MULTIVARIATE RECORDS AND SHAPE

by

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ABSTRACT

For one dimensional data, the notion of a record is natural and universally agreed
upon but this is not the case when the data is in two or more dimensions. We outline
several plausible definitions of a record and settle on one induced by a partial ordering: For
observations $X_1, \ldots, X_n$ we say $X_n$ is a record if each component of $X_n$ is bigger than the
 cororesponding components of previous observations. Some properties of such records of iid
observations are reviewed and used to study the shape of the convex hull of the first $n$
observations, as $n \to \infty$.

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1. INTRODUCTION

The notion of a record in a one-dimensional series of observations is familiar, natural and unambiguous. It means of course an observation bigger (or smaller in appropriate contexts) than previous observations and is commonly used in connection with data describing sports, temperature, rainfall, stress and reliability.

When analyzing multivariate data there is no longer a commonly agreed upon definition and the simplicity resulting from the fact that $\mathbb{R}$ is totally ordered is lost. Consider $\mathbb{R}^2$-valued observations $\{X_n\} = \{(x_n^{(1)}, x_n^{(2)})'\}$. For such data, we can define the notion of record in many ways and the following are all plausible and interesting possibilities:

(a) $X_n$ is a record if $X_n^{(p)} > \bigvee_{i=1}^{n-1} X_i^{(p)}$, $p = 1$ and 2.

(b) $X_n$ is a record if $X_n^{(p)} > \bigvee_{i=1}^{n-1} X_i^{(p)}$, $p = 1$ and 2.

(c) $X_n$ is a record if $X_n$ falls outside the convex hull of $X_1, \ldots, X_{n-1}$.

Definitions (b) and (c) are related as (c) is the infinite dimensional version of (b). One can see this by expressing the definition (c) in terms of the support function of a convex set. Mathematical analysis of (b) and (c) is more difficult than (a) because the definition (a) can be expressed in terms of a partial order, namely that $X_n > X_i$, $i = 1, \ldots, n-1$, where the inequality between vectors is interpreted componentwise.

In section 2 we review some results about records defined according to (a). This material is taken from Goldie and Resnick (1989) where records on general partially ordered sets are considered. For simplicity in this paper we only consider $\mathbb{R}^2$-valued observations. In particular we need to review when an iid sequence has a finite or infinite number of records.

If $X_n$ is a record, then previous observations must be located southwest of $X_n$ in the plane and in this sense $X_n$ sticks out. Another interpretation is that if a right angled template is lowered from $\infty$ until it contacts the sample, $X_1, \ldots, X_n$, then this template contacts the sample in one point iff $X_n$ is a record. Thus we arrive at the idea of probing the sample with various geometric objects (cf. Groeneboom, 1987) and relating the shape of the convex hull of the sample to records. This idea is explored in the last sections where the
fact that a multivariate normal sequence has a finite number of records is used to prove that
the boundary of the convex hull of a normal sample becomes smooth as $n \to \infty$ in the sense
that the exterior angles at vertices of the convex hulls converge to $\pi$ uniformly.

It is rather well known that the shape of a normal sample becomes ellipsoidal
(Geoffroy, 1961; Fisher, 1969; Davis, Mulrow, Resnick, 1987) and hence the boundary
becomes smooth. Using only records, it does not seem possible to obtain detailed
information on the shape of the sample beyond what happens on the boundary of the convex hull.

2. MULTIVARIATE RECORDS

As in the introduction let $\{X_n = (X_n^{(1)}, X_n^{(2)})', n \geq 1\}$ be iid $\mathbb{R}^2$-valued random
variables. $X_n$ is a record if

$$X_n > X_j, \quad j = 1, \ldots, n - 1$$

or equivalently

$$X_n > \bigvee_{j=1}^{n-1} X_j$$

where $(x_1, x_2) \bigvee (y_1, y_2)' = (x_1 \vee y_1, x_2 \vee y_2)'$. As a convention, make $X_1$ a record.

The first order of business is to decide when the sequence has an infinite number of records
so define the counting variables

$$N(A) = \sum_{n=1}^{\infty} 1\{X_n \in A, \ X_n \text{ is a record}\}$$

for $A \in \mathcal{B}(\mathbb{R}^2)$. Let $N = N(\mathbb{R}^2)$ be the total number of records in the sequence. From the
Hewitt–Savage 0–1 Law, $N$ is finite or infinite with probability 1 and either case can
occur. For example, if the bivariate distribution of $X_1$ is a product of two continuous
distributions so that for each $n$ we have $X_n^{(1)}$ and $X_n^{(2)}$ are independent random
variables then

$$EN = \sum_{n=1}^{\infty} P[X_n \text{ is a record}]$$
\[
\sum_{n=1}^{\infty} P(X_n^{(1)} > \bigvee_{j=1}^{n-1} X_j^{(2)}) = \sum_{n=1}^{\infty} \frac{1}{n} < \infty
\]

so that \( P[N < \infty] = 1 \). On the other hand if the distribution of \( X_1 \) concentrates on the diagonal so that \( P[X_n^{(1)} = X_n^{(2)} = 1 \) for \( n \geq 1 \), and the distribution of \( X_n^{(1)} \) is continuous then

\[
\sum_{n=1}^{\infty} P[X_n^{(1)} > \bigvee_{j=1}^{n-1} X_j^{(1)}] = \sum_{n=1}^{\infty} \frac{1}{n} = \infty
\]

and from one dimensional considerations it is clear \( P[N = \infty] = 1 \).

In general we have for \( A \in \mathcal{B}(\mathbb{R}^2) \)

\[
EN(A) = \sum_{n=1}^{\infty} P[X_n \in A, X_n \text{ is a record}]
\]

and if we denote the distribution of \( X_1 \) by \( F \) we obtain

\[
EN(A) = \sum_{n=1}^{\infty} \int_A F(dx)P\left( \bigcap_{j=1}^{n-1} [X_j < x] \right)
\]

\[
= \sum_{n=1}^{\infty} \int_A F(dx)(F(-\infty, x))^{n-1}
\]

\[
= \sum_{n=1}^{\infty} \int_A \frac{F(dx)}{F((-\infty, x)^c)} =: H(A).
\]

The criterion for when the number of records is finite is in terms of \( H \).

**Theorem 2.1.** (Goldie and Resnick, 1989). We have \( P[N(A) < \infty] = 1 \) or \( P[N(A) = \infty] = 1 \) according as \( H(A) < \infty \) or \( H(A) = \infty \).

The converse half of this result is proved using a standard converse to the Borel–Cantelli lemma.

It is not easy to calculate the measure \( H \) explicitly. Cases where this can be done are when (i) components are independent or (ii) the components are totally dependent so that effectively the data are one dimensional. In general, it is a helpful heuristic to think of

\[
H(dx) \approx P[\text{some record occurs at } dx].
\]
Because $H$ is difficult to calculate, examples in $\mathbb{R}^2$ are best checked by not computing $H$ but rather using a notion of asymptotic independence. Earlier we saw that independent components resulted in a finite number of records but in fact asymptotic independence in the following sense is all that is necessary. Let the common distribution of \{$X_n\}$ be $F(x, y)$. Then we say $F$ is in the domain of attraction of a multivariate extreme value distribution $G$ if there exist $a_n^{(i)} > 0$, $b_n^{(i)} \in \mathbb{R}$, $n \geq 1$, $i = 1, 2$ such that
\[
P\left[\bigvee_{j=1}^n X_j^{(i)} - b_n^{(i)} / a_n^{(i)} \leq x_i, \quad i = 1, 2 \right] = F^n(a_n^{(1)} x_1 + b_n^{(1)}, a_n^{(2)} x_2 + b_n^{(2)}) \to G(x_1, x_2)
\]
(cf. Resnick, 1987). Then

**Theorem 2.2.** (Goldie and Resnick, 1989). $N$ is finite or infinite with probability 1 according as $G$ is or is not a product measure.

Criteria based on $F$ which guarantee $G$ is a product measure are well known (Resnick, 1987). Using these, there is a surprising result of Sibuya (1960) (cf. Resnick, 1987) that states that the bivariate normal distribution is in a domain of attraction of a bivariate extreme value distribution which is a product measure, provided the correlation $\rho$ of the bivariate normal density is not equal to 1. Thus we get the next result.

**Corollary 2.3.** (Goldie and Resnick, 1989). If \{$X_n\}$ is iid from a bivariate normal whose correlation $\rho$ satisfies $\rho \neq 1$ then \{$X_n\}$ has a finite number of records:
\[
P[N < \infty] = 1.
\]

Further properties of multivariate records are given in Goldie and Resnick (1989) including expressions for the moments and and Laplace functional of $N(A)$, $A \in \mathcal{B}(\mathbb{R}^2)$ and an expression for the fundamental quantity $Q(A) = P[N(A) = 0]$. In terms of $Q$, the Markovian structure of the records falling in the set $A$ can be discussed and the transition probabilities characterized. A partial independent increment property for the point process $N(\cdot)$ is also given.
3. GEOMETRICAL INTERPRETATION

Let $H_n$ be the convex hull of $X_1, \ldots, X_n$. Whether or not $X_n$ is a record (i.e., whether or not $X_n > \bigvee_{i=1}^{n-1} X_i$) can be examined graphically in the following way.

Consider a right angled template which has edges of infinite length and whose bisecting line has inclination $\pi/4$ with the positive x-axis. Suppose the template is moving down from $\omega = (\infty, \infty)$ to $-\omega = (-\infty, -\infty)$ without changing the inclination and that it cannot pass any point of $H_n$. The template stops when it hits $H_n$. If the sample is from a continuous distribution, there are only two possible outcomes: Either the template hits $H_n$ at one point or two points. These possibilities are shown in Figure 3.1.

![Template hits $H_n$ at one point](image1)

![Template hits $H_n$ at two points](image2)

FIGURE 3.1

Notice that no matter how the template approaches $H_n$, it hits $H_n$ at the same point (or points). Consider the situations in Figure 3.2 and 3.3. In Figure 3.2, the template cannot move down, but still can move to the left until it hits $X_2$. Figure 3.3 shows the opposite case and the template eventually hits $H_n$ at $X_1$ and $X_2$ in both cases.

Remember that the only restriction when we move the template is to keep the inclination $\pi/4$. Alternatively, this procedure may be viewed as moving a horizontal line down until it contacts the sample and then moving a vertical line to the left until the sample is encountered. Do the lines meet the sample at different points or at the same point?
It is then clear that
\[ X_n \text{ is a record iff the template hits } H_n \text{ only at } X_n \]  

(3.1)

and

\[ [\text{Template hits } H_n \text{ at two points}] = \bigcap_{j=1}^{n} [X_j > \bigvee_{i=1, \ i \neq j}^{n} X_i]^c. \]  

(3.2)

Now we consider more general templates. We denote by $\text{Temp}(\theta, \varphi)$ the template with the exterior angle $\theta$ ($\pi < \theta \leq 3\pi/2$) and whose bisecting line makes angle $\varphi$ ($0 \leq \varphi < 2\pi$) with the positive $x$–axis. (See Figure 3.4). The template used in the previous discussion is now denoted $\text{Temp}(3\pi/2, \pi/4)$. Then bringing down (or up depending on $\varphi$) $\text{Temp}(\theta, \varphi)$ from the point at infinity in the direction $\varphi$, we again see that $\text{Temp}(\theta, \varphi)$ hits $H_n$ at either one point or two points. We can use this fact to define another record property as follows.

**Definition 3.1.** $X_n$ is a $(\theta, \varphi)$–record iff $\text{Temp}(\theta, \varphi)$ hits $H_n$ only at $X_n$. 
Let \( \pi < \theta_1 < \theta_2 \leq 3\pi/2 \), then it follows from the definition that if \( X_n \) is a \((\theta_2, \varphi)\)-record, then it is always a \((\theta_1, \varphi)\)-record and that \( X_n \) can possibly be a \((\theta_1, \varphi)\)-record even though it is not a \((\theta_2, \varphi)\)-record. For example, in Figure 3.5, \( X_n \) is not a \((3\pi/2, \pi/4)\)-record (or simply a record) since \( \text{Temp}(3\pi/2, \pi/4) \) hits \( H_n \) at \( X_j \) and \( X_k \), but it is a \((\theta, \pi/4)\)-record since \( \text{Temp}(\theta, \pi/4) \) hits \( H_n \) only at \( X_n \).

The following proposition explains the relation between these two records.

**Proposition 3.2.** \( X_n \) is a \((\theta, \varphi)\)-record \((\pi < \theta \leq 3\pi/2, \ 0 \leq \varphi < 2\pi) \) iff \( Y_n \) is a record among \( Y_1, \ldots, Y_n \) where \( Y_j = S_\theta \cdot T_\varphi \cdot X_j \), \( j = 1, 2, \ldots, n \),

\[
T_\varphi = \begin{bmatrix}
\cos \left( \frac{\pi}{4} - \varphi \right) & -\sin \left( \frac{\pi}{4} - \varphi \right) \\
\sin \left( \frac{\pi}{4} - \varphi \right) & \cos \left( \frac{\pi}{4} - \varphi \right)
\end{bmatrix},
\]

\[
S_\theta = \begin{bmatrix}
\alpha & 1 \\
1 & \alpha
\end{bmatrix},
\]

\( \alpha = \tan \left( \frac{3}{4} \pi - \frac{\theta}{2} \right) \).
**Proof.** Observe by trigonometry that

\[
T_\varphi \begin{bmatrix} r \cos t \\ r \sin t \end{bmatrix} = \begin{bmatrix} r \cos (t + \pi/4 - \varphi) \\ r \sin (t + \pi/4 - \varphi) \end{bmatrix}
\]

so that the mapping \( T_\varphi \) rotates points in the plane through an angle \( \pi/4 - \varphi \). Let

\[ Z_j = T_\varphi X_j. \]

It follows that \( X_n \) is a \((\theta, \varphi)\)-record iff \( Z_n \) is a \((\theta, \pi/4)\)-record. Thus it suffices to show that

\[ Z_n \text{ is a } (\theta, \pi/4)\text{-record iff } Y_n \text{ is a record.} \quad (3.3) \]

Note (3.3) holds when \( \theta = 3\pi/2 \) since in this case \( Y_j = (Z_j^{(2)}, Z_j^{(1)})', \ j = 1, 2, \ldots, n \).

So assume \( \pi < \theta < 3\pi/2 \), which implies \( 0 < \alpha < 1 \). The two edges of Temp \((\theta, \pi/4)\) are given by the lines (see Figure 3.6)

\[
y - Z_n^{(2)} = -\alpha(x - Z_n^{(1)}) \quad \text{for } x \geq Z_n^{(1)}, \text{ and}
\]

\[
y - Z_n^{(2)} = -\frac{1}{\alpha}(x - Z_n^{(1)}) \quad \text{for } x < Z_n^{(1)}. 
\]

Thus, we have

\[
\{Z_n \text{ is a } (\theta, \pi/4)\text{-record}\}
\]

\[
= \cap_{j=1}^{n-1} \{Z_j^{(2)} - Z_n^{(2)} < -\alpha(Z_j^{(1)} - Z_n^{(1)}), Z_j^{(2)} - Z_n^{(2)} < -\frac{1}{\alpha}(Z_j^{(1)} - Z_n^{(1)})\}
\]

\[
= \cap_{j=1}^{n-1} \{\alpha Z_j^{(1)} + Z_j^{(2)} < \alpha Z_n^{(1)} + Z_n^{(2)}, Z_j^{(1)} + \alpha Z_j^{(2)} < Z_n^{(1)} + \alpha Z_n^{(2)}\}
\]

\[
= \cap_{j=1}^{n} \begin{bmatrix} Z_j^{(1)} \\ Z_j^{(2)} \end{bmatrix} < S_\theta \begin{bmatrix} Z_n^{(1)} \\ Z_n^{(2)} \end{bmatrix}
\]

\[
= \cap_{j=1}^{n-1} \{Y_j^{(1)} < Y_n^{(1)}, Y_j^{(2)} < Y_n^{(2)}\}
\]

\[
= \{Y_n \text{ is a record}\}
\]
so that (3.3) holds.

\[ y - Z_n^{(2)} = -\alpha(x - Z_n^{(1)}) \]

\[ y - Z_n^{(2)} = -\frac{1}{\alpha} (x - Z_n^{(1)}) \]

**FIGURE 3.6**

4. APPLICATION TO A RANDOM SAMPLE FROM A BIVARIATE NORMAL DISTRIBUTION

Let \( X_n = (X_n^{(1)}, X_n^{(2)})' \) be an iid sequence of random vectors in \( \mathbb{R}^2 \) from the bivariate normal distribution. We assume \( \text{EX}_n = 0 \) and that the correlation \( \rho \) of \( X_1 \) satisfies \( |\rho| < 1 \). Define events as follows:

\[ A_n = \{ X_n \text{ is a record} \} \]

\[ = \{ X_n^{(i)} > \bigvee_{j=1}^{n-1} X_j^{(i)}, \ i = 1, 2 \} \]

\[ A_n^{(i)} = \{ X_n^{(i)} \text{ is a record} \} \]

\[ = \{ X_n^{(i)} > \bigvee_{j=1}^{n-1} X_j^{(i)}, \ i = 1, 2 \} . \]

\[ B_n = \{ \text{Temp (3}\pi/2, \pi/4 \text{) hits } H_n \text{ at one point} \} \]

\[ B_n^{(\theta, \varphi)} = \{ \text{Temp (\theta, \varphi) hits } H_n \text{ at one point}. \} \]

Notice that \( A_n = A_n^{(1)} \cap A_n^{(2)} \) and \( A_n \subset B_n \). We also define
\[ N = \sum_{n=1}^{\infty} 1_{A_n}, \text{ and} \]

\[ N_i = \sum_{n=1}^{\infty} 1_{A_n(i)}, \quad i = 1, 2. \]

It follows from Corollary 2.3 and one dimensional theory (Resnick, 1987, p. 169) that

\[ P(N < \infty) = 1 \tag{4.1} \]

\[ P(N_1 = \infty) = P(N_2 = \infty) = 1. \tag{4.2} \]

**Lemma 4.1.** Temp \((3\pi/2, \pi/4)\) hits \(H_n\) at two points for all large \(n\), i.e., \(P(B_n \text{ i.o.}) = 0\).

**Proof.** We have (4.1) and (4.2) are respectively equivalent to

\[ P(A_n \text{ i.o.}) = 0, \text{ and} \tag{4.3} \]

\[ P(A_n^{(1)} \text{ i.o.}) = P(A_n^{(2)} \text{ i.o.}) = 1. \tag{4.4} \]

These imply that

\[ 1 = P(\liminf_{n \to \infty} A_n^c \cap \limsup_{n \to \infty} A_n^{(1)}). \]

Now pick \(\omega \in \liminf_{n \to \infty} A_n^c \cap \limsup_{n \to \infty} A_n^{(1)}\) and there exists some \(m\) such that \(\omega \in A_m^{(1)}(A_m^{(2)})^c\) and \(\omega \in A_n^c\) for any \(n \geq m\). If there exists some \(n \geq m\) such that for some \(\ell \leq n\)

\[ X_{\ell} > \bigvee_{i=1}^{n} X_1(\omega) \quad i \neq \ell \]

then if \(\ell < m\), \(X_m^{(1)}(\omega)\) would not be a record (a contradiction to \(\omega \in A_m^{(1)}\)) and if \(n \geq \ell \geq m\), \(X_\ell(\omega)\) is a record (a contradiction to \(\omega \in A_\ell^c\)). From (3.2) we conclude \(\omega \in B_n^c\), \(n \geq m\), whence

\[ 1 = P(\liminf_{n \to \infty} A_n^c \cap \limsup_{n \to \infty} A_n^{(1)} \leq P(\liminf_{n \to \infty} B_n^c) \]
which gives the desired result.

**Lemma 4.2.** For any $\theta \ (\pi < \theta \leq 3\pi/2)$ and $\varphi \ (0 \leq \varphi < 2\pi)$, Temp $(\theta, \varphi)$ hits $H_n$ at two points for all large $n$ a.s.; i.e.,

$$P(B_n(\theta, \varphi) \ i.o.) = 0.$$ 

**Proof.** Let $X_n, Y_n, Z_n$ be as in Proposition 3.2. Let $H(Y_1, \ldots, Y_n)$ be the convex hull of the indicated points. As in Proposition 3.2,

$$[\text{Temp $(\theta, \varphi)$ hits $H_n$ at two points}]$$

$$= [\text{Temp $(3\pi/2, \pi/4)$ hits $H(Y_1, \ldots, Y_n)$ at two points}].$$

Since $\{Y_n\}$ are iid normal, the desired result is a direct application of Lemma 4.1 provided $\rho_Y = \text{Corr} (Y^{(1)}_1, Y^{(2)}_1) < 1$. This is readily checked as follows: Since $\alpha < 1$, det $S_{\theta} = \alpha^2 - 1 \neq 0$ so $S_{\theta}$ is invertible. The rotation $T_{\varphi}$ is also invertible. Since $|\rho| = |\text{corr} (X^{(1)}_1, X^{(2)}_1)| < 1$ the same is true for $\rho_Y$, else there is an a.s. linear relationship between $Y^{(1)}_1$ and $Y^{(2)}_1$ (see Brockwell and Davis, 1987, p. 43) which translates into an a.s. linear relationship between $X^{(1)}_1$ and $X^{(2)}_1$ which violates $|\rho| < 1$.

For any sequences of events $\{C_n\}, \{D_n\}$ we have

$$\limsup_{n \to \infty} C_n \cup D_n = \limsup_{n \to \infty} C_n \cup \limsup_{n \to \infty} D_n$$

and we therefore obtain from Lemma 4.2 the following result.

**Lemma 4.3.** For $\theta_j \ (\pi < \theta_j \leq 3\pi/2)$ and $\varphi_j \ (0 \leq \varphi_j < 2\pi), \ j = 1,2,\ldots,m$, all the $m$ templates Temp $(\theta_j, \varphi_j), \ j = 1,2,\ldots,m$ eventually hit $H_n$ at two points for all large $n$ a.s.; i.e.,

$$P(\bigcup_{j=1}^m B_n(\theta_j, \varphi_j) \ i.o.) = 0.$$
Next consider a vertex \( v \) of \( H_n \) and \( \text{Temp} (\theta, \varphi) \). Let \( \psi \) be the exterior angle of \( v \) and \( \phi \) be the inclination between the bisecting line of \( v \) and the x-axis. (See Figure 4.1). Then we see that

\[
\text{Temp} (\theta, \varphi) \text{ hits } H_n \text{ only at } v \text{ iff } \psi > \theta > \pi \text{ and } |\phi - \varphi| < \frac{\psi - \theta}{2}.
\] (4.5)

We denote by \( V_n \) the number of vertices of \( H_n \) and further define \( v_{n,j} \), \( \psi_{n,j} \), and \( \phi_{n,j} \), \( j = 1,2,...,V_n \) as the vertices of \( H_n \), the exterior angles, and the inclinations, respectively. (See Figure 4.2). The next proposition shows that the exterior angles converge to \( \pi \) uniformly.

**Proposition 4.4.** We have for a bivariate normal sample whose correlation \( \rho \) satisfies \( |\rho| < 1 \):

\[
\lim_{n \to \infty} \frac{V_n}{V} \psi_{n,j} = \pi \text{ a.s.}
\]

**Proof.** Pick any \( \epsilon > 0 \). Then since \( \psi_{n,j} > \pi \) for any \( j \), it suffices to show

\[
\frac{V_n}{V} \psi_{n,j}(\omega) \leq \pi + \epsilon, \quad n \geq n_0(\omega).
\] (4.6)
Choose \( m \) large enough so that \( m > 4\pi/\epsilon \). Then for any \( \phi_{n,j} \), there exists some \( \ell(j) \in \{1,2,\ldots,m\} \) such that
\[
|\phi_{n,j} - \frac{2\pi \cdot \ell(j)}{m}| < \frac{\epsilon}{4}.
\]
(4.7)

For \( m \) so chosen, we take \( m \) templates
\[
\{\text{Temp } (\pi + \frac{\epsilon}{2}, \frac{2\pi}{m} j), \ 1 \leq j \leq m\}
\]
and it follows from Lemma 4.3 that
\[
P(\bigcup_{n \geq N} \bigcap_{j=1}^{m} \{\text{Temp } (\pi + \frac{\epsilon}{2}, \frac{2\pi}{m} j) \text{ hits } H_n \text{ at two points}\}) = 1.
\]
(4.8)

Suppose for \( n \geq N(\omega) \)
\[
\omega \in \bigcap_{j=1}^{m} [\text{temp } (\pi + \frac{\epsilon}{2}, \frac{2\pi}{m} j) \text{ hits } H_n \text{ at two points}].
\]

For any \( j \leq V_n \) there exists \( \ell(j, \omega) \) as in (4.7), whence from (4.5) either
\[
\begin{cases}
\psi_{n,j} \leq \frac{\pi + \frac{\epsilon}{2}}{2} \quad \text{or} \\
|\phi_{n,j}(\omega) - \frac{2\pi \ell(j, \omega)}{m}| > \frac{\psi_{n,j} - (\pi + \frac{\epsilon}{2})}{2}
\end{cases}
\]

and therefore from (4.7)
\[
\frac{\psi_{n,j} - (\pi + \frac{\epsilon}{2})}{2} < \frac{\epsilon}{4},
\]
whence
\[
\psi_{n,j} < \pi + \frac{\epsilon}{2} + \frac{\epsilon}{2} = \pi + \epsilon
\]
giving (4.6) as desired.

\[ \text{\square} \]

**Corollary 4.5.** For a bivariate normal sample whose correlation \( \rho \) satisfies \( |\rho| < 1 \) we have
\[
V_n \to \infty \text{ a.s.}
\]
Proof. We have by Proposition 4.4 that for any $\epsilon > 0$, there exists $n_0$ such that for any $n > n_0$,

$$\max_{1 \leq j \leq V_n} \psi_{n,j} < \pi + \epsilon \text{ a.s.} \quad (4.9)$$

We also have

$$\sum_{j=1}^{V_n} \psi_{n,j} = (V_n + 2) \pi \text{ a.s.} \quad (4.10)$$

It follows from (4.9) and (4.10) that

$$(V_n + 2)\pi \leq V_n(\pi + \epsilon) \text{ a.s.}$$

This implies

$$V_n > \frac{2\pi}{\epsilon} \text{ for any } n \geq n_0 \text{ a.s.}$$

\qed

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