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WEYL–MINKOWSKI DUALITY FOR
INTEGRAL MONOIDS

by

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Abstract

A monoid analogue to the Weyl-Minkowski duality of polar pairs of cones is developed. This duality states that the class of finitely generated integral monoids is identical to the class of monoids arising as integral solution sets of finite systems of constraints defined by Chvátal functions. Here the reverse direction of this equivalence is proved, the forward direction following from earlier work of Blair and Jeroslow. Where \( \mathcal{K}(S) \) and \( \mathcal{M}(S) \) denote, respectively, the cone and \( \mathbb{Z} \)-module generated by the finitely generated integral monoid \( S \), a characterization is given for \( S = \mathcal{K}(S) \cap \mathcal{M}(S) \). The characterization is in terms of a restricted class of Chvátal functions.

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1. Introduction

The geometric duality provided by theorems of Weyl and Minkowski lies at the heart of the classical theory of convex cones. These results establish that the solution set of a finite system of homogeneous linear inequalities may be equivalently described as the set of nonnegative linear combinations of a finite family of vectors. Thus a nonempty cone is polyhedral if (Weyl) and only if (Minkowski) it is finitely generated. Nonhomogeneous, i.e., affine, versions of these results lead easily to the duality theorem of linear programming (see, e.g., Trotter [1982]).

In this paper we consider analogues for Weyl-Minkowski duality in the setting of integer programming. A convex cone is closed under the operation of taking (finite) nonnegative linear combinations; an analogous discrete structure is the monoid, a set of vectors closed under nonnegative integral linear combinations. A monoid is integral if each of its members is integer-valued, finitely generated if it is of the form \( \{ yA : y \in \mathbb{Z}^m_+ \} \) for some \( m \times n \) matrix \( A \) and finitely constrained if it is of the form \( \{ x : Ax \in \mathbb{Z}^m_+ \} \).

Note that an integral monoid is finitely constrained if and only if it is the set of integral vectors in a rational polyhedral cone. It is well-known (see Schrijver [1986], Theorem 16.4) that such a set of integral vectors has an integral Hilbert basis, i.e., is finitely generated over \( \mathbb{Z}_+ \). Thus a direct analogue of the Minkowski result for convex cones holds for integral monoids. The converse condition of Weyl is false; e.g., \{0,2,3,4,\ldots\} is finitely generated, but not finitely constrained. In fact, a finitely generated integral monoid, say \( \{ yA : y \in \mathbb{Z}^m_+ \} \) for \( A \in \mathbb{Z}^{m \times n} \), is finitely constrained if and only if
the linear system \( \{Ax \geq 0\} \) is totally dual integral (see Carvalho and Trotter [1984]). Thus linear Weyl-Minkowski duality for integral monoids is characterized by homogeneous totally dual integral linear systems.

In order to extend this duality relation to all finitely generated integral monoids, one must allow constraints specified by a class of functions that properly includes linear functions. Consider the finitely generated integral monoid \( S = \{ yA: y \in \mathbb{Z}_+^m \} \), where \( A \in \mathbb{Z}^{m \times n} \), and define \( f(x) = \max \{ y \cdot 0: yA = x, y \in \mathbb{Z}_+^m \} \). Then \( f \) is superadditive, i.e., \( f(x+z) \geq f(x) + f(z) \), and clearly \( S = \{ x \in \mathbb{Z}^n: f(x) \geq 0 \} \), so \( S \) is finitely constrained by the single superadditive function \( f \). Whether a suitable converse result holds for superadditive functions, i.e., an analogue to the Minkowski condition, is unknown. Blair and Jeroslow [1982] have strengthened the above by showing that when one restricts to the subclass of superadditive functions known as Chvátal functions (defined in Section 2) one still obtains a finitely constrained description; i.e., \( S = \{ x \in \mathbb{Z}^n: f_i(x) \geq 0, 1 \leq i \leq p \} \), where the \( f_i \) are Chvátal functions. We complete this characterization of finitely generated integral monoids by showing in Section 2 that the converse Minkowski condition also holds. Thus a monoid is the integral solution set of a finite system of Chvátal restrictions if and only if it is finitely generated over \( \mathbb{Z}_+ \) by a finite set of integer-valued vectors.

Any set of points closed under the operation of taking (finite) integral linear combinations is called a \( Z \)-module. Suppose \( S \subset \mathbb{Z}^n \) is a finitely generated integral monoid and let \( \mathcal{K}(S) \) and \( \mathcal{M}(S) \) denote, respectively, the rational convex cone and \( Z \)-module generated by \( S \). Then the following relations hold:

\[
S \subset \mathcal{K}(S) \cap \mathcal{M}(S) \subset \mathcal{K}(S) \cap \mathbb{Z}^n
\]

We have indicated above that \( S = \mathcal{K}(S) \cap \mathbb{Z}^n \) precisely when \( S \) is generated by the rows of an integral Hilbert matrix, i.e., a matrix defining a homogeneous totally dual integral linear system (see Carvalho and Trotter [1984]). In Section 3 we characterize the more general
setting } S = \mathcal{K}(S) \cap \mathcal{M}(S) \text{ in terms of a restricted class of Chvátal functions properly containing the linear functions.}

2. Chvátal functions and integral monoids

The class of \emph{Chvátal functions} is obtained by extending the class of linear functions to allow rounding down to the nearest integer while preserving closure under nonnegative rational linear combinations. Through Corollary 2.5 we now follow the development of Blair and Jeroslow [1982] in defining Chvátal functions and presenting certain of their elementary properties. For any real number } r, \text{ we denote by } \lfloor r \rfloor \text{ the floor of } r, \text{ i.e., the largest integer less than or equal to } r; \text{ if } f \text{ is a real-valued function, then } \lfloor f \rfloor \text{ denotes the function defined by } \lfloor f \rfloor(x) = \lfloor f(x) \rfloor, \forall x. \text{ } \mathbb{Q} \text{ denotes the rationals.}

\textbf{Definition 2.1:} \text{ The class of n-dimensional Chvátal functions from } \mathbb{Q}^n \text{ to } \mathbb{Q}, \text{ denoted } \mathcal{C}_n, \text{ is the smallest class of functions } \mathcal{F} \text{ with the following properties:}

\begin{enumerate}
  \item i) \quad f \text{ linear } \Rightarrow f \in \mathcal{F};
  \item ii) \quad f, g \in \mathcal{F} \text{ and } \alpha, \beta \in \mathbb{Q}_+ \Rightarrow (\alpha f + \beta g) \in \mathcal{F};
  \item iii) \quad f \in \mathcal{F} \Rightarrow \lfloor f \rfloor \in \mathcal{F}. \quad \square
\end{enumerate}

We denote the class of all Chvátal functions by } \mathcal{C} = \bigcup_{n \geq 1} \mathcal{C}_n. \text{ Loosely speaking, the definition suggests that any function in } \mathcal{C}_n \text{ may be viewed as a linear expression of variables } x_1, \ldots, x_n \text{ with round-down operations } \lfloor \cdot \rfloor \text{ consistently inserted. Of course, the coefficient preceding any } \lfloor \cdot \rfloor \text{ in such an expression must be nonnegative; e.g., } -\lfloor x_1 \rfloor, -\lfloor x_1 \rfloor - (3x_2 + \lfloor x_3 \rfloor), \text{ etc. are not Chvátal functions. The restriction in Definition 2.1 to the smallest function class satisfying i) - iii) insures, moreover, that all Chvátal functions may be expressed as just suggested. The association of such expressions with the class of Chvátal functions is many-to-one. The two expressions, for example,
\[ f_1(x) = \lfloor \frac{x}{6} \rfloor - \frac{x}{6}, \quad f_2(x) = \frac{1}{2} \left( \lceil \frac{x}{2} \rceil + \lfloor \frac{5x}{6} \rfloor \right) \]

both describe the Chvátal function

\[
f(x) = \begin{cases} 
  0 & x \equiv 0 \pmod{6} \\
  -1 & \text{otherwise.}
\end{cases}
\]

The \textit{minimum} number of applications of operations ii) and iii) of Definition 2.1 required to express a Chvátal function determines its \textit{rank}, an intuitive measure of its complexity in terms of its "distance" from linearity.

\textbf{Definition 2.2:} If \( f \in \mathcal{C} \) is linear, \( f \) has \textit{rank} 0; \( f \) has \textit{rank} at most \( r \) if and only if there are functions \( g, h \in \mathcal{C} \) with rank less than \( r \) satisfying:

i) \( f = \alpha g + \beta h \), where \( \alpha, \beta \in \mathbb{Q}_+ \), or
ii) \( f = \lfloor g \rfloor \).

To any construction of \( f \in \mathcal{C} \) as described above, one can associate the linear function obtained by removing all round-down operations from this expression. The resulting linear function is, in fact, dependent only on \( f \), in that different expressions for \( f \) will produce the same linear function, called the \textit{carrier} of \( f \).

\textbf{Definition 2.3:} For each \( f \in \mathcal{C}_n \) define (inductively) the set \( S(f) \) as follows:

i) \( f \) linear \( \Rightarrow \) \( f \in S(f) \);

ii) \( f = \alpha g + \beta h \) with \( \alpha, \beta \in \mathbb{Q}_+ \) and \( g, h \in \mathcal{C}_n \)

\[ \Rightarrow (\alpha g' + \beta h') \in S(f) \quad \forall \ g' \in S(g), \ h' \in S(h); \]

iii) \( f = \lfloor g \rfloor \) with \( g \in \mathcal{C}_n \) \( \Rightarrow \) \( g' \in S(f) \) \( \forall \ g' \in S(g) \).

Any function in \( S(f) \) is a \textit{carrier} of \( f \).

By induction on the rank of \( f \), it is straightforward to establish the following elementary properties of Chvátal functions.
Proposition 2.4: Let \( f' \in S(f) \), where \( f \in C_n \). One then has:

i) \( f \) is superadditive; ii) \( f' \) is linear; iii) \( f'(x) - f(x) \geq 0 \), \( \forall \, x \in Q^n \);

iv) \( \exists \, k \in Q_+ \) such that \( f'(x) - f(x) \leq k \), \( \forall \, x \in Q^n \).

An important consequence of bounding \( f' - f \) as in Proposition 2.4(iv) is that \( S(f) \) must contain only one function.

Corollary 2.5: For \( f \in C \), \( S(f) \) contains exactly one function.

One is thus justified in referring to the carrier of \( f \), which will be denoted henceforth by \( f' \).

For example, the Chvátal function

\[
f(x) = \left\lfloor \frac{x_1}{2} + \frac{1}{2} \left\lfloor \frac{x_1}{3} - \frac{x_2}{2} \right\rfloor \right\rfloor
\]

has the carrier \( f'(x) = 2 \cdot \frac{3}{3} x_1 - \frac{1}{4} x_2 \).

We now derive a result stipulating that for certain integral vectors, the value of a Chvátal function and its carrier are equal.

Proposition 2.6: Let \( f \in C_n \), and let \( f' \) be its carrier. Then there exists a positive integer \( k_f \) such that

\[
f(k_f x) = f'(k_f x), \quad \forall \, x \in Z^n.
\]

Proof: The proof proceeds by induction on the rank of \( f \). The statement is true when \( f \) is linear and \( f = f' \). Suppose it is true for all Chvátal functions with rank less than \( r \) and let \( f \) have rank \( r \).

Suppose \( f = \lfloor g \rfloor \), where \( g \in C_n \) has rank less than \( r \). Then \( f' = g' \) and we have by induction that there exists a positive integer \( k_g \) such that \( g(k_g x) = g'(k_g x) \) for all integral vectors \( x \). Thus

\[
f(k_g x) = \lfloor g'(k_g x) \rfloor, \quad \forall \, x \in Z^n.
\]
Let \( m \) be the least common multiple of the denominators of the coefficients (expressed in lowest terms) of the linear function \( g' \), so that \( g'(mx) \in \mathbb{Z} \) for all \( x \in \mathbb{Z}^n \). Then \( f(mk_gx) = g'(mk_gx) = f'(mk_gx) \), for all \( x \in \mathbb{Z}^n \), and \( k_f \) can be set to be \( mk_g \).

On the other hand, suppose \( f = \alpha g + \beta h \), where \( \alpha, \beta \in \mathbb{Q}_+ \) and \( g, h \in \mathcal{C}_n \) have rank less than \( r \). By induction there exist positive integers \( k_g \) and \( k_h \) such that

\[
g(k_gx) = g'(k_gx), \quad \forall \ x \in \mathbb{Z}^n
\]

and

\[
h(k_hx) = h'(k_hx), \quad \forall \ x \in \mathbb{Z}^n.
\]

Thus for any integral vector \( x \),

\[
f(k_gk_hx) = \alpha g(k_gk_hx) + \beta h(k_gk_hx)
\]

\[
= \alpha g'(k_gk_hx) + \beta h'(k_gk_hx)
\]

\[
= f'(k_gk_hx).
\]

So for \( k_f = k_gk_h \), the desired result is obtained. \( \Box \)

It follows from superadditivity that for Chvátal functions \( f_1, \ldots, f_p \in \mathcal{C}_n \), the set \( S = \{ x \in \mathbb{Z}^n : f_i(x) \geq 0, \ 1 \leq i \leq p \} \) is an integral monoid. Using Proposition 2.6 we obtain the following result, establishing a particularly simple relationship between \( S \) and \( \mathcal{K}(S) \), the rational cone generated by \( S \).

**Theorem 2.7:** For \( f_1, \ldots, f_p \in \mathcal{C}_n \), let

\[
S = \{ x \in \mathbb{Z}^n : f_i(x) \geq 0, \ i = 1, \ldots, p \}.
\]

Then for \( f_i^* \) the carrier of \( f_i \), \( 1 \leq i \leq p \), we have

\[
\mathcal{K}(S) = \{ x \in \mathbb{Q}^n : f_i^* (x) \geq 0, \ i = 1, \ldots, p \}.
\]
Proof: First suppose that \( x \in \mathcal{X}(S) \). Then \( x \) is a finite positive rational combination of elements of \( S \), so there exists a positive integer \( m \) such that (by superadditivity) \( f_i(mx) \geq 0 \) for \( i = 1, \ldots, p \). Since \( f'(z) \geq f(z) \) for every \( z \in \mathbb{Z}^n \) and every \( f \in \mathcal{C}_n \), it must be that \( f'_i(mx) \geq 0 \) for \( i = 1, \ldots, p \). Each \( f'_i \) is linear, so \( f'_i(x) \geq 0 \) for \( i = 1, \ldots, p \).

Conversely, for \( x \in \mathbb{Q}^n \), suppose \( f'_i(x) \geq 0 \) for \( i = 1, \ldots, p \). Let \( m \) be an integer such that \( mx \in \mathbb{Z}^n \). By Proposition 2.6 there exists an integer \( k_i \) for each \( i = 1, \ldots, p \), such that

\[
f_i(k_iz) = f'_i(k_iz), \quad \forall \, z \in \mathbb{Z}^n.
\]

In particular, if \( k = k_1k_2\ldots k_p \), then

\[
f_i(kmx) = f'_i(kmx) \geq 0, \quad i = 1, \ldots, p.
\]

Hence \( kmx \in S \) and \( x \in \mathcal{X}(S) \). \( \Box \)

To establish the main result of this section, we also require the following characterization of finitely generated integral monoids.

**Theorem 2.8** (Jeroslow [1978]): An integral monoid \( S \) is finitely generated if and only if its rational cone \( \mathcal{X}(S) \) is polyhedral. \( \Box \)

We will call a constraint of the form \( f(x) \geq 0 \), where \( f \in \mathcal{C} \), a Chvátal restriction. As stated in Section 1, Blair and Jeroslow [1982] have established that any nonempty finitely generated integral monoid is the set of integral vectors satisfying a finite family of Chvátal restrictions. Conversely, the integral solution set for any finite list of Chvátal restrictions is, by Theorem 2.7, an integral monoid whose rational cone is polyhedral; thus Theorem 2.8 implies that this integral monoid must be finitely generated. It has been pointed out to us by an anonymous referee that this converse can also be proved using Corollary 3.14 of Blair and Jeroslow [1982].
Theorem 2.9: A nonempty integral monoid is finitely generated if and only if it is finitely constrained by Chvátal restrictions. □

The duality between finitely generated integral monoids and finite families of Chvátal restrictions expressed in Theorem 2.9 is in direct analogy to the Weyl-Minkowski duality between finitely generated and polyhedral cones.

3. A restricted class of Chvátal functions

Let \( A \in \mathbb{Z}^{m \times n} \) and let \( S \) be the integral monoid generated by the rows of \( A \); i.e., \( S = \{ yA : y \in \mathbb{Z}^m_+ \} \). As before, we denote by \( \mathcal{K}(S) \) and \( \mathcal{M}(S) \), respectively, the rational cone generated by \( S \) and the \( \mathbb{Z} \)-module generated by \( S \). It is easy to see that

\[
\mathcal{K}(S) = \{ yA : y \in \mathbb{Q}^m_+ \} \quad \text{and} \quad \mathcal{M}(S) = \{ yA : y \in \mathbb{Z}^m \} .
\]

Clearly, one has

\[
S \subseteq \mathcal{K}(S) \cap \mathcal{M}(S) \subseteq \mathcal{K}(S) \cap \mathbb{Z}^n ,
\]

and it is not difficult to show that equality holds throughout, i.e., \( S = \mathcal{K}(S) \cap \mathbb{Z}^n \), if and only if the rows of \( A \) constitute an integral Hilbert basis for \( S \). An equivalent restatement of the Hilbert basis condition is that the linear system \( \{ Ax \geq 0 \} \) is totally dual integral, i.e., that

\[
\{ yA = c, y \in \mathbb{Z}^m_+ \} \quad \text{is a consistent system}
\]

\[
\iff
\]

\[
c \in \mathbb{Z}^n
\]

and

\[
\{ yA = c, y \in \mathbb{Q}^m_+ \} \quad \text{is a consistent system.}
\]

This provides a characterization of those finitely generated integral monoids that are finitely constrained by linear functions (see Carvalho and Trotter [1984]).
On the other hand, we have seen in the previous section that any finitely generated integral monoid is finitely constrained by Chvátal restrictions (and conversely). In Carvalho [1984] it is noted that the condition $S = \mathcal{K}(S) \cap \mathcal{M}(S)$ represents a generalization of the notion of total dual integrality for the system $\{Ax \geq 0\}$, since the condition $S = \mathcal{K}(S) \cap \mathcal{M}(S)$ is equivalent to:

$$\{yA = c, y \in \mathbb{Z}^m_+\} \text{ is a consistent system}$$

$$\iff$$

$$\{yA = c, y \in \mathbb{Z}^m\} \text{ is a consistent system}$$

and

$$\{yA = c, y \in \mathbb{Q}^m_+\} \text{ is a consistent system.}$$

In this section we characterize this latter requirement by showing that $S = \mathcal{K}(S) \cap \mathcal{M}(S)$ if and only if $S$ is finitely constrained by the Chvátal functions of type $\mathcal{C}^0$, i.e., $S$ is finitely $\mathcal{C}^0$-constrained.

**Definition 3.1:** Denote $\mathcal{C}^0 = \bigcup_{n \geq 1} \mathcal{C}^0_n$, where

$$\mathcal{C}^0 = \{ f \in \mathcal{C}_n: \text{either } f \text{ is linear or } f = \lfloor g \rfloor - g \text{ with } g \text{ linear} \}.$$  

We indicate two ways to establish this characterization. The first relies on development in Trotter [1982] and Carvalho and Trotter [1984] and the second, providing more insight into the structure of $\mathcal{M}(S)$, is from Ryan [1986, 1987]. For the first approach we require the following two results.

**Proposition 3.2** (Trotter [1982]): Let $M = \{yA + uB: y \in \mathbb{Z}^m, u \in \mathbb{Q}^p\}$, where $A \in \mathbb{Q}^{m \times n}$ and $B \in \mathbb{Q}^{p \times n}$. Then there are integers $r,s$ and matrices $C \in \mathbb{Q}^{r \times n}$ and $D \in \mathbb{Q}^{s \times n}$ such that $M = \{x \in \mathbb{Q}^n: Cx \in \mathbb{Z}^r, Dx = 0\}$.  

Proposition 3.3 (Carvalho and Trotter [1984]): Let $M \subseteq \mathbb{Q}^n$ be a $\mathbb{Z}$-module and let $K \subseteq \mathbb{Q}^n$ be a convex cone. Assume that $M$ and $K$ generate the same subspace. Then $\mathcal{M}(M \cap K) = M$ and $\mathcal{K}(M \cap K) = K$. □

Theorem 3.4: Let $S = \{yA: y \in \mathbb{Z}_+^m\}$ with $A \in \mathbb{Z}^{m \times n}$; i.e., $S$ is a finitely generated integral monoid. Then $S = \mathcal{K}(S) \cap \mathcal{M}(S)$ if and only if $S$ is finitely $\mathcal{C}^0$-constrained.

Proof: (only if) One easily shows $\mathcal{K}(S) = \{yA: y \in \mathbb{Q}_+^m\}$ and $\mathcal{M}(S) = \{yA: y \in \mathbb{Z}^m\}$. By the Weyl condition for convex cones, $\mathcal{K}(S)$ is polyhedral, hence finitely constrained by linear nonnegativity restrictions. By Proposition 3.2, $\mathcal{M}(S)$ is finitely constrained by linear homogeneous restrictions and linear integrality restrictions. The linear homogeneous restrictions can be expressed as linear nonnegativity stipulations and any linear integrality restriction, say $a \cdot x \in \mathbb{Z}$ for $a \in \mathbb{Q}^n$, can be expressed in the form $\lfloor a \cdot x \rfloor - a \cdot x \geq 0$. Thus $S$ is finitely $\mathcal{C}^0$-constrained.

(if) S finitely $\mathcal{C}^0$-constrained implies the existence of matrices $C \in \mathbb{Q}^{p \times n}$ and $B \in \mathbb{Q}^{p \times n}$ for which

$$S = \{x: \ Cx \in \mathbb{Z}_+^r\} \cap \{x: \ Bx \in \mathbb{Q}_+^p\}.$$ 

If the rows of $D \in \mathbb{Q}^{s \times n}$ are a basis for $S^\perp = \{x: \ x \cdot z = 0, \ \forall \ z \in S\}$, then $S \subseteq (S^\perp)^\perp = \{x: \ Dx = 0\}$ and so we may also write

$$S = \{x: \ Cx \in \mathbb{Z}_+^r\} \cap \{x: \ Bx \geq 0\} \cap \{x: \ Dx = 0\}.$$ 

Now take $M = \{x \in \mathbb{Q}^n: \ Cx \in \mathbb{Z}_+^r, \ Dx = 0\}$ and $K = \{x \in \mathbb{Q}^n: \ Bx \geq 0, \ Dx = 0\}$ and let $\mathcal{S}(S), \mathcal{S}(M), \mathcal{S}(K)$ denote, respectively, the rational subspaces generated by $S,M,K$. Then $S \subseteq K$ implies $\mathcal{S}(S) \subseteq \mathcal{S}(K)$ and $K \subseteq (S^\perp)^\perp$ implies $\mathcal{S}(K) \subseteq \mathcal{S}(S)^{\perp\perp} = \mathcal{S}(S)$. Thus $\mathcal{S}(S) = \mathcal{S}(K)$ and similarly $\mathcal{S}(S) = \mathcal{S}(M)$. Since $\mathcal{S}(M) = \mathcal{S}(K)$, we may apply Proposition 3.3 to obtain $\mathcal{M}(M \cap K) = M$ and $\mathcal{K}(M \cap K) = K$. Thus
\[ S = M \cap K = \mathcal{M}(M \cap K) \cap \mathcal{K}(M \cap K) = \mathcal{M}(S) \cap \mathcal{K}(S). \] □

In Theorem 2.7 we have given an explicit description of \( \mathcal{K}(S) \) in terms of the representation of \( S \) as the solution set for a finite list of Chvátal restrictions; one simply passes from the Chvátal functions to their carriers. Once can also obtain from the Chvátal restrictions explicit descriptions of the rational subspace generated by \( S \), denoted \( \mathcal{S}(S) \), and of \( \mathcal{M}(S) \). These descriptions also can be used to prove Theorem 3.4 and we now indicate that development; for proofs the reader is referred to Ryan [1987]. The following proposition gives the explicit description of \( \mathcal{S}(S) \).

**Proposition 3.5:** Let \( S = \{ x \in \mathbb{Z}^n : f_i(x) \geq 0, 1 \leq i \leq p \} \), where \( f_1, \ldots, f_p \in \mathcal{C}_n \), and suppose \( f_i(x) = 0, \forall x \in S \), if and only if \( 1 \leq i \leq k \leq p \). Then \( \mathcal{S}(S) = \{ x \in \mathbb{Q}^n : f_i'(x) = 0, 1 \leq i \leq k \} \). □

Now suppose \( f \in \mathcal{C}_n \) is expressed in the form \( f(x) = \sum_{i=1}^{r} \alpha_i g_i(x) + l(x) \), where \( \alpha_i \in \mathbb{Q}_+ \) and \( g_i(x) = [h_i(x)] \) with \( h_i \in \mathcal{C}_n \), \( \forall i \), and \( l(x) \) is a linear function. Assume inductively that each \( g_i \) is also expressed in this form. We denote this specific representation of \( f \) by \( R \) and define (inductively) the nested linear pieces of \( R, f \), denoted \( N(R,f) \) as

\[
N(R,f) = \{ g_1', \ldots, g_r' \} \cup \left( \bigcup_{i=1}^{r} N(R,h_i) \right).
\]

For example, if \( f(x) = \left[ \frac{1}{2} \left[ \frac{1}{3} x \right] + \frac{1}{5} \left[ \frac{1}{2} x \right] \right] \), then \( N(R,f) = \{ \frac{1}{6} x + \frac{1}{10} x, \frac{1}{3} x, \frac{1}{2} x \} \). The following lemma uses \( N(R,f) \) to characterize those points for which \( f \) and \( f' \) coincide.

**Lemma 3.6:** Suppose \( f \in \mathcal{C}_n \) is specified by some representation \( R \) (as above). Then for \( x \in \mathbb{Z}^n \), \( f'(x) = f(x) \) if and only if \( s(x) \in \mathbb{Z}, \forall s \in N(R,f) \). □
Consider the integral monoid $S = \{ x \in \mathbb{Z}^n : f_i(x) \geq 0, 1 \leq i \leq p \}$ for $f_1, \ldots, f_p \in \mathbb{C}_n$. As in Proposition 3.5 we assume $f_i'(x) = 0, \forall x \in S$, (precisely) for $1 \leq i \leq k (\leq p)$. Finally, let $R_i$ be a representation of $f_i$, say \( f_i(x) = \sum_{j=1}^{r(i)} \alpha_{ij} g_j(x) + t_i(x) \), giving $N(R_i, f_i)$, for $1 \leq i \leq k$. The following theorem indicates that the linear functions in $\bigcup_{i=1}^{k} N(R_i, f_i)$ determine integrality restrictions for $M(S)$.

**Theorem 3.7**: In the above context,

$$M(S) = \{ x \in \mathbb{Z}^n : h(x) \in \mathbb{Z}, \forall h \in \bigcup_{i=1}^{k} N(R_i, f_i) \} \cap S(S).$$

It is straightforward to construct a proof of Theorem 3.4 using the descriptions of $K(S)$ and $M(S)$ stipulated, respectively, by Theorems 2.7 and 3.7. Again, we refer the reader to Ryan [1987] for details.
References


