POINT PROCESSES AND TAUBERIAN THEORY

by

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Abstract. A probabilistic approach to understanding Tauberian theorems is presented which is applicable when the Laplace transform of a monotone function is being considered. Regular variation at $\infty$ (0) of the function is readily shown to be equivalent to regular variation of the transform at 0 ($\infty$) because of the duality between regular variation and weak convergence of Poisson processes whose mean measures are related to the monotone function under consideration. The method is dimensionless and readily extends to $\Pi$-variation.

1. Introduction and Preliminaries.

We present a probabilistic method for understanding the relation between regular variation behavior of monotone functions and corresponding behavior of their Laplace transforms. The method is based on the equivalence of regular variation and the weak convergence of certain Poisson processes. The method has been surveyed in Resnick (1986, 1987) and the present paper is another application. The method is dimensionless and works well in higher dimensions and also is applicable to certain extensions of regular variation.

If $U : (0, \infty) \mapsto (0, \infty)$ is non-decreasing its Laplace transform is defined to be

$$\tilde{U}(\lambda) = \int_0^\infty e^{-\lambda x} U(dx)$$

assuming this integral exists finite for all $\lambda$ sufficiently large. This function $U$ is regularly varying at $\infty$ with index $\alpha \geq 0$ (written $U \in RV_\alpha$) if for all $x > 0$

$$\lim_{t \to \infty} \frac{U(tx)}{U(t)} = x^\alpha.$$ 

(1.1)

Roughly, this says that $U$ behaves like a power for large $x$. Regular variation at 0 is defined similarly and in fact the function $U(x)$ is regularly varying at 0 iff $U(x^{-1})$ is regularly varying at $\infty$. The well known Karamata Tauberian theorems relate the regular variation behavior of $U$ at $\infty$ to that of $\tilde{U}$ at 0. Excellent references for this analytic theory are Geluk and de Haan (1987), Bingham, Goldie and Teugels (1987) and of course Feller (1971).

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We now discuss why point process theory is relevant. Here is the background. Suppose $E$ is a locally compact Hausdorff space. For us it is enough to think of $E$ as some subset of $R^d$ for $d \geq 1$, perhaps compactified in some way. Let the Borel $\sigma$-algebra be denoted by $\mathcal{E}$. For $x \in E$ and $A \in \mathcal{E}$ define

$$
\epsilon_x(A) = \begin{cases} 
1, & \text{if } x \in A \\
0, & \text{if } x \notin A.
\end{cases}
$$

A point measure on $E$ is an atomic measure of the form

$$
\sum_i \epsilon_{x_i},
$$

assuming that the Radon property holds: For a compact set $K$ we have

$$
\sum_i \epsilon_{x_i}(K) < \infty.
$$

So we consider only point measures which have a finite number of points in bounded regions. The collection of all Radon point measures is the set $M_p(E)$ (just $M_p$ when the state space $E$ is unambiguous) and this collection is given a $\sigma$-algebra $\mathcal{M}_p(E)$ (or just $\mathcal{M}_p$) which is the smallest $\sigma$-algebra containing sets of the form

$$
\{m \in M_p : m(G) = n\}
$$

for some relatively compact open set $G$ and some integer $n$ where $0 \leq n \leq \infty$. The set $M_p$ can also be made into a complete separable metric space and the $\sigma$-algebra given is then the Borel $\sigma$-algebra. The notion of convergence corresponding to the metric is vague convergence: Let $C^+_K(E)$ be the non-negative continuous functions on $E$ vanishing off a compact set. Then vague convergence of a sequence $m_n \in M_p$ to $m \in M_p$ means for all $f \in C^+_K(E)$

$$
m_n(f) \to m(f)
$$

where

$$
m(f) = \int_E f(x)m(dx).
$$

For further information on these and related concepts the following are recommended: Kallenberg (1983), Neveu (1976), Resnick (1987).

A point process is a measurable mapping $N$ from a probability space $(\Omega, \mathcal{F}, P)$ to $(M_p, \mathcal{M}_p)$. In Billingsley’s (1968) terminology, $N$ is a random element of $M_p$. The distribution of this random element is $P \circ N^{-1}$ which is a measure on $M_p$. The distribution of $N$ is determined by the Laplace functional which is a mapping $\Psi$ from bounded, non-negative functions on $E$ to $[0, \infty)$ defined by ($f \geq 0$ and bounded with domain $E$)

$$
\Psi(f) = E \exp\{-N(f)\} = E \exp\{-\int_E f(x)N(\omega, dx)\}
$$

$$
= \int_{M_p} \exp\{-\int_E f(x)m(dx)\}P \circ N^{-1}(dm).
$$
Weak convergence of random elements in $M_p(E)$ is characterized by convergence of Laplace functionals at all functions in $C^+_k(E)$.

A central role is played by the Poisson process. If $\mu$ is a Radon measure on $(E, \mathcal{E})$ we will say the random element $N$ of $M_p$ is Poisson Random Measure on $E$ with mean measure $\mu$ (PRM($\mu$) for short) if for any $A \in \mathcal{E}$

$$P[N(A) = k] = \begin{cases} \frac{e^{-\mu(A)}(\mu(A))^k}{k!}, & \text{if } \mu(A) < \infty \\ 0, & \text{if } \mu(A) = \infty \end{cases},$$

and if $A_1, \ldots, A_j$ are disjoint in $\mathcal{E}$ then $N(A_1), \ldots, N(A_j)$ are independent random variables. A Poisson process can be recognized by the characteristic form of its Laplace functional; $N$ is PRM($\mu$) iff

$$\Psi_N(f) = \exp\{-\int_E \left(1 - e^{-f(x)}\right) \mu(dx)\}.$$

From this and the fact that weak convergence is characterized by convergence of Laplace functionals we get the following simple but useful result. Let "$\Rightarrow$" denote weak convergence.

**Proposition 1.1.** Suppose for $n \geq 0$ that $N_n$ is PRM($\mu_n$). Then

$$N_n \Rightarrow N_0$$

in $M_p(E)$ iff

$$\mu_n \Rightarrow \mu_0$$

where $\Rightarrow$ is vague convergence of measures.

There are some useful transformation results which will help us in later work. These are quoted from Resnick (1987):

**Proposition 1.2.** Let $(E_i, \mathcal{E}_i), i = 1, 2$ be two state spaces (locally compact, Hausdorf with countable base).

(a) Suppose $T$ is a measurable map

$$T : E_1 \mapsto E_2$$

with the property that $T^{-1}(B_2)$ is bounded in $E_1$ whenever $B_2$ is bounded in $E_2$. If $N_1$ is PRM($\mu_1$) on $E_1$ then $N_1 \circ T^{-1}$ is PRM($\mu \circ T^{-1}$) on $E_2$.

(b) Suppose

$$N = \sum_n \xi_n$$

is PRM($\mu$) on $E_1$ and suppose $\{J_n\}$ are iid random elements of $E_2$ with common distribution $F$ which are independent of $N$. Then

$$\sum_n \xi_n J_n$$

is PRM($\mu \times F$ on $E_1 \times E_2$).

The last preliminary we need to dispose of concerns when a continuous map on the state space $E$ induces a continuous map in $M_p(E)$. The following is also from Resnick (1987).
PROPOSITION 1.3. Let \((E_i, E_i), i = 1, 2\) be two state spaces (locally compact, Hausdorff with countable base). Suppose \(T : E_1 \leftrightarrow E_2\) is continuous and satisfies \(T^{-1}(K_2)\) is compact in \(E_1\) whenever \(K_2\) is bounded in \(E_2\). Then \(\hat{T} : M_p(E_1) \rightarrow M_p(E_2)\) defined by

\[
\hat{T}m = m \circ T^{-1}
\]

is continuous. Note that if \(m = \sum_n \epsilon_{a_n}\) then \(\hat{T}m = \sum_n \epsilon_{T_a}\).

The compactness condition in Proposition 1.3 needs to be checked carefully. Lack of attention to this point can lead to errors. (The author speaks from experience.)

2. Warmup: Karamata's Theorem.

Let us illustrate the techniques involved by letting the point process method flex its muscles on a well known result which is a special case of Karamata's Theorem which says that the integral of a regularly varying function of index \(\alpha > 0\) is regularly varying with index \(\alpha + 1\).

For what follows, if \(U\) is non-decreasing on \([0, \infty)\) we denote the measure on \(([0, \infty), B([0, \infty)))\) also by \(U\). We denote Lebesgue measure by \(dt\).

The definition of regular variation given in (1.1) has a well known and readily derived sequential formulation (de Haan, 1970; Feller, 1971; Resnick, 1987): \(U \in RV_\alpha\) iff there exist constants \(a_n \rightarrow \infty\) with \(a_n \sim a_{n+1}\) and

\[
\lim_{n \to \infty} U(a_n x)/x = x^\alpha.
\]

We first formulate a probabilistic equivalent to the statement that \(U\) is regularly varying.

PROPOSITION 2.1. Suppose \(U\) is non-decreasing on \([0, \infty)\) and that

\[
\sum_k \epsilon_{(t_k, u_k)}
\]

is \(PRM(dt \times U)\) on \(M_p([0, \infty) \times [0, \infty))\). Then \(U \in RV_\alpha, \alpha > 0\) iff there exist constants \(a_n \rightarrow \infty\) with \(a_n \sim a_{n+1}\) and

\[
\sum_k \epsilon_{(t_k, u_k)} \Rightarrow PRM(dt \times dx^\alpha)
\]

in \(M_p([0, \infty) \times [0, \infty))\). We may take \(a_n = U^{-1}(n)\).

PROOF: We observe that

\[
\sum_k \epsilon_{(t_k, u_k)} \Rightarrow PRM(dt \times dx^\alpha)
\]

is equivalent to convergence of Laplace functionals; i.e. equivalent to

\[
\exp\{-\int_{[0, \infty) \times [0, \infty)} \left(1 - e^{-f(t, u)}\right) dsU(du)\}
\]

\[
= \exp\{-\int_{[0, \infty) \times [0, \infty)} \left(1 - e^{-f(t, u)}\right) dt \, n^{-1} U(a_n \, du)\}
\]

\[
\rightarrow \exp\{-\int_{[0, \infty) \times [0, \infty)} \left(1 - e^{-f(t, u)}\right) dt \, dv^\alpha\}
\]

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for all \( f \in C_X^1([0, \infty) \times [0, \infty)) \) which is equivalent to

\[
U(a, v)/n \rightarrow v^\alpha, \quad \forall v > 0.
\]

We may now give a point process explanation of why \( U \in RV_\alpha, \alpha > 0 \) implies that the integral of \( U \) is also regularly varying. This is a special case of Karamata's Theorem (Feller, 1971; de Haan 1970; Geluk and de Haan, 1987; Bingham, Goldie, Teugels, 1987; Resnick, 1987) and is presented for didactic purposes.

**Proposition 2.2.** Suppose \( U \) is non-decreasing on \([0, \infty)\) and integrable in a neighborhood of 0. If \( U \in RV_\alpha, \alpha > 0 \) then \( \int_0^1 U(s) ds \in RV_{\alpha+1} \) and

\[
\lim_{s \to 0} \frac{\int_0^s U(s) ds}{sU(x)} = \frac{1}{\alpha + 1}.
\]

**Proof:** From Proposition 2.1 we know that \( U \in RV_\alpha \) implies

\[
N_n := \sum_k \epsilon_{(n_k, \infty)} \Rightarrow N_0 := \text{PRM}(dt \times dx^\alpha)
\]

in \( M_p ((0, \infty) \times [0, \infty)) \). Consider the map

\[
T : (0, \infty) \times [0, \infty) \mapsto (0, \infty) \times [0, \infty)
\]

defined by

\[
T(t, x) = (t, x/t).
\]

\( T \) satisfies the compactness condition of Proposition 1.3 since for \( 0 < a < b < \infty, y > 0 \) we have

\[
T^{-1}[a, b] \times [0, y] = \{(t, x) : a \leq x \leq b, s/t \leq y\}
\]

\[
= \{(t, x) : a \leq t \leq b, x \leq ty\}
\]

which is compact. Then we can conclude from (2.2) that the Poisson processes

\[
N_n \circ T^{-1} \rightarrow N_0 \circ T^{-1}
\]

in \( M_p ((0, \infty) \times [0, \infty)) \) and therefore from Proposition 1.1 the mean measures converge: For \( \eta > 0 \)

\[
EN_n \circ T^{-1}[\eta, 1] \times [0, 1] \rightarrow EN_0 \circ T^{-1}[\eta, 1] \times [0, 1]
\]

which translates to

\[
\begin{align*}
E\{\# \& : n^{-1} \eta \leq t_k \leq n^{-1}, u_k / t_k \leq n a_n\} &= \int_{n^{-1} \eta}^{n^{-1}} ds U(sn a_n) \\
&= a_n^{-1} n^{-1} \int_{a_n \eta}^{a_n} U(y) dy \\
&\rightarrow \int_{\{(t, y) : y/t \leq 1\}} dt dy^\alpha = \frac{\alpha^\alpha}{\alpha + 1}
\end{align*}
\]
and recalling that \( a_n = U^-(n) \) and changing variables \( x = a_n \) yields

\[
\lim_{\varepsilon \to \infty} \frac{\int_{\varepsilon}^{\infty} U(y) \, dy}{xU(x)} \to \frac{1 - \eta}{\alpha + 1}.
\]

Therefore

\[
\liminf_{\varepsilon \to \infty} \int_{\varepsilon}^{\infty} U(y) \, dy / (xU(x)) \geq \liminf_{\varepsilon \to \infty} \int_{\varepsilon}^{\infty} U(y) \, dy / (xU(x)) = \frac{1 - \eta}{\alpha + 1}
\]

and since \( \eta \) is arbitrary this lim inf is bounded below by \( 1 / (\alpha + 1) \). On the other hand

\[
\int_{0}^{\infty} U(y) \, dy / (xU(x)) = \int_{0}^{\eta x} + \int_{\eta x}^{\infty} U(y) \, dy / (xU(x)) \leq \eta x U(\eta x) / (xU(x)) + \int_{\eta x}^{\infty} U(y) \, dy / (xU(x)) \to \eta^{\alpha + 1} + (1 - \eta) / (\alpha + 1)
\]

and after letting \( \eta \to 0 \) we get the desired result. 

3. Tauberian Theorems for Regularly Varying Functions.

We now consider how the point process method illuminates the following classical result (Feller, 1971; Bingham, Goldie, Teugels, 1987; Geluk and de Haan, 1987):

**Proposition 3.1.** Suppose \( U \) is non-decreasing on \( [0, \infty) \) with Laplace transform \( \hat{U}(\lambda) \), for \( \lambda > 0 \). Then for \( \alpha > 0 \),

\[ U \in RV_{\alpha} \]

iff

\[ \hat{U}(\frac{1}{\lambda}) \in RV_{\alpha} \]

in which case as \( x \to \infty \)

\[ \hat{U}(\frac{1}{x}) \sim U(x) \Gamma(\alpha + 1). \]

**Proof:** Suppose \( U \in RV_{\alpha} \) and let

\[
\sum_k \epsilon_{(t_k, u_k)}
\]

be PRM(\( dt \times U \)) on \( M_p \((0, \infty) \times [0, \infty)\)). Then from Proposition 2.1 we have with \( a_n = U^-(n) \) that

\[
\sum_k \epsilon_{(t_k, u_k / a_n)} \Rightarrow PRM(\, dt \times dx^{\alpha})
\]

in \( M_p \((0, \infty) \times [0, \infty)\)). Suppose \( \{E_n\} \) are iid unit exponential random variables independent of the Poisson process. From Proposition 1.2 we have

\[
\sum_k \epsilon_{(t_k, u_k / a_n, E_k)} \Rightarrow PRM(\, dt \times dx^{\alpha} \times e^{-\gamma d})
\]

(3.1)
in $M_p([0, \infty) \times [0, \infty) \times (0, \infty])$. The mapping

$$T : [0, \infty) \times [0, \infty) \times (0, \infty) \mapsto [0, \infty) \times [0, \infty)$$

defined by

$$T(t, u, y) = (t, u/y)$$

is continuous and we show that applying this mapping to (3.1) yields

$$N_n := \sum_k \varepsilon_{\left(\frac{nt_k}{a_n}, \frac{u_k}{a_n E_k}\right)} \Rightarrow \mathcal{P}(\alpha + 1) dx$$

in $M_p([0, \infty) \times [0, \infty))$. In proving (3.2) the only real issue is the compactness criterion of Proposition 1.3 because it is clear how to compute the limiting mean measure in (3.2) since for instance at $[0, 1] \times [0, z]$ this limiting mean measure is

$$\int_{\{(u, y), u/y \leq z\}} dx e^{-y} dy = \int_0^\infty (zy)^a e^{-y} dy$$

$$= z^a \int_0^\infty (y)^a e^{-y} dy$$

$$= z^a \Gamma(\alpha + 1).$$

To verify (3.2) apply $T$ to the convergence in (3.1) restricted to the compact set $[0, M] \times [0, M] \times [M^{-1}, \infty)$. If the domain of the point processes is compact the compactness condition of Proposition 1.3 evaporates and so we have

$$N_{n,M} := \sum_k 1_{\left[n t_k \leq M, a_k \leq M, E_k \geq M^{-1}\right] \varepsilon_{\left(\frac{nt_k}{a_n}, \frac{u_k}{a_n E_k}\right)}}$$

$$\Rightarrow \mathcal{P}(\alpha + 1) dx e^{-y} dy 1_{[0, M] \times [0, M] \times [M^{-1}, \infty)}(t, x, y) \circ T^{-1}.$$

As $M \to \infty$ the right most process of (3.3) converges to the correct Poisson limit and so by Billingsley (1968; Theorem 4.2) we will have proven (3.2) if we prove

$$\lim_{M \to \infty} \lim_{n \to \infty} \sup P[\rho(N_{n,M}, N_n) > \eta] = 0$$

where $\eta > 0$ and $\rho$ is the vague metric on $M_p$. Because of the form of the metric (cf. Kallenberg, 1983; Resnick, 1987) it suffices to show

$$\lim_{M \to \infty} \lim_{n \to \infty} \sup P[|N_{n,M}(f) - N_n(f)| \geq \eta] = 0$$

for $f \in C_K^+$. Assume for concreteness that $sup_{[0, \infty) \times [0, \infty)} f(t, x) = 1$ and that the support of $f$ is contained in $[0, 1] \times [0, 1]$. The probability in (3.4) times $\eta$ is

$$\eta P\left[\lim \sum_k \left|f(nt_k, \frac{u_k}{a_n E_k}) - f(nt_k, \frac{u_k}{a_n E_k})1_{\left[n t_k \leq M, a_k \leq M, E_k \geq M^{-1}\right]}\right| > \eta\right]$$

$$\leq \sum_k \left|f(nt_k, \frac{u_k}{a_n E_k}) - f(nt_k, \frac{u_k}{a_n E_k})1_{\left[n t_k \leq M, a_k \leq M, E_k \geq M^{-1}\right]}\right|$$

$$\leq \sum_k f(nt_k, \frac{u_k}{a_n E_k}) \left(1_{\left[n t_k > M\right]} + 1_{\left[u_k > a_n M\right]} + 1_{\left[E_k < M^{-1}\right]}\right)$$

$$= I + II + III.$$
Now if $M > 1$ then $I = 0$ because of the hypothesized support of $f$. For evaluating $III$ we note that what is crucial in the expectation is how many terms are being summed. We have $III$ bounded by

$$n^{-1} U(a_n M^{-1})(1 - e^{-M^{-1}}) \leq M^{-1} n^{-1} U(a_n M^{-1})$$

and as $n \to \infty$ this converges to

$$M^{-1} M^{-\alpha} \to 0$$

as $M \to \infty$. It remains to deal with $II$. We have $II$ bounded by

$$E \sum_k 1_{\{n_k \leq 1, u_k/a_n > M, u_k/a_n \leq B_k\}}$$

$$= n^{-1} \int \int_{\{M < x/a_n < y\}} dU(x) e^{-y} dy$$

$$\leq \int_M^{\infty} e^{-y} \frac{U(a_n y)}{n} dy.$$ 

For given $\delta > 0$ we have by Potter's inequalities (Bingham, Goldie, Teugels, 1987; Geluk and de Haan, 1987; Resnick, 1987) that for all large $n$ and $y > M$

$$U(a_n y)/n \leq (1 + \delta)y^{\alpha+\delta}$$

and putting this in the previous inequality shows that $\lim_n \limsup_m = 0$ as required. Thus (3.2) is verified.

Now (3.2) gives convergence of Poisson processes and by Proposition 1.1 the corresponding mean measures converge. Evaluating the mean measures on $[0, 1] \times [0, x]$ yields

$$E \sum_k \epsilon_{(n_k, x \in E_k)}([0, 1] \times [0, x])$$

$$= \int \int_{\{t \leq n^{-1}, x/t \leq y\}} dt dU(s) e^{-y} dy$$

$$= n^{-1} \int_0^{\infty} e^{-t/x} dU(s)$$

$$= \frac{\dot{U}(\frac{1}{x})}{n} \to \Gamma(\alpha + 1)x^\alpha$$

which is the equivalent sequential version of the assertion of the Proposition.

Now for the converse, suppose $\dot{U}(1/x) \in RV_{\alpha}$. Since the mean measure of

$$\sum_k \epsilon_{(n_k, u_k \in E_k)}$$

on $[0, t] \times [0, x]$ is $t\dot{U}(1/x)$ we get from Proposition 2.1 that there exist constants $a_n \to \infty, a_n \sim a_{n+1}$ such that

$$\sum_k \epsilon_{(t_k, \frac{x}{a_{n_k}})} \Rightarrow PRM(dt \times dx^\alpha).$$
This implies that
\[
\left\{ \sum_k \epsilon_{(t_k, \frac{x_k}{E_k}, E_k), n \geq 1} \right\}
\]
is a tight sequence in \( M_p ((0, \infty) \times [0, \infty] \times [0, \infty)) \) since if \( f(t, x, y) \in C_K^+ \) with
\[
g(t, x) = \sup_y f(t, x, y)
\]
we have
\[
\sum_{k=1}^\infty \epsilon_{(t_k, \frac{x_k}{E_k}, E_k)}(f) \leq \sum_{k=1}^\infty \epsilon_{(t_k, \frac{x_k}{E_k})}(g)
\]
and the rightmost sequence, being convergent, is tight in \( R \); hence so is the left hand sequence and this implies tightness in the \( M_p \) space (Kallenberg, 1983; Resnick, 1987). Tightness implies every subsequence has a further subsequence where convergence takes place. For a Poisson sequence, the only possible limits are Poisson processes. If along a subsequence we have convergence of
\[
\sum_k \epsilon_{(t_k, \frac{x_k}{E_k}, E_k)}
\]
then by Proposition 1.3 we get that
\[
(3.6) \quad \sum_k \epsilon_{(t_k, \frac{x_k}{E_k})}
\]
is also convergent to a Poisson limit with mean measure say \( dt \times V \). (The mapping of Proposition 1.3 is \( (t, x, y) \mapsto (t, xy) \). The compactness criterion must be dealt with in a manner similar to what was done in the first half of the proof of this proposition.) From the proof of the first half we have along this subsequence
\[
\sum_k \epsilon_{(t_k, \frac{x_k}{E_k})} \Rightarrow PRM(dt \times V \times e^{-y} dy)
\]
and
\[
\sum_k \epsilon_{(t_k, \frac{x_k}{E_k})} \Rightarrow PRM(dtd\hat{V}(1/x))
\]
whence from \((3.5)\) we conclude
\[
\hat{V}(1/x) = x^\alpha.
\]
So in \((3.6)\) every convergent subsequence has the same Poisson limit and hence the full sequence converges. By Proposition 2.1 we conclude \( U \in RV_\alpha \) as desired. \( \blacksquare \)
4. Multivariate Regular Variation and Tauberian Theorems.

We now consider the multivariate version of Proposition 3.1. Suppose \( U \) is a Radon measure on \([0, \infty)^d, d > 1\) whose Laplace transform \( \hat{U} \) exists:

\[
\hat{U}(\lambda) = U(\lambda_1, \ldots, \lambda_d) = \int_{[0, \infty)^d} e^{-\sum_{i=1}^d \lambda_i x_i} U(dx), \quad \lambda > 0.
\]

Relations and operations between vectors are to be interpreted componentwise; for instance if

\[
x = (x_1, \ldots, x_d)
\]

we interpret

\[
1/x = (1/x_1, \ldots, 1/x_d).
\]

For vectors \( a < b \) we denote the rectangle

\[
[a, b] = \{x : a_i < x_i \leq b_i, i = 1, \ldots, d\}.
\]

If \( U(x) \) is the distribution corresponding to the measure \( U \) we say \( U \) is regularly varying with limit measure \( V \) and index \( \alpha > 0 \) if there exists \( g \in RV_\alpha \) such that for \( x \geq 0 \), we have

\[
\lim_{t \to \infty} \frac{U(tx)}{g(t)} = V(x).
\]

We assume \( V(1) = 1 \) as a convenient normalization. Then for \( t > 0 \)

\[
V(tx) = t^\alpha V(x).
\]

This scaling property transforms under a polar coordinate transformation as follows: Let \( x \mapsto \|x\| \) be a convenient norm and let

\[
\mathbb{R} = \{x \in [0, \infty)^d : \|x\| = 1\}
\]

be the unit sphere. There exists a finite measure \( S \) on \( \mathbb{R} \) such that

\[
V\{x : \|x\| \leq r, \frac{x}{\|x\|} \in A\} = r^\alpha S(A)
\]

where \( A \) is a Borel subset of \( \mathbb{R} \). The equivalent sequential version of (4.1) is that there exists a regularly varying sequence \( \{a_n\} \), the retraction to the integers of a function \( a(\cdot) \in RV_{1/\alpha} \) such that for \( x \geq 0 \),

\[
\frac{U(a_n x)}{n} \to V(x)
\]

which is equivalent to vague convergence of measures

\[
\frac{U(a_n \cdot)}{n} \rightharpoonup V.
\]
Suppose
\[ \sum_k \varepsilon_{(t_k, u_k)} \]
is PRM\((dt \times U)\) with state space \([0, \infty) \times [0, \infty)^d\). Exactly as in Proposition 2.1 we have that (4.3) is equivalent to
\[ (4.4) \quad \sum_k \varepsilon_{(n t_k, u_k/a_n)} \Rightarrow PRM(dt \times V) \]
in \(M_e ([0, \infty) \times [0, \infty)^d)\).

We may now follow the steps of Proposition 3.1. Suppose \(\{E_n = (E_n^{(1)}, \ldots, E_n^{(d)}), n \geq 1\}\) are iid \(d\)-dimensional random vectors each of whose components are iid unit exponential random variables. If (4.3) holds then appending the \(\{E_n\}\) in (4.3) yields
\[ (4.5) \quad N_n := \sum_k \varepsilon_{(n t_k, u_k/a_n, E_k)} \Rightarrow PRM(dt \times V \times \prod_{i=1}^d e^{-y_i} dy_i) \]
Define the mapping \(T : [0, \infty) \times [0, \infty)^d \times (0, \infty)^d \mapsto [0, \infty) \times [0, \infty)^d\) by
\[ T(t, x_1, \ldots, x_d, y_1, \ldots, y_d) = (t, x_1/y_1, \ldots, x_d/y_d). \]
We need to show that (4.5) implies
\[ (4.6) \quad N_n \circ T^{-1} := \sum_k \varepsilon_{(n t_k, \frac{x_k}{a_n}, E_k^{(i)})} \Rightarrow PRM(dt \times V \times \prod_{i=1}^d e^{-y_i} dy_i) \circ T^{-1}. \]
Assuming (4.6) is true, the mean measures of the converging Poisson processes converge also and we conclude as in Proposition 3.1 that for \(z \in [0, \infty)^d\)
\[ EN_n \circ T^{-1} ([0, 1] \times [0, z]) = \int \int \{ (s, z, y) : z_i/y_i \leq z_i, i = 1, \ldots, d \} ds \frac{dU(a_n x)}{n} \prod_{i=1}^d e^{-y_i} dy_i \]
\[ = \int_{[0, \infty)^d} \prod_{i=1}^d e^{-x_i/y_i} U dx \frac{dU(a_n x)}{n} \]
\[ = \int \prod_{i=1}^d \exp\left\{ -\frac{x_i}{a_n z_i}\right\} U dx \frac{1}{a_n z} \]
\[ = \mathcal{U}(1/z) \]
\[ \rightarrow \int \int \{ (s, z, y) : z_i/y_i \leq z_i, i = 1, \ldots, d \} ds dV(z) \prod_{i=1}^d e^{-y_i} dy_i \]
\[ = \int \prod_{i=1}^d \exp\left\{ -\frac{z_i}{z_i}\right\} V(dx) \]
\[ = \mathcal{V}(1/z). \]
Thus we conclude that if $U$ is regularly varying,

\[
\frac{U(a_n \cdot)}{n} \overset{\ast}{\to} V,
\]

then $\hat{U}(1/x)$ is also regularly varying:

\[
\hat{U}(\frac{1}{a_n x})/n \overset{\ast}{\to} \hat{V}(1/x)
\]

provided we verify (4.6). This verification parallels that of (3.2). For $M > 0$ define

\[
N_{n,M} := \sum_k \sum_{[n,t_k \leq M, u_k(i) \leq M, E_k(i) \geq M^{-1}, i = 1, \ldots, d]} \delta_a \left( \frac{u_k(i)}{a_n E_k(i)} \right), i = 1, \ldots, d)
\]

and we must show

\[
\lim_{M \to \infty} \lim_{n \to \infty} \sup P[\rho(N_{n,M}, N_n) > \eta] = 0
\]

for any $\eta > 0$. Suppose $f$ is continuous with compact support and suppose without loss of generality for convenience that $\sup f = 1$ and that the support of $f$ is contained in $[0, 1] \times [0, 1]^d$. Mimicking the details following (3.4) we need to check

\[
E \sum_k f(n t_k, \frac{u_k(i)}{a_n E_k(i)}, i = 1, \ldots, d) \left( \sum_{j=1}^{d} \sum_{j=1}^{d} \sum_{j=1}^{d} 1_{[u_k(i)/a_n E_k(i) \geq M]} \right)
\]

has a lim, lim sup, = 0. Now $I$ is zero if $M > 1$ and $III_1$, for example, is bounded by

\[
n^{-1} U(a_n, M^{-1}, a_n, \ldots, a_n)(1 - e^{-M^{-1}})
\]

\[
= \nu(M^{-1}, 1, \ldots, 1) M^{-1}
\]

\[
\leq V(1) M^{-1} = M^{-1}
\]

\[
\to 0
\]

as $n$ and then $M$ go to $\infty$. For the terms in $II$ we have for example $II_1$ bounded by

\[
E \sum_k \sum_{[n,t_k \leq M, u_k(i) \leq E_k(i), E_k(i) \leq E_k(i), i = 1, \ldots, d]} \delta_a \left( \frac{u_k(i)}{a_n E_k(i)} \right)
\]

\[
= n^{-1} \int_{M \leq \sum_{i=1}^{d} y_i} \prod_{i=1}^{d} e^{-y_i} dy U(a_n dx)
\]

\[
= n^{-1} \int_{M \leq y_1} \prod_{i=1}^{d} e^{-y_i} dy (U(a_n dy) - U(a_n, M, a_n y_2, \ldots, a_n y_d))
\]

\[
= n^{-1} \int_{M \leq \sum_{i=1}^{d} y_i} \prod_{i=1}^{d} e^{-y_i} dy U(a_n dy)
\]

\[
= n^{-1} \int_{M \leq \sum_{i=1}^{d} y_i} e^{-\sum_{i=1}^{d} y_i} dy \ldots dy U(a_n \left( \sum_{i=1}^{d} y_i \right) 1)
\]

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and since $U(t1)$ is a univariate regularly varying function of index $\alpha$ we get again by Potter's inequalities that for some $\delta > 0$ and large $n$ the foregoing is bounded above by

$$\int \sum_{i=1}^{d} y_i \exp^{-\sum_{i=1}^{d} y_i} dy_1 \ldots dy_d$$

which converges to 0 as $M \to \infty$.

The converse follows the steps of the converse in Proposition 3.1. We have proved the following theorem (cf. Stam, 1977; de Haan, Omey and Resnick, 1984).

**Proposition 4.1.** Suppose $U$ is a measure on $[0, \infty)^d$. Then $U$ is regularly varying with positive index; i.e.

$$\frac{U(a_n \cdot)}{n} \Rightarrow V,$$

iff the distribution function $\hat{U}(1/x)$ is regularly varying:

$$\hat{U}(\frac{1}{a_n x})/n \Rightarrow \hat{V}(1/x).$$

**Remark:** Note that $\hat{V}(1/x)$ is the distribution function corresponding to a measure since if

$$\sum_h e_{(t_h, v_h)}$$

is $PRM(dt \times V)$ then

$$\hat{V}(1/x) = E \sum_h e_{(t_h, v_h)}/g(t), [0,1] \times [0, x].$$

A similar remark applies to $\hat{U}(\frac{1}{a_n x})$.

5. II-Variation, Point Processes and Tauberian Theory.

A second kind of variation which has proved useful in probabilistic limit theory is II-variation. A non-decreasing function $U : (0, \infty) \to (0, \infty)$ is called II-varying (written $U \in II$ if for some slowly varying function $g$ and for all $x > 0$ we have

$$\lim_{t \to \infty} \frac{U(tx) - U(t)}{g(t)} = \log x.$$  

The function $g$ is called the auxiliary function. An equivalent sequential form of (5.1) for measures is

$$\frac{\sum_{n}^{} \frac{a_n}{n} U(a_n \cdot)}{a(n)} \Rightarrow L$$

on $(0, \infty)$ where $L$ is the measure on $(0, \infty)$ with density $1/x$ so that for $0 < a < b$ we have $L(a, b) = \log b - \log a$ and where the function $a(\cdot)$ is $RV$. The relation of $a(\cdot)$ to $g$ is that $a(\cdot)$ is an asymptotic inverse of the function $t/g(t)$. (Cf. Resnick, 1986.)

Now we give some relationships between II-variation and point processes.
Proposition 5.1. Suppose $U$ is non-decreasing on $[0, \infty)$. Let

$$
\sum_k \epsilon(t_k, u_k)
$$

be $PRM(dt \times U)$. The following are equivalent:

(a) $U \in \Pi$.

(b) For some function $a(\cdot) \in RV_1$ we have

$$
N_n := \frac{1}{a(n)} \sum_k \epsilon(n t_k, \frac{u_k}{a(n)}) \Rightarrow dt \times L
$$

in the space of Radon measures on $[0, \infty) \times (0, \infty)$.

(c) For some function $a(\cdot) \in RV_1$ we have

$$
\sum_k \epsilon(\frac{a(n) t_k}{a(n)}, \frac{u_k}{a(n)}) \Rightarrow PRM(dt \times L)
$$

in $M_p([0, \infty) \times (0, \infty))$.

(d) For some function $a(\cdot) \in RV_1$ the $1/a(n)$-thinning of

$$
\sum_k \epsilon(\frac{u_k}{a(n)}, \frac{a(n)}{t_k})
$$

converges weakly in $M_p([0, \infty) \times (0, \infty))$ to $PRM(dt \times L)$.

Remark: If $0 < p < 1$, the $p$-thinning of $PRM(\mu)$ for some measure $\mu$ yields $PRM(p \mu)$. By the phrase "$p$-thinning" we mean that each point of the point process is independently inspected and discarded with probability $1 - p$ and retained with probability $p$.

Proof: For (b) it is necessary to prove

$$
\lim_{n \to \infty} E \exp\{-\sum_k a(n)^{-1} f(nt_k, u_k/a(n))\} = \exp\{-\iint f(t, y)dtL(dy)\}. \tag{5.3}
$$

for $f$ continuous with compact support. Because the left side of (5.3) is the Laplace functional of a Poisson process we have the left side equal to

$$
\exp\{-\iint \left(1 - e^{-a(n)^{-1} f(nt_k, u_k/a(n))}\right) dsU(dz)\}
$$

$$
= \exp\{-\iint a(n) \left(1 - e^{-a(n)^{-1} f(t, y)}\right) dt \frac{n}{a(n)} U(a(n)dy)\}
$$

and it is easy to see (cf. Resnick, 1986, Corollary 5.4) that (5.2) holds iff

$$
\frac{n}{a(n)} U(a(n)dy) \rightarrow L(dy)
$$

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which is equivalent to (a). For (c) note that
\[ E \sum_{k} \epsilon_{(\frac{x}{a(n)}t_{k}, \frac{x}{a(n)}T_{k})}([0,t] \times [0,z]) \]
\[ = \frac{n}{a(n)} U(a(n)z) \]
and since a sequence of Poisson processes converges iff their mean measures converge we get (c) equivalent to (5.2) and hence to (a).

The proof of (d) is similar. \[ \Box \]

We have the following result of de Haan (de Haan, 1970; Geluk and de Haan, 1987) which is parallel to Proposition 2.2 in method.

**Proposition 5.2.** Suppose the non-decreasing function U is II-varying with auxiliary function \( g \). Then

\[ V(z) := \frac{1}{z} \int_{0}^{z} U(s)ds \]

is also in II with auxiliary function \( g \).

**Proof:** From Proposition 5.1 (c) we have
\[ \sum_{k} \epsilon_{(\frac{x}{a(n)}t_{k}, \frac{x}{a(n)}T_{k})} \Rightarrow \sum_{k} \epsilon_{(t_{k}, z_{k})} := PRM(dt \times L) \]
in \( \mathcal{M}_{\nu} ([0,\infty) \times (0,\infty)) \) whence we get from Proposition 1.3
\[ \sum_{k} \epsilon_{(\frac{x}{a(n)}t_{k}, \frac{x}{a(n)}T_{k})} \Rightarrow \sum_{k} \epsilon_{(t_{k}, z_{k}/t_{k})} \]
in \( \mathcal{M}_{\nu} ((0,\infty) \times (0,\infty)) \). So the mean measures converge and for \( c < 1 < z \)
\[ E \sum_{k} \epsilon_{(\frac{x}{a(n)}t_{k}, \frac{x}{a(n)}T_{k})}[c,1] \times [1,z] \]
\[ = E \sum_{k} \epsilon_{(t_{k}, z_{k})} \left[ \frac{n}{a(n)c}, \frac{n}{a(n)} \right] \times \left[ \frac{a(n)^2}{n}, \frac{a(n)^2}{n} \cdot \frac{a(n)^2}{n} \right] \]
\[ = \int \int_{\{(s,z) : \frac{a(n)}{c} \leq s \leq \frac{a(n)}{z}, \frac{a(n)}{c} \leq z \leq \frac{a(n)^2}{s} \}} dsdU(z) \]
\[ = \int_{\frac{a(n)}{c}}^{\frac{a(n)}{z}} \left( U\left( \frac{a(n)^2}{n} \cdot s \right) - U\left( \frac{a(n)^2}{n} \cdot s \right) \right) ds \]
\[ = \int_{c}^{1} \frac{n}{a(n)} \left( U(a(n)zs) - U(a(n)s) \right) ds \]
\[ \Rightarrow (1 - c) \log z. \]

(5.4)

Suppose we know that

\[ \lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{0}^{\varepsilon} \frac{n}{a(n)} \left( U(a(n)zs) - U(a(n)s) \right) ds = 0. \]

(5.5)
Then it would follow that as \( n \to \infty \)

\[
\frac{n}{a(n)} V(a(n)(1, x)) = \frac{n}{a(n)} \left( \frac{1}{a(n)x} \int_0^{a(n)x} U(s) ds - \frac{1}{a(n)} \int_0^{a(n)} U(s) ds \right)
\]

\[
= \int_0^1 \frac{n}{a(n)} (U(a(n)x s) - U(a(n)s)) ds = 0
\]

\[
\to \log x
\]

as required.

From the definition of \( \Pi \)-variation it can be readily checked that given \( \delta > 0 \) there exist constants \( k > 0, t_0 > 0 \) such that for \( t \geq t_0, x \geq 1 \)

\[
(5.6) \quad \frac{U(tz) - U(t)}{g(t)} \leq kx^\delta
\]

(Geluk and de Haan, 1987; Resnick, 1987). This enables us to readily show (5.5) as follows. Write

\[
\int_0^c \frac{n}{a(n)} (U(a(n)x s) - U(a(n)s)) ds
\]

\[
\leq \int_0^{t_0/a(n)} \frac{n}{a(n)} U(a(n)x s) ds + \int_{t_0/a(n)}^c \frac{n}{a(n)} (U(a(n)x s) - U(a(n)s)) ds
\]

\[
= I + II.
\]

Now

\[
I \leq \frac{n}{a(n)^2} U(xt_0) \to 0
\]

as \( n \to \infty \) since the denominator is regularly varying with index 2. For \( II \) we have

\[
II = \int_{t_0/a(n)}^c \frac{n}{a(n)} (U(a(n)x s) - U(a(n)s)) ds + \int_{t_0/a(n)}^c \frac{n}{a(n)} (U(a(n)) - U(a(n)s)) ds
\]

\[
= IIa + IIb.
\]

Now

\[
IIa \leq \int_{t_0/a(n)}^c \frac{n}{a(n)} (U(a(n)x c) - U(a(n))) ds
\]

\[
\leq \frac{n}{a(n)} (U(a(n)x c) - U(a(n))) c
\]

\[
\to c \log c \to 0
\]

as \( n \to \infty \) and then \( c \to \infty \). For large \( n \) we have the integrand of \( IIb \) equal to

\[
\frac{n}{a(n)} \left( \frac{U(a(n)x s^{-1}) - U(a(n)s)}{g(a(n)s)} \right) g(a(n)s)
\]

\[
\leq kx^\delta \frac{g(a(n)s)}{g(a(n))}
\]
and Potter's inequalities assure us that \( g(a(n) s)/g(a(n)) \) has an upper bound in \( s \) free of \( n \) thereby showing that the integrand of \( IIb \) has a uniform upper bound which is integrable. Therefore

\[
\limsup_{n \to \infty} IIb = \int_0^c -\log sds \to 0
\]
as \( c \to 0 \) as desired. \( \blacksquare \)

One may note that much of the effort of the previous proof was devoted to overcoming the weakness that \( \Pi \)-variation is not well suited to controlling behavior near 0.

The main focus of this section is the Tauberian theorem for \( \Pi \)-varying functions first given by de Haan (1976).

**Proposition 5.3.** Suppose \( U : [0, \infty) \mapsto (0, \infty) \) is non-decreasing with Laplace transform

\[
\hat{U}(\lambda) = \int_0^\infty e^{-\lambda x} U(dx),
\]
which is finite for \( \lambda > 0 \). Then

\[
U(x) \in \Pi
\]
iff

\[
\hat{U}(\frac{1}{a}) \in \Pi.
\]

**Proof:** We follow the steps used to prove Proposition 3.1. Suppose \( U \in \Pi \). Combining Proposition 5.1 and Proposition 1.2 yields

\[
\sum_k \varepsilon_{\left(\frac{x}{n} t_k, \frac{x}{n} (x_n)ight)} \Rightarrow \sum_k \varepsilon_{(t_k, j_n, e_k)} := PRM(dt \times L \times e^{-y} dy)
\]
in \( M_p ([0, \infty) \times (0, \infty) \times (0, \infty)) \). If we can show

(5.7)

\[
N_n := \sum_k \varepsilon_{\left(\frac{x}{n} t_k, \frac{x}{n} (x_n)ight)} \Rightarrow N_\infty := \sum_k \varepsilon_{(t_k, j_n/e_k)}
\]
in \( M_p ([0, \infty) \times (0, \infty)) \) then the mean measures would converge and we would have for \( x > 1 \)

\[
E \sum_k \varepsilon_{\left(\frac{x}{n} t_k, \frac{x}{n} (x_n)ight)} ([0, 1] \times [1, x])
\]

\[
= -\frac{n}{a(n)} \int_{a(n)/x \leq u \leq a(n)x} U(du)e^{-y} dy
\]

\[
= -\frac{n}{a(n)} \int_0^\infty \left( e^{-\frac{u}{a(n)x}} - e^{-\frac{u}{a(n)x}} \right) U(du)
\]

\[
= -\frac{n}{a(n)} \left( \hat{U}(\frac{1}{a(n)x}) - \hat{U}(\frac{1}{a(n)x}) \right)
\]

\[
= -E \sum_k \varepsilon_{(t_k, j_n/e_k)} ([0, 1] \times [1, x])
\]

\[
= \int \int_{t \leq 1, 1 \leq u/y \leq x} dt dL(u)e^{-y} dy
\]

\[
= \int_0^\infty e^{-y} dy L(y, yx) = \log x
\]

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as desired.

So it remains to prove (5.7). As in the proof of Proposition 3.4 we define the approximations

\[ N_{n,M} = \sum_k \mathbb{1}_{\{\frac{\tau(n)}{n} \leq M, M^{-1} \leq \frac{u_k}{a(n)}, \frac{u_k}{a(n)} \leq M^{-1} \leq M\}} e^{\frac{\tau(n)}{a(n)} \cdot \frac{u_k}{a(n)}} \]

and we need to prove (3.4). The expectation following (3.4) that needs to be dealt with is now \((f \leq 1 \text{ with support in } [0,1] \times [\delta, \delta^{-1}])\)

\[
E \sum_k f\left(\frac{a(n)}{n} t_k, \frac{u_k}{a(n)} \right) \left(\mathbb{1}_{\{\frac{\tau(n)}{a(n)} \geq M\}} + 1_{\{\frac{u_k}{a(n)} < M^{-1}\}} + 1_{\{\frac{u_k}{a(n)} > M\}} + 1_{\{\frac{u_k}{a(n)} > M^{-1}\}} + 1_{\{\frac{u_k}{a(n)} > M\}}\right) \\
= I + II + III + IV + V.
\]

If \(M > \delta^{-1}\) then \(I = 0\). We have \(IV\) bounded by

\[
\frac{n}{a(n)} \int_{\delta \leq \frac{a(n)}{n} \leq \delta^{-1}, y > M} \mathcal{U}(u)e^{-y} dy \\
= \frac{n}{a(n)} \int_0^\infty (\mathcal{U}(a(n)y/\delta) - \mathcal{U}(a(n)\delta y)) e^{-y} dy \\
= \frac{n}{a(n)} \int_0^\infty (\mathcal{U}(a(n)y/\delta) - \mathcal{U}(a(n))) e^{-y} dy + \frac{n}{a(n)} (\mathcal{U}(a(n)) - \mathcal{U}(a(n)\delta)) |e^{-y} dy \\
\]

and using (5.6) and letting \(n \to \infty\) yields an upper bound of

\[
\int_0^\infty k\delta e^{-y} dy + \log e M e^{-M} \to 0
\]
as \(M \to \infty\).

The term \(V\) can be dealt with similarly. Now consider \(II\). An upper bound is

\[
E \sum_k 1_{\{\frac{u_k}{a(n)} \leq M^{-1}, a(n)\frac{u_k}{a(n)} \leq 1, \frac{\tau(n)}{a(n)} \leq \delta^{-1}\}} \\
= \frac{n}{a(n)} \int_0^\infty e^{-y} \left(\int_{a(n)y \leq u_k < a(n)y^{M^{-1}}, u_k \leq a(n)M^{-1}} \mathcal{U}(y) dy \right) dy \\
= \frac{n}{a(n)} \int_0^\infty e^{-y} \mathcal{U}(a(n)y, a(n)\left(\frac{y}{\delta} \wedge M^{-1}\right)) dy \\
\leq \frac{n}{a(n)} \int_0^\infty e^{-y} \mathcal{U}(a(n)y, a(n)M^{-1}) dy \\
= \frac{n}{a(n)} \int_0^{\frac{\pi}{2}} e^{-y} \mathcal{U}(a(n)y, a(n)M^{-1}) dy \\
= \frac{n}{a(n)} \int_0^{\frac{\pi}{2}} e^{-y} |U(a(n)M^{-1}) - U(a(n))| + |U(a(n)) - U(a(n)y\delta)| dy \\
\leq \frac{n}{a(n)} |U(a(n)M^{-1}) - U(a(n))|/(\delta M) + \frac{n}{a(n)} \int_0^{\pi/2} e^{-y} |U(a(n)) - U(a(n)y\delta)| dy.
\]
The first piece converges to 0 as \(n\) and then \(M\) go to \(\infty\). The second piece is handled by the argument that dealt with (5.5). Details for \(III\) are almost identical.

The converse follows as for Proposition 3.1.
6. Multivariate II-Variation.

Multivariate II-variation was considered first by de Haan and Omey (1983). Let $C \subset [0, \infty)^d$ be a cone in the positive quadrant. For instance we could have $C = (0, \infty)^d$ or $C = [0, \infty)^d \setminus \{0\}$. Consider a Radon measure $U$ defined on $C$, the Borel subsets of $C$, which is not identically 0. Call this measure $U$ II-varying on $C$ if there exists a slowly varying function $g(t)$ and a limit measure $L$ on $C$ such that as $t \to \infty$

\[
\frac{U(t)}{g(t)} \to L.
\]

Since $g$ is slowly varying, $L$ has an invariance property:

\[
L(sA) = L(A), \quad s > 0, A \in C
\]

and this has certain ramifications. Let $x \mapsto ||x||$ be a convenient norm on $R^d$ and as in section 4 let $\mathcal{N}$ be

\[
\mathcal{N} := \{x \in C : ||x|| = 1\}.
\]

For $y_1, y_2 > 1$ and $A$ a Borel subset of $\mathcal{N}$ we have using (6.2)

\[
L\{x : ||x|| \in (1, y_1 y_2], x/||x|| \in A\} = L\{x : ||x|| \in (1, y_1], x/||x|| \in A\} + L\{x : ||x|| \in (y_1, y_1 y_2], x/||x|| \in A\} + L\{x : ||x|| \in (y_1 y_2, 1], x/||x|| \in A\}
\]

which means for some measure $S$ on $B(\mathcal{N})$ we have

\[
L\{x : ||x|| \in (1, y], x/||x|| \in A\} = S(A) \log y
\]

with $y > 1$ and

\[
S(A) = L\{x : ||x|| \in (1, e], x/||x|| \in A\}.
\]

In general we see that if $0 < a < b$ then

\[
L\{x : ||x|| \in (a, b], x/||x|| \in A\} = S(A) \log \frac{b}{a}
\]

Furthermore, for any fixed relatively compact $A \subset \mathcal{N}$ we have that the measure

\[
U(\{x \in C : ||x|| \in \cdot, x/||x|| \in A\})
\]

is a one dimensional II-varying measure.

The sequential version of (6.1), namely (5.2) with the proper vector interpretations, continues to hold as in the last section and in fact Proposition 5.1 is also true in the multivariate setting. Simply interpret convergences as taking place in $M_p ((0, \infty) \times (0, \infty)^d)$.

In case $C = (0, \infty)^d$, we have the following further interpretations of what it means for a Radon measure on $(0, \infty)^d$ to be II-varying. These are analogous to results in Theorem 2 in de Haan and Omey (1983).
PROPOSITION 6.1. Suppose $U$ is a Radon measure on $(0, \infty)^d$. The following are equivalent.

(a) $U$ is II-varying.
(b) There exist a function $a(\cdot) \in RV_1$ and a measure $L$ satisfying (6.2) such that for $i = 1, \ldots, d$ and $x > 1 := (1, \ldots, 1)$
\[
\frac{n}{a(n)^2} \int_{(a(n), b(n)]} u_i U(du) \to \int_{(1, x]} u_i L(du)
\]
where $u = (u_1, \ldots, u_d)$.
(c) There exist a function $a(\cdot) \in RV_1$ and a measure $L$ satisfying (6.2) such that
\[
\frac{n}{a(n)^2} \int_{(a(n), b(n)]} \wedge_{i=1}^d u_i U(du) \to \int_{(1, x]} \wedge_{i=1}^d u_i L(du).
\]
(d) There exist a function $a(\cdot) \in RV_1$ and a measure $L$ satisfying (6.2) such that
\[
\frac{n}{a(n)^2} \int_{(a(n), b(n)]} \sum_{i=1}^d u_i U(du) \to \int_{(1, x]} \sum_{i=1}^d u_i L(du).
\]

Remark: de Haan and Ome (1983) assume that $U(0, x] < \infty$ and thus the form of their Theorem 2 is slightly different from our Proposition 6.1. Dealing with measures is slightly more natural in the multivariate context than treating distribution functions.

PROOF: We show (a) and (c) are equivalent, the rest of the verifications being very similar. If (a) holds then from the multivariate analogue of Proposition 5.1(c) we have
\[
\sum_k \varepsilon_{(\frac{x(n)u_k}{a(n)u_k}, \frac{x(n)u_k}{a(n)u_k})} \Rightarrow \sum_k \varepsilon_{(t, j_k)} = PRM(dt \times L)
\]
in $M_p ((0, \infty) \times (0, \infty)^d)$ and where of course the point process
\[
\sum_k \varepsilon_{(t, u_k)}
\]
is PR$(dt \times U)$. Define the map
\[
T : M_p ((0, \infty) \times (0, \infty)^d) \mapsto M_p ([0, \infty)^d \times (0, \infty)^d)
\]
by (notation reminder: $t/u = (t/u_1, \ldots, t/u_d)$)
\[
T(t, u) = (t/u, u).
\]
Since for example
\[
T^{-1} ([0, 1] \times [a, b]) = \{(t, u) : t/u \leq 1, a \leq ub\}
\]
is compact in $M_p ((0, \infty) \times (0, \infty)^d)$ the compactness criterion in Proposition 1.3 is satisfied and we conclude from (6.4) that
\[
\sum_k \varepsilon_{(\frac{x(n)u_k}{a(n)u_k}, \frac{x(n)u_k}{a(n)u_k})} \Rightarrow \sum_k \varepsilon_{(t, j_k)}
\]

(6.5)
in $M_p \left( [0, \infty)^d \times (0, \infty)^d \right)$. Since Poisson processes are converging, so do their mean measures and thus

$$
E \sum_k \epsilon \left( \frac{\alpha(n)^2 \lambda_k}{a(n)^2} \right) \left( [0, 1] \times [1, x] \right)
= \int \left\{ a(n) \leq u \leq a(n)x \cap s \leq \frac{1}{a(n)^2} \tau_{u_i, i = 1, \ldots, d} \right\} U(du) dt
= \int \left\{ a(n) \leq u \leq a(n)x \right\} \Lambda^d \left( u_i \right) U(du)
= E \sum_k \epsilon \left( \frac{\lambda_k}{\alpha(n)} \right) \left( [0, 1] \times [1, x] \right)
= \int \left\{ 1 \leq u \leq x, t \leq u_i, i = 1, \ldots, d \right\} L(du) dt
= \int \left\{ 1 \leq x \right\} \Lambda^d \left( u_i \right) L(du)
$$
as desired.

Conversely suppose \( (c) \) holds. Then \( (6.5) \) holds. Define the inverse map to the one discussed prior to \( (6.5) \). This is the map from

$$
M_p \left( [0, \infty)^d \times (0, \infty)^d \right) \mapsto M_p \left( [0, \infty) \times (0, \infty)^d \right)
$$
defined by

$$
(t/u, u) \mapsto (t, u)
$$
and it can be readily checked that this map satisfies the compactness criterion of Proposition 1.3. Applying this map to \( (6.5) \) yields \( (6.4) \) which is equivalent to \( \Pi \)-variation.

We now state the Tauberian theorem given first by de Haan and Omey (1983). Here we need to assume $C = [0, \infty)^d \setminus \{0\}$. In \( (6.1) \) we assume convergence on the boundaries of the positive quadrant. This makes sets of the form $\{ u \geq 0 : a < \|u\| < b \}$ bounded in $[0, \infty)^d \setminus \{0\}$ for any $0 < a < b$. Thus the measure $S$ is finite on $\mathcal{H}$ and if $U$ is $\Pi$-varying in the multivariate sense then $U\{ x \in C : \|x\| \in \cdot \}$ is $\Pi$-varying in the one dimensional sense.

**Proposition 6.2.** Suppose $U$ and $L$ are Radon measures on $C = [0, \infty)^d \setminus \{0\}$ such that for some slowly varying function $g(t)$

$$
(6.1) \quad \frac{U(t)}{g(t)} \underset{\lambda \downarrow 0}{\rightarrow} L
$$
vaguely on $C$. Assume that $L(\partial C) = 0$ and for $\lambda > 0$ suppose that

$$
\tilde{U}(\lambda) := \int_{[0, \infty)^d} e^{-\lambda x} U(x) < \infty
$$

\[21\]
with the same true for $L$. Then the measure corresponding to the distribution function $\hat{U}(1/x)$ is $\Pi$-varying on $(0, \infty)^d$ with auxiliary function $g$.

Notational reminder: For vectors $x, y$ we perform operations componentwise so that for instance

$$x/y = (x_i/y_i, \ldots, x_d/y_d).$$

Proof: As in Proposition 5.3 we have

$$\sum_h \epsilon_{(z_h, z_h^*)} \Rightarrow \sum_h \epsilon_{(t_h^*, 1_h^*)}$$

on $M_p \left( [0, \infty) \times ([0, \infty)^d \setminus \{0\} \right)$ where

$$\sum_h \epsilon_{(t_h^*, 1_h^*, E_h)} = PRM(dt \times U \times \prod_{i=1}^d e^{-y_i} dy_i)$$

and

$$\sum_h \epsilon_{(t_h^*, 1_h^*, E_h)} = PRM(dt \times L \times \prod_{i=1}^d e^{-y_i} dy_i).$$

Fix $x > 0$ and define two norms

$$\|u\|_1 = \sqrt{\sum_{i=1}^d u_i^2}, \quad \|u\|_2 = \sum_{i=1}^d u_i.$$

The subset of $C$ defined by

$$\Lambda := \{u \in C : \|u\|_1 \leq 1, \|u\|_2 > 1\}$$

is bounded in $C$ with zero $L$-measure on the boundary. Since (6.6) implies the mean measures converge we get by plugging the set $\Lambda$ into the mean measure convergence that

$$E \sum_h \epsilon_{(z_h, z_h^*)} ([0, 1] \times \Lambda)$$

$$= \frac{n}{a(n)} \int_C U(du) \left( e^{-\sum_{i=1}^d z_i^*} - e^{-\sum_{i=1}^d z_i} \right)$$

$$= \frac{n}{a(n)} \hat{U} \left( \frac{1}{a(n)^{1/2}}, \frac{1}{a(n)x} \right)$$

$$\rightarrow E \sum_h \epsilon_{(t_h^*, 1_h^*)} ([0, 1] \times \Lambda)$$

$$= \hat{\mathcal{L}}(1, \frac{1}{x}).$$
7. Concluding Remarks.

The method of proof of Proposition 6.2 gives rise to the feeling that Laplace transforms are not natural and convenient transforms for II-varying measures. A measure $U$ on, for example, $C = (0, \infty)^d$ need not have the property that $U(0,x] < \infty$ for $x > 0$. Witness the measure $U(dx, dy) = L(dx, dy) = x^{-1} y^{-1} dzdy$. However when dealing with Laplace transforms it is natural to require that the transform $\hat{U}(1/x) < \infty$ for $x > 0$. So there is asymmetry between what is required for $U$ compared with $\hat{U}$.

One dimensional II-varying functions have found an essential role in limit theory for extremes (de Haan, 1970; Bingham, Goldie, Teugels, 1987; Resnick, 1987) and in dealing with stable laws and processes of index $-1$ (de Haan and Resnick, 1979). The applications of multivariate II-variation have yet to be explored.

In order to consider the Laplace transform, our technique appends exponential random variables to a convergent sequence of Poisson process. It may be possible to explore other kinds of transforms by our method by appending random variables with distributions other than the exponential.

Functions of dominated variation (Geluk and de Haan, 1987; Bingham, Goldie, Teugels, 1987) can undoubtedly be treated by our methods. Instead of convergent sequences of Poisson processes, one deals with sequences of Poisson processes every subsequence of which has a further subsequence which is convergent.

References


**Keywords.** Tauberian theory, point processes, Poisson processes, regularly varying functions, II-variation, weak convergence

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