CONVERGENCE OF SCALED RANDOM SAMPLES IN $\mathbb{R}^d$

by

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Abstract

Let $\{X_j, 1 \leq j \leq n\}$ be a sequence of iid random vectors in $\mathbb{R}^d$ and $S_n = \{X_j/b_n, 1 \leq j \leq n\}$. When do there exist scaling constants $b_n \to \infty$ such that $S_n$ converges to some compact set $S$ in $\mathbb{R}^d$ almost surely (in probability)? We show that a limit set $S$ is star shaped (i.e., $x \in S$ implies $tx \in S$ for $0 \leq t \leq 1$) so that after a polar coordinate transformation the limit set is the hypograph of an upper semicontinuous function. We specify necessary and sufficient conditions for convergence to a particular limit set. Some examples are also given.

Keywords and phrases: random sets, extremes, regular variation, upper semicontinuous functions, almost sure convergence.

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Running head: Convergence of Scaled Random Samples

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ABSTRACT

Let \( \{X_j, 1 \leq j \leq n\} \) be a sequence of iid random vectors in \( \mathbb{R}^d \) and \( S_n = \{X_j / b_n, 1 \leq j \leq n\} \). When do there exist scaling constants \( b_n \to \infty \) such that \( S_n \) converges to some compact set \( S \) in \( \mathbb{R}^d \) almost surely (in probability)? We show that a limit set \( S \) is star shaped (i.e., \( x \in S \) implies \( t \cdot x \in S \) for \( 0 \leq t \leq 1 \)) so that after a polar coordinate transformation the limit set is the hypograph of an upper semicontinuous function. We specify necessary and sufficient conditions for convergence to a particular limit set. Some examples are also given.
1. INTRODUCTION.

Let \( \{X_n, n \geq 1\} \) be an independent and identically distributed (iid) sequence of random vectors in \( \mathbb{R}^d \). For a (nonrandom) compact set \( S \) in \( \mathbb{R}^d \), when does there exist a sequence of real constants \( b_n \to \infty \) such that as \( n \to \infty \)

\[
S_n := \{X_n/b_n, 1 \leq j \leq n\} \to S \text{ a.s.}
\]

with respect to the Hausdorff metric? What is the class of possible limit sets \( S \)? Note that if \( \{S_n\} \) has an almost sure limit, this limit by the Hewitt–Savage 0–1 Law must be almost surely constant.

A related but more restricted problem was first considered by Fisher (1966, 1969), who assumed that \( d = 2 \) and that the components \( X_{n1} \) and \( X_{n2} \) of \( X_n \) are nonnegative, independent, and identically distributed. In this setting, Fisher showed that limit sets were of the form

\[
S = \{(x,y) \geq 0: x^\alpha + y^\alpha \leq 1\} \text{ for some } 0 \leq \alpha \leq \infty
\]

and that a necessary and sufficient condition for convergence to such a set was regular variation of the function \(-\log P\{X_{n1} > x\}\). This result is partially generalized by Davis, Mulrow, and Resnick (1988), whose result is stated in section 5. Eddy (1982) considers a related problem of limits in the sense of convergence in probability for scaled unions of iid random sets and derives a convergence criterion based on the inclusion functional.

In this paper, we give a complete solution to our problem. We characterize the class of limit sets \( S \) as being transforms of hypographs of upper semicontinuous functions and give necessary and sufficient conditions for convergence to a particular limit set.

Section 2 discusses some preliminaries and in section 3 some results on relative stability are stated, which will be needed in section 4.
In section 4, we first show that limit sets are star shaped (i.e., \( x \in S \) implies \( \epsilon \cdot x \in S \) for \( 0 \leq \epsilon \leq 1 \)) which suggests that a polar coordinate transformation is a natural transformation and we give necessary and sufficient conditions for almost sure convergence in the transformed coordinates. Our condition can be modified to cover convergence in probability as well.

In section 5 we give a construction which shows that the class of limit sets in polar coordinates is the class of hypographs of upper semicontinuous functions on the unit sphere. We give some additional examples and show that the log-regular variation condition used in Davis, Mulrow, Resnick (1988) can be checked to be sufficient by showing it implies our sufficient condition.

Based on our conditions, it is easy to construct examples of samples which converge in probability but not almost surely, something not seen in the literature until now.

2. PRELIMINARIES.

Let \( \mathcal{K}_d \) be the closed subsets of \( \mathbb{R}^d \) topologized by the usual topology as discussed in Matheron (1975) or Vervaat (1988). Let \( \mathcal{K}_d \) denote the collection of non-empty compact subsets of \( \mathbb{R}^d \) and \( \mathcal{C}(\mathcal{K}_d) \) the collection of non-empty convex compact subsets of \( \mathbb{R}^d \). Suppose that \( d \) is a metric on \( \mathbb{R}^d \) and for \( S, T \in \mathcal{K}_d \), define Hausdorff metric (Matheron, 1975) \( D: \mathcal{K}_d \times \mathcal{K}_d \to \mathbb{R}_+ \) by

\[
D(S, T) = \inf \{ \delta : S \subseteq T^\delta, T \subseteq S^\delta \}
\]

where

\[
S^\delta = \{ x : d(x, y) < \delta \text{ for some } y \in S \}.
\]

The following convergence criteria are useful (cf. Matheron, 1975, page 6, Theorem 1-2-2).
Lemma 2.1. For \( F_n, F \in \mathcal{F}_d \), \( n \geq 1 \), \( F_n \to F \) as \( n \to \infty \) if and only if the following two conditions hold.

(2.1) For any \( y \in F \), there exists for all large \( n \) \( y_n \in F_n \) such that \( d(y_n, y) \rightarrow 0 \) as \( n \to \infty \).

(2.2) For any subsequence \( \{n_k\} \), if \( y_{n_k} \in F_{n_k} \) converges, then \( \lim_{k \to \infty} y_{n_k} \in F \).

Furthermore, convergence of sets \( S_n \to S \) in \( \mathcal{K}_d \) is equivalent to the analogues of (2.1) and (2.2) holding as well as \( \sup_{n \geq 1} \sup \{ \| x \| : x \in S_n \} < \infty \) for some norm \( \| \cdot \| \) on \( \mathbb{R}^d \).

Note that if the sets \( S_n \) are random elements of \( \mathcal{K}_d \) and \( S \in \mathcal{K}_d \) is nonrandom, then Lemma 2.1 can be used to characterize almost sure convergence or convergence in probability.

A measurable function \( U : \mathbb{R}_+ \to \mathbb{R}_+ \) is called regularly varying at \( \infty \) with index \( \rho \in [-\infty, \infty] \) (written \( U \in RV(\rho) \)) if for \( x > 0 \)

\[
\lim_{t \to \infty} \frac{U(tx)}{U(t)} = x^\rho.
\]

(Interpret \( x^{\infty} \) by \( x^{\infty} = 0 \) if \( x < 1 \); \( x^{\infty} = 1 \) if \( x = 1 \) and \( x^{\infty} = \infty \) if \( x > 1 \). A similar interpretation applies to \( x^{-\infty} \).) We call \( \rho \) the exponent of variation. If \( \rho = 0 \), \( U \) is slowly varying. A frequently used property of regular varying functions (cf. Resnick, 1987, page 23, Proposition 0.8, (v)) is that if \( U \) is nondecreasing, \( U(\infty) = \infty \), and \( U \in RV(\rho) \), \( 0 \leq \rho \leq \infty \), then

\[
U^r \in RV(\rho^{-1})
\]

where \( U^r(y) = \inf \{ x : U(x) \geq y \} \).

Let \( \{X_n, n \geq 1\} \) be iid random variables with common distribution function \( F(\cdot) \), \( F(x) < 1 \) for all \( x \), and set \( M_n = \vee_{j=1}^n X_j \). The sequence \( \{M_n\} \) is almost surely relatively stable (relatively stable in probability) if there exist constants \( b_n \) such that
\[ M_n / b_n \to 1 \text{ almost surely (in probability) as } n \to \infty. \]

It is known that if \( \{M_n\} \) is relatively stable in probability then \( b_n \sim \frac{1}{F} (1 - \frac{1}{n}) \) (cf. Resnick and Tomkins, 1973). Necessary and sufficient conditions for relative stability are given as follows (cf. Barndorff–Nielsen, 1963; Resnick and Tomkins, 1973; Gnedenko, 1943; de Haan, 1970):

**Lemma 2.2.** \( \{M_n\} \) is a.s. relatively stable iff

\[
\int_1^{\infty} \frac{dF(x)}{1 - F(x)} < \infty
\]

for all \( 0 < \epsilon < 1 \). \( \{M_n\} \) is relatively stable in probability iff

\[
\lim_{t \to \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\omega}.
\]

The normal and exponential distributions satisfy (2.4) and \( M_n \) is a.s. relatively stable. The Cauchy and Pareto distributions do not satisfy (2.5). An interesting example is when \( F \) is given by

\[
-(\log(1 - F(x))) = (\log \ell)^{-1} \log x \log \log x \text{ for } x > e, \quad \ell > 1
\]

(Resnick and Tomkins, 1973). This \( F \) satisfies (2.5) but not (2.4) and hence \( \{M_n\} \) is relatively stable in probability but not a.s. relatively stable. (For this example in fact \( \limsup_{n \to \infty} M_n / b_n = \ell \text{ a.s.} \)

Note that if \( \{M_n\} \) is relatively stable in probability, then it follows from (2.5) that \( 1/(1 - F) \in RV(\infty) \). Since \( b_n \sim \frac{1}{F} (1 - 1/n) = (\frac{1}{1 - F})^{-1}(n) \), we have from (2.3) that

\[
(1/(1 - F))^{-1} \in RV(0)
\]
and furthermore (Resnick, 1987, page 17, Proposition 0.5)

\[(2.8) \quad \lim_{n \to \infty} \frac{b \lfloor nt \rfloor}{b_n} = 1 \text{ locally uniformly on } (0, \omega).\]

For a non-negative function \( f \) defined on a topological space \( E \), we define the hypograph of \( f \) by

\[\text{hypo} (f) := \{(t, x) \in E \times (0, \omega): 0 < x \leq f(t)\}.\]

\((E \times (0, \omega) \text{ has the relative topology inherited from } E \times [0, \omega]).\) We call \( f \) upper semicontinuous (USC) if for all \( t \in E \)

\[f(t) = \bigwedge_{G \in \mathcal{G}_t} \bigvee_{u \in G} f(u)\]

where \( \mathcal{G}_t \) is the collection of all open sets containing \( t \). In practice, the collection \( \mathcal{G}_t \) can be thinned considerably. The following lemma provides characterizations of USC functions and hypographs.

**Lemma 2.3.** (Vervaat, 1988). The following are equivalent:

(i) \( f \in \text{USC}; \)

(ii) \( \text{hypo} (f) \) is closed in \( E \times (0, \omega). \)

(iii) \( f^{-1} [0, x) = \{ t \in E, f(t) < x \} \) is open in \( E \) for all \( 0 < x \leq 1; \)

If there exists some metric \( d \) on \( E \) then \( f \) is usc iff \( \limsup_{n \to \infty} f(t_n) \leq f(t) \) for all \( t_n, t \in E \) such that \( d(t_n, t) \to 0 \) as \( n \to \omega \).

3. FURTHER RESULTS ON RELATIVE STABILITY.

Throughout this section, we assume that \( \{X_n, n \geq 1\} \) and \( \{X_n^{(i)}, n \geq 1\} \) are sequences of iid random variables from common d.f.'s \( F \) and \( F_i \) \((i = 1, 2)\) respectively and that \( \{X_n^{(1)}, n \geq 1\} \) and \( \{X_n^{(2)}, n \geq 1\} \) are independent. For convenience we also assume
that all the d.f.'s have support $\mathbb{R}_+$. Set

$$M_n = \sum_{j=1}^{n} X_j, \quad M_n^{(i)} = \sum_{j=1}^{n} X_n^{(i)}.$$ 

Whether or not relative stability is assumed we always define

$$b_n = F (1 - 1/n), \quad b_n^{(i)} = F_i^{(i)} (1 - 1/n), \quad i = 1, 2.$$ 

Finally define $\bar{F} = 1 - F, \quad \bar{F}_i = 1 - F_i$.

We first show that relative stability in $\mathbb{R}$ is equivalent to the scaled sample converging to the unit interval.

**Proposition 3.1.** $M_n$ is a.s. relatively stable iff

$$S_n := \{X_j/b_n, \quad 1 \leq j \leq n\} \rightarrow [0, 1] =: S$$

a.s. with respect to the Hausdorff metric on $\mathcal{F}_d$.

**Proof.** We first show that if $\{M_n\}$ is a.s. relatively stable, then for any $p \in [0, 1]$ there exists a sequence $k_n = k_n(p)$ such that $1 \leq k_n \leq n$ and

$$M_{k_n}/b_n \rightarrow p \text{ a.s.} \quad (3.1)$$

If $p = 0$, then (3.1) trivially holds with $k_n = 1$, so assume $p > 0$. Since

$$\frac{M_{k_n}}{b_n} = \frac{M_{k_n}}{b_{k_n}} \cdot \frac{b_{k_n}}{b_n},$$

it suffices to show that there exists a sequence $k_n \rightarrow \infty$ such that

$$b_{k_n}/b_n \rightarrow p \text{ as } n \rightarrow \infty. \quad (3.2)$$

This follows in a standard way from the fact that $\{b_n\}$ is a slowly varying sequence. (Cf. Loeve, 1977, p. 335.)
Now we are ready to prove Proposition 3.1. The sufficiency is obvious so assume \( \{M_n\} \) is a.s. relatively stable. To prove \( S_n \rightarrow S \), it suffices to show (2.1) and (2.2). Pick \( y \in S \) and it follows from (3.1) there exists \( k_n \rightarrow \infty \) such that \( M_{k_n} / b_n \rightarrow y \) a.s. as \( n \rightarrow \infty \). Since \( M_{k_n} / b_n \in S_n \), (2.1) is satisfied.

Now suppose there exists \( y_{n_j} \in S_{n_j} \) such that \( y_{n_j} \rightarrow y_0 \) a.s. and \( y_0 > 1 \). Then

\[
\frac{M_{n_j}}{b_{n_j}} \text{ a.s. } \frac{X_i}{b_{n_j}} \text{ a.s. } \frac{y_{n_j}}{y_0} > 1 \text{ as } n \rightarrow \infty,
\]

which violates relative stability, so that (2.2) follows. \( \square \)

We now discuss preservation of relative stability under an asymptotic equivalence. Recall \( b_n^{(i)} = F_i^{x^{-1}}(1-n_{i-1}^{-1}) \), \( i = 1,2 \).

**Proposition 3.2.** Suppose \( \{M_n^{(1)}\} \) is a.s. relatively stable and \( \ell b_n^{(1)} \sim b_n^{(2)} \) for some \( \ell > 0 \). Then \( \{M_n^{(2)}\} \) is also a.s. relatively stable.

**Proof.** We need to show for any convenient \( A > 0 \)

\[
\int_0^\infty \frac{F_2(dx)}{A F_2(\epsilon x)} = \int_1^1 \frac{ds}{F_2(A) F_2(\epsilon F_2(s))} < \infty
\]

for \( 0 < \epsilon < 1 \). Since by assumption \( F_2^{x^{-1}}(s)/F_1^{x^{-1}}(s) \rightarrow \ell \), \( s \rightarrow 1 \), it suffices to prove

\[
(3.3) \int_0^1 \frac{ds}{A_1 F_2(\epsilon (\ell+\epsilon_1) F_1(s))} = \int_0^\infty \frac{F_1(dx)}{A_2 F_2(\epsilon (\ell+\epsilon_1)x)} < \infty
\]

for \( 0 \leq A_1 \leq 1 \), \( 0 \leq A_2 < \infty \) and \( \epsilon_1 > 0 \) arbitrarily small. From the integration by parts
formula

\[ \int \frac{dF_1(y)}{\tilde{F}_2(ky)} = \int \frac{\tilde{F}_1(y/k)}{\tilde{F}_2(y)} dF_2(y), \]

a change of variable and a second use of \( F_2^r(s)/F_1^r(s) \to \ell \), we get that the finiteness of the integral on the right side of (3.3) is equivalent to

\[ \int_0^1 \frac{F_1(\frac{\ell-\epsilon_2}{\epsilon(\ell+\epsilon_1)})}{(1-s)^2} ds = \int_{F_1^r(0)}^\infty \frac{(\ell-\epsilon_2)x}{\epsilon(\ell+\epsilon_1)} \frac{dF_1(x)}{\tilde{F}_2(x)} < \infty, \]

where \( \epsilon_2 > 0 \) can be chosen arbitrarily small. Since \( 0 < \epsilon < 1 \) and \( \epsilon_1, \epsilon_2 \) can be chosen arbitrarily small, we may assume \( C := (\ell-\epsilon_2)/(\epsilon(\ell+\epsilon_1)) > 1 \) and from the integration by parts formula (3.4) the finiteness follows from the assumed stability of \( F_1^r \). \( \square \)

The next sequence of comparison results are needed in section 4.

**Lemma 3.3.** \( M_n^{(1)}/M_n^{(2)} \to 0 \) a.s. as \( n \to \infty \) iff for any \( t > 0 \)

\[ \int_0^\infty \frac{dF_1(x)}{1 - F_2(tx)} < \infty. \]

**Proof.** For \( \epsilon > 0 \), let \( E_n = \{ X_n^{(1)}/M_n^{(2)} > \epsilon \} \). Suppose \( M_n^{(1)}/M_n^{(2)} \to 0 \) a.s. Then \( P(E_n \text{ i.o.}) = 0 \) and using the independence of \{\( X_n^{(1)} \)\} and \{\( X_n^{(2)} \)\}, we have for \( m < n \)

\[ P(E_mE_n) = P\{X_m^{(1)}/M_m^{(2)} > \epsilon, X_n^{(1)}/M_n^{(2)} > \epsilon \} \]

\[ \leq P\{X_m^{(1)}/M_m^{(2)} > \epsilon, X_n^{(1)}/\bigvee_{j=m+1}^n X_j^{(2)} > \epsilon \} \]

\[ = P\{X_m^{(1)}/M_m^{(2)} > \epsilon \} \cdot P\{X_n^{(1)}/\bigvee_{j=m+1}^n X_j^{(2)} > \epsilon \} \]

\[ = P(E_m)P(E_{n-m}) \]
and from an extension of the Borel–Cantelli Lemma (Spitzer, 1964), we deduce that

$$\omega > \sum \frac{P(E_n)}{\sum \int_0^\infty F_2^n(\epsilon^{-1}x) dF_1(x)} = \int_0^\infty \frac{dF_1(x)}{1 - F_2(\epsilon^{-1}x)}$$

and we have the desired result.

Conversely, if (3.5) holds, then $\Sigma P(E_n) < \omega$ and it follows from the Borel–Cantelli Lemma that $P(E_n \ i.o.) = 0$, which implies

$$X_n^{(1)}/M_n^{(2)} \to 0 \ a.s. \ as \ n \to \infty.$$ 

Define $L_n^{(1)} = \inf \{k \leq n; \frac{X_k^{(1)}}{n} = \max_{j=1}^n X_j^{(1)}\}$. Since $L_n^{(1)} \to \infty$ a.s., it follows from the above that

$$M_n^{(1)}/M_n^{(2)} = \frac{X_n^{(1)}}{M_n^{(2)}} \cdot \frac{M_n^{(2)}}{L_n^{(1)}} \cdot \frac{L_n^{(1)}}{M_n^{(2)}} \leq \frac{X_n^{(1)}}{L_n^{(1)}} \to 0 \ a.s. \ \ \ \ \square$$

**Proposition 3.4.** Suppose $M_n^{(2)}$ is a.s. relatively stable. Then $M_n^{(1)}/b_n^{(2)} \to 0$ a.s. iff $b_n^{(1)}/b_n^{(2)} \to 0$ as $n \to \infty$.

**Proof.** If $M_n^{(1)}/b_n^{(2)} \to 0$ a.s. then for $\delta > 0$ $P\{M_n^{(1)}/b_n^{(2)} < \delta\} \to 1$, which is

$$P_n(\delta b_n^{(2)}) \to 1 \ as \ n \to \infty.$$ 

Taking logarithms on both sides and using the relation $\log(1-u) \sim -u$ as $u \downarrow 0$, we have

$$n(1 - F_1(\delta b_n^{(2)})) \to 0,$$

which is equivalent to

$$\frac{1 - F_1(\delta b_n^{(2)})}{1 - F_2(b_n^{(2)})} \to 0 \ as \ n \to \infty.$$
This implies that for any \( \epsilon > 0 \),

\[
\frac{1}{1 - F_2(u)} \leq \frac{\epsilon}{1 - F_1(u)} \quad \text{for} \quad u > u_0(\epsilon).
\]

Taking inverses we get

\[
\frac{b^{(1)}(t)}{b^{(2)}(t)} \leq \delta \quad \text{for} \quad t \geq t_0(\epsilon).
\]

Since \( b^{(2)}(\cdot) \) is slowly varying, \( b^{(2)}(t) \sim b^{(2)}(\frac{t}{\epsilon}) \), we get from (3.6) that

\[
\limsup_{t \to \infty} \frac{b^{(1)}(t)}{b^{(2)}(t)} \leq \delta.
\]

Since \( \delta > 0 \) is arbitrary, the result follows.

Conversely, assume \( b^{(1)}_n / b^{(2)}_n \to 0 \) as \( n \to \infty \). Then for \( \delta > 0 \), we have

\[
0 < b^{(1)}_n / b^{(2)}_n < \delta \quad \text{for} \quad n \geq n_0(\delta).
\]

For the desired result, by Lemma 3.3 it suffices to show that for \( t > 0 \),

\[
J_1 := \int_0^\infty \frac{dF_1(y)}{1 - F_2(ty)} = \int_0^1 \frac{ds}{1 - F_2(tF_1^{-1}(s))} < \infty.
\]

Because of (3.7) \( J_1 < \infty \) if

\[
J_2 := \int_0^1 \frac{ds}{1 - F_2(tF_2^{-1}(s))} < \infty.
\]

However \( J_2 < \infty \) because we assume \( \{M_n^{(2)}\} \) is a.s. relatively stable. \( \square \)

The following proposition translates asymptotic equivalence of inverse functions into a more readily verified condition on distribution tails.
**Proposition 3.5.** Suppose \( \{M_n^{(2)}\} \) is relatively stable in probability. Then

\[
b_n^{(1)} / b_n^{(2)} \to \ell \geq 0 \quad \text{as} \quad n \to \infty
\]

iff

\[
\lim_{t \to \infty} \frac{1 - F_1(tx)}{1 - F_2(t)} = \begin{cases} 
0 & \text{if } x > \ell \\
\infty & \text{if } x < \ell.
\end{cases}
\]

**(Proof.)** We concentrate on the case \( \ell > 0 \); the case when \( \ell = 0 \) will be clear. So suppose \( b_n^{(1)} / b_n^{(2)} \to \ell > 0 \). Then by the relative stability of \( M_n^{(2)} \), we have

\[
\frac{b_n^{(1)}}{M_n^{(2)}} = \frac{b_n^{(2)}}{M_n^{(2)}} \cdot \frac{b_n^{(1)}}{b_n^{(2)}} \xrightarrow{P} 1 \cdot \ell.
\]

Thus for \( \epsilon > 0 \),

\[
P\{b_n^{(1)}/M_n^{(2)} > \ell + \epsilon\} = \frac{b_n^{(1)}}{\ell + \epsilon} \to 0 \quad \text{as} \quad n \to \infty.
\]

Remembering that \( \log(1-x) \sim -x \) as \( x \downarrow 0 \), we have from the above that

\[
n(1 - F_2(b_n^{(1)}/(\ell + \epsilon))) \to \infty \quad \text{as} \quad n \to \infty
\]

but this is equivalent to

\[
\frac{1 - F_2(b_n^{(1)}/(\ell + \epsilon))}{1 - F_1(b_n^{(1)})} \to \infty \quad \text{as} \quad n \to \infty,
\]

which gives one half of (3.8). The other half is similar.

To show the converse, we use the following lemma from Resnick (1971): For

\( 0 \leq L \leq \infty \) we have

\[
\lim_{t \to \infty} \frac{1 - F_1(t)}{1 - F_2(t)} = L
\]

iff

\[
\lim_{n \to \infty} \int_0^\infty F_1^n(x) dF_2^n(x) = \frac{1}{1 + L}.
\]

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If (3.8) holds, then
\[
\lim_{t \to \infty} \frac{1 - F_2(t/(\ell + \epsilon))}{1 - F_1(t)} = \omega,
\]
whence from (3.10)
\[
\lim_{n \to \infty} \int_0^\infty F_2^n(\frac{x}{\ell + \epsilon})dF_1^n(x) = 0.
\]

Since \(M_n^{(1)}\) and \(M_n^{(2)}\) are independent, the above is the same as
\[
(3.11) \quad \lim_{n \to \infty} \mathbb{P} \left[ \frac{M_n^{(1)}}{M_n^{(2)}} > \ell + \epsilon \right] = 0.
\]

A reverse inequality follows in the same way yielding \(M_n^{(1)}/M_n^{(2)} \xrightarrow{\mathbb{P}} \ell\). Noting that
\[
\frac{M_n^{(1)}}{M_n^{(2)}} = \frac{b_n^{(1)}}{b_n^{(2)}} \cdot \frac{M_n^{(1)}}{M_n^{(2)}}
\]
and \(M_n^{(2)}\) is relatively stable, we have that \(\frac{M_n^{(1)}}{b_n^{(2)}} \xrightarrow{\mathbb{P}} \ell\). Thus \(M_n^{(1)}\) is relatively stable in probability and therefore \(b_n^{(1)} \sim b_n^{(2)} \cdot \ell\).

\[\square\]

4. LIMIT SETS OF SCALED RANDOM SAMPLES.

From now on, we assume \(X_1, X_2, \ldots\) is a sequence of iid random vectors in \(\mathbb{R}^d\).

When do there exist scaling constants \(b_n > 0\) such that the sequence of random sets \(S_n := \{X_j/b_n, 1 \leq j \leq n\}\) converges to some non-empty compact set \(S\) in \(\mathbb{R}^d\)? Since \(S = \{0\}\) is a degenerate case (the sequence \(\{b_n\}\) grows too fast), we exclude this from consideration. Thus \(S \neq \{0\}\) and without loss of generality, we may assume
\[
(4.1) \quad \sup_{x \in S} \|x\| = 1,
\]

\[12\]
where $\|x\|$ is a norm on $\mathbb{R}^d$; any convenient norm suffices for our purposes. The relation between the scaling constants $\{b_n\}$ used in the definition of $S_n$ and those used to scale maxima in section 3 will soon become apparent.

We say that a set $S \subset \mathbb{R}^d$ is star shaped if $x \in S$ implies $\epsilon \cdot x \in S$ for any $0 \leq \epsilon < 1$. We will show later that any possible limit set is always star shaped. This is an important property of limit sets and suggests that our problem is better understood in polar coordinates rather than in the usual cartesian coordinates. The relevance of star shaped sets is noted in Eddy's (1982) work as well.

**Proposition 4.1.** Let $\{X_j, j \geq 1\}$ be iid in $\mathbb{R}^d$. If there exist $b_n \to \infty$ and a non-empty compact set $S$ such that

$$S_n := \{X_j/b_n, j \leq n\} \to S$$

then

(i) $\{M_n\} := \{\vee_{i=1}^n \|X_i\|\}$ is a.s. relatively stable and

(ii) $S$ is star shaped.

**Proof:** (i) Since $S_n \subset S_{\epsilon^n}$ for large $n$ we get eventually

$$\vee_{j=1}^n \|b_n^{-1} X_j\| = M_n/b_n \leq \sup_{x \in S} \|x\| + \delta(\epsilon) = 1 + \delta(\epsilon)$$

where $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. A reverse inequality comes from the fact that eventually $S \subset S_{\epsilon^n}$.

(ii) Pick $s \in S$. Since $S_n \to S$ there exist $\ell(n) \leq n$ with $X_{\ell(n)}/b_n \to s$. From (3.2) there exist $\{k_n\}$ such that for given $\epsilon$, $0 < \epsilon < 1$ we have as $n \to \infty$

$$b_{k_n}/b_n \to \epsilon$$

and $k_n \to \infty$ since $\epsilon > 0$. Therefore
which by (2.2) shows $\epsilon S \in S$. Hence $S$ is star shaped.

In order to pass to more convenient representations of our sets, we need the following result which is a variant of Proposition 3.18 in Resnick (1987).

Consistent with previous notation, let $\mathcal{F}(E)$ be the closed subsets of a topological space $E$ and $\mathcal{K}(E)$ be the compact subsets.

**Proposition 4.2.** Let $E_1, E_2$ be locally compact with countable bases and suppose $T: E_1 \to E_2$ is continuous. For $S \subset E_1$, define $T^*S = \{T(s), s \in S\} \subset E_2$.

(i) If

\[
(4.2) \quad T^{-1}(\mathcal{K}(E_2)) \subset \mathcal{K}(E_1),
\]

then $T^*: \mathcal{K}(E_1) \to \mathcal{K}(E_2)$ is continuous: If $S_n \in \mathcal{K}(E_1), 1 \leq n \leq \omega$ and $S_n \to S_\omega$ in $\mathcal{K}(E_1)$, then $T^*S_n \to T^*S_\omega$ in $\mathcal{K}(E_2)$.

(ii) If $T$ is a homeomorphism, then $T^*: \mathcal{F}(E_1) \to \mathcal{F}(E_2)$ is continuous: If $S_n \in \mathcal{F}(E_1), 1 \leq n \leq \omega$ and $S_n \to S_\omega$ in $\mathcal{F}(E_1)$, then $T^*S_n \to T^*S_\omega$ in $\mathcal{F}(E_2)$.

**Remark:** (4.2) holds if $T$ is a homeomorphism. Variant formulations are possible. For instance if the domain $D$ of $T$ is compact, $T$ is continuous and $S_n \subset D, 1 \leq n \leq \omega$, then the conclusion holds since (4.2) is automatic.

**Proof.** We only check (i) as (ii) is almost the same. One can proceed by Lemma 2.1 or alternatively by showing inverse images of open sets are open. Basis sets in $\mathcal{K}(E_2)$ are of the form $(K_2 \in \mathcal{K}(E_2), G_2$ open in $E_2)$

\[
\{F_2 \in \mathcal{K}(E_2): F_2 \cap K_2 = \emptyset\}, \{F_2 \in \mathcal{K}(E_2): F_2 \cap G_2 \neq \emptyset\}.
\]
Now
\[(T^*)^{-1}\{F_2 \in \mathcal{H}(E_2): F_2 \cap K_2 = \emptyset\} = \{F_1 \in \mathcal{H}(E_1): T^*F_1 \cap K_2 = \emptyset\}.\]

Since \(K_2 = T^*(T^{-1}K_2)\) and from (4.2) \(T^{-1}K_2 \in \mathcal{H}(E_1)\), we have this inverse image equal to
\[\{F_1 \in \mathcal{H}(E_1): T^*(F_1 \cap T^{-1}K_2) = \emptyset\} = \{F_1 \in \mathcal{H}(E_1): F_1 \cap T^{-1}K_2 = \emptyset\}\]
which is open in \(\mathcal{H}(E_1)\). Likewise
\[(T^*)^{-1}\{F_2 \in \mathcal{H}(E_2): F_2 \cap G_2 \neq \emptyset\} = \{F_1 \in \mathcal{H}(E_1): F_1 \cap T^{-1}G_2 \neq \emptyset\},\]
which is open in \(\mathcal{H}(E_1)\) since \(T^{-1}G_2\) is open in \(E_1\) as a consequence of the continuity of \(T\). □

Pick a convenient norm \(\|x\|\) on \(\mathbb{R}^d\) and let \(\Theta = \{x \in \mathbb{R}^d: \|x\| = 1\}\) be the unit sphere in \(\mathbb{R}^d\) with respect to this norm and define a polar coordinate transformation
\[T: \mathbb{R}^d \setminus \{0\} \to \Theta \times (0,\alpha)\]
by
\[Tx = (\theta, r) = (\theta(x), r(x))\]
\[= (x/\|x\|, \|x\|)\]
for \(x \neq 0\). Sometimes we write \(Tx = x^*\). If \(T\) is applied to the random vector \(X_i \neq 0\) we write \(T(X_i) = (\theta_i, \|X_i\|)\). Note that for \(b > 0, \ x \neq 0\)
\[T(x/b) = \left(\frac{x/b}{\|x/b\|}, \frac{\|x\|}{\|b\|}\right) = \left(\frac{x}{\|x\|}, \frac{\|x\|}{\|b\|}\right)\]
and this suggests our problem is simpler if phrased in polar coordinates. The fact that \(T\) is not defined at \(0\) necessitates working in \(\mathbb{R}^d \setminus \{0\}\) with the relative topology specified by
\[\mathcal{S}(\mathbb{R}^d \setminus \{0\}) = \{F^\# \cap \{0\}^C: F^\# \in \mathcal{S}(\mathbb{R}^d)\}\].
T: \mathbb{R}^d \setminus \{0\} \to \Theta \times (0,\infty) is a homeomorphism provided we specify \( \Theta \times (0,\infty) \) has the product topology and

\[
\mathcal{F}(0,\infty) = \{ F^\# \cap (0,\infty) : F^\# \in \mathcal{F}(\mathbb{R}^d) \}.
\]

**Lemma 4.3.** Suppose \( \{X_j : j \geq 1\} \) are iid in \( \mathbb{R}^d \). If \( S \in \mathcal{K}_d \) is star shaped and \( S_n \in \mathcal{K}_d \) then

\[
S_n \to S \text{ in } \mathcal{K}_d
\]

iff

\[
S_n \cap \{0\}^c \to S \cap \{0\}^c \text{ in } \mathcal{F}(\mathbb{R}^d \setminus \{0\}),
\]

and

\[
\sup_n \sup \{ \|x\| : x \in S_n \} < \infty.
\]

**Proof.** Given \( S_n \to S \). If \( s \in S \setminus \{0\}^c \), then there exist \( s_n \in S_n \), \( s_n \to s \). Eventually \( s_n \neq 0 \) so \( s_n \in S_n \cap \{0\}^c \) and (2.1) holds. If \( s_n' \in S_n' \cap \{0\}^c \) and \( \lim_{n' \to \infty} s_n' = s_\infty \) exists in \( \mathbb{R}^d \setminus \{0\} \) then \( s_\infty \neq 0 \), \( s_\infty \in S \), so \( s_\infty \in S \cap \{0\}^c \) as required for (2.2).

Conversely suppose \( S_n \cap \{0\}^c \to S \cap \{0\}^c \). If \( s \neq 0 \) there exist \( s_n \neq 0 \), \( s_n \in S_n \), \( s_n \to s \) as required by (2.1). If \( s = 0 \), pick \( s^\# \in S \). Since \( S \) is star shaped, for every positive integer \( k \) there exist \( s_n^{(k)} \in S_n \cap \{0\}^c \) such that \( s_n^{(k)} \to s^\#/k \). There exists a function \( f(j) \to \infty \) as \( j \to \infty \) with

\[
d(s_j^{(f(j))}, s^\#/f(j)) < 1/f(j)
\]

We have \( s_j^{(f(j))} \in S_j \) and

\[
d(s_j^{(f(j))}, 0) \leq d(s_j^{(f(j))}, s^\#/f(j)) + d(s^\#/f(j), 0)
\]

\[
\leq 1/f(j) + d(s^\#/f(j), 0) \to 0.
\]

So (2.1) is satisfied in this case as well.
Suppose \( s_{n'} \in S_{n'} \) and \( \lim_{n' \to \infty} s_{n'} = s_\infty \) exists. If \( s_\infty = 0 \) then (2.2) is satisfied since 0 is in the closed, star shaped set \( S \). If \( s_\infty \neq 0 \) then since we assume \( S_n \cap \{0\}^c \to S \cap \{0\}^c \) we get from (2.2) that \( s_\infty \in S \cap \{0\}^c \subset S \) as required. \( \square \)

As before let \( T \) be the polar coordinate transform. Define \( T^* : \mathcal{F}(\mathbb{R}^d \setminus \{0\}) \to \mathcal{F}(\Theta \times (0,\infty)) \) by

\[
T^* S = \{ T s, s \in S, s \neq 0 \} = S^* .
\]

We now show \( S \) is a limit set for a scaled random sample iff after a polar coordinate transform \( S^* = \text{hypo} \ell \) where \( \ell \in \text{USC} \). This characterizes the class of limit sets.

**Theorem 4.4.** There exists \( b_n \to \infty \) such that in \( \mathcal{K}_d \)

\[
S_n = \{ X_i / b_n, 1 \leq i \leq n \} \to S,
\]

with \( S \) compact and \( \forall \| x \| = 1 \), iff there exists \( \ell : \Theta \to (0,\infty) \) with \( \ell \) USC and \( \forall \ell(\Theta) = 1 \) and in \( \mathcal{F}(\Theta \times (0,\infty)) \)

\[
S_n^* = \{ (\ell, \| X_i / b_n \|) : 1 \leq i \leq n, \| X_i / b_n \| \neq 0 \}
\]

\( \to \text{hypo}(\ell) = S^* = \{ (\ell(x), r(x)) : x \in S, x \neq 0 \}. \)

Furthermore, for any \( S \in \mathcal{K}_d \) such that \( T^* S = \text{hypo} \ell \) for some \( \ell \in \text{USC} \), there exists an iid sequence \( \{ X_j, j \geq 1 \} \) such that for appropriate scaling constants \( b_n \to \infty \)

\[
\{ X_j / b_n, 1 \leq j \leq n \} \to S
\]

a.s. in \( \mathcal{K}_d \).
Proof. Since \( S^*_n = T^*(S_n \cap \{0\}^c) \) and \( S^* = T^*(S \cap \{0\}^c) \) and \( T \) is a homeomorphism, we conclude \( S_n \rightarrow S \) iff \( S^*_n \rightarrow S^* \) from Propositions 4.1, 4.2 and Lemma 4.3 above. The star shaped property of \( S \) means for \( x \in S \) and \( 0 < t < 1 \)

\[(\theta, tr) \in S^* \text{ if } (\theta, r) \in S^*.\]

Let \( \Theta' = \{ \theta(x), x \in S, x \neq 0 \} \). Define \( \ell : \Theta \rightarrow [0, \omega) \) by

\[
\ell(\theta) = \sup \{ r : (\theta, r) \in S^* \} \text{ if } \theta \in \Theta' \\
= 0 \hspace{1cm} \text{ if } \theta \in \Theta \setminus \Theta'.
\]

Then

\[S^* = \text{hypo}(\ell).\]

Since \( S^* \) is closed, \( \ell \) is USC from Lemma 2.3.

Given any USC function \( \ell \), we explain how to construct a scaled sample \( S^*_n \) converging to \( \text{hypo}(\ell) \). This will show that the class of limit sets in polar coordinates is indeed \( \{ \text{hypo}(\ell), \ell \in \text{USC} \} \).

Given \( \ell(\theta), \theta \in \Theta \) where \( \Theta \) is the unit sphere in \( \mathbb{R}^d \) with respect to the norm \( ||x|| \). Then \( \text{hypo}(\ell) \) is closed in \( \Theta \times (0, \omega] \). Assume \( \sup_{\theta \in \Theta} \ell(\theta) = 1 \). We first construct a distribution \( \mu \) on \( \Theta \times (0, \omega] \) whose support is \( \text{hypo}(\ell) \).

Fix \( \epsilon > 0 \) and let \( (\text{hypo}(\ell))^{\epsilon} \) be the \( \epsilon \)-swelling of \( \text{hypo}(\ell) \) and define \( \pi:\)

\[(\text{hypo}(\ell))^{\epsilon} \rightarrow \text{hypo}(\ell) \text{ by }\]

\[\pi(\theta, r) = \text{closest point of } \text{hypo}(\ell) \text{ to } (\theta, r).\]

So \( \pi \) is the identity on \( \text{hypo}(\ell) \). On \( (\text{hypo}(\ell))^{\epsilon\setminus \text{hypo}(\ell)} \), \( \pi \) maps points to the closest element of \( \text{hypo}(\ell) \). If there is more than one closest element, choose one arbitrarily (but measurably). Let \( L \) be Lebesgue measure on \( (\text{hypo}(\ell))^{\epsilon} \) and define \( \mu \) by \( \mu = L \circ \pi^{-1}.\)
We now verify that the support of $\mu$ is $\text{hypo}(\ell)$. Let $(\theta_0, r_0) \in \text{hypo}(\ell)$ and suppose $G \subset \Theta \times (0, \infty)$ is open and $(\theta_0, r_0) \in G$. We must show $\mu(G) > 0$. From the definition of $\mu$, it suffices to show $\pi^{-1}(G)$ has positive Lebesgue measure and for this it is enough if we prove $\pi^{-1}G$ contains an open set. We seek an open $U \subset (\text{hypo}(\ell))^\epsilon$ such that $\pi(U) \subset G \cap (\text{hypo}(\ell))$. We may set

$$U = (\text{hypo}(\ell))^\epsilon \cap \{(\theta, r) \in G : d((\theta, r), (\theta_0, r_0)) < 1/2 \quad d((\theta_0, r_0), G^C)\}.$$ 

If $(\theta, r) \in U$ then because $\pi$ maps $(\theta, r)$ to the closest point of $\text{hypo}(\ell)$:

$$d(\pi(\theta, r), (\theta_0, r_0)) \leq d(\pi(\theta, r), (\theta, r)) + d((\theta, r), (\theta_0, r_0))$$

$$\leq 2d((\theta, r), (\theta_0, r_0)) < d((\theta_0, r_0), G^C)$$

from the definition of $U$. Thus $\pi(\theta, r) \in G$, whence $\pi(U) \subset G$. Since $U$ is open and non-empty, the verification that support $\mu = \text{hypo}(\ell)$ is complete.

Now let $\{(\theta_i, R_i), i \geq 1\}$ be iid with distribution $\mu$. A sample from any distribution will converge to the support of the distribution so

$$\{(\theta_i, R_i), i \leq n\} \longrightarrow \text{hypo}(\ell)$$

in $\mathcal{F}(\Theta \times (0, \infty))$. (If $G$ is open and $G \cap \text{hypo}(\ell) \neq \emptyset$ then $\mu(G) > 0$ and

$$P[\liminf_{n \to \infty} (G \cap \{(\theta_i, R_i), i \leq n\} \neq \emptyset)] = 1.$$ 

If $K$ is compact in $\Theta \times (0, \infty)$ and $K \cap \text{hypo}(\ell) = \emptyset$ then $P[\{(\theta_i, R_i), i \leq n\} \cap K = \emptyset \forall n] = 1$.)

Finally let $\{E_i, i \geq 1\}$ be iid unit exponential random variables independent of $\{(\theta_i, R_i), i \geq 1\}$. Then we claim

$$\{\theta_i, R_i, E_i / \log n \quad i \leq n\} \longrightarrow \text{hypo}(\ell)$$

as $n \to \infty$. To check this suppose $K$ is compact and $K \subset ((\text{hypo}(\ell))^\epsilon)^C$. Since

$$P[\{(\theta_i, R_i), i \leq n\} \cap K \text{ i.o.}] = 0$$

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and
\[ \forall E_i / \log n \to 1 \]

we get for all large \( n \)
\[ \{ (\theta_i, R_i E_i / \log n), \ i \leq n \} \cap K = \emptyset. \]

Also let \( G \) be open with \( G \cap \text{hypo}(\ell) \neq \emptyset \). We may suppose \( G \) is of the form
\[ G = I_1 \times I_2 \]
where \( I_1 \subset \Theta \) is open and \( I_2 = (r_1, r_2), \ 0 < r_1 < r_2 \). We need to show
\[ \{ (\theta_i, R_i E_i / \log n), \ i \leq n \} \cap G \neq \emptyset \]
for all large \( n \). Pick \( \delta \) small so that \( I_2' = (r_1 + \delta, r_2 - \delta) \subset I_2 \). Then
\[ G' = I_1 \times I_2' \cap \text{hypo}(\ell) \neq \emptyset. \]
There exist indices \( K(n) \) such that \( (\theta_{K(n)}, R_{K(n)}) \in G' \) and \( K(n) \sim n / \mu(G') \). (Let \( K(1) = \inf \{ j \geq 1 : (\theta_j, R_j) \in G' \} \); \( K(2) = \inf \{ j > K(1) : (\theta_j, R_j) \in G' \} \), etc.). Then \( \{ K_n \} \) is independent of \( \{ E_i \} \) and \( \{ E_{K(n)} \} = \{ E'_n \} \) is iid unit exponential.

Let \( L(n) \) be the record value times of the sequence \( \{ E'_n \} \) so that \( E'_{L(n)} \sim n \) and \( \log L(n) \sim n \) (Resnick, 1987). Then
\[ \log K(L(n)) \sim \log L(n) \sim n. \]

Since \( E'_{L(n)} \sim n \), for all \( n \) large
\[ (\theta_{K(L(n))}, R_{K(L(n))} \frac{E'_{L(n)}}{n}) \in G. \]
If
\[ K(L(n)) \leq m < K(L(n+1)) \]
then
\[ \log K(L(n)) \leq \log m \leq \log K(L(n+1)) \]
\[ \log K(L(n)) \sim n \sim \log m \]
as \( n \to \infty \). So for all large \( m \)

\[ \{(\theta_i, R_i E_i / \log m), i \leq m\} \cap G \neq \emptyset \]
since if \( K(L(n)) \leq m < K(L(n+1)) \)

\[ (\theta_{K(L(n))}^*, R_{K(L(n))}^*) \frac{E_{L(n)}'}{\log m} = (\theta_{K(L(n))}^*, R_{K(L(n))}^*) \frac{E_{L(n)}'}{n} \cdot \frac{n}{\log m} \in G. \]

This proves the claim and completes the construction. \( \square \)

Having identified the class of limits, we turn attention to criteria for convergence to a specified limit. We continue to work in polar coordinates. Define open and closed balls in \( \Theta \) as follows: For \( \theta \in \Theta, \ \epsilon > 0 \) and \( d(\cdot, \cdot) \), the metric on \( \mathbb{R}^d \), set

\[ B(\theta, \epsilon) := \{ \omega \in \Theta: d(\theta, \omega) < \epsilon \}, \]
\[ B(\theta, \epsilon+) := \{ \omega \in \Theta: d(\theta, \omega) \leq \epsilon \}, \]

For \( B \subset \Theta \) and \( S \subset \mathbb{R}^d \), the restriction of \( S \) to \( B \) is defined by

\[ S(B) := \{ x \in S \cap \{0\}^c: \theta(x) \in B \} \]

and in polar coordinates we have

\[ S^*(B) := T^*S(B). \]

Finally, for \( B \subset \Theta \) and a function \( \ell: \Theta \to \mathbb{R}_+ \), we define the sup measure \( \ell^V: 2^\Theta \to \mathbb{R}_+ \) by

\[ \ell^V(B) := \sup_{\theta \in B} \ell(\theta) \text{ and} \]
\[ \ell^V(\phi) \equiv 0. \]
The next result reduces our set convergence problem to convergence of compact subsets of $\mathbb{R}_+$.

**Proposition 4.5.** (Cf. Vervaat, 1988, section 10). Let $\ell: \Theta \to \mathbb{R}_+$ be an USC function with $\ell^V(\Theta) = 1$. Set $S^* = \text{hypo}(\ell)$, $S = (T^*)^{-1}(S^*) \cup \{0\}$. If

$$S_n^* := \{(\theta_{n,j}, r_{n,j}) : 1 \leq j \leq j_n\}$$

is a set of points in $\Theta \times (0,\infty)$, $j_n \to \infty$, then

$$S_n := (T^*)^{-1}(S_n^*) \to S$$

in $\mathcal{K}_d$ iff

$$K_n = K_n(B(\theta,t)) := \{\|x\| : x \in S_n(B(\theta,t))\}$$

$$\to K = K(B(\theta,t)) := [0,\ell^V(B(\theta,t))]$$

in $\mathcal{K}[0,\infty)$ for any $\theta \in \Theta$ and $t > 0$ such that

$$(4.5) \quad \ell^V(B(\theta,t)) = \ell^V(B(\theta,t+)).$$

One might expect that $S_n \to S$ implies $S_n(B) \to S(B)$ as $n \to \infty$ provided (4.5) holds. However this is not true in general.

**Example.** In $\mathbb{R}^2$, let $\Theta$ be the unit circle which we parameterize by $\Theta = \{\theta : 0 \leq \theta < 2\pi\}$. Let

$$\ell_n(\theta) = \begin{cases} \frac{\pi}{n} & \pi - \frac{1}{n} \\ \frac{\pi}{2} & \pi \end{cases} (\theta), \quad \ell(\theta) = \begin{cases} \frac{\pi}{n} & \pi \\ \frac{\pi}{2} & \pi \end{cases} (\theta),$$

and $S_n^* = \text{hypo}(\ell_n)$, and $S^* = \text{hypo}(\ell)$. Then (4.5) is satisfied with $\theta = t = \pi/2$ and one checks readily that $S_n^* \to S^*$ as $n \to \infty$. But $S_n^*(B(\frac{\pi}{2}, \frac{\pi}{2} +))$ does not converge to $S^*(B(\frac{\pi}{2}, \frac{\pi}{2} +))$ since

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\[ S^*(B(\frac{\pi}{2}, \frac{\pi}{2} + \rangle) = \{(r, \theta) : 0 < r \leq 1, \theta \in \{\frac{\pi}{2}\}, 0 \leq \theta \leq \pi\} \text{ and} \]

\[ S^*(B(\frac{\pi}{2}, \frac{\pi}{2} + \rangle) = \{(r, \theta) : 0 < r \leq 1, \theta \in \{\frac{\pi}{2}, \pi\}, 0 \leq \theta \leq \pi\}. \]

Combining Proposition 4.5 and the results in the previous section, we have the following criterion for the convergence of scaled random samples to a limiting compact set. Recall \{X_n\} are i.i.d. in \(\mathbb{R}^d\). Set \(q(B) = P[\theta(X_1) \in B]\).

**Theorem 4.6.** Let \(\ell : \Theta \rightarrow [0,1]\) be a USC function with \(\ell^*(\Theta) = 1\), \(S^* = \text{hypo}(\ell)\), \(S = (T^*)^{-1}(S^*) \cup \{0\}\). Then there exists a sequence \(b_n \rightarrow \infty\) such that

\[ S_n := \{X_j/b_n, 1 \leq j \leq n\} \rightarrow S \text{ a.s. as } n \rightarrow \infty \]

in \(\mathcal{H}_d\) iff the following two conditions hold:

(i) \(\{M_n\} = \{\sum_{j=1}^{n} \|X_j\|\} \) is a.s. relatively stable.

(ii) For any \(B(\theta,t)\) satisfying (4.5),

if \(q(B(\theta,t)) = 0\) then \(\ell^*(B(\theta,t)) = 0\) and

if \(q(B(\theta,t)) > 0\) then

\[ \lim_{s \rightarrow \infty} \frac{P\{\|X_1\| > s, \theta(X_1) \in B(\theta,t)\}}{P\{\|X_1\| > s\}} = \begin{cases} 0 & \text{if } x > \ell^*(B(\theta,t)) \\ \infty & \text{if } x < \ell^*(B(\theta,t)) \end{cases} \]

**Proof.** Necessity. (i) follows from Proposition 4.1. Assume that \(B(\theta,t)\) satisfies (4.5). If \(q(B(\theta,t)) = 0\) then clearly \(\ell^*(B(\theta,t)) = 0\), so assume \(q(B(\theta,t)) > 0\). Observing Proposition 3.5, we have that (4.6) is equivalent to

\[ \lim_{n \rightarrow \infty} \frac{b_n^\#}{b_n} = \ell^*(B(\theta,t)) \]

where \(b_n^\# = G^+(1 - \frac{1}{n})\) and

\[ 23 \]
(4.8) \[ G(x) = P(\|X_1\| \leq x | \alpha(X_1) \in B(\theta, t)) \].

Thus it suffices to show that (4.7) holds. Since \( S_n \rightarrow S \) a.s. we have from Lemma 4.3 and Proposition 4.5 that

(4.9) \[ \{ \frac{1}{b_n} \cdot \|X_j\| : \alpha(X_j) \in B(\theta, t), 1 \leq j \leq n \} \rightarrow [0, \ell^V(B(\theta, t))] \] a.s. as \( n \rightarrow \infty \).

Define random indices as follows:

\[
\begin{align*}
K_0 &= 0, \\
K_i &= \inf\{j > K_{i-1} : \alpha(X_j) \in B(\theta, t)\}, \\
L_n &= \sup\{j : K_j \leq n\}, \\
N_n &= \sum_{j=1}^{n} 1[\alpha(X_j) \in B(\theta, t)]; \text{ and} \\
X_1^\# &= X_{K_1}, X_2^\# = X_{K_2}, \ldots, X_{N_n}^\# = X_{L_n}.
\end{align*}
\]

The "découpage de Lévy" (see for example, Resnick, 1987) says \( \{X_i^\#, i \geq 1\} \) are iid with distribution \( P[X_1 \in \cdot | \alpha(X_1) \in B(\theta, t)] \) and (4.9) is equivalent to

(4.10) \[ \{ \frac{1}{b_n} \cdot \|X_j^\#\|, 1 \leq j \leq N_n \} \rightarrow [0, \ell^V(B(\theta, t))] \] a.s.,

implying

(4.11) \[ \frac{M_N^\#}{b_n} := \frac{1}{b_n} \cdot \sum_{j=1}^{N_n} \|X_j^\#\| \rightarrow \ell^V(B(\theta, t)) \] a.s. as \( n \rightarrow \infty \).

Since \( N_n/n \rightarrow q(B(\theta, t)) > 0 \) a.s. as \( n \rightarrow \infty \) and \( b_n \in RV(0) \), it follows from (4.11) that

\[ \frac{M_N^\#}{b_n} = \frac{M_N^\#}{b_n} \cdot \frac{b_n}{b_n \cdot \frac{N_n}{n}} \rightarrow \ell^V(B(\theta, t)) \cdot 1 \] a.s. as \( n \rightarrow \infty \),

implying

(4.12) \[ \frac{M_N^\#}{b_n} \rightarrow \ell^V(B(\theta, t)) \] a.s. as \( n \rightarrow \infty \).
If \( \ell^V(B(\theta,t)) > 0 \), then (4.12) implies that \( \{M_n^{\#}\} \) is a.s. relatively stable with normalizing constant \( b_n \cdot \ell^V(B(\theta,t)) \). However, since \( \|X_j^{\#}\| \) has the distribution given by (4.8), it follows that \( b_n^{\#} \sim b_n \cdot \ell^V(B(\theta,t)) \), which is (4.7). If \( \ell^V(B(\theta,t)) = 0 \), then (4.7) immediately follows from (4.12) and Proposition 3.4.

**Sufficiency.** It again follows from Proposition 4.5 that it suffices to show that (4.10) holds for \( B(\theta,t) \) satisfying (4.5). We consider two cases:

(a) \( \ell^V(B(\theta,t)) > 0 \): Note that \( \ell^V(B(\theta,t)) > 0 \) clearly implies \( q(B(\theta,t)) > 0 \). Then we have from (4.7) and Proposition 3.2 that \( \{M_n^{\#}\} \) is a.s. relatively stable, which is equivalent to

\[
\frac{M_n^{\#}/b_n}{b_n} \rightarrow \ell^V(B(\theta,t)) \quad \text{a.s. as } n \rightarrow \infty.
\]

From Proposition 3.1

\[
\{\|X_j^{\#}\|/(b_n \ell^V(B(\theta,t))), \ 1 \leq j \leq n\} \rightarrow [0,1]
\]

whence

\[
\{\|X_j^{\#}\|/b_n, \ 1 \leq j \leq n\} \rightarrow [0, \ell^V(B(\theta,t))].
\]

Therefore since \( b[.] \) is slowly varying

\[
\{\|X_j^{\#}\|/b_n, \ 1 \leq j \leq N_n\} = (b_{N_n}/b_n)\{\|X_j^{\#}\|/b_{N_n}, \ 1 \leq j \leq N_n\} \rightarrow [0, \ell^V(B(\theta,t))] \quad \text{a.s.}
\]

and we get (4.10) as required.

(b) \( \ell^V(B(\theta,t)) = 0 \): If \( q(B(\theta,t)) = 0 \) then \( S_n(B(\theta,t)) = \emptyset \) a.s. and hence \( S(B(\theta,t)) = \emptyset \), so \( \ell = 0 \) on \( B(\theta,t) \). Assume \( q(B(\theta,t)) > 0 \). Then it follows from (4.7) and Proposition 3.4 that

\[
\frac{M_n^{\#}/b_n}{b_n} \rightarrow 0 \quad \text{a.s. as } n \rightarrow \infty,
\]

implying

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\[ \{ \frac{1}{b_n} \cdot \| X_j^\# \|, \ 1 \leq j \leq N_n \} \longrightarrow \{ 0 \} \text{ a.s.,} \]

which is (4.8).

As a special case of the previous proposition, we have the following.

**Corollary 4.7.** Suppose \( X_1 \) is spherically symmetric so that \( \theta_1, \| X_1 \| \) are independent and \( \theta_1 \) is uniformly distributed on \( \Theta \). Then

\[ S_n \longrightarrow S = \{ (\theta, r) : r \leq 1 \} \text{ a.s.} \]

iff \( \{ M_n \} = \{ \bigwedge_{j=1}^{n} \| X_j \| \} \) is a.s. relatively stable.

5. COMPLEMENTS AND EXAMPLES.

We first show in this section that the condition of Theorem 4.6 is implied by the log regular variation condition in Davis, Mulrow, and Resnick (1987).

**Proposition 5.1.** (David, Mulrow, Resnick, 1987) Let \( X_1, X_2, \ldots \) be an iid sequence of random vectors in \( \mathbb{R}^d \) having a common joint distribution which has support in \( \{ x \geq 0 \} \). Assume that the components are identically distributed.

Suppose for \( x \in [0, \infty)^d \setminus \{ 0 \} =: C \), as \( t \to \infty \)

\[ f(t, x) := \frac{-\log P\{ X_1 \geq t \cdot x \}}{-\log P\{ X_1 \geq t \cdot 1 \}} \to \lambda(x) \]

(5.1)

where \( \lambda(x) \) satisfies for \( t > 0, \ x \in C \), and some \( \alpha > 0 \)

\[ \lambda(t \cdot x) = t^\alpha \lambda(x). \]

(5.2)

Assume \( \lambda(x) \) is strictly increasing on \( C \), i.e.
\( \lambda(x) < \lambda(y) \) when \( x \preceq y \) and \( x \neq y \).

Then in \( \mathcal{K}_d \), as \( n \to \infty \), for some \( b_n > 0 \)

\[
S_n := \{X_j/b_n : 1 \leq j \leq n\} \longrightarrow S := \{x \geq 0 : \lambda(x) \leq 1\} \quad \text{a.s.}
\]

We now verify that the assumptions in Proposition 5.1 are sufficient to guarantee the limit set is \( [\lambda \leq 1] \) by showing that (4.6) holds. We restrict attention to the case \( d = 2 \) and show that (4.6) holds with the norm \( \|(x,y)\| = x \vee y \). Assume for example that \( B(\theta, \epsilon) \) is of the form:

\[
(5.3) \quad B(a,b) := \{(1,y) : a < y < b\} \text{ for } 0 \leq a < b \leq 1.
\]

From (5.2)

\[
S = \{(x,y) : x \geq y \geq 0, \ \lambda((x,y)) \leq 1\} \cup \{(x,y) : y > x \geq 0, \ \lambda((x,y)) \leq 1\}
\]

\[
= \{(x,y) : x \geq y \geq 0, \ \alpha \cdot \lambda((1,y)) \leq 1\} \cup \{(x,y) : y > x \geq 0, y \cdot \lambda((x,1)) \leq 1\}
\]

which is expressed in polar coordinates as

\[
S^* = \{\theta, r) : \theta \in \{1\} \times [0,1] \cup [0,1] \times \{1\}, \ r \cdot \lambda(\theta) \leq 1\}.
\]

This identifies \( \ell(\theta) \) as

\[
(5.4) \quad \ell(\theta) = \left( \frac{\lambda((1,0))}{\lambda(\theta)} \right)^{1/\alpha}.
\]

Since \( \lambda \) is strictly increasing, it follows from (5.3) and (5.4) that

\[
\ell^V(B(a,b)) = \left( \frac{\lambda((1,0))}{\lambda((1,a))} \right)^{1/\alpha}.
\]

We also have that
\{\| (x, y) \| > tu, \emptyset ((x, y)) \in B(a, b) \}

= \{ x \vee y > tu, (x, y)/(x \vee y) \in B(a, b) \}

= \{ x \vee y > tu, (1, y/x) \in B(a, b) \}

= \{ x > tu, a < y/x < bx \}.

Because (5.1) implies

\[ \lim_{t \to \infty} \frac{P\{X \vee Y > t\}}{P\{X > t\}} = 1, \]

it suffices to show

\[ \lim_{t \to \infty} \frac{P\{X > ut, aX < Y < bX\}}{P\{X > t\}} = \begin{cases} 0 & \text{if } u > \lambda_0 \\ \infty & \text{if } u < \lambda_0 \end{cases} \]

where

\[ \lambda_0 = \left[ \frac{\lambda((1,0))}{\lambda((1,a))} \right]^{1/\alpha}. \]

This is a fairly straightforward consequence of the assumptions. \qed

For the rest of the section, we examine some examples of Theorem 4.6 for the case d = 2. We denote again by \( S_n \) and \( S \) the set of sample points and a limit set in the Cartesian coordinates and by \( S_n^* \) and \( S^* \) those in the polar coordinates. Note that Proposition 5.1 does not apply to the following example since \( \lambda \) is not strictly increasing (cf. Davis, Mulrow, and Resnick, 1988, example 3.6).

**Example 5.2.** Suppose the distribution is given by \( P\{X > x, Y > y\} = (e^x + e^y - 1)^{-1} \).

Then we shall show that with \( b_n = \log n \), \( S_n \to S \) a.s. where

\[ S = \{(x, y): 0 < x, y < 1, 2x - 1 < y \leq (x+1)/2 \}. \]
As in Proposition 5.1, choosing \( \| (x,y) \| = x \vee y \) as the norm, and using the symmetry, we need to show that

\[
T_n^* \to T^* \quad \text{a.s. as } n \to \infty
\]

where

\[
T_n^* = \{(\sigma(x,y), r(x,y)): (x,y) \in S_n, \ x \geq y \} = \{((1,Y_j/X_j), X_j/b_n): X_j \geq Y_j, \ 1 \leq j \leq n \}
\]

and

\[
T^* = \{(\sigma(x,y), r(x,y)): (x,y) \in S, \ x \geq y \} = \{((1, y/x), x): 0 \leq y \leq x \leq (y+1)/2 \}.
\]

Since (5.5) is equivalent to

\[
U_n^* \to U^* \quad \text{a.s. as } n \to \infty
\]

where

\[
U_n^* = \{(Y_j/X_j, X_j/b_n): X_j \geq Y_j, \ 1 \leq j \leq n \}
\]

and

\[
U^* = \{ (\sigma, r): 0 \leq \theta \leq 1, \ 0 < r \leq 1/(2-\theta) \},
\]

we apply Theorem 4.6 to (5.6). Observe that since

\[
P\{X \vee Y > t\} = P[X > t] + P[Y > t] - P[X > t, Y > t] = 2e^{-t} - (2e^{t} - 1)^{-1} \sim 3/2 e^{-t}
\]

we have \( \{v_{1}^{n} \|(X_i, Y_i)\|, n \geq 1 \} \) is relatively stable and hence the first condition in Theorem 4.6 is satisfied. Thus since \((X,Y)\) has support \(\mathbb{R}^{2}_{+}\), we only need to show that

(4.8) holds. Now the corresponding USC function \( \ell(\theta) = 1/(2-\theta) \) is strictly increasing, and therefore, (4.8) is equivalent to \( 0 \leq a < b \leq 1 \),
\[
\lim_{t \to \infty} f(t,a,b,x) := \lim_{t \to \infty} \frac{P\{X > tx, \ a < Y/X < b\}}{P\{X \vee Y > t\}}
\]
\[
= \begin{cases} 
0 & \text{if } x > 1/(2-b) \\
\infty & \text{if } x < 1/(2-b)
\end{cases}.
\]

(5.8)

Since the Jacobian of the transformation \((x,y) \to (r = x, \ \theta = y/x)\) is \(J = r\), we have

\[
P\{X > tx, \ a < Y/X < b\} = \int_{r=tx}^{\infty} \int_{\theta=a}^{b} \frac{2re^{r(1+\theta)}}{(e^r + e^{r\theta} - 1)^2} \, d\theta \, dr.
\]

(5.9)

For \(x > 1/(2-b)\), it follows from (5.7) and (5.8) that as \(t \to \infty\),

\[
f(t,a,b,x) \leq \frac{2}{3} e^t \int_{r=tx}^{\infty} 2 \cdot e^{-2r} \, dr \int_{\theta=a}^{b} re^{\theta} \, d\theta
\]

\[
= \frac{4}{3} e^t \int_{tx}^{\infty} e^{-2r} \cdot (e^{br} - e^{ar}) \, dr
\]

\[
\leq \frac{4}{3} e^t \int_{tx}^{\infty} e^{-(2-b)r} \, dr
\]

\[
= \frac{4}{3(2-b)} \cdot e^{-(2-b)x} \cdot t
\]

\[
\to 0,
\]

which is one half of (5.8). For \(x < 1/(2-b)\) similar calculations show

\[
f(t,a,b,x) \to \infty,
\]

which is the other half of (5.8).

\[\square\]

For the following examples, we make the usual polar coordinate transformation

\[
\|X\| = (X^2 + Y^2)^{1/2}, \ \ \theta(X) = \arctan(Y/X).
\]

Example 5.3. Suppose the distribution of \((\theta(X), \|X\|)\) is

\[
P\{\theta(X) \in d\theta, \|X\| \in dr\} = F(dr) \cdot \frac{d\theta}{2\pi}
\]

Example 5.3
where $F$ is specified by (2.6). Then since $\{M_n\}$ is not a.s. relatively stable, $S_n$ does not converge a.s. However, since $\{M_n\}$ is relatively stable in probability, we have that
\[
S_n \xrightarrow{P} S = \{(x,y): x^2 + y^2 \leq 1\} \text{ as } n \to \infty. \quad \Box
\]

**Example 5.4.** Let $B_n$ be a random variable and $Y_n^* = (\theta(Y_n), ||Y_n||)$ and $Z_n^* = (\theta(Z_n), ||Z_n||)$ be random vectors. Suppose $P(B_n = 0) = P(B_n = 1) = 1/2$, $\theta(Y_n)$ and $\theta(Z_n)$ are uniformly distributed on $[0, \pi)$ and $[\pi, 2\pi)$ respectively, $P(||Y_n|| > r) = P(||Z_n|| > r/2) = e^{-r}$, and these are all independent. Then define $X_n^*$ as $X_n^* = (\theta(X_n), ||X_n||) := B_n \cdot Y_n^* + (1-B_n) \cdot Z_n^*$. The distribution of $X_n^*$ clearly satisfies the conditions of Theorem 4.6 with $b_n = \log n$ and the limit set is the union of two rectangles. It can be written
\[
S^* = \text{hypo } \ell
\]
where
\[
\ell(\theta) = \begin{cases} 1, & 0 \leq \theta \leq \pi \\ 2^{-1}, & \pi < \theta < 2\pi \end{cases}.
\]

For this example, the d.f. of $\theta(X_n)$ is continuous since $\theta(X_n)$ is uniformly distributed on $[0, 2\pi)$, but the corresponding USC function is discontinuous at $\theta = 0, \pi$.

**Example 5.5.** As in the previous example, set
\[
X_n^* = B_n \cdot Y_n^* + (1-B_n) \cdot Z_n^*
\]
where $B_n$ is the same as before, $\theta(Y_n)$ is uniformly distributed on $[0,2\pi)$, $P(||Y_n|| > r) = e^{-r}$ and $Z_n$ is degenerate at $(\pi,0)$, i.e. $P(\theta(Z_n) = \pi, ||Z_n|| = 0) = 1$. Then we see that with $b_n = \log n$, as $n \to \infty$
\[
S_n^* \to S^* = \{(\theta,r): 1_{[0,2\pi]}(\theta) \cdot 1_{(0,1)}(r)\}.
\]
Now the d.f. of $\theta(X_n)$ is discontinuous at $\theta = \pi$ even though the USC $\ell(\theta) = 1$ is continuous.
From the last two examples, we see that the continuity of the d.f. of \( \theta(X_n) \) has nothing to do with that of the corresponding USC function.

**Example 5.6.** Let \( C \) be the Cantor set and \( F_C(\cdot) \) be the Cantor distribution. For the constructions of \( C \) and \( F_C \), see for example, Billingsley, 1979. \( F_C \) can also be defined as the d.f. corresponding to the random variable \( Z = \sum_{n=1}^{\infty} B_n / 3^n \) where \( B_n \) are iid and \( P(B_n = 0) = P(B_n = 2) = 1/2 \).

Now suppose \( P(\|X_n\| > r) = e^{-r}, \theta(X_n) \) has the distribution \( F_C \) and they are independent. Then it follows that with \( b_n = \log n \), as \( n \to \infty \)

\[
S_n^* \overset{a.s.}{\longrightarrow} S^* = \{ (\theta, r): 0 \leq \theta \leq 1, \ 0 < r \leq 1_C(\theta) \}.
\]

Note that since \( C \) is closed, \( 1_C(\theta) \) is USC and hence by Lemma 2.3, \( \hypo(1_C) \) is closed. Since \( \{M_n\} \) is clearly a.s. relatively stable, we only need to check the second condition in Theorem 4.6. For \( 0 \leq a < b \leq 1 \), let \( B(a, b) = \{ \theta: a < \theta < b \} \) and \( B[a, b] = \{ \theta: a \leq \theta \leq b \} \). It is seen that \( P(\theta(X_n) \in B(a, b)) \geq 0 \) and \( E^Y(B(a, b)) = 1 \) whenever (4.5) is satisfied. Thus we must show that (4.6) holds. However, since \( \|X_n\| \) and \( \theta(X_n) \) are independent, we have as \( t \to \infty \),

\[
P\{\|X_n\| > tx, \ \theta(X_n) \in B(a, b)\} / P\{\|X_n\| > t\}
= \frac{P\{\theta(X_n) \in B(a, b)\}}{P\{\|X_n\| > t\}} \to \begin{cases} 0 & \text{if } x > 1 \\ \infty & \text{if } x < 1 \end{cases},
\]

as desired.

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