Erratum: Probabilistic Models for Linear Programming

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In this paper, published in *Mathematics of Operations Research*, Vol. 16, No. 4, pp. 671–693, I proposed various models for generating random linear programming problems and investigated several properties of these models, including the probability that the feasible region is bounded, the distribution of the distance from a particular interior point to each constraint hyperplane, and some properties of the vertices of the feasible region. Unfortunately, one of the results about these vertices is incorrect, and this affects some of the corollaries. Here I describe what is invalid, and explain where the difficulty arises in the proof.

Theorem 3.6 of the paper claims to give the distribution of the components of a vertex of the feasible region of a linear programming problem generated according to Model 1. This feasible region has the form $\{x \in \Re^n : Ax = A\hat{x}, x \geq 0\}$, where each entry of the $m \times n$ matrix A is independently distributed as a standard normal variable. For the theorem, I assumed $\hat{x} = e$, the vector of ones. The proof examined a particular basic solution, and found the distribution of its components conditioned on its being feasible. However, this distribution unfortunately does not agree with that of a random vertex, defined as one drawn from the uniform distribution on the vertices of the feasible region, because of subtle conditioning difficulties.

Consider the situation in the theorem with $m = 1$ and $n = 2$. Then the feasible region is $\{(x_1, x_2) : a_1x_1 + a_2x_2 = a_1 + a_2, x_1 \geq 0, x_2 \geq 0\}$.

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Both a_j's are standard normal, so nonzero with probability one, and they are independent. Geometrically, the feasible region is the intersection of the nonnegative orthant in $\Re^2$ with a line which goes through $(1, 1)$ and has angle uniformly distributed in $[0, 2\pi]$. Suppose we want to calculate the probability that a random vertex has norm at most one.

Let us argue as in the proof of Theorem 3.6. Then we choose a basic solution, say that with x_1 basic. This has $x_1 = 1 + a_2/a_1$. It is feasible if either a_1 and a_2 have the same sign (probability $1/2$), in which case its norm is greater than one, or they have different signs (probability $1/2$) and a_2 is smaller in absolute value than a_1 (probability $1/2$, independent, giving a probability for this case of $1/4$), in which case its norm is at most one, for a total probability of $3/4$. Conditioned on this being a basic feasible solution, its norm is at most one with probability $1/3$.

Next suppose we consider a random vertex. With probability $1/2$, a_1 and a_2 have the same sign, and then there are two vertices, each with norm greater than one. If not, then there is just one vertex, with norm less than one. Thus the norm of a random vertex is at most one with probability $1/2$, which is different from the result obtained before! Hence considering a particular basis and conditioning on its being feasible conditions the sign patterns of A, making it twice as likely to have two entries of the same sign, and therefore decreasing the probability that the norm of the basic solution is at most one.

This error in Theorem 3.6 luckily does not have too many consequences. Corollary 3.7 remains true, since the "proof" of the theorem shows that every basic solution, hence a fortiori every vertex, is nondegenerate with probability one. Corollary 3.8, concerning the expected distance of a vertex from e, is now unproved, and quite possibly false, and similarly for the analogous statement at the end of the second paragraph of Section 4, corresponding to a slightly modified model. However, the subsection on the expected number of vertices remains valid, since here we only need the probability that a given basic solution is feasible, and this is computed correctly.

Theorem 6.1, which is concerned with Model 3 and random linear programming problems on the simplex, relevant to Karmarkar's algorithm, is likewise unproved and probably false. The remarks following the theorem, which suggest that there are likely to be circumscribing and inscribing balls around e whose radii are in the ratio $O(\sqrt{n})$ to 1, are still valid if one interprets $\bar{x}$ as a particular basic solution, conditioning on its being feasible, rather than as a random vertex. Thus the argument, while not rigorous, still provides some indication of why Karmarkar's algorithm tends to work much better than the worst-case analysis guarantees.