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ARBITRAGE AND MARTINGALES

by

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Chapter 1

Introduction

The classical martingale systems theorem formalizes the intuitive notion that no winning betting systems are available to a gambler playing a 'fair' game. If the process on which the gambler is betting is a martingale, then the gambler's fortune also follows a martingale and no winning strategy can exist.

In two fundamental papers, Harrison & Kreps [17] and Harrison & Pliska [18] construct theoretical security market models which develop the link between the fundamental economic notion of the absence of arbitrage opportunities, and the probabilistic concept of a martingale. We give a brief sketch of the approach taken by Harrison & Pliska, and how it explains the Black-Scholes option pricing formula (in the simple zero interest rate case).

Suppose \( W = \{W_t : 0 \leq t \leq T\} \) is a Brownian motion defined on a probability space \( (\Omega, \mathcal{F}, Q) \), and adapted to the filtration \( (\mathcal{F}_t : 0 \leq t \leq T) \). Let \( S = \{S_t : 0 \leq t \leq T\} \) represent the price evolution of a certain risky stock, and assume \( S \) obeys the stochastic differential equation:

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dW_t. 
\] (1.1)
A European call with strike price $K$ is a contract giving its holder the right to purchase the stock for $K$ at time $T$. At time $T$ such a contract is worth $(S_T - K)^+$; but what should it be worth at time 0?

Girsanov's change of measure theorem provides a new probability measure, $Q^*$, equivalent to $Q$, (for each event $A$, $Q(A) = 0 \iff Q^*(A) = 0$), under which

$$W^*_t \triangleq W_t + \frac{\mu}{\sigma} t$$

is a Brownian motion. Simple substitution shows that the price process satisfies

$$\frac{dS_t}{S_t} = \sigma dW^*_t$$

(1.2)

and is a $Q^*$ martingale.

Harrison & Pliska then use the new measure $Q^*$ to define a class of trading strategies available to investors. A 'riskless' security $B = \{B_t : 0 \leq t \leq T\}$ is introduced. We set $B \equiv 1$ to reflect a zero interest rate, and call $B$ the bank account. We shall not precisely define the class of trading strategies allowed, but only remark that a trading strategy is a complete 'investment plan' for managing a portfolio of the stock and the bank account, without cash inflow or outflow over $[0, T]$. It is intuitively clear that the class of trading strategies should not contain a plan for making a sure profit with no possibility of loss, an arbitrage opportunity. Harrison & Pliska prohibit arbitrage opportunities by insisting (along with other conditions) that for each strategy, the value process (which tracks the strategy's worth over time) be a $Q^*$ martingale. Nevertheless, the class of strategies is large enough to include a strategy whose value at time $T$ replicates the payoff, $(S_T - K)^+$, of the call option. The unique rational price for the option at time 0 must be the 'starting capital' needed by this special trading strategy.
The value process of such a replicating strategy is a $Q^*$ martingale whose value at $T$ is $(S_T - K)^+$; hence the starting capital needed is (with $E^*[\cdot]$ denoting expectation with respect to $Q^*$)

$$E^*[ (S_T - K)^+]$$  \hspace{1cm} (1.3)

It follows from (1.2) that $S_T$ has a lognormal distribution under $Q^*$, and a simple calculation yields a closed form expression for (1.3), first discovered (using a different approach) by Black & Scholes [5].

Central to Harrison & Pliska’s method is the ‘artificial’ construction of a new probability measure $Q^*$, the martingale measure, under which the price process $S$ is a $Q^*$ martingale. This new probability measure has two uses. First, it enables the construction of a wide class of trading strategies containing no arbitrage opportunities. Second, the price at time 0 of any contingent claim attainable by a trading strategy is simply its expected value at time $T$, under $Q^*$.

The continuous time trading model sketched above began with the complete specification of the behavior of the price process $S$, given by (1.1). In a general setting, for what processes $S$ can a market model be constructed which contains no arbitrage opportunities, yet a wide class of trading strategies?

In chapter 2, we answer this question in the context of a discrete time security model. After defining a natural class of trading strategies, we show that under general conditions, for arbitrage to be precluded it is necessary and sufficient that the underlying price process admit an equivalent martingale measure.

The sufficiency of the martingale measure condition is essentially the usual martingale systems theorem couched in the language of a security market model. This fundamental idea that gambling systems (or the more respectable investment systems) cannot work when the underlying process is martingale dates back to Doob’s
definition of martingale [12], and to early work of Halmos [16].

The main new result of chapter 2 is that the sufficient condition is also necessary, and can therefore be interpreted as a converse to the martingale systems theorem: If no gambling systems work then the underlying process must be a martingale (under some equivalent measure).

The rest of the thesis focuses on stochastic models for the evolution of bond prices and interest rates, using the martingale measure approach. Finance practitioners are particularly interested in bond pricing models which enable calculation of contingent claim (e.g. option) prices, but no such model has met with the widespread acceptance that greeted the Black-Scholes stock option model.

A theory of bond option pricing must be built on a sensible stochastic model for the evolution of the term structure of interest rates, which describes the relationship among prices of bonds of varying maturities (or, equivalently, among interest rates on loans of varying maturities). In chapter 3, we classify some of the existing approaches to modelling the term structure of interest rates, discuss in some detail a particularly common approach which focuses primarily on the evolution of the spot interest rate, and draw attention to some specific drawbacks to these models.

One feature of the Black-Scholes formula not shared by spot rate models is apparent from our development above. By (1.2), the $Q^*$ distribution of $S_T$ is independent of $\mu$, and hence the Black-Scholes formula (1.3) is also. The stock price drift $\mu$ reflects investors' attitudes towards risk, since more risk averse investors demand higher expected returns on a given risky security. Such a parameter is difficult to estimate, and its absence from the formula was a reason for the success of the Black-Scholes formula.

In chapter 4, we propose a new class of models for the term structure of interest
rates. We use the martingale measure technique to preclude arbitrage, and to price contingent claims. These models have none of the drawbacks associated with the spot rate models discussed in chapter 3, and in fact share most of the desirable properties of the Black-Scholes model.
Chapter 2

Discrete Time Security Market Models

2.1 Introduction

In this chapter we set up a mathematical model of a financial market which meets at a discrete and finite set of times. In this market, the ‘securities’ prices are represented by a given $d + 1$ dimensional stochastic process. We define admissible trading strategies, the means by which investors trade, then arbitrage opportunities, trading strategies which generate profits with no possibility of loss. We give a necessary and sufficient condition for arbitrage opportunities to be excluded: that under some probability measure equivalent to the given measure the price process is a martingale.

The study of security market models using martingales began with Harrison & Kreps [17] and Harrison & Pliska [18]. In these papers, and in Taqqu and Willinger [35] similar discrete time models are constructed, but under the severe restriction that the probability space is finite. Under this restriction, the necessity of the equivalent martingale measure condition is a direct consequence of Farkas’ Lemma (the finite
dimensional separating hyperplane theorem).

Here we prove the equivalence of the two conditions in full generality. In particular, we impose no conditions on our processes which are not preserved under substitution of an equivalent measure (such as $L^2$ or even integrability restrictions).

This chapter also serves to motivate the discussions in chapter 4, in which a new class of models for the evolution of interest rates is developed. There the martingale approach is used to ensure that the models developed are arbitrage-free, and as a tool in calculating contingent claims prices.

### 2.2 Construction of the Market Model

Let $(\Omega, \mathcal{F}, Q)$ be a probability space equipped with an increasing family of sub-$\sigma$-fields of $\mathcal{F}$ (a filtration)

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \mathcal{F}_n = \mathcal{F}.$$  

For technical reasons, we assume that $\mathcal{F}_0$ is complete, that is it contains all subsets of elements of $\mathcal{F}$ having $Q$ probability 0.

Let $S = \{S_t : t = 0, 1, \ldots, n\}$ be a $d+1$ dimensional stochastic process adapted to $(\mathcal{F}_t)$. The components of $S$, i.e. $S^0, S^1, \ldots, S^d$ represent the prices of the individual securities. For concreteness, we assume all prices are in dollars, and that there are no taxes, transactions costs, etc., so at time $t$, an investor can buy or sell one unit of security $j$ for $\mathbb{S}S_t^j$.

Securities $S^1, S^2, \ldots S^d$ may take any real values, but we make the following assumption about the 0’th security.

**Assumption 2.2.1** $S^0 \equiv 1$.

This seemingly strong assumption allows investors to borrow and lend their money
with no interest. However, a simple corollary will show that under the more realistic condition that the 0'th process is strictly positive:

\[
Q\{S_t^0 > 0 \text{ for all } 0 \leq t \leq n\} = 1,
\]

to preclude arbitrage it is necessary and sufficient that the discounted price process \(S/S^0\) be a martingale under some equivalent measure. (Of course, if some other component \(S^j\) is always positive, we can use it as the discount factor, and get the necessary and sufficient condition in terms of \(S/S^j\).)

The essence of the security market model is the description of the means by which investors can trade, and hence transfer money from time 0 to \(n\).

**Definition 2.2.1** A trading strategy is a \(d + 1\) dimensional process \(\phi = \{\phi_t : t = 1, 2, \ldots, n\}\) such that

- \(\phi\) is predictable, that is \(\phi_t \in \mathcal{F}_{t-1}\) for \(t = 1, 2, \ldots, n\).
- \(\phi_t \cdot S_t = \phi_{t+1} \cdot S_t \) Q a.s. for \(t = 0, 1, \ldots, n-1\).

**Definition 2.2.2** The value process associated with a trading strategy \(\phi\), \(V(\phi)\) is

\[
V_t(\phi) = \begin{cases} 
\phi_t \cdot S_t & t = 1, 2, \ldots, n \\
\phi_1 \cdot S_0 & t = 0
\end{cases}
\]

Observe that \(V(\phi)\) is an adapted, real valued process. During the time interval \([t - 1, t]\), an investor following trading strategy \(\phi\) holds \(\phi_t^j\) units of security \(j\). At \(t\), the investor forms a new portfolio \(\phi_{t+1}\), using new information available in \(\mathcal{F}_t\). The first condition ensures that the investor forms portfolio \(\phi_t\) using only information available up to and including time \(t - 1\), and the second is the so-called self-financing condition, which ensures that the investor neither adds nor withdraws funds when he forms his new portfolio at trading dates.
The process whose value at $t$ is the gain realized by time $t$, $V_t(\phi) - V_0(\phi)$, is actually a *martingale transform*, as introduced by Burkholder [8]. For each $t \geq 1$, we can write

\[
V_t(\phi) = \phi_t \cdot S_t \\
= \phi_t \cdot (S_t - S_{t-1}) + \phi_t \cdot S_{t-1} \\
= \phi_t \cdot (S_t - S_{t-1}) + \phi_{t-1} \cdot S_{t-1} \\
= \ldots \\
= V_0(\phi) + \sum_{k=1}^{t} \phi_k \cdot (S_k - S_{k-1}).
\]

**Definition 2.2.3** An *arbitrage opportunity* is an admissible trading strategy $\phi$ satisfying

- $Q\{V_0(\phi) = 0\} = 1$.
- $Q\{V_n(\phi) \geq 0\} = 1$ and $Q\{V_n(\phi) > 0\} > 0$.

An investor following an arbitrage opportunity begins with no capital, never ends up with a loss, and with positive probability realizes a profit.

Since any positive multiple of an arbitrage opportunity is also an arbitrage opportunity, it is clear that in any kind of well functioning market, such arbitrage opportunities cannot exist (or they would be exploited by greedy investors).

**Remark 2.2.1** In the branch of economics known as general equilibrium theory, models are built for which prices of commodities are determined within the model by the forces of supply and demand generated by the interaction of many self-interested agents. In the language of the economist, prices are *endogenous* rather than *exogenous* variables. In contrast, it is customary in finance to assume that individual investors are 'price takers', and that prices seen by these investors are the output from some
larger equilibrium. Even without a careful definition of an economic equilibrium, it seems clear that provided economic agents ‘prefer more to less’, no arbitrage opportunities could exist. This distinction between the approach of the economist and the finance theorist will reappear when we discuss models for the evolution of bond prices.

A natural question to ask is ‘What conditions on the underlying price process $S$ are necessary and sufficient to rule out arbitrage opportunities?’ The rest of this chapter provides an answer to this question.

We begin by imposing another restriction on the trading strategies available to investors.

**Definition 2.2.4** An *admissible* trading strategy is a trading strategy satisfying the following (non-negative wealth) constraint:

- $V_t(\phi) \geq 0 \text{ a.s. for } t = 1, 2, \ldots, n$

Henceforth we shall restrict investors to admissible trading strategies; in other words they can never have negative net worth. This restriction does not prevent investors from selling individual securities short, only from using strategies whose entire worth could become negative.

In the finite $\Omega$ world of Harrison & Pliska [18] (where the non-negative wealth restriction is enforced) and Taqqu & Willinger [35] (where it is not), it has no real effect on the theory. Here, we use the restriction to ensure existence of a certain expectation in Lemma 2.2.2 below. As the following Lemma explains, in our discrete time model, the imposition of the restriction cannot exclude any arbitrage opportunities, and so we can make the assumption without changing the character of our subsequent theory.
Remark 2.2.2 In continuous time models, the natural analog of the class of trading strategies presented here does contain arbitrage opportunities. Dybvig [14] shows that a non-negative wealth constraint precludes them.

Lemma 2.2.1 The class of admissible trading strategies contains no arbitrage opportunities if and only if the class of trading strategies contains no arbitrage opportunities.

Proof: The ‘if’ part of this statement is trivial. An argument adapted from a finite \( \Omega \) result of Harrison & Pliska [18] shows how to construct an admissible arbitrage opportunity out of a non-admissible one, proving the converse.

Let \( \phi \) be a (non-admissible) arbitrage opportunity. For some \( i \) (1 \( \leq \) i \( \leq \) n), some \( A \in \mathcal{F}_i \) with \( Q(A) > 0 \), we must have

\[
V_i(\phi) < 0 \text{ on } A, \quad V_j(\phi) \geq 0 \text{ (all } i < j \leq n). \]

(In other words, \( i \) is the last time this strategy can have negative value.) We use \( \phi \) to build an admissible arbitrage opportunity \( \psi \). In the following definition, the first component in each parenthesis is the (scalar) bond holding; the second is a \( d \)-vector representing the holdings of the risky stocks. Let

\[
\psi_t = \begin{cases} 
(0, 0) & t \leq i \\
(\phi_t^0 - V_i(\phi), \phi_t) & t > i \text{ on } A \\
(0, 0) & t > i \text{ on } A^c 
\end{cases}
\]

The strategy is clearly predictable, and we only need check the self-financing condition at \( i \) on the set \( A \):

\[
\psi_{i+1} \cdot S_i = (\phi_{i+1}^0 - V_i(\phi)) + \sum_{k=1}^{d} \phi_{i+1}^k S_i
\]
\[ \psi_{i+1} \cdot S_i - V_i(\phi) = 0 = \psi_i \cdot S_i. \]

For \( j > i \), on \( A \)

\[ \psi_j \cdot S_j = \phi_j \cdot S_j - V_i(\phi) \geq 0. \]

In particular, \( \psi_n \cdot S_n \geq 0 \), and since \( Q(A) > 0 \), \( \psi \) is an (admissible) arbitrage opportunity.

\[ \square \]

**Remark 2.2.3** Henceforth the term *arbitrage opportunity* refers to admissible arbitrage opportunities.

**Definition 2.2.5** Probability measures \( Q \) and \( Q^* \) defined on \( (\Omega, F) \), are said to be *equivalent* if \( Q(A) = 0 \Leftrightarrow Q^*(A) = 0 \) for each \( A \in F \). We write \( Q \sim Q^* \).

**Remark 2.2.4** Notice that the class of admissible trading strategies and the class of arbitrage opportunities remain invariant if \( Q \) is replaced by an equivalent \( Q^* \). We therefore speak of arbitrage opportunities and admissible strategies with respect to equivalence classes of measures.

The following sufficient condition for arbitrage to be ruled out is essentially a restatement of the fundamental martingale systems theorem (gambling systems cannot work against martingales). Theorems of this sort date back to Halmos [16], and Doob's introduction of the concept of martingale [12].
Lemma 2.2.2 Suppose that $S$ is a (vector) martingale under some $Q^* \sim Q$. Then no arbitrage opportunities exist.

Proof: By the definition of a martingale, we have (denoting expectation with respect to $Q^*$ by $E^*[]$).

- $E^*|S_t^j| < \infty$ for $t = 0, 1, \ldots, n$, $j = 0, 1, \ldots, d$.

- $E^*[S_{t+1}|\mathcal{F}_t] = S_t$ for $t = 0, 1, \ldots, n - 1$.

Suppose $\phi$ is an arbitrage opportunity. Since $V_t(\phi) \geq 0$ for each $t$, $Q$ a.s. and hence $Q^*$ a.s., $E^*[V_t(\phi)]$ is defined, although possibly infinite. This, together with the integrability of $S$ under $Q^*$, allows the conditioning operation in the second equality below (see, for example, section 6.5 of Ash [2]). We have

$$E^*[V_n(\phi)] = E^*[\phi_n \cdot S_n]$$

$$= E^*[\phi_n \cdot E^*[S_n|\mathcal{F}_{n-1}]]$$

(since $\phi_n \in \mathcal{F}_{n-1}$)

$$= E^*[\phi_n \cdot S_{n-1}]$$

($S$ is a $Q^*$ martingale)

$$= E^*[\phi_{n-1} \cdot S_{n-1}]$$

(by the self financing property)

$$= \cdots$$

$$= E^*[\phi_1 \cdot S_0]$$

$$= E^*[V_0(\phi)]$$

$$= 0,$$
and so \( Q^* \{ V_n(\phi) = 0 \} = 1 \), a contradiction. \( \square \)

### 2.3 Necessity of the Equivalent Martingale Measure Condition

This section is devoted to proving the converse of Lemma 2.2.2:

**Theorem 2.3.1** Suppose no arbitrage opportunities exist. Then \( S \) is a (vector) martingale under some \( Q^* \sim Q \).

In a recent paper, Pliska and Back [3] show necessity of the martingale measure condition, in the case of one risky stock \((d = 1)\). Their constructive proof does not extend to more than one dimension, however. Also Delbaen [11] has produced a different proof for the case \( d = 1 \). Delbaen’s idea is to use continuous densities in order to apply the Measurable Selection Theorem. Here we exploit this idea to get a proof for general \( d \).

#### 2.3.1 Centering a Bounded Random Variable

We will begin with a simple ‘Theorem of the Alternative’, and work our way to the proof of Theorem 2.3.1. Let \( Y \) be a bounded \( d \)-dimensional random variable defined on some \((\Omega, \mathcal{A}, P)\). (Assume \( Y \in \mathcal{K} \), for some compact \( \mathcal{K} \subseteq \mathbb{R}^d \).)

**Lemma 2.3.1** Exactly one of the following two conditions holds:

1. There exists \( \alpha \in \mathbb{R}^d \) with \( P\{ \alpha \cdot Y \geq 0 \} = 1 \) and \( P\{ \alpha \cdot Y > 0 \} > 0 \) or

2. There exists a positive \( g \in C[\mathcal{K}] \) (the space of continuous, real valued functions on \( \mathcal{K} \)) with

\[
E[g(Y)Y] = 0 \quad (\in \mathbb{R}^d).
\]
Proof: If both conditions held, we would have

\[ 0 = \alpha \cdot 0 \]
\[ = \alpha \cdot E[g(Y)Y] \]
\[ = E[g(Y)\alpha \cdot Y] \]
\[ > 0, \]

a contradiction. It remains to show that at least one of the two conditions is met.

Before doing this, we recall some definitions (see Rockafellar [32], pages 95-97).

**Definition 2.3.1** Let \( C_1 \) and \( C_2 \) be non-empty sets in \( \mathbb{R}^d \). A hyperplane \( H \) is said to separate \( C_1 \) and \( C_2 \) if \( C_1 \) is contained in one of the closed half-spaces associated with \( H \), and \( C_2 \) lies in the opposite closed half-space. It is said to separate \( C_1 \) and \( C_2 \) properly if \( C_1 \) and \( C_2 \) are not both actually contained in \( H \) itself.

Consider the two convex subsets of \( \mathbb{R}^d \):

\[ \{ E[g(Y)Y] : g \in C[\mathbb{R}], g > 0 \} \text{ and } \{ 0 \}. \]

If they are not disjoint, then condition 2. holds. If they are, then the (finite dimensional) separating hyperplane theorem (Theorem 11.3 of Rockafellar) asserts that there exists a hyperplane \( H \) which properly separates them. Writing \( H \) as \( \{ x \in \mathbb{R}^d : \alpha \cdot x = 0 \} \), we have (possibly after substituting \(-\alpha \) for \( \alpha \))

\[ \alpha \cdot E[g(Y)Y] \geq 0 \text{ for all } g \in C[\mathbb{R}], g > 0. \]

Let \( \{ h_n(\cdot) \} \) be a sequence of bounded, positive elements of \( C[\mathbb{R}] \) so that \( h_n(x) \to 1_A \), where \( A = \{ x \in \mathbb{R} : \alpha \cdot x < 0 \} \). Then

\[ 0 \leq \alpha \cdot E[h_n(Y)Y] \]
\[ \rightarrow \alpha \cdot E[1_A Y] \text{ (Dominated Convergence)} \]
\[ \leq 0 \]

and so \( P\{\alpha \cdot Y < 0\} = 0 \), i.e. \( P\{\alpha \cdot Y \geq 0\} = 1 \). We must also have \( P\{\alpha \cdot Y > 0\} > 0 \) since \( \alpha \) properly separated our two convex sets. \( \Box \)

## 2.3.2 Statements of Measurability Lemmas

Proving an extension of the result of the previous section requires some technical measurability lemmas, which we state here. Their proofs are relegated to the Appendix of this chapter.

Let \( Y \) be a \( d \)-dimensional random variable defined on \((\Omega, \mathcal{A}, P)\). Let \( A_0 \subseteq \mathcal{A} \).

**Definition 2.3.2** A regular conditional probability for \( Y \) given \( A_0 \) is a map 
\( C : (\Omega \times \mathcal{B}(\mathbb{R}^d)) \to \mathbb{R} \), satisfying

- For each fixed \( A \in \mathcal{B}(\mathbb{R}^d) \), \( C(\omega, A) = P\{Y \in A|A_0\}(\omega) \) for almost all \( \omega \).

- For each \( \omega \), \( C(\omega, \cdot) \) is a probability measure on \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\).

It is well known (see Theorem 6.6.4 of Ash [2]) that such a \( C \) exists.

**Lemma 2.3.2** Let \((S, \mathcal{G})\) be a measurable space, and \( C \) be a regular conditional probability for \( Y \) given \( A_0 \). If \( F : (\mathbb{R}^d \times S) \to \mathbb{R}^k \) is a bounded \( \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G} \)-measurable function, then

1. \( F^*(\omega, \gamma) = \int C(\omega, dy) F(y, \gamma) \) is \( A_0 \otimes \mathcal{G} \)-measurable.

2. If we view \( \gamma \) as an \( A_0 \)-measurable random variable (i.e., a \((\Omega, A_0) \to (S, \mathcal{G})\) measurable map), then the map

\[ \omega \to F^*(\omega, \gamma(\omega)) \]
is a version of $E[F(Y, \gamma)|A_0]$.

Lemma 2.3.3 Let $\mathcal{K}$ be a compact subset of $\mathbb{R}^d$. Let $C[\mathcal{K}]$ denote the metric space of continuous real-valued functions on $\mathcal{K}$ with norm $\|g\| = \sup_{y \in \mathcal{K}} g(y)$. Then the map $(g, y) \rightarrow g(y)$ from $(C[\mathcal{K}] \times \mathcal{K}, \mathcal{B}(C[\mathcal{K}]) \otimes \mathcal{B}(\mathcal{K}))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable. ($\mathcal{B}(C[\mathcal{K}])$ is the $\sigma$-algebra generated by open sets under the sup-norm metric.)

2.3.3 Conditionally Centering a Random Variable

We now extend our simple result of subsection 2.3.1 to allow for an ‘initial’ $\sigma$-field, $A_0$. Roughly, we assert that if $A_0$ provides no information enabling a successful bet on $Y$, then we can center the conditional mean of $Y$ given $A_0$.

Let $Y$ be a $d$-dimensional random variable defined on $(\Omega, \mathcal{A}, P)$. Let $A_0 \subseteq \mathcal{A}$ be complete.

Theorem 2.3.2 Exactly one of the following two conditions holds:

1. There exists a $d$-dimensional $A_0$ measurable random variable $\alpha$ with

   $$P\{\alpha \cdot Y \geq 0\} = 1 \text{ and } P\{\alpha \cdot Y > 0\} > 0$$

2. There exists a bounded $h \in \mathcal{A}$ with $h > 0$ a.s., $E[h||Y||] < \infty$ and $E[h Y | A_0] = 0$

Remark 2.3.1 A consequence of this theorem is the necessity of the equivalent martingale measure condition for our security market model, when $n = 1$ (i.e. if trading takes place only on one day). To see this set

$$Y = (S_1^1 - S_0^1, S_1^2 - S_0^2, \ldots, S_1^d - S_0^d).$$
An arbitrage opportunity is an \( \alpha \) satisfying condition 1., so the absence of arbitrage ensures the existence of an \( h \) satisfying condition 2. The martingale measure \( Q^* \) is defined by
\[
dQ^*/dQ = ch
\]
for some normalizing constant \( c \).

**Proof:** The proof that both conditions cannot hold simultaneously is nearly identical to the proof of the analogous statement in Lemma 2.3.1. To show that at least one must hold, we begin by remarking that it suffices to prove it in the case when \( Y \) is bounded. Observe that
\[
\alpha \cdot Y \geq 0 \iff \alpha \cdot \frac{Y}{1 + \|Y\|} \geq 0
\]
and
\[
\alpha \cdot Y > 0 \iff \alpha \cdot \frac{Y}{1 + \|Y\|} > 0.
\]

So a proof of the theorem for bounded random variables would assure the existence of a bounded \( \tilde{h} > 0 \), with
\[
\int_{A_0} \tilde{h} \frac{Y}{1 + \|Y\|} dP = 0 \quad \text{for each } A_0 \in \mathcal{A}_0
\]

Let
\[
h \triangleq \frac{\tilde{h}}{1 + \|Y\|}.
\]

\( h \) is bounded, positive and
\[
E[h\|Y\|] = E \left[ \frac{\tilde{h}}{1 + \|Y\|\|Y\|} \right] < \infty.
\]

Also,
\[
\int_{A_0} h \ Y \ dP = 0 \quad \forall A_0 \in \mathcal{A}_0,
\]
so $E[h \ Y | A_0]$ exists and is 0.

We proceed with the proof assuming $P\{Y \in K\} = 1$ for some compact $K \subseteq R^d$.

Let $C : \Omega \times B(K) \to R$ be a regular conditional probability for $Y$ given $A_0$. For $\alpha \in R^d$, set

$$H_\geq(\omega, \alpha) = \int C(\omega, dy)1_{\{\alpha \cdot y \geq 0\}} \text{ and}$$

$$H_\gt(\omega, \alpha) = \int C(\omega, dy)1_{\{\alpha \cdot y > 0\}}.$$ 

$H_\geq$ and $H_\gt$ are measurable in the following ways:

1. Each is a $A_0 \otimes B(R^d)$ measurable function.

2. If $\alpha$ is an $A_0$-measurable function from $\Omega \to R^d$, then

   - $H_\geq(\omega, \alpha(\omega))$ is a version of $P\{\alpha \cdot Y \geq 0 | A_0\}$.

   - $H_\gt(\omega, \alpha(\omega))$ is a version of $P\{\alpha \cdot Y > 0 | A_0\}$.

To see this, apply Lemma 2.3.2 with $(S, G)$ as $(R^d, B(R^d))$ and $F(y, \alpha) = 1_{\{\alpha \cdot y \geq 0\}}$ (respectively $1_{\{\alpha \cdot y > 0\}}$).

Hence we can define the $A_0 \otimes B(R^d)$-measurable set

$$A \overset{\Delta}{=} \{(\omega, \alpha) : H_\geq(\omega, \alpha) = 1 \text{ and } H_\gt(\omega, \alpha) > 0\}.$$ 

For each $\omega$, $C(\omega, \cdot)$ is a probability measure on $(K, B(K))$. By Lemma 2.3.1, we conclude that for each $\omega$ exactly one of the following holds:

1. There exists $\alpha \in R^d$ with $(\omega, \alpha) \in A$.

2. There exists a positive, continuous $g : K \to R$ with

$$\int C(\omega, dy)g(y)y = 0. \tag{2.1}$$
So, roughly speaking, for each state of the world (each \( \omega \)) there is either a winning strategy (\( \alpha \)) which works in that state of the world, or there exists a continuous 'centering' function \( g \). (Note that if \( g \) satisfies 2., so does every positive multiple of \( g \).) We now show that the set of 'winning' \( \omega \)'s satisfying condition 1. is 'small', by showing that otherwise we could construct an arbitrage opportunity using their associated \( \alpha \)'s.

Let

\[
W \text{ (for winning)} \overset{\Delta}{=} \{ \omega : \exists \alpha \text{ with } (\omega, \alpha) \in A \}
\]

(the set satisfying condition 1.). This is the projection of \( A \) onto its first coordinate. In general, a projection of a product measurable set need not be measurable, but in this case the fact that \( \mathbb{R}^d \) is a complete separable metric space, and \( (A, A_0, P) \) is complete, assures that \( W \in A_0 \). Rockafellar [33] states this without proof. A proof can be found in Sainte-Beuve [34].

If \( d = 1 \) it is easy to show directly that \( W \) is \( A_0 \)-measurable. In this case, \( W \) is exactly equal to \( B \cup S \), where

\[
B = \{ \omega : H(\omega, 1) = 1 \} \cap \{ \omega : H_>(\omega, 1) > 0 \}
\]

and

\[
S = \{ \omega : H(\omega, -1) = 1 \} \cap \{ \omega : H_>(\omega, -1) > 0 \}.
\]

\( B \) and \( S \) stand for 'Buy' and 'Sell', and the interpretation is that with only 1 risky stock, and 1 time period, there are really only two trading strategies available: buying and selling.

Returning to the \( d \)-dimensional case, we now show that \( P(W) = 0 \). Define the
correspondence \( \varphi : (\Omega, \mathcal{A}_0, P) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \) by

\[
\varphi(\omega) = \begin{cases} 
\{ \alpha : (\omega, \alpha) \in A \} & \text{if } \omega \in W \\
\{0\} & \text{if } \omega \in W^C
\end{cases}
\]

The set of \((\omega, \alpha)\) with \(\alpha \in \varphi(\omega)\) is exactly

\[
((\Omega \times \{0\}) \cap A^C) \cup A
\]

and so is a \(\mathcal{A}_0 \otimes \mathcal{B}(\mathbb{R}^d)\)-measurable set. The sections are non-empty by construction. \(\mathbb{R}^d\) is a complete separable metric space and so the Measurable Selection Theorem provides us with an \(\mathcal{A}_0\)-measurable \(\alpha^*: \Omega \to R^k\) for which \(\alpha^*(\omega) \in \varphi(\omega)\) for almost all \(\omega\).

The measurability of \(\alpha^*\) ensures that

\[
H_{\geq}(\omega, \alpha^*(\omega)) = P\{\alpha^* \cdot Y \geq 0|\mathcal{A}_0\}(\omega) \ a.s.
\]

By the definition of \(\alpha^*\) the left hand side equals 1 a.s. Furthermore

\[
H_{>}(\omega, \alpha^*(\omega)) = P\{\alpha^* \cdot Y > 0|\mathcal{A}_0\}(\omega) \ a.s.
\]

is non-negative a.s. and positive on \(W\) so \(P\{\alpha^* \cdot Y > 0\}\) is positive if \(P(W) > 0\). This would mean that \(\alpha^*\) is an arbitrage opportunity and hence we must have \(P(W) = 0\).

Summarizing, the application of Lemma 2.3.1 enabled us to find, for each \(\omega \in W^C\), a continuous \(g_\omega > 0\) satisfying

\[
\int C(\omega, dy)g_\omega(y) y = 0.
\]

We have just shown that \(P(W) = 0\), so there is a function \(g_\omega\) of the above form for \(P\)-almost all \(\omega\). It remains only to combine \(g\)'s of this form to obtain a measurable map \(h\) for which \(E[h Y|\mathcal{A}_0] = 0\).
Definition 2.3.3 Let
\[ D = \{ g \in C[\mathcal{K}] : \| g \| \leq 1 \}. \]

\( D \) is a closed subset of the complete separable metric space \( C[\mathcal{K}] \) and so is a complete separable metric space itself. (To get a countable dense subset, simply truncate each member of a countable dense subset of \( C[\mathcal{K}] \).) Let \( \mathcal{D} \) be the Borel \( \sigma \)-field of \( D \) (the \( \sigma \)-field generated by the open sets under the sup norm metric).

Define \( \theta : \Omega \times D \to \mathbb{R}^d \) by
\[
\theta(\omega, g) = \int C(\omega, dy) \, g(y) \, y.
\]

In the notation of Lemma 2.3.2, this time we have \((S, \mathcal{G}) = (D, \mathcal{D}) \) and \( F(y, g) = g(y) \, y \). Since \( \mathcal{K} \) is compact, and \( \| g \| \leq 1 \) for each \( g \in D \), this map is bounded. Lemma 2.3.3 shows that the map \((g, y) \to g(y)\) from \((C[\mathcal{K}] \times \mathcal{K}, \mathcal{B}(C[\mathcal{K}]) \otimes \mathcal{B}([\mathcal{K}]))\) to \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) is measurable. We thus conclude that \( \theta \) is a \( \mathcal{A}_0 \otimes \mathcal{D} \) measurable map, and that if the map
\[ \omega \to g_\omega \]
is \( \mathcal{A}_0 \)-measurable, then
\[ \omega \to \theta(\omega, g_\omega) \]
is a version of \( E[g(Y)Y|\mathcal{A}_0] \).

We need another application of the Measurable Selection Theorem, to measurably select certain \( g \)'s with \( \theta(\omega, g) = 0 \).

Let
\[
B = \{ (\omega, g) \in (\Omega \times D) : g > 0 \}
\]
\[
\cap \{ (\omega, g) \in (\Omega \times D) : \theta(\omega, g) = 0 \}.
\]
The measurability of $\theta(\cdot, \cdot)$ ensures that $B$ is a $\mathcal{A}_0 \otimes \mathcal{D}$-measurable set. Define the correspondence $\psi : (\mathcal{A}, \mathcal{A}_0, P) \to (D, \mathcal{D})$ by

$$
\psi(\omega) = \begin{cases} 
\{1\} & \text{if } \omega \in W, \\
\{g \in D : (\omega, g) \in B\} & \text{if } \omega \in W^C.
\end{cases}
$$

The set of $(\omega, g)$ with $g \in \psi(\omega)$ is exactly $((\mathcal{A} \times \{1\}) \cap B^C) \cup B$ and so is $A_0 \otimes D$-measurable.

To see that the sections are non-empty, observe that, by definition, $W^C$ consists of those $\omega$ for which there exists centering functions $g$ of the proper form. As we remarked earlier, every positive multiple of a centering $g$ (a $g$ satisfying (2.1)) also has this property. Here we pick the representative with norm 1.

Applying the Measurable Selection Theorem gives a measurable map $\omega \to g^*_\omega$. Since $g^*_\omega \in \psi(\omega)$ a.s., and $P(W^C) = 1$, we have

$$
g^*_\omega > 0 \text{ and } \|g^*_\omega\| = 1 \text{ a.s.}
$$

Furthermore,

$$
\int C(\omega, dy)g^*_\omega(y)y = 0 \text{ a.s.}
$$

and is a version of $E[g^*(Y)Y|\mathcal{A}_0]$. So

$$
E[1_{A_0}g^*(Y)Y] = 0 \text{ for each } A_0 \in \mathcal{A}_0.
$$

Finally, we define the random variable $h$, by

$$
h(\omega) = g^*_\omega(Y(\omega)).
$$

To see that $h$ is $A$-measurable observe that the map

$$
\omega \to (g^*_\omega, Y(\omega))
$$
is a measurable map from \((\Omega, \mathcal{A})\) to \((C[K] \times K, \mathcal{B}(C[K]) \otimes \mathcal{B}(K))\), and apply Lemma 2.3.3. We have \(h > 0\) a.s., and \(\|h\| \leq 1\) a.s. Since \(Y\) is bounded, we are assured that

\[
E[h \|Y\|] < \infty.
\]

Finally, we constructed \(h\) so that

\[
E[1_{A_0} h Y] = 0, \quad \text{for each } A_0 \in \mathcal{A}_0,
\]

meaning that

\[
E[h Y | A_0] = 0.
\]

\[\square\]

### 2.3.4 Extension to Finite Time Horizon

We are now ready to prove the necessity of the equivalent martingale measure condition, when \(n\) (the number of trading days) is arbitrary.

**Proof:** To build the equivalent martingale measure, it turns out that we need only rule out 'one day' arbitrage. More precisely, given \(\alpha_{i-1} \in \mathcal{F}_{i-1}\), define a strategy \(\phi\) as follows. (The first component in each parenthesis is the (scalar) bank account holding, and the second is a \(d\)-vector representing the holdings of the risky stocks.)

\[
\phi_t = \begin{cases} 
(0, 0) & t \leq i - 1 \\
(-\alpha_{i-1} \cdot S_{i-1}, \alpha_{i-1}) & t = i \\
(\alpha_{i-1} \cdot (S_i - S_{i-1}), 0) & t \geq i + 1
\end{cases}
\]

Define

\[
\Delta_i \triangleq S_i - S_{i-1}.
\]
It is easy to check that $\phi$ is an admissible trading strategy if and only if

$$Q\{\alpha_{i-1} \cdot \Delta_i \geq 0\} = 1.$$ 

If admissible, it is an arbitrage opportunity if and only if

$$Q\{\alpha_{i-1} \cdot \Delta_i > 0\} > 0.$$ 

So the assumption of no arbitrage opportunities implies that for each $1 \leq i \leq n$, there is no $\alpha_{i-1} \in \mathcal{F}_{i-1}$ with

$$Q\{\alpha_{i-1} \cdot \Delta_i \geq 0\} = 1 \text{ and } Q\{\alpha_{i-1} \cdot \Delta_i > 0\} > 0. \quad (2.2)$$

To generate the equivalent martingale measure, $Q^*$, we only need be assured that no strategy of this form is an arbitrage opportunity. We will define the Radon-Nykodym derivative

$$\frac{dQ^*}{dQ} = c \ h_0 h_1 \ldots h_n \quad (2.3)$$

where $c$ is a normalizing constant, and each $h_k$ is a positive, bounded, $\mathcal{F}_k$-measurable random variable. Also, for each $k \geq 1$, we shall ensure that

$$E[h_k \| \Delta_k \| E[h_{k+1} \ldots h_n | \mathcal{F}_k]] < \infty \quad (2.4)$$

and

$$E[h_k \Delta_k E[h_{k+1} \ldots h_n | \mathcal{F}_k] | \mathcal{F}_{k-1}] = 0. \quad (2.5)$$

We define the sequence $\{h_1, \ldots, h_n\}$ inductively, beginning with $h_n$ and working back to $h_1$. Directly combining (2.2) (with $i = n$) and Theorem 2.3.2 gives a positive, bounded, $\mathcal{F}_n$-measurable r.v., $h_n$, with

$$E[h_n \| \Delta_n \|] < \infty$$
and

\[ E[h_n \Delta_n | \mathcal{F}_{n-1}] = 0. \]

Supposing that \( h_{k+1} \ldots h_n \) have been constructed and satisfy the conditions specified, we show how to construct \( h_k \) (\( 1 \leq k \leq n-1 \)). Observe that for \( \alpha \in \mathcal{F}_{k-1} \),

\[ \alpha \cdot \Delta_k \geq 0 \iff \alpha \cdot \Delta_k E[h_{k+1} \ldots h_n | \mathcal{F}_k] \geq 0 \]

and

\[ \alpha \cdot \Delta_k > 0 \iff \alpha \cdot \Delta_k E[h_{k+1} \ldots h_n | \mathcal{F}_k] > 0. \]

So, again combining (2.2) (with \( i = k \)) and Theorem 2.3.2 gives a positive, bounded, \( \mathcal{F}_k \)-measurable \( h_k \) with

\[ E[h_k \| \Delta_k \| E[h_{k+1} \ldots h_n | \mathcal{F}_k]] < \infty \]

and

\[ E[h_k \Delta_k E[h_{k+1} \ldots h_n | \mathcal{F}_k] | \mathcal{F}_{k-1}] = 0 \]

confirming (2.4) and (2.5). Continue in this way until \( h_1 \) has been defined.

Finally, set

\[ h_0 = \frac{1}{1 + \|S_0\|}, \]

and define \( Q^* \) by 2.3. Each \( h_k \) is bounded and positive, so there is a unique positive \( c \) making \( Q^* \) a probability measure.

We begin by showing that \( S \) is integrable under \( Q^* \).

\[
E^*[\|S_0\|] = c E[h_0 h_1 \ldots h_n \|S_0\|] \\
\leq c E[h_1 \ldots h_n] \\
< \infty.
\]
For each $k \geq 1$,

$$E^*[||\Delta_k||] = cE[h_0 h_1 \ldots h_n||\Delta_k||]$$

$$= cE[h_0 h_1 \ldots h_k||\Delta_k||E[h_{k+1} \ldots h_n|\mathcal{F}_k]]$$

(the expectation exists since all terms are positive)

$$< \infty$$

(by (2.4) and the boundedness of $h_0 h_1 \ldots h_{k-1}$)

So

$$E^*[||S_k||] \leq E^*[||S_0||] + E^*[||\Delta_1||] + \ldots + E^*[||\Delta_k||] < \infty.$$ 

Now let $A \in \mathcal{F}_{k-1}$ for some $1 \leq k \leq n$.

$$E^*[1_A \Delta_k] = cE[h_0 h_1 \ldots h_n 1_A \Delta_k]$$

$$= cE[h_0 h_1 \ldots h_k 1_A \Delta_k E[h_{k+1} \ldots h_n|\mathcal{F}_k]]$$

(by (2.4) and the boundedness of $h_0 h_1 \ldots h_{k-1}$.)

$$= cE[h_0 h_1 \ldots h_k 1_A E[h_k \Delta_k E[h_{k+1} \ldots h_n|\mathcal{F}_k]|\mathcal{F}_{k-1}]]$$

$$= 0$$

(by (2.5))

In other words,

$$E^*[S_k - S_{k-1}|\mathcal{F}_{k-1}] = 0,$$

and so $S = \{S_k : k = 0, 1, \ldots, n\}$ is a $Q^*$ martingale. \hfill \Box

### 2.4 Stochastic Bond Prices

Suppose we relax the assumption that $S^0 \equiv 1$, and replace it with:
Assumption 2.4.1

\[ Q\{S_t^0 > 0 \text{ for all } 0 \leq t \leq n\} = 1. \]

One can construct trivial examples to show that the martingale measure condition is no longer necessary to preclude arbitrage. The proof given previously breaks down because the trading strategies defined in the beginning of section 2.3.4 are no longer self-financing. Instead, given \( \alpha_{i-1} \in \mathcal{F}_{i-1} \), consider the strategy \( \phi_t \) defined by

\[
\phi_t = \begin{cases} 
(0,0) & t \leq i - 1 \\
(-\alpha_{i-1} \cdot S_{i-1}/S_{i-1}^0, \alpha_{i-1}) & t = i \\
(\alpha_{i-1} \cdot (S_i/S_i^0 - S_{i-1}/S_{i-1}^0), 0) & t \geq i + 1
\end{cases}
\]

This strategy is admissible if and only if

\[ Q\{\alpha_{i-1} \cdot (S_i/S_i^0 - S_{i-1}/S_{i-1}^0) \geq 0\} = 1, \]

and if so, is an arbitrage opportunity if and only if

\[ Q\{\alpha_{i-1} \cdot (S_i/S_i^0 - S_{i-1}/S_{i-1}^0) > 0\} > 0. \]

(The condition \( S^0 > 0 \) is employed again here, to guarantee that the investor's positive wealth at time \( i \) is still positive at \( n \).)

Repeating our previous argument for the discounted process \( S/S^0 \), yields the corollary:

**Corollary 2.4.1** Under assumption 2.4.1, a necessary and sufficient condition for the preclusion of arbitrage opportunities in the market model, is that there exist an equivalent measure \( Q^* \) under which

\[ S/S^0 = \{S_t/S_t^0 : t = 0, 1, \ldots, n\} \]

is a \( Q^* \) martingale.
2.5 Infinite Time Horizon

If we extend the model described in section 2.2 so that trading takes place at times \( \{1, 2, \ldots \} \) then a simple example shows that the equivalent martingale measure condition is no longer necessary for arbitrage to be precluded. Let \( d = 1, \ S^0 \equiv 1, \) and

\[
S^1_t = X_1 + X_2 + \ldots + X_t,
\]

where for some \( 0 < p < 1, \ p \neq 1/2, \ \{X_i : i = 0, 1, \ldots \} \) is a sequence of i.i.d. random variables defined on some \((\Omega, \mathcal{F}, Q)\), with

\[
X_i = \begin{cases} 
1 & \text{with probability } p \\
-1 & \text{with probability } 1 - p
\end{cases}
\]

The only measure which could possibly be an equivalent martingale measure for \( \{S_n\} \) has each \( X_i \) independently take the values 1 and -1 with probability 1/2. But the strong law of large numbers shows that this measure and \( Q \) are not equivalent. The model contains no arbitrage opportunities since the only admissible strategy \( \phi \) with \( V_0(\phi) = 0 \) and \( V_t(\phi) \geq 0 \) for all \( t \) is the zero strategy. (In the infinite time case, we define an arbitrage opportunity as a \( \phi \) for which the two conditions in Definition 2.2.3 hold for any (non-random) \( n \).)

2.6 Appendix

Proof (of Lemma 2.3.2):

1. Let \( \mathcal{H} \) denote the set of measurable \( F \) for which \( \int C(\omega, dy) F(y, \gamma) \) is defined and \( A_0 \otimes \mathcal{G} \)-measurable. \( \mathcal{H} \) is a vector space of real valued functions on \((\mathbb{R}^d \times S)\).

Let

\[
\mathcal{M} = \{B \times G : B \in \mathcal{B}(\mathbb{R}^d) \text{ and } G \in \mathcal{G}\}.
\]
\[ \mathcal{H} \text{ contains } 1_M \text{ for each } M \text{ in } \mathcal{M}, \text{ since} \]
\[ \int C(\omega, dy) 1_B(y) 1_G(\gamma) = 1_G(\gamma) C(\omega, B). \]

\[ \mathcal{M} \text{ is obviously closed under intersection, and contains } \mathcal{A} \times \mathbb{R}^d. \text{ If } F_n \uparrow \text{ F.a.s.} \]

and the \( F_n \)'s are non-negative elements of \( \mathcal{H} \), then let

\[ F_n^*(\omega, \gamma) = \int C(\omega, dy) F_n(y, \gamma) \text{ and} \]
\[ F^*(\omega, \gamma) = \int C(\omega, dy) F(y, \gamma). \]

By the dominated convergence theorem, \( F^*(\omega, \gamma) = \lim_{n \to \infty} F_n^*(\omega, \gamma) \) for each \( (\omega, \gamma) \). Hence \( F^* \), being the limit of \( \mathcal{A}_0 \otimes \mathcal{G} \)-measurable functions is itself \( \mathcal{A}_0 \otimes \mathcal{G} \)-measurable. Having shown that \( \mathcal{H} \) is closed under bounded convergence, we use the Multiplicative System Theorem (stated below) to conclude that \( \mathcal{H} \)

contains all bounded \( \sigma(\mathcal{M}) \)-measurable functions, namely all bounded \( \mathcal{A}_0 \otimes \mathcal{G} \)-measurable functions.

2. The map

\[ \omega \rightarrow (\omega, \gamma(\omega)) \]

is an \( (\Omega, \mathcal{A}_0) \rightarrow ((\Omega \times S), (\mathcal{A}_0 \otimes \mathcal{G})) \)-measurable map, and so, using the result of Part 1., the composite map

\[ \omega \rightarrow F^*(\omega, \gamma(\omega)) \]

is \( \mathcal{A}_0 \) measurable. It only remains to show that for each \( A_0 \in \mathcal{A}_0, \)

\[ \int_{A_0} F(Y(\omega), \gamma(\omega)) \, dP(\omega) = \int_{A_0} dP(\omega) \int C(\omega, dy) \ F(y, \gamma(\omega)). \]

Let \( \mathcal{H} \) denote the set of \( F \)'s for which both sides of the equation are defined and equal. \( \mathcal{H} \) is a vector space containing 1. Suppose \( F(y, \gamma) = 1_B(y) 1_G(\gamma), \)
for some $B \in \mathcal{H}^d$, $G \in \mathcal{G}$. The right hand side is

$$
\int_{A_0} dP(\omega)C(\omega, B) \ I_{\{\gamma \in G\}}
= \int_{A_0 \cap \{\gamma \in G\}} dP(\omega)C(\omega, B)
= P\{A_0 \cap \{\gamma \in G\} \cap \{Y \in B\}\}
$$

which equals the left hand side. So $\mathcal{H}$ contains the multiplicative system $\mathcal{M}$ as defined in Part 1. As above, it can be shown that $\mathcal{H}$ is closed under bounded convergence. Applying the multiplicative system theorem again, we conclude that $\mathcal{H}$ contains all bounded $A_0 \otimes \mathcal{G}$-measurable functions.

\[\Box\]

**Proof (of Lemma 2.3.3):** Since both $\mathcal{H}$ and $C[\mathcal{K}]$ are separable, Billingsley [4] (page 225) applies to show that the $\sigma$ field $\mathcal{B}(C[\mathcal{K}]) \otimes \mathcal{B}(\mathcal{K})$ is generated by the metric

$$
d((f, x), (g, y)) = \max(\|f - g\|, |x - y|),
$$

under which I claim our map is continuous.

To see this, suppose $(g, y) \in C[\mathcal{K}]$ and $\epsilon > 0$, let $\delta$ guarantee $|g(u) - g(y)| < \epsilon$ when $|u - y| < \delta$. If $(f, x) \in C[\mathcal{K}] \times \mathcal{K}$ with $d((g, y), (f, x)) < \min(\epsilon, \delta)$, then

$$
|f(x) - g(y)| \leq |f(x) - g(x)| + |g(x) - g(y)| \leq 2\epsilon.
$$

\[\Box\]

**Measurable Selection Theorem**

Let $\varphi$ be a correspondence of a measure space $(A, \mathcal{F}, \nu)$ into a complete separable metric space $S$ such that the graph of $\varphi$ is a $(\mathcal{F} \otimes \mathcal{B}(S))$ measurable set. Then there
exists a measurable function $f$ of $A$ into $S$ such that $f(\omega) \in \varphi(\omega)$, $\nu$-a.e.


Multiplicative System Theorem

Theorem Let $\mathcal{A}$ be a collection of subsets of $\Omega$ which contains $\Omega$ and is closed under intersection. Let $\mathcal{H}$ be a vector space of real valued functions on $\Omega$ satisfying

- If $A \in \mathcal{A}$ then $1_A \in \mathcal{H}$.
- If $\{f_n\}$ is a sequence of non-negative functions in $\mathcal{H}$ which increase to a bounded $f$, then $f \in \mathcal{H}$.

Then $\mathcal{H}$ contains all bounded $\sigma(\mathcal{A})$-measurable functions.

Chapter 3

Modelling Interest Rate Movements

3.1 Introduction

In 1979, the Federal Reserve Board abandoned its long standing policy of attempting to keep interest rates stable through manipulation of the money supply. It shifted its attention to the level of the money supply itself, and adopted a goal of long term stable, controlled growth of the money supply.

Perhaps as a result of this policy change, interest rate volatilities reached record highs in the early 1980's. Investors became aware of the concept of interest rate risk: the possibility of sustaining large losses or gains due to unforeseen changes in the level of interest rates. There has been recent interest in the development of theoretical models whose application could help practitioners manage interest rate risk.

In this chapter, we briefly summarize some of the existing approaches to modelling interest rate evolution under uncertainty. We begin with the more general theories, the earliest of which dates back to the 19th century. However, we focus on a class of
more specific and quantitative models, in which the approach is to choose a diffusion process representing the evolution of the 'spot' interest rate. We close the chapter by pointing out several drawbacks to the spot rate approach. In chapter 4, we present a class of interest rate models which avoid many of the drawbacks of the 'spot-rate' approach.

3.1.1 Definitions

The term structure of interest rates describes the relative prices of default-free bonds of varying maturity. A model of the term structure is an explanation of how the term structure is determined and/or how it may evolve over time.

Two equivalent ways to specify the term structure are used throughout this chapter and the next. We begin by providing definitions, and exhibiting the simple formulae used to translate from one description to the other.

Let $T$ be a fixed positive constant, considered to be the end of all economic activity. Let $T$ represent the triangular region $\{(s, t) : 0 \leq s \leq t \leq T\}$.

**Definition 3.1.1** Let $(s, t) \in T$. A contract guaranteeing a $\$1$ payment at time $t$ is called a (pure discount) bond due at $t$. At time $s$, the market price for such a contract is denoted by $P(s, t)$. Assume, for our present discussion, that $P$ is a (deterministic) strictly positive, $C^2$ function.

**Definition 3.1.2** For any $(s, t) \in T$, the interest rate in force at time $s$ for 'instantaneous' loans over the time interval $[t, t + dt]$ is called the forward rate at $t$ as seen from $s$, and is denoted by $f(s, t)$. Contracting at time $s$, the lender agrees to lend $\$1$ at time $t$, and the borrower agrees to repay $\$1 + f(s, t)dt$ at $t + dt$. Assume for now that $f$ is deterministic and jointly continuous.
Consider the cash flows to an investor who, at time \( s \), sells short a bond due at \( t \) and uses the proceeds to buy \( P(s, t)/P(s, t + dt) \) bonds due at \( t + dt \). At time \( s \) he breaks even, at \( t \) he pays \$1, and at \( (t + dt) \) he receives \( P(s, t)/P(s, t + dt) \). The interest earned over the period \((t, t + dt)\) is

\[
\frac{P(s, t)}{P(s, t + dt)} - 1 \approx \frac{-P_2(s, t)dt}{P(s, t + dt)}.
\]

So by making two trades at time \( s \), the investor can 'lock in' the rate \( f(s, t) \) on an instantaneous loan to be made at \( t \). It is clear that we must have

\[
f(s, t) = \frac{-P_2(s, t)}{P(s, t)}.
\]

The formula for converting from forward rates to equivalent bond prices is also well known:

\[
P(s, t) = \exp\left( - \int_s^t f(s, v) \, dv \right).
\]

Finally, we give a special name to the rate at which investors are contracting at time \( s \) for instantaneous loans at \( s \).

**Definition 3.1.3** The 'spot' rate at time \( s \) is

\[
r(s) = f(s, s).
\]

Since forward rates are the rates at which investors contract for future loans, it is natural to think of them as being the market's 'prediction' of future spot rates. For example, if all investors know with certainty the complete future course of interest rates, it must be that \( r(t) = f(s, t) \) for each \((s, t)\). Otherwise, by borrowing at the lower rate, and lending at the higher, any investor could reap unlimited profits. In an uncertain world, the relationship between forward rates and future spot rates is not so clear, and has been the subject of debate among economists for years.
3.1.2 Early Approaches

The expectations hypothesis was one of the earliest explanations of the determinants of the term structure under uncertainty. Building on the certainty model, the pure expectations hypothesis states that the forward rate \( f(s, t) \) is simply the expected future spot rate \( r(t) \) (given the information available at \( s \)) (see Malkiel [28], page 22 or Meiselman [29].) The idea can be traced back to Fisher’s work [15] in 1896.

The expectations hypothesis was originally intended to express some kind of risk neutrality on the part of investors, so is sometimes defined as the hypothesis that at time \( t \), each available bond offers the same instantaneous expected return (see, for example, Haugen [19], page 295). The two versions are not equivalent, in general, as was pointed out by Merton [30] and Jarrow [25]. Concrete examples of the discrepancy will appear in models developed in Chapter 4.

The liquidity premium theory was put forward by Hicks [20]. It asserts that since over short time horizons, long term bonds are riskier than short term ones, a risk premium (in the form of a higher expected return) must be paid to holders of such bonds as compensation. Such a premium would decrease long bond prices, and hence increase long forward rates. Taking the first version of the expectations hypothesis as a benchmark, Hicks argued that long forward rates were thus greater than expected future spot rates.

The market segmentation theory advanced by Culbertson [10] notes that the liquidity premium theory rests on the assumption that investors have short time horizons. Culbertson asserts that investors with different time horizons will in fact trade in separate markets, which will have little interaction. Hence term premiums will be unpredictable, depending on the supply and demand of bonds within each market.
3.2 The Spot Rate Approach

The theories described in the previous section apparently were intended as explanations of the general behaviour of the term structure, and not as specific models leading to actual computations. For example, the term 'expected' seems to be used as a synonym for 'anticipated', rather than to represent the mathematical expectation of some random variable.

In this section we describe a class of models for which the stochastic evolution of the term structure is specified in a more precise way. Specifically, we consider models developed by Vasicek [36], Dothan [13], Richard [31], Artzner & Delbaen [1], Cox, Ingersoll & Ross [9] and Brennan & Schwartz [7]. The Cox, Ingersoll & Ross model is an equilibrium model, while the others are known as arbitrage models. Later we shall comment on the distinction between the arbitrage and equilibrium methods, but we begin by focusing on the common feature of all the models mentioned: their emphasis on the stochastic behavior of the spot rate.

Let \{W_s : 0 \leq s \leq T\} be a Brownian motion defined on a probability space \((\Omega, F, Q)\), and adapted to a right continuous filtration \((F_s)\).

**Assumption 3.2.1** The spot interest rate process, \(r(\cdot)\), satisfies the stochastic differential equation

\[
    dr(s) = \mu(r(s)) \, ds + \sigma(r(s))dW_s.
\]

(3.2)

where \(\mu(\cdot)\) and \(\sigma(\cdot) > 0\) are known, real valued functions.

For the exact meaning of 3.2, and conditions on \(\mu(\cdot)\) and \(\sigma(\cdot)\) sufficient to ensure existence of a diffusion \(r(\cdot)\) satisfying it, see Karatzas & Shreve [27].

Spot rate advocates choose particular functions \(\mu(\cdot)\) and \(\sigma(\cdot)\) on the basis of mathematical convenience, economic intuition, empirical estimates, etc. The follow-
ing table shows the particular values of $\mu(\cdot)$ and $\sigma(\cdot)$ used in some of the models previously cited. (Parameters are assumed to be fixed, known constants.)

$$
\begin{array}{ccc}
\mu(r) & \sigma(r) \\
\kappa(\theta - r) & \sigma_0 \sqrt{r} \\
\alpha(\gamma - r) & \rho \\
0 & \sigma_0 r
\end{array}
$$

3.2.1 Arbitrage-Free Evolution

After selecting particular functions $\mu(\cdot)$ and $\sigma(\cdot)$, spot rate proponents make the following assumption.

**Assumption 3.2.2** The price at time $s$ of a discount bond expiring at $t$, when the spot rate is $r(s)$ is $P(s, t, r(s))$, for some (as yet unspecified) twice continuously differentiable $P$.

As usual, the choice of $P$ must be consistent with the no-arbitrage hypothesis, so one restriction on the form of $P$ is immediately obvious: for each $t$, $P(t, t, \cdot) \equiv 1$. An argument based on Merton's explanation [30] of the Black-Scholes formula, leads to a second necessary condition, that $P$ satisfy a certain parabolic differential equation. Because it leads to a definition of the market price of risk, we present it here.

At time $s$, consider a portfolio consisting of one $T_1$-bond and $\alpha$ $T_2$-bonds, where $(s, T_1)$ and $(s, T_2) \in T$. Let

$$V(s) = P(s, T_1, r(s)) + \alpha P(s, T_2, r(s)).$$
For notational ease, let $P^i = P(s, T_i, r(s))$, $P_r = \frac{\partial P}{\partial r}$, $P_{rr} = \frac{\partial^2 P}{\partial r^2}$, $P_s = \frac{\partial P}{\partial s}$. Set $\alpha = -\frac{\sigma(r(s))P^1_r}{\sigma(r(s))P^2_r} = -\frac{P^1_r}{P^2_r}$, use Ito’s Lemma to get:

$$dV(s) = \left( P^1_s + \mu(r(s))P^1_r + 1/2\sigma^2(r(s))P^1_{rr} \right) ds - \frac{\sigma(r(s))P^1_r}{\sigma(r(s))P^2_r} \left( P^2_s + \mu(r(s))P^2_r + 1/2\sigma^2(r(s))P^2_{rr} \right) ds.$$ 

We have chosen $\alpha$ so that $dV(s)$ has no $dW_s$ term. Although trading strategies and thus arbitrage opportunities are not specified precisely, Merton’s argument proceeds by noting that the portfolio is ‘riskless’ and so must earn the spot rate, $r(s)$. Equating the right hand side to $r(s)V(s)ds$ gives the condition

$$\frac{P^1_s + \mu(r(s))P^1_r + \frac{1}{2}\sigma^2(r(s))P^1_{rr} - r(s)P^1}{\sigma(r(s))P^1_r} = \frac{P^2_s + \mu(r(s))P^2_r + \frac{1}{2}\sigma^2(r(s))P^2_{rr} - r(s)P^2}{\sigma(r(s))P^2_r}. \quad (3.3)$$

Since $T_1$ and $T_2$ are arbitrary, the quantity

$$\frac{P_s + \mu(r(s))P_r + \frac{1}{2}\sigma^2(r(s))P_{rr} - rP}{\sigma(r(s))P_r} \quad (3.4)$$

must be independent of bond maturity (although it may depend on $s$ and $r$). Denote the common value by $\lambda(s, r)$. We have arrived at another necessary condition for $P$, that it solve the parabolic differential equation

$$P_s + \mu(r)P_r + \frac{1}{2}\sigma^2(r)P_{rr} - rP = \lambda(s, r)\sigma(r)P_r \quad (3.5)$$

with boundary condition $P(s, s, \cdot) = 1$.

Across all maturities, the ratio of ‘excess’ (over $rP$) expected return to dispersion coefficient is forced to be constant. The common ratio is often called the market price of risk, because it reflects the excess instantaneous expected return paid by each asset in exchange for its component of risk (its $dW_s$ coefficient.) Thus the necessary
condition expressed by the p.d.e. is that a market price of risk, independent of 
maturity, must exist.

Examination of this argument shows that we really only used the fact that $P$
gives bond prices to determine the boundary conditions of (3.5). Hence the price 
function of any other asset (such as a call option) satisfies the same equation, with 
different boundary conditions.

From (3.5) we also see that if $(\mu_1(\cdot), \lambda_1(\cdot, \cdot), \sigma(\cdot))$ and $(\mu_2(\cdot), \lambda_2(\cdot, \cdot), \sigma(\cdot))$ are 
choices of model parameters satisfying

$$\mu_1(r) - \lambda_1(s, r)\sigma(r) \equiv \mu_2(r) - \lambda_2(s, r)\sigma(r),$$

then for each $(s, r)$ bond prices and contingent claims prices will be identical in the 
two models. Unfortunately, the form of this relation does not allow us to calculate 
without knowing $\lambda(\cdot, \cdot)$, as can be done in the Black-Scholes model. (This and other 
drawbacks of the spot rate approach are discussed at greater length in the final 
section of this Chapter.)

Artzner & Delbaen [1] study the spot rate approach using martingales. In their 
model, investors trade in an economy with only two assets: a bond expiring at $T$, 
and a ‘bank account’ process $B(s)$, where

$$B(s) = \exp \left( \int_0^s r(u) \, du \right).$$

Allowable trading strategies are the simple trading strategies of Harrison & Kreps [17].

**Theorem 3.2.1 (Artzner & Delbaen)** To preclude arbitrage, it is necessary and 
sufficient that $P(s, r(s))/B(s)$ be a $Q^*$ martingale where $Q^* \sim Q$. Under assumption 
3.2.2, the Radon-Nykodym derivative can be written

$$dQ^*/dQ = \exp \left( \int_0^T \tilde{q}(s, r(s))dW_s - \frac{1}{2} \int_0^T \tilde{q}(s, r(s))^2 ds \right), \quad (3.6)$$
where \( \bar{q}(s, r(s)) \) satisfies

\[
\int_0^T \bar{q}(s, r(s))^2 ds < \infty \quad a.s.
\]

An application of Girsanov's theorem shows that, under \( Q \), the spot rate is still a diffusion, and satisfies

\[
dr(s) = (\mu(r(s)) + \sigma(r(s))\bar{q}(r(s), s))ds + \sigma(r(s))dW^*_s,
\]

where \( W^* \) is a \( Q^* \) Brownian motion. Under the further assumption that \( \bar{q}(\cdot, \cdot) \) is \( C^\infty \), they apply the Feynman-Kac formula to (3.7), and conclude that \( \bar{P}(s, r) \) satisfies the parabolic partial differential equation

\[
\bar{P}_s + (\mu(r) + \sigma(r)\bar{q}(r, s))\bar{P}_r + \frac{1}{2} \sigma(r)^2 \bar{P}_{rr} - r\bar{P} = 0
\]

with boundary conditions \( \bar{P}(T, \cdot) \equiv 1 \).

Aside from the fact that in Artzner & Delbaen's economy only one bond trades, there is an obvious similarity between differential equations (3.5) and (3.8). In the latter, the function \( \bar{q}(\cdot, \cdot) \) is acting as a price of risk.

### 3.2.2 Arbitrage vs. Equilibrium

We continue with the development of a generic 'spot rate' model. In view of the necessary condition that \( P \) must satisfy (3.5), to complete the specification of the model, each author is left to choose a market price of risk function \( \lambda(\cdot, \cdot) \).

Some authors, for tractability, take \( \lambda(\cdot, \cdot) \) to be constant. Cox, Ingersoll & Ross (CIR), on the other hand, emphasize that since Merton's arbitrage argument only yields a necessary condition, an arbitrary choice of \( \lambda(\cdot, \cdot) \) might not lead to the preclusion of arbitrage. In fact, they give an example of what can go wrong.
An equilibrium model begins with a description of the tastes, endowments and production possibilities of agents, and the mechanism through which they can trade, and seeks to find a pricing scheme, at which 'markets clear'. Being much more comprehensive than an arbitrage model, a typical equilibrium model also relies on more assumptions, particularly with regard to agents' risk attitudes and preferences. CIR develop an equilibrium model of the 'Spot Rate' type, with \( \mu(r) = \kappa(\theta - r) \), \( \sigma(r) = \sigma_0 \sqrt{r} \). They derive the differential equation

\[
P_s + \kappa(\theta - r)P_r + \frac{1}{2} \sigma_0^2 r P_{rr} - rP = \psi(r, s) P_r,
\]

which is (3.5) with \( \psi(r, s) = \lambda(r, s) \sigma(r) \). The model generated by their equilibrium corresponds to setting \( \psi(r, s) = \lambda r \) (using \( \lambda \) to distinguish their constant \( \lambda \) from our function \( \lambda \)). In our notation,

\[
\lambda(r) = \frac{\lambda r}{\sigma_0 \sqrt{r}} = \frac{\lambda \sqrt{r}}{\sigma_0}.
\]

They then remark that a deficiency of the no-arbitrage approach (versus the equilibrium approach) is that certain choices of \( \psi(\cdot, \cdot) \) lead to models guaranteeing, not precluding, arbitrage. As an example, they pick

\[
\psi(r, s) = \psi_0 + \lambda r,
\]

which corresponds to

\[
\lambda(r) = \frac{\psi_0 + \lambda r}{\sigma_0 \sqrt{r}}. \tag{3.9}
\]

Under our notation, the market price of risk \( \lambda(s, r) \) matches the \( \tilde{q}(s, r) \) appearing in the Radon-Nykodym derivative expression (3.6).

Choosing \( \lambda(\cdot, \cdot) \) from (3.9), if \( \psi_0 \neq 0 \), it is easy to check that

\[
E \left[ \exp \left( \int_0^T \tilde{q}(s, r(s)) dW_s - \frac{1}{2} \int_0^T \tilde{q}(s, r(s))^2 ds \right) \right] < 1
\]
and so no equivalent measure makes the discounted price of a $T$-bond a martingale.  

Thus, the problem does not lie with the arbitrage approach itself, but with the fact that the existence of a market price of risk is necessary, but not sufficient to preclude arbitrage. The martingale approach leads to a sufficient condition.

3.3 Problems with Spot Rate Models

3.3.1 Matching Observable Prices

The first criticism of the spot rate approach is a practical one. Once $\mu(\cdot), \sigma(\cdot),$ and $\lambda(\cdot, \cdot)$ are chosen, the function $P$ is determined by (3.5). In particular, $P(0, s, r(0))$ is determined. But at time 0, we may observe bond prices in the market. What if they do not match those generated by the model?

One solution is to specify the functional forms of $\mu(\cdot), \sigma(\cdot, \cdot),$ and $\lambda(\cdot, \cdot)$ in terms of enough parameters so that the generated prices $P(0, \cdot, r(0))$ can be fit closely to the observable prices. This is unsatisfactory from a modelling point of view. A typical parameter now has two roles: to help fit the initial curve, and to determine the form of the random evolution. Parameter estimates would thus be unstable, due to the changing shape of the curve being fit.

3.3.2 The Market Price of Risk

Recall that under the martingale measure of the Black-Scholes stock option model, the form of the stock price evolution does not involve the drift $\mu$, and the market price of risk $\mu/\sigma$. In fact, the independence of the Black-Scholes formula from risk reflecting parameters (which are not easily estimated) was one of the main reasons for its great success.

Unfortunately, the spot rate models for interest rate evolution do not share the
same property. Under the martingale measure $Q^*$, (3.7) shows that the character of the spot rate evolution depends on the price of risk function $q(\cdot, \cdot)$. Consequently, to use a spot rate model, even to price contingent claims, a market price of risk function $\bar{q}$ must be specified. In other words, like the Black-Scholes model, the price of risk function determines which of the many equivalent measures makes asset prices martingales, but unlike the Black-Scholes model the evolution of asset prices under that measure depends on the price of risk.

The reason for this is the role of the parameters under a spot rate model. The functions $\mu(\cdot)$ and $\sigma(\cdot)$ are natural choices to describe the behaviour of a diffusion such as $r$, but this parameterization does not lead to the desirable ‘cancelling out’ effect of Black-Scholes. As we remarked earlier, however, varying $\mu(\cdot)$ and $\lambda(\cdot, \cdot)$ without changing $\mu(\cdot) + \lambda(\cdot, \cdot)\sigma(\cdot)$ preserves contingent claim prices. Another difference between Black-Scholes and spot rate models is that there is no parameter which when varied, changes the price of a general contingent claim while preserving the prices of bonds.

3.3.3 Markovian Spot Rates

If a unique, weak solution to (3.2) exists, then under regularity conditions ($\mu(\cdot)$ and $\sigma(\cdot)$ bounded on compact sets) the $r(\cdot)$ process will have the Markov property. (see Theorem 5.4.20 of Karatzas and Shreve [26].) An implication of a Markovian spot rate model is that all information about the future course of interest rates is contained in the current spot rate. In particular, all other forward rates have no additional influence, contradicting economic intuition that forward rates are market predictions of future spot rates.
Chapter 4

A New Class of Bond Pricing Models

4.1 Introduction

In this chapter we present a new class of models for the evolution of the term structure of interest rates. Our models are designed to enable calculation of 'contingent claim' values (options on bonds, interest rate 'caps', etc.), while avoiding the problems associated with the spot rate approach discussed in the previous chapter.

We begin by contrasting our modelling approach to that of the spot rate approach discussed in chapter 3. We then set out the general form of the stochastic differential equations governing the movement of forward rates in our model. The drift coefficients in these equations are at first unspecified; later we show that the condition that (discounted) bond prices be martingales uniquely determines the drift coefficients. Some simple examples of models within the class illustrate the basic ideas, and permit calculation of closed form expressions for the prices of contingent claims. We provide an existence and uniqueness theorem for solutions to the resulting fam-
ily of stochastic differential equations. Finally, we show by a counterexample that the strong restrictions on the coefficients imposed in the existence and uniqueness theorem cannot be substantially weakened.

4.2 Modelling Approach

Two fundamental differences distinguish our approach from the spot rate approach discussed in chapter 3. The first is that we directly specify the stochastic evolution of the entire forward rate curve. The second is that we deliberately only specify the form of the evolution under the martingale measure.

A complete model for the evolution of bond prices implicitly describes the evolution of forward rates, through the mapping from bond prices to rates. In our models, we directly specify the evolution of forward rates.

One reason for doing this is that the forward curve has a natural economic interpretation, and a particularly simple evolution in the degenerate case of no uncertainty - the curve remains unchanged over time. Generating bond prices from forward rates by equation 3.1 avoids the potentially troublesome constraint $P(t,t,\cdot) \equiv 1$.

A second reason relates to the qualitative nature of the models which we construct. In spot rate models, $\mu(\cdot)$ and $\sigma(\cdot)$ are chosen to yield a desired evolution for the spot interest rate. The effect of these choices on other quantities of interest is not as clear. For example, Jamshidian [24] notes that under the Cox, Ingersoll and Ross [9] model, the volatility of longer forward rates approaches zero. This may or may not be a desirable property, but it is not an obvious consequence of their choice $\mu(r) = \kappa(\theta - r)$ and $\sigma(r) = \sigma_0 \sqrt{r}$. Directly modelling the effect of randomness on the forward curve, makes the qualitative nature of various models more transparent.

We allow $d$ sources of uncertainty, modelled by a $d$-dimensional Brownian motion,
to affect the evolution of the forward curve over time. Our goal is to build models suitable for contingent claims pricing, so parameters controlling the influence of randomness enter via the dispersion coefficients in the stochastic differential equations for \( f \). Recall that under the Black-Scholes model, the parameter determining option prices, \( \sigma \), enters as a dispersion coefficient. Unlike parameters affecting the 'drift' of a process, the role of dispersion parameters is preserved under substitution of an equivalent probability measure, explaining their importance in contingent claims pricing.

Our object of study, the forward rate process, is a two time parameter stochastic process: \( \{ f(s, t, \omega) : 0 \leq s \leq t \leq T \} \) is the path of forward rates observed when the state of the world is \( \omega \). Given an arbitrary (Lipschitz continuous) function \( I(\cdot) \), representing forward rates observable at time 0, we construct an \( f \) having three properties: the implied evolution of bond prices creates no arbitrage opportunities, the form of the movement corresponds to pre-specified volatility parameters, and \( f(0, t) = I(t) \). In general, the spot rate process \( f(t, t) \) will not be Markovian.

As our focus is on contingent claim pricing, we need only study the interest rate evolution under the probability measure for which (discounted) asset prices (i.e. bond prices) are martingales. Under this measure, the pricing formula for a contingent claim is particularly simple, being just the (discounted) expected value of the future payment. For each such martingale measure, there is a family of equivalent measures, under which the nature of the interest rate evolution is different, but contingent claims prices are the same. We give examples of what kind of 'real-world' evolutions are possible, but the focus of our theory is on the evolution under the martingale measure.
4.3 Specification of the Model

Let $T$ be a fixed positive constant, considered to be the end of all economic activity. Let $T$ represent the triangular region $\{(s, t) : 0 \leq s \leq t \leq T\}$.

Let $\{W^s_s = W^s_0, \ldots, W^s_T : 0 \leq s \leq T\}$ be a $d$-dimensional Brownian motion defined on a probability space $(\Omega, F, Q)$. Let $(F_s)$ be the $Q^*$ augmentation of the filtration $(\sigma(W_u : 0 \leq u \leq s) : 0 \leq s \leq T)$. $(F_s)$ then satisfies the usual conditions: $F_0$ contains all null sets, and $F_s$ is right continuous.

Our aim is to construct a stochastic process, an indexed family of random variables, representing the evolution of forward interest rates. A slight complication is introduced by the fact that our processes are indexed by two time parameters: $f(s, t, \omega)$ is the forward rate for time $t$ as seen from $s$, when the state of the world is $\omega$. At each time $s$, $f(s, \cdot)$ is the observed forward curve. On the other hand, the process $f(\cdot, t)$ represents the fluctuations over time of the rate being charged for instantaneous loans to be made at $t$.

For descriptive convenience, we extend the single time parameter notion of an adapted process.

**Definition 4.3.1** A jointly measurable random field $h : T \times \Omega \rightarrow \mathbb{R}$ is adapted if, for each $(s, t) \in T$, $h(s, t, \cdot) \in F_s$.

Let $\sigma_i : i = 1, \ldots, d$ be positive, bounded, Borel measurable functions from $\mathbb{R} \times T$ to $\mathbb{R}$. Let $I(\cdot)$ be a (non-random) Lipschitz continuous function on $[0, T]$. Let $\alpha : T \times \Omega \rightarrow \mathbb{R}$ be an adapted, jointly continuous, non-negative process.

A process $f : T \times \Omega \rightarrow \mathbb{R}$ is said to be a solution of the family of equations

$$f(s, v) = I(v) + \int_0^s \alpha(u, v)du + \sum_{i=1}^d \int_0^s \sigma_i(f(u, v), u, v)dW^s_u \quad \forall (s, v) \in T$$  \hspace{1cm} (4.1)
if

1. $f$ is adapted, and has jointly continuous sample paths.

2. For each $(s, v) \in T$, the random variable defined by the right hand side is $(Q \ a.s.)$ equal to $f(s, v)$. The first integral is interpreted as a path-by-path Lebesgue integral, and the others are defined for each $(s, v)$ by the usual definition of the stochastic integral. If $f$ satisfies condition 1., these stochastic integrals are well-defined by the boundedness and measurability of $\sigma_i$.

The $i$'th Brownian motion component influences $f$ through the volatility function $\sigma_i$. Writing $\sigma_i$ as $\sigma_i(f(u, v), u, v)$ reflects an assumption that the volatility of a particular forward rate depends only on that rate value, and either time coordinate. This is a modelling assumption which seems to match economic intuition.

The drift process $\alpha$ is as yet unspecified. We next prove our assertion that $\alpha$'s only role is to preclude arbitrage opportunities.

### 4.4 Condition for Martingale Bond Prices

The relation between forward rates and bond prices now reappears. Having argued that it is most natural to model the influence of randomness on forward rates, we set out a model doing this. However, the equivalent martingale measure techniques for precluding arbitrage apply not to interest rates, but to prices of traded assets. In this section, we show that a unique choice of the drift process, $\alpha(\cdot, \cdot)$, makes the (discounted) price of each bond a martingale. In a corollary, we invoke Girsanov's theorem to describe the allowable drift processes if we view the process under some $Q \sim Q^*$. 
To reflect the fact that investors can borrow and lend at the spot interest rate, we define a bank account process \( B(\cdot) \) by

\[
B(s) = \exp \left( \int_0^s f(v, v) dv \right).
\]

Buying the ‘bank account’ process corresponds to investing money at the spot rate. We shall preclude arbitrage by insisting that discounted (by the bank account) asset prices be martingales.

We will need the following special case of a lemma of Ikeda and Watanabe [23], page 116.

**Lemma 4.4.1** With the probability space and filtration as defined above, and with any \( 1 \leq i \leq d \), let \( \{ h(u, v, \omega) \} \) be a family of real random variables such that

1. The map \( ((u, \omega), v) \rightarrow h(u, v, \omega) \) from \( ([0, T] \times \Omega) \times [0, T] \) to \( \mathbb{R} \otimes B(\mathbb{R}) \) measurable, where \( \mathcal{P} \) is the usual predictable \( \sigma \)-algebra of \( ([0, T] \otimes \Omega) \), generated by sets of the form \( \{ A \times (t_1, t_2] : A \in \mathcal{F}_{t_1} \} \).

2. \( h \) is bounded.

3. The map

\[
(v, \omega) \rightarrow \int_0^s h(u, v, \omega) dW^i_u
\]

is \( B([0, T]) \otimes \mathcal{F} \)-measurable for each \( s \in [0, T] \). (If conditions 1. and 2. hold, then the right hand side is well defined, and a continuous square integrable martingale, for each \( v \).)

Then

\[
u \rightarrow \int_0^T h(u, v, \omega) dv
\]

is predictable and

\[
E \int_0^T \left( \int_0^T h(u, v) dv \right)^2 du < \infty
\]
and
\[ \int_0^s \int_0^T h(u, v, \omega) dv dW_u^v = \int_0^T \int_0^s h(u, v, \omega) dW_u^v dv. \]

**Corollary 4.4.1** With the assumptions of the Lemma, define, for fixed t,
\[ M^i(s) \triangleq \int_0^t \int_0^s h(u, v) dW_u^v dv. \]

Then each \((M^i(s), \mathcal{F}_s)\) is a continuous, square integrable martingale, and
\[ <M^i, M^j>_s = \delta_{ij} \int_0^s \left( \int_0^t h(u, v) dv \right)^2 du. \]

**Proof:** If \(h\) satisfies the conditions of the lemma, then so does \(h(u, v)1_{\{v \leq t\}}\). The rest of the proof is immediate from the lemma and properties of stochastic integrals.
\[ \square \]

The discounted price at time \(s\) of a bond expiring at \(t\) is
\[ Z(s, t) \triangleq \frac{P(s, t)}{B(s)} = \frac{\exp(- \int_s^t f(s, v) dv)}{\exp(\int_0^s f(v, v) dv)} \]

**Theorem 4.4.1** Let \(f\) be a solution to (4.1). For each fixed \(t\), \((Z(s, t), \mathcal{F}_s)\) is a \(Q^*\) martingale if and only if
\[ \alpha(u, v) = \sum_{i=1}^d \sigma_i(f(u, v), u, v) \int_u^v \sigma_i(f(u, y), u, y) dy. \quad (4.2) \]

**Proof:** Fix \(t \in [0, T]\). Define
\[ S(s) \triangleq \int_0^s f(v, v) dv + \int_s^t f(s, v) dv, \]
so
\[ Z(s, t) = \exp(-S(s)). \]
We begin by showing that $S(s)$ is a semimartingale, so that an application of Ito’s lemma is justified.

\[
S(s) = \int_0^t f(s \land v, v) dv = \int_0^t I(v) dv + \int_0^t \int_0^{s \land v} \alpha(u, v) dudv + \sum_{i=1}^d \int_0^t \int_0^{s \land v} \sigma_i(f(u, v), u, v) dW^i_u.
\]

If $f$ is a solution to (4.1), then the conditions of Lemma 4.4.1 are met when $h(u, v) = \sigma_i(f(u, v), u, v) 1_{\{u \leq v\}}$. Apply Corollary 4.4.1 for each $i : 1 \leq i \leq d$ in turn, to see that

\[
M(s) \triangleq \sum_{i=1}^d \int_0^t \int_0^{s \land v} \sigma_i(f(u, v), u, v) dW^i_u dv
\]

is a martingale which can be written

\[
\sum_{i=1}^d \int_0^t \int_0^{s \land v} \sigma_i(f(u, v), u, v) 1_{\{u \leq v\}} dv dW^i_u.
\]  \hspace{1cm} (4.3)

Furthermore

\[
\langle M \rangle_s = \sum_{i=1}^d \int_0^s \left( \int_0^t \sigma_i(f(u, v), u, v) 1_{\{u \leq v\}} dv \right)^2 du
\]

\[= \sum_{i=1}^d \int_0^s \left( \int_0^t \sigma_i(f(u, v), u, v) dv \right)^2 du.
\]

Also,

\[
A(s) \triangleq \int_0^t \int_0^{s \land v} \alpha(u, v) dudv
\]

\[= \int_0^t \int_0^s \alpha(u, v) 1_{\{u \leq v\}} dudv
\]

\[= \int_0^s \int_0^t \alpha(u, v) 1_{\{u \leq v\}} dvdu
\]

(by Fubini’s Theorem, since $\alpha \geq 0$)

is of bounded variation, so the decomposition

\[
S(s) = S(0) + M(s) + A(s),
\]
with

\[ S(0) = \int_0^t I(v) dv, \]

shows that \( S \) is a semimartingale. Applying Itô's lemma to \( g(x) = \exp(-x) \) gives

\[
g(S(s)) - g(S(0)) = \int_0^s g'(S(u))dS(u) + 1/2 \int_0^s g''(S(u))d<S>_u
\]
\[
= -\int_0^s \exp(-S(u))dS(u) + 1/2 \int_0^s \exp(-S(u))d<S>_u
\]
\[
= -\int_0^s \exp(-S(u))dM(u) - \int_0^s \exp(-S(u))dA(u) + 1/2 \int_0^s \exp(-S(u))d<M>_u
\]

The first term here is a local martingale, and is actually a martingale if

\[ E \int_0^T \exp(-2S(u))d<M>_u < \infty. \quad (4.4) \]

By the decomposition of \( S \), the boundedness of \( S(0) \), the non-negativity of \( \alpha \), and the boundedness of each \( \sigma_i \), we need only verify that

\[ E \int_0^T \exp(-2M(u))du < \infty. \]

For notational simplicity, write (4.3) as

\[ M(u) = \sum_{i=1}^d \int_0^u b_i^i dW^i_y, \]

where each \( b_i^i \) is bounded. Then

\[
E \exp(-2M(u)) = E \exp(-2 \sum_{i=1}^d \int_0^u b_i^i dW^i_y)
\]
\[
= E \left( \exp(-2 \sum_{i=1}^d \int_0^u b_i^i dW^i_y - 1/2 \sum_{i=1}^d \int_0^u 4(b_i^i)^2 dy) \right) \cdot \exp \left( 1/2 \sum_{i=1}^d \int_0^u (b_i^i)^2 dy \right).
\]
We can now apply Novikov’s condition (see Karatzas and Shreve [27], page 198) which guarantees that if \( N = \{ N_s : 0 \leq s \leq T \} \) is a continuous local martingale, the exponential supermartingale \( \exp(N_s - 1/2 <N>_s) \) is a martingale if \( E \exp(1/2 <N>_T) < \infty \). Applying this result here (using the boundedness of \( b \)) shows that the first term alone has expectation 1. The second term is bounded, and we have verified (4.4).

For \( \exp(-S(s)) \) to be a martingale, it is therefore necessary and sufficient that

\[
A(s) = 1/2 <M>_s \quad \text{Q} \times \lambda \ a.s.
\]

From the definitions, we get

\[
\int_0^t \int_0^t \alpha(u,v)1_{\{u \leq v\}} du dv = 1/2 \sum_{i=1}^d \int_0^t \left( \int_v^t \sigma_i(f(u,v),u,v) dv \right)^2 du
\]

which holds if and only if

\[
\int_0^t \alpha(u,v)1_{\{u \leq v\}} dv = 1/2 \sum_{i=1}^d \left( \int_v^t \sigma_i(f(u,v),u,v) dv \right)^2
\]

\[\square\]

**Corollary 4.4.2** For each \( t \), the bond price process \( P(\cdot,t) \) is a semimartingale and

\[
P(s,t) = E^\nu \left[ \exp \left( - \int_s^t f(v,v) dv \right) \big| \mathcal{F}_s \right]
\]

**Proof:** From the identity \( P(s,t) = Z(s,t)B(s) \), the bounded variation of \( B(s) \), and Ito’s lemma we can conclude that \( P(\cdot,t) \) is also a semimartingale. Also

\[
P(s,t) = Z(s,t)B(s)
\]

\[
= E^\nu[Z(t,t)|\mathcal{F}_s]B(s)
\]
\[ E^* \left[ \frac{1}{B(t)} | \mathcal{F}_s \right] B(s) \]
\[ = E^* \left[ \frac{B(s)}{B(t)} | \mathcal{F}_s \right] B(s) \]
\[ = E^* \left[ \exp \left( - \int_{s}^{t} f(v, v) dv \right) | \mathcal{F}_s \right] \]

Notice the natural analog between this expression, and the corresponding expression under certainty. \(\square\)

**Corollary 4.4.3** Let \( f \) solve (4.1), with

\[ \alpha(u, v) = \sum_{i=1}^{d} \sigma_i(f(u, v), u, v) \int_{u}^{v} \sigma_i(f(u, y), u, y) dy. \]

Suppose \( Q \) is another probability measure on \((\Omega, \mathcal{F})\), equivalent to \( Q^* \), and for some adapted, measurable, process \( \phi(\cdot) \)

\[ dQ^*/dQ = \exp \left( \int_{0}^{T} \phi(s) \cdot dW_s - 1/2 \int_{0}^{T} ||\phi(s)||^2 ds \right). \]

Let

\[ W_s \triangleq W^*_s + \int_{0}^{s} \phi(u) dW^*_u. \]

\( W = \{ W_s : 0 \leq s \leq T \} \) is a \( Q \) Brownian motion, and under \( Q \), \( f \) satisfies

\[ f(s, v) = I(v) + \int_{0}^{s} \alpha(u, v) du + \sum_{i=1}^{d} \int_{0}^{s} \sigma_i(f(u, v), u, v) dW^*_u \quad \forall (s, v) \in \mathcal{T} \]

with

\[ \alpha(u, v) = \phi(u) \cdot \sigma(f(u, v), u, v) + \sum_{i=1}^{d} \sigma_i(f(u, v), u, v) \int_{u}^{v} \sigma_i(f(u, y), u, y) dy. \]

**Proof:** Immediate from Girsanov's theorem. \(\square\)
4.5 Constant Volatility

The simplest model of our class is the one obtained by setting $d = 1$ and $\sigma(f, s, t) \equiv \sigma_0$, for some positive constant $\sigma_0$. We shall call this the constant volatility model, because under it, the volatility of a point on the forward curve does not depend on the forward rate value, or either time coordinate. The influence of randomness is simply to shift the entire forward curve up or down, in an additive fashion (i.e. independent of the level of rates). This example illustrates many of the properties of our models, and is certainly the simplest of the class. Since we obtain, by inspection, solutions to the evolution equations, we do not yet need the general existence and uniqueness results of the next section.

Ho & Lee [22] develop a discrete time model for the evolution of bond prices, and in that context introduced the idea of viewing an observed curve as the starting point of a stochastic process. In the Appendix to this chapter, we show that our constant volatility model is the natural continuous time limit of Ho & Lee’s model. The simple form of the forward rate evolution is another argument for using forward rates rather than bond prices as the vehicle for describing the term structure. Easy calculations with this model give useful closed form expressions for the prices of many contingent claims. We demonstrate how to do the calculations to price European options on pure discount and coupon bonds; formulae giving prices of ‘caps’, ‘floors’, and other claims can also be obtained.

The forward rate evolves, then, according to

$$df(s, t) = \alpha(s, t) \, ds + \sigma_0 dW_s.$$  

For simplicity, we first examine the evolution under the martingale measure, $Q^*$. We then give an example to show what kind of forward rate evolution is possible under
equivalent 'real-world' probability measures $Q$.

The drift formula (4.2) shows that for bond prices to be $Q^*$ martingales we must have

$$\alpha(s, t) = \sigma_0^2 (s - t).$$

The solution is easily seen to be

$$f(s, t) = f(0, t) + \sigma_0^2 (st - \frac{s^2}{2}) + \sigma_0 W^*_s.$$

Observe that under this model, forward rates become negative with positive probability (under $Q^*$, and hence under any equivalent $Q$). The spot rate behaviour can be read off from the above expression,

$$r(s) = f(0, s) + \sigma_0 W^*_s + \sigma_0 \frac{s^2}{2}. \quad (4.6)$$

Under the martingale measure, the constant volatility model results in the spot interest rate at time $s$ being its predicted value at time 0, plus the value of a Brownian motion at time $s$, plus a deterministic term of $\sigma_0 \frac{s^2}{2}$. This last term may appear strange, but it must be remembered that this expression describes the behaviour of the spot rate under $Q^*$.

The discrepancy between different versions of the expectations hypothesis discussed in Chapter 3 can be seen in (4.6). Under $Q^*$, expected future spot interest rates are larger than current forward rates, so the pure expectations hypothesis described by Malkiel [28] does not hold. However under $Q^*$, each bond has the same instantaneous expected return $r(s)$.

As remarked earlier, contingent claims prices are determined solely by the evolution of the rate process under the martingale measure. Associated with a particular evolution under a martingale measure $Q^*$, are many ‘real world’ evolutions for which
claims prices would be identical. These ‘real world’ measures permit a surprisingly wide class of behaviors consistent with a given set of claims prices. As an illustration, we exhibit an equivalent measure for the constant volatility model under which the spot rate is mean reverting. Define

\[ X(s) \triangleq \frac{\kappa(\theta - r(s))}{\sigma_0} - \sigma_0 s. \]

It is easy to check that

\[ \int_0^T X(u)^2 du < \infty \quad Q^* \text{ a.s.,} \]

and so we can define

\[ R(s) \triangleq \exp \left( \int_0^s X(u) dW_u^* - \frac{1}{2} \int_0^s X(u)^2 ds \right). \]

It is also easy to check that

\[ E^* \left[ \exp \left( 1/2 \int_0^T X(u)^2 du \right) \right] < \infty, \]

so Novikov’s condition assures us that the exponential supermartingale \((R(s), F_s)\) is in fact a martingale, and \(R(T)\) a valid Radon-Nykodym derivative. By Girsanov’s theorem

\[ W_s \triangleq W_s^* - \int_0^s X(u) du \]

is a Brownian motion under the equivalent probability measure \(Q\) defined by

\[ dQ/dQ^* = R(T). \]

How does the spot rate evolve under \(Q\)? For simplicity assume that \(f(0, t) \equiv f_0\) for some \(f_0\). Then

\[
\begin{align*}
  r(s) &= f_0 + \frac{\sigma_0^2 s^2}{2} + \sigma_0 W_s^* \\
  &= f_0 + \frac{\sigma_0^2 s^2}{2} + \sigma_0 \left( W_s + \int_0^s \frac{\kappa(\theta - r(u))}{\sigma_0} du - \frac{\sigma_0 s^2}{2} \right) \\
  &= f_0 + \sigma_0 W_s + \int_0^s \kappa(\theta - r(u)) du
\end{align*}
\]
So, under $\tilde{Q}$, $r(\cdot)$ satisfies the stochastic differential equation

$$dr(t) = \kappa(\theta - r(t))dt + \sigma_0 d\tilde{W}_t,$$

and so is mean reverting.

### 4.5.1 European Call Option Formulae

Under the constant volatility model, the time $t$ price of a $\$1$ pure discount bond expiring at $T$ is

$$P(t, T) = \exp \left( -\int_t^T f(0, s)ds - \frac{\sigma_0^2}{2} \int_t^T sds + (T - t)\frac{\sigma_0^2 t^2}{2} - \sigma_0(T - t)W_t^* \right)$$

$$= \exp \left( -\int_t^T f(0, s)ds - \frac{\sigma_0^2(T - t)}{2} T - \sigma_0(T - t)W_t^* \right)$$

Using this formula, we can derive simple expressions for the price of a European call option on pure discount bonds, and coupon bonds. Let $X(t)$ denote the time $t$ price of an instrument which pays to its holder $\$C_i$ at time $T_i$, $1 \leq i \leq n$, with

$$0 \leq t \leq T_1 \leq \cdots \leq T_i \leq \cdots \leq T_n.$$  

Such an instrument is called a coupon bond.

Obviously, we have

$$X(t) = \sum_{i=1}^n C_i P(t, T_i).$$

**Definition 4.5.1** A European call option with strike price $K$ and expiration date $t$ is a contract giving its holder the right to purchase the coupon bond for $\$K$ at time $t$.

We will need the following calculation.

**Lemma 4.5.1** Let $\tilde{W}_s^*$ be a Brownian bridge conditioned to hit $u$ at time $t$. Then

$$E^*[\exp(-\sigma_0 \int_0^t \tilde{W}_s^* ds)] = \exp(\frac{\sigma_0^2 t^3}{24} - \frac{ut\sigma_0}{2}).$$
Proof: One can define the Brownian bridge $\tilde{W}_u^* : 0 \leq u \leq t$ by

$$\tilde{W}_u^* = W_u^* - \frac{u}{t}(W_t^* - u).$$

Now

$$\int_0^t \tilde{W}_s^* ds$$

is Gaussian. A simple calculation and an application of Fubini’s theorem gives

$$E^*[\int_0^t \tilde{W}_s^* ds] = \frac{ut}{2}$$

and

$$E^*[(\int_0^t \tilde{W}_s^* ds)^2] = \frac{t^3}{12} + \frac{u^2t^2}{4}.$$  

Using the moment generating function for a one dimensional Gaussian random variable, we get

$$E^*[\exp(-\sigma_0 \int_0^t \tilde{W}_s^* ds)] = \exp(\frac{\sigma_0^2 t^3}{24} - \frac{ut \sigma_0}{2}).$$

□

We can now calculate the call price, $C$.

$$C = E^* \left[ \frac{(X(t) - K)^+}{\exp(\int_0^t r(s) ds)} \right]$$

$$= \exp(-\int_0^t f(0,s)ds - \sigma_0^2 t^3/6) E^* \left[ \frac{(X(t) - K)^+}{\exp(\int_0^t \sigma_0 W_s^*)} \right]$$

$$= \exp(-\int_0^t f(0,s)ds - \sigma_0^2 t^3/6) \int_{-\infty}^a E^*[\exp(-\sigma_0 \int_0^t \tilde{W}_s^* ds) | \tilde{W}_t^* = u]$$

$$\left( \sum_{i=1}^n \exp(-\int_0^{T_i} f(0,s)ds - \frac{\sigma_0^2(T_i - t)}{2} t T_i - \sigma_0(T_i - t) u - K) \right) p(u) du$$

where $p(u) = 1/\sqrt{2\pi} \exp(-u^2/2t)$ and $a$ satisfies

$$\sum_{i=1}^n \exp(-\int_0^{T_i} f(0,s)ds - \frac{\sigma_0^2(T_i - t)}{2} t T_i - \sigma_0(T_i - t) a) = K.$$
It is clear that there is exactly one such \( a \), since the left hand side is strictly decreasing in \( a \) and unbounded above and below. It is therefore very easy to get fast numerical approximations of \( a \), and in the case \( n = 1 \) (a pure discount bond), we can solve to get

\[
a = \frac{-\log(K) - \int_0^{T_1} f(0, s)ds - \frac{\sigma_0^2(T_1 - t)}{2} tT_1}{\sigma_0(T_1 - t)}
\]

In both cases, of course

\[
Q^*\{ \text{Option is exercised} \} = Q^*\{ X(t) \geq K \}
= Q^*\{ W_t^* \leq a \}
\]

Use of the result of Lemma 4.5.1 and simplification gives

\[
C \exp(\sigma_0 \frac{t^3}{8})
= \sum_{i=1}^n C_i P(0, T_i) \int_{-\infty}^a \exp\left(-\frac{\sigma_0^2(T_i - t)}{2} tT_i - \sigma_0(T_i - t)u \frac{\sigma_0 ut}{2}\right) p(u)du
- K P(0, t) \int_{-\infty}^a \exp\left(-\frac{\sigma_0 ut}{2}\right) p(u)du
\]

Completing the square in each integral and simplifying gives

\[
\sum_{i=1}^n C_i P(0, T_i) \Phi \left( a + \frac{1}{2} \sigma_0 \frac{(2T_i - t)}{\sqrt{t}} \right) - K P(0, t) \Phi \left( a + \frac{1}{2} \sigma_0 t^2 \right).
\]

### 4.6 Existence and Uniqueness of Solutions

#### 4.6.1 Existence

Fix \( T > 0 \). Let \( T \) denote the triangular region \( \{(s, t) : 0 \leq s \leq t \leq T\} \).

**Theorem 4.6.1** Let \( \sigma_1 \) and \( \sigma_2 \) map \( \mathbb{R} \times T \to \mathbb{R} \). Assume each is measurable, non-negative, bounded (by \( M \)) and Lipschitz continuous (with parameter \( K \)) in its first
argument. Let I(·) be a non-negative Lipschitz continuous (with parameter K) function on \([0,T]\). Let \((\Omega, F, Q)\) be a probability space on which is defined a \(d\)-dimensional Brownian motions \(W\), adapted to a right continuous filtration \(\mathcal{F}_s\). Then there is a jointly continuous process \(\{f(s,v) : (s,v) \in \mathcal{T}\}\), with \(f(s,v) \in \mathcal{F}_s\), such that except on a null set, for all \((s,v) \in \mathcal{T}\),

\[
f(s,v) = I(v) + \sum_{i=1}^{d} \int_{0}^{s} \sigma_i(f(u,v),u,v) \int_{u}^{v} \sigma_i(f(u,y),u,y) dy \, du + \sum_{i=1}^{d} \int_{0}^{s} \sigma_i(f(u,v),u,v) dW_u^i.
\]

(4.7)

**Proof:** The proof proceeds by the usual successive approximations approach, however, some care is needed because the process is indexed by two time parameters.

For \((s,v)\) in \(\mathcal{T}\), let

\[
f^0(s,v) = I(v).
\]

We want to define a sequence of processes \(\{f^0, f^1, \ldots\}\) by setting

\[
f^{n+1}(s,v) = f^n(0,v) + \sum_{i=1}^{d} \int_{0}^{s} \sigma_i(f^n(u,v),u,v) \int_{u}^{v} \sigma_i(f^n(u,y),u,y) dy \, du + \sum_{i=1}^{d} \int_{0}^{s} \sigma_i(f^n(u,v),u,v) dW_u^i.
\]

(4.8)

If \(f^n\) is jointly continuous, (and hence measurable), with \(f^n(s,v) \in \mathcal{F}_s\) for each \((s,v)\), then the boundedness and measurability of each \(\sigma_i\) assures that the integrals on the right hand side are well defined for each \((s,v)\). For our sequence to be well-defined, our first task is to verify that a version of the right hand side is a jointly continuous process.

First, we establish a bound to be applied several times during the proof of existence.
Lemma 4.6.1 Let $\sigma : \mathbb{R} \times T \rightarrow \mathbb{R}$ be measurable, non-negative, bounded (by $M$) and Lipschitz continuous (with parameter $K$) in its first argument. Let $g$ and $\bar{g}$ be measurable functions on $T$. Then

$$
\left( \sigma(g(u, v), u, v) \int_u^v \sigma(g(u, y), u, y)du - \sigma(\bar{g}(u, v), u, v) \int_u^v \sigma(\bar{g}(u, y), u, y)du \right)^6 \leq L \int_u^v (g(u, y) - \bar{g}(u, y))^2 du + (g(u, v) - \bar{g}(u, v))^2
$$

for some $L$ depending only on $M, T, K$ (and not on $\omega, u, v$.)

Proof: We first establish a simple inequality.

If $A, B, C, D$ are non-negative with $|A - C| \leq \epsilon_1$ and $|B - D| \leq \epsilon_2$, then

$$
|AB - CD| \leq \max(A\epsilon_2 + B\epsilon_1, C\epsilon_2 + D\epsilon_1) + \epsilon_1 \epsilon_2.
$$

To see this observe that $0 \leq A \leq C + \epsilon_1$ and $0 \leq B \leq D + \epsilon_2$ gives

$$
AB - CD \leq C\epsilon_2 + D\epsilon_1 + \epsilon_1 \epsilon_2.
$$

Similarly $0 \leq C \leq A + \epsilon_1$ and $0 \leq D \leq B + \epsilon_2$ gives

$$
AB - CD \geq -(A\epsilon_2 + B\epsilon_1 + \epsilon_1 \epsilon_2).
$$

The desired inequality now follows. Apply it with

$$
\begin{align*}
A &= \sigma(g(u, v), u, v) \\
B &= \int_u^v \sigma(g(u, y), u, y)du \\
C &= \sigma(\bar{g}(u, v), u, v) \\
D &= \int_u^v \sigma(\bar{g}(u, y), u, y)du \\
\epsilon_1 &= \min(2M, K|g(u, v) - \bar{g}(u, v)|) \\
\epsilon_2 &= K \int_u^v |g(u, y) - \bar{g}(u, y)|du
\end{align*}
$$

to get

$$
|AB - CD| \leq MK \int_u^v |g(u, y) - \bar{g}(u, y)|du + MTK|g(u, v) - \bar{g}(u, v)| + 2MK \int_u^v |g(u, y) - \bar{g}(u, y)|du
$$

$$
\leq L_1 \int_u^v |g(u, y) - \bar{g}(u, y)|du + L_1|g(u, v) - \bar{g}(u, v)|
$$
for $L_2 = \max(3MK, MTK)$, and hence

$$(AB - CD)^6 \leq L_2 \int_u^v (g(u, y) - \hat{g}(u, y))^6 \, du + L_2 (g(u, v) - \hat{g}(u, v))^6$$

for $L_2$ depending only on $M, T, K$. 

**Lemma 4.6.2** If $f^n$ is jointly continuous, (and hence measurable), with $f^n(s, v) \in F_s$ for each $(s, v)$, then $f^{n+1}$ as defined by (4.8) has a modification which is jointly continuous. Hence the sequence $\{f^0, f^1, \ldots\}$ is well defined, and each element can be taken jointly continuous.

**Proof:** It is clear that $f^0$ satisfies the conditions, so to complete the proof we only need show that if $f^n$ does then so does $f^{n+1}$. We apply a two dimensional analog of Kolmogorov's criterion, due to Yadrenko [37], and stated as Problem 2.2.9 of Karatzas and Shreve [27].

**Theorem 4.6.2** A random field is a collection of random variables $\{X_t; t \in A\}$, where $A$ is a partially ordered set. Suppose $\{X_t; t \in [0, T]^k\}, k \geq 2$ is a random field satisfying

$$E|X_t - X_s|^p \leq C||t - s||^{k+\beta}$$

for some positive constants $\alpha$, $\beta$, and $C$. Then there exists a continuous modification of $X$ which is locally Holder continuous with exponent $\gamma$ for every $\gamma \in (0, \beta/\alpha)$.

We shall apply this theorem with $\alpha = 6$ and $\beta = 3$ (and of course $k = 2$.) To simplify the notation, let $\sigma^n_i(u, v)$ denote $\sigma_i(f^n(u, v), u, v)$. Let $(s, v_1), (s, v_2)$ be points in $T$, with $v_1 < v_2$.

$$f^{n+1}(s, v_2) - f^{n+1}(s, v_1)$$
\[ I(v_2) - I(v_1) + \sum_{i=1}^{d} \int_0^s \sigma_i^n(u, v_2) \int_{u}^{v_2} \sigma_i^n(u, y) dy - \sigma_i^n(u, v_1) \int_{u}^{v_1} \sigma_i^n(u, y) dy \ du \\
+ \sum_{i=1}^{d} \int_0^s \sigma_i^n(u, v_2) - \sigma_i^n(u, v_1) \ dW^i_u \]

So

\[ E(f^{n+1}(s, v_2) - f^{n+1}(s, v_1))^6 \leq L(I(v_2) - I(v_1))^6 \]
\[ + \sum_{i=1}^{d} LE \int_0^s \left( \sigma_i^n(u, v_2) \int_{u}^{v_2} \sigma_i^n(u, y) dy - \sigma_i^n(u, v_1) \int_{u}^{v_1} \sigma_i^n(u, y) dy \right)^6 \ du \]
\[ + \sum_{i=1}^{d} LE \int_0^s (\sigma_i^n(u, v_2) - \sigma_i^n(u, v_1))^6 du \]

(by the Burkholder-Davis-Gundy and H"older inequalities)

\[ = (a) + (b) + (c) \]

The Lipschitz continuity of \( I(\cdot) \) ensures that

\[ (a) \leq K^6 (v_2 - v_1)^6. \]

An application of the 'ABCD' inequality of Lemma 4.6.1 with

\[ A = \sigma^n_i(u, v_2) \quad B = \int_{u}^{v_2} \sigma^n_i(u, y) dy \]
\[ C = \sigma^n_i(u, v_1) \quad D = \int_{u}^{v_1} \sigma^n_i(u, y) dy \]
\[ c_1 = K|f^n(u, v_2) - f^n(u, v_1)| \quad c_2 = M(v_2 - v_1) \]

shows

\[ (b) \leq L(v_2 - v_1)^6 + LE(f^n(u, v_2) - f^n(u, v_1))^6 \]

for some \( L \). Also,

\[ (c) \leq 2K^6 LE \int_0^s (f^n(u, v_2) - f^n(u, v_1))^6 du \]
\[ = L_2 E \int_0^s \left( f^n(u, v_2) - f^n(u, v_1) \right)^6 \, du. \]

Combining these, we have

\[ E(f^{n+1}(s, v_2) - f^{n+1}(s, v_1))^6 \leq L_1 (v_2 - v_1)^6 + L_2 E \int_0^s \left( f^n(u, v_2) - f^n(u, v_1) \right)^6 \, du. \]

Letting

\[ u^n(s) = E(f^n(s, v_2) - f^n(s, v_1))^6, \]

we have by Fubini,

\[ u^{n+1}(s) \leq L_1 (v_2 - v_1)^6 + L_2 \int_0^s u^n(v) \, du \]

so \( u^{n+1}(s) \leq L_1 (v_2 - v_1)^6 \exp(L_2 s) \) as can easily be proved by induction. In other words,

\[ E(f^{n+1}(s, v_2) - f^{n+1}(s, v_1))^6 \leq L_1 (v_2 - v_1)^6 \exp(L_2 t) \leq L_3 (v_2 - v_1)^3 \]

\( \text{(since } v_1, v_2 \leq T). \)

We next prove that

\[ E(f^{n+1}(s_2, v) - f^{n+1}(s_1, v))^6 \leq L(s_2 - s_1)^3. \]

\[ f^{n+1}(s_2, v) - f^{n+1}(s_1, v) = \sum_{i=1}^d \int_{s_1}^{s_2} \sigma^n_i(u, v) \int_v^s \sigma^n_i(u, y) dy \, du \]

\[ + \sum_{i=1}^d \int_{s_1}^{s_2} \sigma^n_i(u, v) dW^i_u. \]
So

\[ E(f^{n+1}(s_2, v) - f^{n+1}(s_1, v))^6 \leq L(s_2 - s_1)^6 + L \sum_{i=1}^{d} E \left( \int_{s_1}^{s_2} \sigma_i^n(u, v) dW_u^i \right)^6 \]

\[ \leq L(s_2 - s_1)^6 + L_1(s_2 - s_1)^6 \]

\[ \leq L_2(s_2 - s_1)^3 \]

Finally,

\[ E(f^{n+1}(s + h_1, v + h_2) - f^{n+1}(s, v))^6 \]

\[ \leq L E(f^{n+1}(s + h_1, v + h_2) - f^{n+1}(s + h_1, v))^6 \]

\[ + L E(f^{n+1}(s + h_1, v) - f^{n+1}(s, v))^6 \]

\[ \leq L_1 h_2^3 + L_1 h_1^3 \]

\[ \leq L_2 (h_1^2 + h_2^2)^{3/2} \]

\[ = L_2 \|h\|^3 \]

By Yadrenko's Theorem 4.6.2, \( f^{n+1} \) has a continuous modification. \( \Box \)

Having shown that our sequence is well defined, and each element jointly continuous, we now show that for each \((s, v), f^n(s, v)\) is a Cauchy sequence in \(L^6\). Define

\[ D^n(s, v) \]

\[ = E(f^{n+1}(s, v) - f^n(s, v))^6 \]

\[ \leq 4^6 \sum_{i=1}^{d} E \left( \int_{0}^{s} \sigma_i^n(u, v) \int_{u}^{v} \sigma_i^n(u, y) dy du - \int_{0}^{s} \sigma_i^{n-1}(u, v) \int_{u}^{v} \sigma_i^{n-1}(u, y) dy du \right)^6 \]

\[ + 4^6 \sum_{i=1}^{d} E \left( \int_{0}^{s} \left( \sigma_i^n(u, v) - \sigma_i^{n-1}(u, v) \right) dW_u \right)^6 \]

(since \((a + b + c + d)^6 \leq 4^6(a^6 + 4b^6 + 4c^6 + 4d^6.)\) We have

\[ E \left( \int_{0}^{s} \sigma_i^n(u, v) \int_{u}^{v} \sigma_i^n(u, y) dy du - \int_{0}^{s} \sigma_i^{n-1}(u, v) \int_{u}^{v} \sigma_i^{n-1}(u, y) dy du \right)^6 \]
\begin{align*}
\leq & \quad T^5 E \int_0^8 \left( \sigma_i^n(u, v) \int_u^v \sigma_i^n(u, y) dy - \sigma_i^{n-1}(u, v) \int_u^v \sigma_i^{n-1}(u, y) dy \right)^6 du \\
& \text{(Jensen's inequality)} \\
\leq & \quad LE \int_0^8 \left( \int_u^v |f^n(u, y) - f^{n-1}(u, y)|^6 dy \right) \int_u^v |f^n(u, v) - f^{n-1}(u, v)|^6 du \\
& \text{(using Lemma 4.6.1 with } \sigma = \sigma_i, \ g = f^n, \ \bar{g} = f^{n-1}) \\
\leq & \quad L \int_0^8 \left( \int_u^v D^{n-1}(u, y) dy + D^{n-1}(u, v) \right) du \\
& \text{(Fubini's theorem)} \\
\end{align*}

and

\begin{align*}
E \left( \int_0^8 \left( \sigma_i^n(u, v) - \sigma_i^{n-1}(u, v) \right) dW_u \right)^6 \\
= & \quad E \int_0^8 \left( \sigma_i^n(u, v) - \sigma_i^{n-1}(u, v) \right)^6 du \\
& \text{(by the Burkholder-Davis-Gundy inequalities)} \\
\leq & \quad K^6 E \int_0^8 \left( f^n(u, v) - f^{n-1}(u, v) \right)^6 du \\
= & \quad K^6 \int_0^8 E \left( f^n(u, v) - f^{n-1}(u, v) \right)^6 du \\
= & \quad K^6 \int_0^8 D^{n-1}(u, v) du \\
\end{align*}

Combining all this shows that, for some constant \( L \),

\[ D^n(s, v) \leq L \int_0^8 \left( \int_u^v D^{n-1}(u, y) dy + D^{n-1}(u, v) \right) du. \]

By induction we now prove that

\[ D^n(s, v) \leq L^n(T + 1)^n \frac{s^n}{n!}. \]

By increasing \( L \) if necessary, this is true for \( n = 1 \). If

\[ D^{n-1}(s, v) \leq L^{n-1}(T + 1)^{n-1} \frac{s^{n-1}}{(n-1)!}, \]

then

\begin{align*}
D^n(s, v) \leq & \quad LL^{n-1}(T + 1)^{n-1} \int_0^8 T \frac{u^{n-1}}{(n-1)!} + \frac{u^{n-1}}{(n-1)!} du \\
& \text{(by the dominance of } T \text{ and } u^{n-1} \text{ for large } n) \\
\end{align*}
\[ = L^n(T + 1)^n s^n \]

So \( D^n(s, v) \to 0 \) uniformly and geometrically fast, so \( \{ f^n(s, v) \} \) is a Cauchy sequence in \( L^6 \). Hence, for each \( (s, v) \) there exists a \( (\mathcal{F}_s\)-measurable) \( f(s, v) \) so that

\[ f^n(s, v) \to f(s, v) \text{ in } L^6. \]

Furthermore, it is easy to see that

\[ E( f(s, v) - f^n(s, v) )^6 \to 0 \quad \text{uniformly over } T. \]

For each \( (s, v) \), \( f(s, v) \) is defined up to \( L^6 \) equivalence. For integration (both stochastic and otherwise) of \( f \) to make sense we need to choose versions of these \( L^6 \) limits in such a way that the resulting \( f \), when viewed as a stochastic process, is a measurable map from \( \Omega \times T \) to \( R \). We do this by showing something stronger, that there is a modification of the random field \( f \), whose sample paths are continuous. Most of the work has already been done, in the proof showing that each \( f^n \) has a continuous modification. Recall that, for each \( n \),

\[ E( f^n(s, v_2) - f^n(s, v_1) )^6 \leq L_3(v_2 - v_1)^3 \]

and

\[ E( f^n(s_2, v) - f^n(s_1, v) )^6 \leq L_2(s_2 - s_1)^3. \]

Since for each \( (s, v) \), \( f^n(s, v) \to f(s, v) \) in \( L^6 \), it is easy to show that these relations also hold for \( f \). As before Yadrenko's theorem then provides a continuous modification of \( f \). In other words versions of the random variables \( f(s, v) \) obtained as \( L^6 \) limits can be chosen so that the resulting process is continuous. It is easy to see that \textit{a fortiori} this modification is a measurable process. (Approximate \( f \) by processes
flat on intervals.) Relabelling the modification by \( f \), it only remains to show that that \( f \) actually solves the system (4.7), namely that for each \((s, v) \in T\),

\[
f(s, v) = I(v) + \sum_{i=1}^{d} \int_{0}^{s} \sigma_{i}(f(u, v), u, v) \int_{u}^{v} \sigma_{i}(f(u, y), u, y) dy \, du
\]

\[
+ \sum_{i=1}^{d} \int_{0}^{s} \sigma_{i}(f(u, v), u, v) dW_{v}^{i} \quad a.s.
\]

Let \( R(s, v) \) denote the right hand side of this expression. Let \( \sigma_{i}(u, y) \) represent \( \sigma_{i}(f(u, y), u, y) \). To show that \( f(s, v) = R(s, v) \) a.s., we will show that as \( n \to \infty \), both terms on the right hand side of the following inequality converge to 0.

\[
E \left( f(s, v) - R(s, v) \right) \]

\[
\leq 2^{6} E \left( f(s, v) - f^{n+1}(s, v) \right) ^{6} + 2^{6} E \left( f^{n+1}(s, v) - R(s, v) \right) ^{6}.
\]

We already know that

\[
E \left( f(s, v) - f^{n+1}(s, v) \right) ^{6} \to 0.
\]

As for the second term,

\[
E \left( f^{n+1}(s, v) - R(s, v) \right) ^{6}
\]

\[
\leq 4^{6} \sum_{i=1}^{d} E \left( \int_{0}^{s} \sigma_{i}^{n}(u, v) \int_{u}^{v} \sigma_{i}^{n}(u, y) dy - \sigma_{i}(u, v) \int_{u}^{v} \sigma_{i}(u, y) dy du \right)^{6}
\]

\[
+ 4^{6} \sum_{i=1}^{d} E \left( \int_{0}^{s} \sigma_{i}^{n}(u, v) - \sigma_{i}(u, v) dW_{v}^{i} \right)^{2}
\]

\[
\leq 4^{6} T^{5} \sum_{i=1}^{d} E \int_{0}^{s} \left( \sigma_{i}^{n}(u, v) \int_{u}^{v} \sigma_{i}^{n}(u, y) dy - \sigma_{i}(u, v) \int_{u}^{v} \sigma_{i}(u, y) dy \right)^{2} du
\]

(by Jensen's inequality)

\[
+ 4^{6} \sum_{i=1}^{d} E \int_{0}^{s} \left( \sigma_{i}^{n}(u, v) - \sigma_{i}(u, v) \right)^{6} du
\]

(by the Burkholder-Davis-Gundy inequalities)

\[
\leq LE \int_{0}^{s} \left( f^{n}(u, y) - f(u, y) \right)^{6} du + \left( f^{n}(u, v) - f(u, v) \right)^{6} du
\]
\[ +2 \cdot 4^6 K^6 E \int_0^s \left( f^n(u, v) - f(u, v) \right)^6 du \]

(by Lemma 4.6.1 with \( \sigma = \sigma_i, \ g = f^n, \ \bar{g} = f \))

\[ = L \int_0^s \int_u^v E \left( f^n(u, y) - f(u, y) \right)^6 du + E \left( f^n(u, v) - f(u, v) \right)^6 du \]

\[ + 2 \cdot 4^6 K^2 \int_0^s E \left( f^n(u, v) - f(u, v) \right)^6 du \]

(by Fubini)

But this expression tends to 0 as \( n \to \infty \), completing the existence proof.

\[
4.6.2 \quad \text{Uniqueness}
\]

**Theorem 4.6.3** There is at most one solution to (4.7).

Suppose \( \tilde{f} \) is another solution. With the notation, \( \tilde{\sigma}_i(u, y) = \sigma_i(\tilde{f}(u, y), u, y) \) this means, for each \( (s, v) \),

\[
\tilde{f}(s, v) = I(v) + \sum_{i=1}^d \int_0^t \tilde{\sigma}_i(u, v) \int_u^s \tilde{\sigma}_i(u, y) dy \ du
\]

\[ + \sum_{i=1}^d \int_0^t \tilde{\sigma}_i(u, v) dW^i_v \quad \text{a.s.} \]

Let

\[
H(s, v) = E \left( f(s, v) - \tilde{f}(s, v) \right)^2
\]

\[ \leq 4 \sum_{i=1}^d E \left( \int_0^t \sigma_i(u, v) \int_u^s \sigma_i(u, y) dy - \tilde{\sigma}_i(u, v) \int_u^s \tilde{\sigma}_i(u, y) dy \ du \right)^2
\]

\[ + 4 \sum_{i=1}^2 E \left( \int_0^t \sigma_i(u, v) - \tilde{\sigma}_i(u, v) \ dW^i_v \right)^2
\]

\[ \leq 4T \sum_{i=1}^d E \int_0^s \left( \sigma_i(u, v) \int_u^s \sigma_i(u, y) dy - \tilde{\sigma}_i(u, v) \int_u^s \tilde{\sigma}_i(u, y) dy \right)^2 du
\]
\begin{align*}
+4 \sum_{i=1}^{2} E \int_{0}^{s} (\sigma_i(u, v) - \tilde{\sigma}_i(u, v))^2 \, du \\
\leq L E \int_{0}^{s} \int_{v}^{s} (f(u, y) - \tilde{f}(u, y))^2 \, du + (f(u, v) - \tilde{f}(u, v))^2 \, du \\
+ 8K^2 E \int_{0}^{s} (f(u, v) - \tilde{f}(u, v))^2 \, du \\
\quad \text{(by Lemma 4.6.1)} \\
= L \int_{0}^{s} \int_{v}^{s} H(u, y) \, du + H(u, v) \, du \\
+ 8K^2 \int_{0}^{s} H(u, v) \, du \\
\quad \text{(by Fubini)} \\
\leq L_1 \int_{0}^{s} \int_{v}^{s} H(u, y) \, du + H(u, v) \, du
\end{align*}

Remark 4.6.1 Since \( f \) and \( \tilde{f} \) have continuous sample paths, \( H \) is continuous and hence bounded on \( T \). The following Lemma shows that the only bounded, non-negative function satisfying a relationship of this kind is the zero function.

Hence \( \tilde{f}(s, v) = f(s, v) \), as claimed.

Lemma 4.6.3 Suppose \( H : T \rightarrow \mathbb{R} \) is non-negative (bounded) and satisfies

\[ H(s, v) \leq L \int_{0}^{s} \int_{v}^{s} H(u, y) \, du + H(u, v) \, du \]

for all \((s, v) \in T\). Then \( H \equiv 0 \).

Proof: \quad Assume \( H(s, v) \leq M \) for all \((s, v) \in T\). We prove by induction that

\[ H(s, v) \leq (T + 1)^{k} L^{k} M \frac{t^{k}}{k!} \]

for each positive integer \( k \). This is true for \( k = 0 \). Assume it is true for \( k = n \). Then

\begin{align*}
H(s, v) & \leq L \int_{0}^{s} \int_{u}^{s} (T + 1)^{n} L^{n} M \frac{v^{n}}{n!} \, du + (T + 1)^{n} L^{n} M \frac{v^{n}}{n!} \, du \\
& \leq (T + 1)^{n+1} L^{n+1} M \int_{0}^{s} \frac{v^{n}}{n!} \, du \\
& = (T + 1)^{n+1} L^{n+1} M \frac{t^{n+1}}{(n + 1)!} \, du,
\end{align*}
proving the assertion for \( k = n + 1 \). Hence \( H \equiv 0 \). \( \square \)

## 4.7 Proportional Volatility

The simplest model of our class which would result in non-negative interest rates arises from setting \( \sigma(f) = \sigma_0 f \), for some positive constant \( \sigma_0 \). In this section we show that, perhaps surprisingly, there is no solution to the no arbitrage stochastic differential equation corresponding to this choice of \( \sigma(\cdot) \). (So the boundedness condition in the existence theorem of the previous section cannot be dropped.)

The no-arbitrage differential equation for \( f \) reduces to

\[
df(t, s) = f(t, s) \left( \sigma_0^2 \int_t^s f(t, v) dv dt + \sigma_0 dW_t \right)
\]

By Ito’s Lemma, any solution to the equation for \( f \) will be of the form

\[
f(t, s) = f(0, s) \exp(\sigma_0^2 \int_0^t \int_0^s f(u, v) dv du - \sigma_0^2 t/2 + \sigma_0 W_t).
\]

For simplicity, take \( \sigma_0 = 1 \), and \( f(0, \cdot) \equiv 1 \), giving

\[
f(t, s) = \exp(\int_0^t \int_u^s f(u, v) dv du) \exp(W_t - t/2).
\]

We now show that no process \( f \) solves this differential equation. The proof rests on the fact that the function \( 2a/(a - st)^2 \) satisfies a similar looking non-stochastic equation,

\[
h(t, s) = \exp \left( \int_0^t \int_u^s h(u, v) dv du \right) \frac{2}{a - t^2}.
\]

Before proceeding, a uniqueness and a monotonicity lemma must be established.
Lemma 4.7.1 Let $R$ and $T$ be given with $0 \leq R < T$. Let $K = \{(t, s) : R \leq t \leq s \leq T\}$. If $g : K \to \mathbb{R}$ is continuous then there is at most one function $f : K \to \mathbb{R}$ satisfying

$$ f(t, s) = \exp \left( \int_R^t \int_u^s f(u, v) dv du \right) g(t, s) $$

for all $(t, s) \in K$.

**Proof:** If $f_1$ and $f_2$ are solutions, then each is a continuous function on the compact set $K$, and so $\|f_1\|, \|f_2\| \leq L$, for some $L$. So for $(t, s) \in K$,

$$ |f_1(t, s) - f_2(t, s)| \\ \leq |g(t, s)| \left| \exp \left( \int_R^t \int_u^s f_1(u, v) dv du \right) - \exp \left( \int_R^t \int_u^s f_2(u, v) dv du \right) \right| \\ \leq \|g\| \exp(T^2 L) \int_R^t \int_u^s |f_1(u, v) - f_2(u, v)| dv du $$

using the inequality $|e^x - e^y| \leq \max(e^x, e^y)|y - x|$.

With $d(t, s) = \|f_1(t, s) - f_2(t, s)\|$, and $L_1 = \|g\| \exp(T^2 L)$, we have

$$ d(t, s) \leq L_1 \int_R^t \int_u^s d(u, v) dv du $\
\leq L_1 \int_0^t \int_u^s d(u, v) dv + d(u, s) du $$

for all $(t, s) \in K$. Applying the Lemma in the previous section, we can immediately conclude that $d \equiv 0$ on $K$. \hfill \Box

Lemma 4.7.2 With $K$ as in Lemma 4.7.1, suppose $f_1(\cdot, \cdot)$, $f_2(\cdot, \cdot)$, $g_1(\cdot, \cdot)$ and $g_2(\cdot, \cdot)$ each map $K \to \mathbb{R}$, with $g_1$ and $g_2$ continuous, positive and satisfying $g_1 \geq g_2 > 0$. Furthermore

$$ f_1(t, s) \geq \exp \left( \int_R^t \int_u^s f_1(u, v) dv du \right) g_1(t, s) $$
and
\[ f_2(t, s) = \exp \left( \int_R^t \int_u^s f_2(u, v)dvdu \right) g_2(t, s) \]
for all \((t, s)\) in \(K\). Then \(f_1 \geq f_2\).

**Proof:** Define the mapping \(M : C[K] \to C[K]\) by
\[
(M \cdot f)(t, s) = \exp \left( \int_R^t \int_u^s f(u, v)dvdu \right) g_2(t, s).
\]
Notice that this map is monotone, i.e. \(h_1 \geq h_2 \Rightarrow Mh_1 \geq Mh_2\).

\[
(M \cdot f_1)(t, s) = \exp \left( \int_R^t \int_u^s f_1(u, v)dvdu \right) g_2(t, s) \\
\leq \exp \left( \int_R^t \int_u^s f_1(u, v)dvdu \right) g_1(t, s) \\
\leq f_1(t, s)
\]

So the sequence \(f_1, M \cdot f_1, M^2 \cdot f_1\) is decreasing and bounded below by 0, and so has a limit, \(f\). The dominated convergence theorem shows
\[
f(t, s) = \exp \left( \int_R^t \int_u^s f(u, v)dvdu \right) g_2(t, s).
\]
By the uniqueness of solutions demonstrated in Lemma 4.7.1, in fact \(f_2 \equiv f \leq f_1\) on \(K\).

Suppose \(f\) satisfies, for \((t, s) \in T\),
\[
f(t, s) = \exp(\int_0^t \int_u^s f(u, v)dvdu) \exp(W_t - t/2).
\]
For the moment, let \(a, \epsilon_1, \epsilon_2\) be positive constants with \(\epsilon_2^2 < a\) and \(T\epsilon_1 < a\). Let
\[
A = \{ \omega : \exp(W_t - t/2) \geq \frac{2a(a - \epsilon_2^2)}{(a - t^2)(a - T\epsilon_1)^2} \text{ for all } \epsilon_1 \leq t \leq \epsilon_2 \}.
\]
For \((t, s): \epsilon_1 \leq t \leq \epsilon_2, \ t \leq s \leq T\), and \(\omega \in \Lambda\), we have

\[
\begin{align*}
f(t, s) & \geq \exp \left( \int_0^t \int_u^s f(u, v) dvdu \right) \frac{2a(a - \epsilon_1^2)}{(a - t^2)(a - s\epsilon_1)^2} \\
& \geq \exp \left( \int_{\epsilon_1}^t \int_u^s f(u, v) dvdu \right) \frac{2a(a - \epsilon_1^2)}{(a - t^2)(a - s\epsilon_1)^2}
\end{align*}
\]

By Lemma 4.7.2, on \(\Lambda\), \(f \geq h\), where \(h\) solves

\[
h(t, s) = \exp \left( \int_{\epsilon_1}^t \int_u^s h(u, v) dvdu \right) \frac{2a(a - \epsilon_1^2)}{(a - t^2)(a - s\epsilon_1)^2}.
\]

It is easy to check that

\[
h(t, s) = \frac{2a}{(a - st)^2}.
\]

solves this. It is the unique solution by Lemma 4.7.1. Now set \(a = T^2/2, \epsilon_1 = T/4, \epsilon_2 = T/2\). We have \(\epsilon_1^2 < a\) and \(T\epsilon_1 < a\) so \(P(\Lambda) > 0\). The desired contradiction follows from the fact that \(h(T/2, T) = +\infty\).

### 4.8 Appendix

We now show by direct calculation that by appropriately scaling a sequence of Ho & Lee models, the interpolated path of the spot rate (viewed as a random element of \(C[0, T]\)) converges weakly to the path of the spot rate under our constant volatility model.

Two parameters determine the evolution in Ho & Lee’s model: an implied probability \(\pi\), and a ‘spread’ parameter \(\delta\). Randomness is introduced through an i.i.d. sequence of Bernoulli random variables \(\{B_1, B_2, \ldots\}\) with \(P\{B_i = 1\} = \pi\).

In both the Ho & Lee model, and our constant volatility model, the initial forward rate curve enters additively, so without loss we take it to be identically 0. Equation (24) of Ho & Lee [22] then says that the spot rate (the one period interest rate in
their discrete model) in effect after \( n \) steps is

\[
\ln(\pi \delta^{-n} \left( 1 - \pi \right)) + \ln(\delta) \sum_{i=1}^{n} B_i.
\]

Set \( l \overset{\Delta}{=} \ln(\delta) \). The \( m' \)th Ho & Lee model is one in which \( m \) discrete steps are taken per 'day'. The appropriate scaling is to take \( l \) in the \( m' \)th model as \( l/m\sqrt{m} \).

After scaling and linear interpolation, the spot rate at \( l \) is

\[
m\log\left( \pi \exp\left( \frac{l[m]}{m\sqrt{m}} \right) + (1 - \pi) \right) + m\frac{l}{m\sqrt{m}} \sum_{i=1}^{[m]} B_i \\
+ m(m[m] - [m]) \left( \log\left( \frac{\pi \exp\left( \frac{-l[ml]+l}{m\sqrt{m}} \right) + (1 - \pi)}{\pi \exp\left( \frac{-l[ml]}{m\sqrt{m}} \right) + (1 - \pi)} \right) + \frac{l}{m\sqrt{m}} B[ml] + 1 \right) \\
= \ m \log\left( \pi \exp\left( \frac{-l[ml]}{m\sqrt{m}} \right) + (1 - \pi) \right) + \pi \sqrt{m} l \\
+ m(m[m] - [m]) \log\left( \frac{\pi \exp\left( \frac{-l[ml]+l}{m\sqrt{m}} \right) + (1 - \pi)}{\pi \exp\left( \frac{-l[ml]}{m\sqrt{m}} \right) + (1 - \pi)} \right) \\
+ \frac{l}{\sqrt{m}} \sum_{i=1}^{[m]} B_i + (m[m] - [m]) \frac{l}{\sqrt{m}} B[ml+1] - \pi \sqrt{m} l \\
= \ m \log\left( \pi \exp\left( \frac{-l[ml]}{m\sqrt{m}} \right) + (1 - \pi) \right) + \pi \sqrt{m} l \\
+ m(m[m] - [m]) \log\left( \frac{\pi \exp\left( \frac{-l[ml]+l}{m\sqrt{m}} \right) + (1 - \pi)}{\pi \exp\left( \frac{-l[ml]}{m\sqrt{m}} \right) + (1 - \pi)} \right) \\
+ \frac{1}{\sqrt{m}} \sum_{i=1}^{[m]} X_i + \frac{(m[m] - [m])}{\sqrt{m}} X[ml+1] \\
= \ (a) + (b) + (c)
\]

where \( X_i \overset{\Delta}{=} B_i - \pi \), so \( \{X_1, X_2, \ldots\} \) is a sequence of i.i.d. random variables with mean 0, and variance \( \sigma^2 \overset{\Delta}{=} l^2 \pi (1 - \pi) \). Now, as \( m \to \infty \),

\[
(a) \ = \ m \log\left( \pi \exp\left( \frac{-l[ml]}{m\sqrt{m}} \right) + (1 - \pi) \right) + \pi \sqrt{m} l
\]
\[ \begin{align*}
&= m \log \left( 1 + \pi \left( \exp \left( -\frac{l[lml]}{m \sqrt{m}} \right) - 1 \right) \right) + \pi \sqrt{m} t l \\
&= m \left( \pi \left( \exp \left( -\frac{l[lml]}{m \sqrt{m}} \right) - 1 \right) - \frac{\pi^2}{2} \left( \exp \left( -\frac{l[lml]}{m \sqrt{m}} \right) - 1 \right)^2 + o \left( \frac{1}{m} \right) \right) + \pi \sqrt{m} t l \\
&= m \left( \pi \left( -\frac{l[lml]}{m \sqrt{m}} + \frac{1}{2} \left( \frac{l[lml]}{m \sqrt{m}} \right)^2 \right) - \frac{\pi^2}{2} \left( \frac{-l[lml]}{m \sqrt{m}} \right)^2 + o \left( \frac{1}{m} \right) \right) + \pi \sqrt{m} t l \\
&= m \left( -\frac{\pi l[lml]}{m \sqrt{m}} + \frac{\pi (l[lml])^2}{2 m \sqrt{m}} \right) + \pi \sqrt{m} t l \\
&\rightarrow \pi(1 - \pi) l^2 t^2 / 2 \\
&= \sigma^2 t^2 / 2
\end{align*} \]

Also, as \( m \rightarrow \infty \),

\[ \begin{align*}
(b) & = m(mt - [lml]) \log \left( \frac{\pi \exp \left( -\frac{l[lml]}{m \sqrt{m}} \right) + (1 - \pi)}{\pi \exp \left( -\frac{l[lml]}{m \sqrt{m}} \right) + (1 - \pi)} \right) \\
&= m(mt - [lml]) \log \left( \frac{1 + \pi \left( \exp \left( -\frac{l[lml]}{m \sqrt{m}} \right) - 1 \right)}{1 + \pi \left( \exp \left( -\frac{l[lml]}{m \sqrt{m}} \right) - 1 \right)} \right) \\
&= m(mt - [lml]) \log \left( \frac{1 + \frac{\pi \left( \exp \left( -\frac{l[lml]}{m \sqrt{m}} \right) - \exp \left( -\frac{l[lml]}{m \sqrt{m}} \right) \right)}{1 + \frac{\pi \left( \exp \left( -\frac{l[lml]}{m \sqrt{m}} \right) - 1 \right)}}}{1 + \frac{\pi \left( \exp \left( -\frac{l[lml]}{m \sqrt{m}} \right) - 1 \right)}} \right) \\
&\approx m(mt - [lml]) \log \left( 1 + \frac{l[lml]}{m \sqrt{m}} - 1 + \frac{l[lml]}{m \sqrt{m}} \right) \\
&= m(mt - [lml]) \log \left( 1 - \frac{\pi l}{m \sqrt{m}} \right) \\
&\approx \frac{\pi l}{m \sqrt{m}} \\
&\rightarrow 0
\end{align*} \]

By Donsker's Theorem

\[ (c) \rightarrow \sigma W_t \]

Hence the linearly interpolated path of spot rates in a scaled sequence of Ho &
Lee models converges weakly to the evolution (4.6) given in the description of the constant volatility model.
Bibliography


