other scheduling considerations can enter to complicate the problem. In such cases, it may be necessary to carry additional inventory in order to make the schedule feasible. We refer to this inventory as scheduling stock.

Q15. Suppose the production schedule on the stage 1 machine had been ABCBCACAB. How much additional initial inventory of each product would be required to ensure that stage 2 for each product is kept busy? Redraw Figure 17 to reflect this schedule.
Q16. Suppose an upstream machine cycles through all products at least once a month but the sequence changes from month to month in seemingly random patterns. How much inventory must be kept in front of a downstream stage for a single product to ensure that it is never starved?

Q17. Suggest possible causes of and remedies for the need for scheduling stock.

9. Anticipation Stock

Similar to cycle stock is the concept of anticipation stock. In this case, suppose a high capacity machine (stage 2) is located downstream from a low capacity machine (stage 1). To avoid wasting capacity on the downstream machine, it may be run at full capacity but at periodic intervals. It can be used to process other products during the idle periods. Figure 18 illustrates. This stage 2 inventory could be referred to as cycle stock because of its periodic buildup and depletion but could also be referred to as anticipation stock. Stage 2 inventory is built up in anticipation of being used down-stream at a faster rate than it can be produced. Production planning is needed to ensure that stage 2 has sufficient anticipation stock to meet its production schedule.
Figure 19. Nested Cyclic Production Upstream of the Bottleneck

Q18. Stage 2 of a calendar manufacturer is the finished goods warehouse which ships all of its stock during the month of August of each year. Stage 1 is the calendar printing and assembly process which can produce at most 1 million calendars per month. To be flexible, in case actual orders exceed projections, the company plans to have 80% of total projected shipments in the warehouse by the end of May of each year. When must the company begin production if total projected annual shipments are 10 million calendars? (Answer: October 1.) Draw a two year cumulative flow plot, September to August, to illustrate this production plan.

We think of cycle stock, scheduling stock, and anticipation stock as stored capacity: it is created because capacity is not always available exactly when it is needed.

10. Echelon Stock vs. Installation Stock

Consider three stages in series with decreasing production capacities. Production at stage 3, the bottleneck, is continuous and maximal. Production at stages 1 and 2 is cyclical with stage 2 producing twice as often as stage 1. (Stage 2 production frequency is an integer multiple of the stage 1 production frequency: we say that the
Figure 20. Installation Stock Pattern

Figure 21. Echelon Stock Pattern
frequencies are nested.) Figure 19 illustrates. As in Figures 15 and 17, the input and output curves for stage 1 coincide. Inventory is carried only in stages 2 and 3.

Figure 20 displays the inventory plot for stage 2 inventory, sometimes called the stage 2 installation stock. Note that there exist intervals of time in which stage 2 installation stock is zero. It is clear from Figure 19 that such periods cause no problem for the bottleneck stage provided there is excess inventory at stage 3. Figure 21 displays the inventory plot for the total of inventory at stages 2 and 3. This is referred to as stage 2 echelon inventory: the total of inventory at stage 2 and at all stages downstream of stage 2. Figures 20 and 21 illustrate that echelon inventory patterns are often simpler than installation stock patterns.

Q19. The maximum production rates in stages 1, 2, and 3 are 13, 7.5, and 5 units per period, respectively. Formulate the ordering rules (MIN and MAX levels) for stages 1 and 2 that give rise to the behavior of Figure 19. (Answer: Stage 1 MIN level = 10 units of stage 2 echelon inventory; stage 1 MAX level = 50 units of stage 2 echelon inventory; stage 2 MIN level= 10 units of stage 3 inventory; and stage 2 MAX level = 30 units of stage 3 inventory.)

Q20. How much scheduling stock would be required if the production cycles of stage 2 were not coordinated with the cycles of stage 1? Draw a sample cumulative flow plot that illustrates a lack of coordination and show the additional initial inventory necessary to prevent stockouts.

II. Safety Stock

In this section, we consider the need for inventory that arises because production rates are not perfectly predictable. Randomness in production rates can arise, for example, because process times are random, because process capacity is variable (a worker in an assembly department may not show up for work, a machine may break down), or because of a combination of these reasons.

To illustrate, we consider a two stage system in which the second stage is the bottleneck and experiences a random production rate. The expected production rate is 5 units per period. Stage 1 has unlimited production capacity but a fixed process time of two periods. Stage 2 has a zero process time. Both stages have unlimited storage capacity. Initial inventories in stages 1 and 2 are 10 and 5 units, respectively. To control the input rate into stage 1 we impose an upper limit of 10 units on inventory in
stage 1. This effectively sets the input rate to stage 2 at 5 units per period, the same as the expected production rate in stage 2. Figure 22 is a sample cumulative flow plot for this system. Stage 2 has an inventory level that varies with the random productive capacity. Occasionally, stage 2 runs out of inventory and the output rate is limited by the input rate.

It is difficult to demonstrate graphically the long run behavior of such a system but it can be argued easily that the average amount of inventory in the system is unbounded, in an expected value sense. This is simply because the expected output rate is less than the input rate. Whenever the production rate is above its expected value of 5 units per period and there is no inventory in the system, the output rate is bounded by the input rate, which is equal to the expected production rate. Whenever the production rate is below its expected value, the output rate is bounded by the production rate. The expected output rate is therefore less than the expected production rate. Since the input rate equals the expected production rate, expected average inventory must be unbounded.

If inventory is to be kept under control, the input rate must be tied somehow to the output rate. This can be done, for example, by dropping the limit on the amount of
Figure 23. Random Production Rate and Low Reorder Level

Figure 24. Random Production Rate and High Reorder Level
inventory in stage 1 and, instead, controlling the input to stage 1 by means of an order policy based on stage 1 echelon inventory (total system inventory). As stage 1 echelon inventory is depleted, the order policy will release material into stage 1 to replenish it. In this way, the bottleneck will pull inventory into the system according to its ability to process it.

Figure 23 shows the impact of a MIN level of 10 units of stage 1 echelon inventory. The MAX level is the same as the MIN level: this is one-for-one replenishment. Figure 24 show the impact of increasing this MIN level to 15 units. Figures 22-24 use exactly the same sequence of (randomly generated) production rates. Note that the larger MIN level permits a greater output rate (133 units in total) than the smaller MIN level (107 units in total) but means that more inventory is in the system on average.

The expected average stage 1 inventory is pipeline stock: it is given by the product of the stage 1 process time and the expected output rate. The expected output rate depends on the MIN level that is chosen (it is no greater than the expected production rate). The expected average amount of stage 2 inventory is referred to as safety stock. The expected time that material spends in stage 2 is referred to as safety time. Note that the input rate to stage 2 in a period is determined two periods earlier, the process time of stage 1. The purpose of safety stock is to buffer the stage 2 production rate from this input rate while inventory is being replenished through the stage 1 “pipeline.” If the process time through stage 1 were zero there would be no need for safety stock.

Q21. Suppose the process time for stage 1 is 48 hours. The process capacity of stage 2 each hour is random so that the production rate is uniformly distributed over the range from 10 to 30 (all real values in the range are equally likely). The process capacity in one hour is independent of the process capacities in all previous hours of operation. (a) What should be the MIN level to guarantee no loss in throughput? (Answer: 1470 units) (b) What should the MIN level be to assure, with 97.5% probability, no throughput loss in an hour? (Hint: use the normal approximation. Answer: 1059 units) Observe that the answer to (b) is considerably less than the answer to (a).

12. The Value Dimension

Figure 25 presents a cumulative flow plot for a four-stage system in which the stages are labelled as RM (Raw Material Acquisition), PF (Parts Fabrication), AS (Assembly), and FG (Finished Goods Distribution), respectively. Input and output flow rates are constant so that all inventory levels and flow times are equal to their
average values (flow times of the units in initial and terminal inventories are assumed to extend outside the range of the plot). Imagine a third dimension to the cumulative flow plot that indicates the per unit value of inventory in a stage. This dimension is approximately represented in Figure 25 by shading: the darker the shade, the higher the unit value. In general, value is added to the product at each stage of the process, so the third dimension can be thought of as cumulative value added (analogous to the cumulative flow dimension).

The unit value dimension is represented more accurately by cross-sectional analysis than by shading. Figure 26 plots the flow time and unit value dimensions (left hand axis) corresponding to a horizontal slice at the 95th unit of the cumulative flow plot. In this example, raw material value is determined at the time the material enters the system, value during fabrication and assembly is added uniformly over the corresponding flow times, and there is no value added during finished goods distribution.

Figure 27 plots the inventory and value dimensions corresponding to a vertical slice of the cumulative flow plot at time 10. The area under the curve in Figure 27 is
Figure 26. Cumulative Value Added Over Total Manufacturing and Distribution Flow Time

Figure 27. Distribution of Unit Value Over Inventories
the total dollar investment in inventory at time 10. The area under the curve over the relevant stages is the dollar investment in inventory in that stage:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Raw materials</td>
<td>$ 40,000</td>
</tr>
<tr>
<td>Fabrication WIP</td>
<td>150,000</td>
</tr>
<tr>
<td>Assembly WIP</td>
<td>57,500</td>
</tr>
<tr>
<td>Finished Goods</td>
<td>120,000</td>
</tr>
<tr>
<td><strong>Total Inventories</strong></td>
<td><strong>$367,500</strong></td>
</tr>
</tbody>
</table>

This investment data is reported on the financial statements of the company at time 10. By way of contrast, there is no financial statement that summarizes the information of Figure 26.

The shaded areas in Figures 26 and 27 are closely related. Since inventory levels and flow times are equal to their respective average values, Figure 27 is equivalent to Figure 26 with the horizontal axis scaled by the output rate (sales measured in units), by Little’s Law. Alternatively, we can scale the vertical axis (the left hand axis) in Figure 26 by the average output rate. The right hand axis in Figure 26 shows the resulting scale. By Little’s formula, the area under the curve in Figure 26, measured against the right hand scale is the average total dollar investment in inventory. The area under the curve over the raw material acquisition time is the dollar investment in raw materials ($20,000/period x 2 periods = $40,000) and similarly for the other inventory investment categories. Figure 26 with the right hand axis is referred to as an *inventory profile*. The inventory profile can be used as a basis for discussing the relative value of different purchasing, manufacturing, or distribution options.

Q22. Which effect is more beneficial: a 10% reduction in raw material prices or a 10% reduction in finished goods flow time? (Answer: raw material price reduction.)

Q23. An investment in a material handling system is expected to reduce average fabrication flow time by 0.5 periods. By how much will this improvement reduce total dollar inventory investment? (Answer: $18,750.)

We have measured value in this section in the accounting sense of cost added to the product: the cost of materials, the cost of labor, and the overhead cost of equipment and facilities. An economist might object to this treatment because traditional methods of overhead allocation have sometimes led to seriously incorrect assessments
of product value. Nevertheless, the inventory profile can play an important role in discussions of cost-cutting by introducing the element of flow time.

13. Further Questions

Most inventory problems can be seen to involve one or more of the elements discussed in the preceding sections. Determining the appropriate amount of inventory to carry in a system typically involves distinguishing between pipeline, cycle, scheduling, and safety stocks, establishing appropriate levels for each category of need, and implementing control policies that limit inventory and coordinate activities. More fundamentally, however, improving a system involves understanding the causes of inventory in each category of need and working to eliminate those causes. The inventory profile plot, which summarizes the value dimension of the cumulative flow plot, can be useful in identifying at what stage in the system improvements will yield the greatest economic return.

This discussion of basic concepts in material flow is not intended to be exhaustive. The following questions should stimulate thought on how to extend the ideas already presented. Suggestions for further reading are presented in Section 15.

Q24. How is a random customer demand process like a machine with a random production rate? Relate the admonition "Don't starve the bottleneck!" with the warning "You can't sell from an empty truck."

Q25. Draw a cumulative flow plot that illustrates cycle stock, pipeline stock and safety stock.

Q26. A problem in designing water reservoirs is to determine a dam size so that the reservoir can supply downstream demand for water at a constant rate in spite of the fact that stream inflows to the reservoir are random from month to month. If the dam is empty then downstream demands are unmet. If the dam is full, additional inflows are spilled and cannot be used to satisfy future demand. Draw a cumulative flow plot to illustrate this problem.

Q27. Suppose production rates are infinite at stages 1 and 2 of the system described in Section 10. Redraw Figures 19, 20, and 21 for this case.
Q28. A central warehouse supplies inventory to a collection of retailers. Demand for the product occurs randomly at each of the retailers. The retailers order from the central warehouse on a one-for-one replenishment basis: every time a unit is sold by a retailer, the retailer issues a replenishment order at the central warehouse for one unit. Periodically, the central warehouse places orders with a factory. Orders from the factory take a fixed amount of time to be filled and shipped to the warehouse. Shipments from the central warehouse to the retailers take a fixed amount of time. Draw a cumulative flow plot that illustrates the amount of inventory in transit from the factory, the amount of inventory on hand at the warehouse, the total amount of inventory in transit to the retailers, and the total amount of inventory on hand at the retailers. Discuss the form of the ordering policy used by the central warehouse: does it matter whether it is based on echelon stock or installation stock? Why?

Q29. The need for safety stock in Section 11 arose because stage 1 had a positive process time and stage 2 had random production rates. If stage 1 had a zero process time but random production rates then there would still be a need for inventory in stage 2 to buffer the stage 2 production rates from the stage 1 production rates. Such inventory is referred to as decoupling stock. Illustrate the need for decoupling stock with a cumulative flow plot.

14. Limitations of the Cumulative Flow Plot

The previous sections are intended to show how the cumulative flow plot can be used to illustrate and unify a variety of concepts in material logistics. It has to be admitted, however, that the plot has several limitations to its usefulness.

As with all plots, the visual interpretation of areas and slopes is sensitive to scaling of the axes. This becomes a problem for the cumulative flow plot because the curves are non-decreasing over time. One is plotting cumulative output against the same axis that one wants to measure inventory levels. The longer the time horizon, the larger the scale on the vertical axis will need to be. Inventories and flow times become insignificant relative to the cumulative output and the length of the time horizon. The cumulative flow plot is useful only over time horizons that are short relative to the average material flow time.

A related limitation is that perception of relative magnitudes is much more difficult in a curve difference chart such as the cumulative flow plot than it is in an x-y chart (compare Figures 26 and 27 in Cleveland and McGill [1984]). That is, if the
objective is to display the pattern of inventory over time, then the x-y chart of Figure 2(a) is superior to the curve difference chart of Figure 2(b). Similarly, if the objective is to show the evolution of flow time, then Figure 2(c) is superior to Figure 2(b). On the other hand, the power of the cumulative flow plot lies in the relationships it reveals between inventory, flow rates, and flow times. In general, it will be used in conjunction with other graphical views of data, such as x-y charts, bar charts, and Gantt charts, as we have done in Figures 2, 11, and 17.

It is also a problem to visualize limits in a cumulative flow plot that relate directly to inventory or to flow time. MIN/MAX levels and buffer capacity limits are defined relative to inventory levels. On an inventory plot such as Figure 2(a), these points and limits are constant levels against the inventory axis. On a cumulative flow plot, they are seen as curves parallel to a cumulative output curve. Similarly, imposing a limit on the maximum age of inventory would be seen as a constant level in Figure 2(c) against the flow time axis but as a curve parallel to a cumulative input curve on a cumulative flow plot. The lessons illustrated by Figures 23 and 24 would be clearer if they were supplemented with inventory plots.

Another limitation is that input, output, and inventory must all be measured in the same units. In multi-stage manufacturing environments, changes in units are common both within and between stages. To create a cumulative flow plot, unit conversions of either input or output are frequently required.

A problem related to unit conversion is the problem of yield. Figure 1 asserts a conservation of flow. Losses due to scrap mean that the total output from the system is less than the total input. In this case, the cumulative input and output curves would diverge and the interpretation of horizontal and vertical distances between the curves as flow times and inventories, respectively, would no longer be valid. If the exact yields are known for each period then the outputs could be expressed in equivalent units of input. This would correct the problem of interpreting horizontal and vertical distances but the slope of the cumulative output curve would now measure the throughput rate of the system in terms of the input material rather than the output material.

The restriction that inventory be a non-negative quantity can be relaxed. Negative inventory levels can be interpreted as backorders: output that has been promised but not yet delivered. When there are backorders, the output curve crosses and exceeds the input curve in the cumulative flow plot. Unfortunately, the multi-stage version of the
cumulative flow plot becomes confusing as curves begin to cross one another. For a unified treatment of single and multi-stage models, we have restricted attention to non-negative inventory levels (equivalent to the lost sales case in inventory modeling).

To interpret horizontal differences between input and output curves as flow times, one must use the FIFO assumption. This may not hold true in many situations. If there are several identical machines in a stage, for example, one unit entering the system may gain access to and complete production at one machine while a unit that entered the system ahead of it is still waiting to gain access to a different machine. Newell [1971] has proposed a different cumulative flow plot that accurately portrays flow times without the FIFO assumption but it requires detailed tracking of each unit in process.

Finally, the multi-stage version of the cumulative flow plot is effective but it is not easily extended to non-serial production systems. Question Q28 indicates that stages or echelons are natural in distribution systems and that the cumulative flow plot can be adapted to such systems. In such an adaptation, however, the FIFO assumption clearly fails. Interpreting horizontal distance in the plot as flow time is valid only in an average sense. In manufacturing settings, the cumulative flow plot can be used to trace the flow of a single raw material into many final products provided there are natural stages of transformation. We have implemented this adaptation within COSMOS, a prototype computer-aided logistics modeling system (Jackson, Muckstadt, and Jones [1987]).

15. Summary and Related Publications

An early use of the cumulative flow plot is by Eagle [1957] to show the seasonal bulge in Hawaiian pineapple inventories. The cumulative flow plot is used widely in texts on production planning and inventory control but its use appears to be restricted solely to an explanation of the single stage production planning problem: matching cumulative supply to cumulative requirements of a single product. It is used to introduce the concepts of capacity, anticipation stock, and backorders. The purpose of this paper has been to motivate wider use of the plot. In particular, it is well suited to represent many phenomena that arise in multi-stage systems and it provides a natural link to inventory value profiles.
The cumulative flow plot is used in the semiconductor fabrication industry to monitor, in the aggregate, the progress of semiconductor wafers through the production line. The plot is particularly appropriate in this industry because the production process is serial in nature and flow times are a critical concern to production managers. Research on adapting the plot to other environments would be useful.

The cumulative flow plot is featured in Wiendahl [1987 and 1988] as a tool for operational analysis. Wiendahl is the first to extend the plot to multi-stage systems. Wiendahl also enhances the plot with overlays that relate job release date with earliest job completion date and actual job completion date.

Little's Law and the use of cumulative flow plots in queueing analysis (Section 1) are discussed in greater detail in Newell [1971]. An excellent introduction to the Just-In-Time philosophy (Section 2) can be found in Hall [1983]. Formal models of pipeline inventory, cycle stock, anticipation stock, and safety stock (Sections 3, 4, 7, and 11) can be found in all the production planning textbooks listed in the references. The classic text on inventory models is by Hadley and Whitin [1963]. Bottleneck analysis in multi-stage systems (Section 5) is presented in the form of a novel by Goldratt and Cox [1984]. The graphical treatment of scheduling in Section 8 complements the presentation of cyclic scheduling in McClain and Thomas [1985]. The concept of echelon stock (Section 10) is critical in advanced work in production and distribution planning (see Maxwell and Muckstadt [1985]). Section 12 draws heavily on the paper by Clark [1984] but may mark the first use of the cumulative flow plot to introduce these ideas. The inadequacy of traditional cost accounting measures of value is discussed in Cooper and Kaplan [1987]. Decoupling stock in serial systems (Question Q29) is studied in Conway, Maxwell, McClain, and Thomas [1988]. The cumulative flow plot is used in a microcomputer game of multi-echelon distributions systems developed by Jackson, Muckstadt, and Rushmeier [1984].

The use of graphical aids to enhance the modeling and understanding of complex systems is growing rapidly. Computer graphics has made routine the graphical analysis of large data sets and it has made possible innovative approaches to human interaction with models and data. The three-dimensional Gantt chart by Jones [1988] is an example of such innovation. The list of references includes several papers that explore issues in the visualization of data (Bell [1985], Cleveland and McGill [1984], DeSanctis [1984], Jackson et al [1987], Jones [1986], Jones and Maxwell [1986], and Tufte [1983]).
References


Appendix: A Multi-Stage Model of Material Flow

This appendix describes the model that was used to generate the cumulative flow plots used in the paper. It is suitable for implementation as an Excel or Lotus 1-2-3 spreadsheet and useful for classroom demonstrations.

Except for indices, lower case letters refer to model parameters. Except for limits of indices, upper case letters refer to model variables.

Let \( n \) index the stage of production: \( n = 0, 1, 2, \ldots, N+1 \). Cumulative input and output curves are plotted only for stages 1 through \( N \). Stages 0 and \( N+1 \) are used to impose boundary conditions.

Let \( t \) index the time period: \( t = -L, -L+1, \ldots, 0, 1, 2, \ldots, T \), where \( L \) is an upper bound on the process time used in any of the \( N \) stages. Cumulative input and output curves are plotted only for periods 0 through \( T \).

Denote the inventory in stage \( n \) at the end of period \( t \) by \( I_{nt} \). Let \( R_{nt} \) and \( S_{nt} \) denote, respectively, the total receipts into and total shipments out of stage \( n \) during period \( t \). Conservation of flow implies:

\[
I_{n,t-1} + R_{nt} - S_{nt} = I_{nt},
\]

for \( n = 0, \ldots, N \), and \( t = 1, \ldots, T \). Initial inventories are model parameters:

\[
I_{n0} = i_n,
\]

for \( n = 0, \ldots, N \). The receipts of material into stage 0 are model parameters:

\[
R_{0t} = r_0,
\]

for \( t = -L, \ldots, T \). Receipts into subsequent stages prior to period 1 are determined by the initial inventory in that stage and the production lead time for that stage. Let \( p_n \) denote the production lead time for stage \( n \). Then,

\[
R_{nt} = \{ i_n \text{ if } t = p_n = 0; \ i_n/p_n \text{ if } -p_n < t \leq 0; \ 0 \text{ otherwise} \}.
\]
for \( t = -L, ..., 0, \) and \( n = 1, ..., N+1. \) For all remaining stages and time periods, the receipts in one stage are simply the shipments from the previous stage:

\[
R_{nt} = S_{n-1,t},
\]

for \( n = 1, ..., N, \) and \( t = 1, ..., T. \)

It remains to specify the output variables, \( S_{nt}. \) A variety of factors act to limit output. For example, because of production process times, not all the inventory in a stage is available for output at the same time. Let \( W_{nt} \) denote the inventory in stage \( n \) that is available for output during period \( t. \) Similarly, let \( C_{nt} \) denote maximum production rate possible in stage \( n \) during period \( t. \) Let \( B_{nt} \) denote the maximum amount of inventory that can be moved from stage \( n \) to stage \( n+1 \) in period \( t \) without violating maximum inventory limits in stage \( n+1. \) Let \( O_{nt} \) denote the maximum production authorized in stage \( n, \) period \( t, \) according to the order policy. Then,

\[
S_{nt} = \min \{ e_n W_{nt}, C_{nt}, B_{nt}, O_{nt} \},
\]

for \( n = 0, ..., N, \) and \( t = 1, ..., T, \) where \( e_n \) is the behavioral efficiency factor for stage \( n, \) a model parameter. \( S_{n0} = 0 \) for all \( n. \)

To determine inventory available for output, \( W_{nt}, \) let \( R_{n}^t \) denote cumulative receipts into stage \( n \) up to and including period \( t: \)

\[
R_{n}^t = R_{n,-L} + R_{n,-L+1} + ... + R_{n,t}.
\]

Define \( S_{n}^t \) similarly as the cumulative shipments from stage \( n \) up to and including period \( t. \) Then,

\[
W_{nt} = R_{n}^t - S_{n}^{t-1},
\]

for \( n = 0, ..., N, \) and \( t = 1, ..., T. \)

To determine production capacity, \( C_{nt}, \) let \( U_{nt} \) denote a random variable chosen according to the uniform distribution on the interval \([0,1]. \) Let \( u_n \) denote the average production rate and let \( v_n \) denote the maximum spread about this average. Let \( c_{nt} \) denote the number of shifts (or number of machines) available in stage \( n \) in period \( t. \) The variables \( u_n, v_n, \) and \( c_{nt} \) are model parameters. Then,
\[ C_{nt} = (u_n - 0.5U_{nt}v_{nt})c_{nt}, \]

for \( n = 0, ..., N, \) and \( t = 1, ..., T. \)

To determine maximum authorized production, \( O_{nt} \), let \( r_n \) and \( g_n \) denote the MIN and MAX levels, resp., for stage \( n \) production based on stage \( n+1 \) echelon inventory. Let \( E_{mt} \) denote stage \( n \) echelon inventory at the end of period \( t \):

\[ E_{mt} = I_{mt} + I_{mt+1} + \ldots + I_{Lt}, \]

for \( n = 0, ..., N. \) Also, let \( M_{mt} \) denote cumulative authorized production:

\[ M_{mt} = \{ S_{n-1} + g_n - E_{n+1,t-1}, \text{ if } E_{n+1,t-1} \leq r_n; M_{n,t-1}, \text{ otherwise} \} \]

for \( n = 0, ..., N, \) and \( t = 1, ..., T, \) where \( M_{m0} = 0 \) for all stages \( n. \) Then,

\[ O_{nt} = M_{mt} - S_{n-1}, \]

for \( n = 0, ..., N, \) and \( t = 1, ..., T. \)

The most complicated formula is that required to determine buffer constrained output, \( B_{mt}. \) Let \( b_n \) denote the maximum inventory allowed in stage \( n, \) for \( n = 0, ..., N. \) Since input and output are assumed to occur continuously within a period, the maximum inventory that can be moved from stage \( n \) to stage \( n+1 \) in period \( t \) is given by:

\[ B_{nt} = b_{n+1} - I_{n+1,t-1} + S_{n+1,t}. \]

The complication arises because \( S_{n+1,t} \) may be determined by \( e_{n+1} W_{n+1,t} \) and \( W_{n+1,t}, \) in turn, will depend on \( S_{nt} \) and, hence, \( B_{nt}, \) if \( p_{n+1} = 0. \) For that circular chain of relationships to hold it must also be true that \( S_{n+1,t} = b_{n+1}/(1-e_{n+1}). \) Consequently, let

\[ Q_{nt} = \{ b_n/(1-e_n), \text{ if } p_n = 0; W_{nt}, \text{ otherwise} \} \]

for \( n = 0, ..., N, \) and \( t = 1, ..., T. \) If \( e_n = 1 \) then \( Q_{nt} \) can be undefined: we use an arbitrarily large number for \( Q_{nt} \) in this case. Then,
\[ B_{nt} = b_{n+1} - I_{n+1,t-1} + \min \{ e_{n+1,t}, Q_{n+1,t}, C_{n+1,t}, B_{n+1,t}, O_{n+1,t} \} \]

for \( n = 0, \ldots, N \), and \( t = 1, \ldots, T \).

All stage \( N+1 \) variables are identically zero with the exception of \( b_{N+1} \), which is set to an arbitrarily large number.

The input curve for stage \( n \) is given by the plot of \( E_{n+1,0} + R_n \) over time \( t = 0, 1, \ldots, T \). The output curve for stage \( n \) is given by the plot of \( E_{n+1,0} + S_n \) over time \( t = 0, 1, \ldots, T \).

It is an interesting student exercise to use the spreadsheet model to reproduce the various plots used in this paper. For example, the WIP wave of Figure 12 is generated by the parameters listed in Table A1 and the following:

\[ r_{nt} = 999 \quad \text{for all } t; \text{ and} \]
\[ c_{nt} = 1 \quad \text{for all } n \text{ and all } t, \text{ except} \]
\[ c_{00} = 3. \]

<table>
<thead>
<tr>
<th>Stage</th>
<th>Initial Inventory</th>
<th>Behavioral Efficiency</th>
<th>Process Time</th>
<th>Prod. Rate Average</th>
<th>Prod. Rate Spread</th>
<th>Buffer Size</th>
<th>Reorder Point</th>
<th>Reorder Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>999</td>
<td>999</td>
<td>999</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0.8</td>
<td>1</td>
<td>30</td>
<td>0</td>
<td>999</td>
<td>999</td>
<td>999</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0.8</td>
<td>1</td>
<td>8.5</td>
<td>0</td>
<td>999</td>
<td>999</td>
<td>999</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0.8</td>
<td>1</td>
<td>7</td>
<td>0</td>
<td>999</td>
<td>999</td>
<td>999</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0.8</td>
<td>1</td>
<td>6</td>
<td>0</td>
<td>999</td>
<td>999</td>
<td>999</td>
</tr>
</tbody>
</table>

Table A1. Parameter Settings for Capacitated WIP Wave