ON ANSTREICHER'S COMBINED PHASE I—PHASE II PROJECTIVE ALGORITHM FOR LINEAR PROGRAMMING

by

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*Research supported in part by NSF Grant ECS—8602534 and ONR Contract N00014—87—K—0212.
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Anstreicher has proposed a variant of Karmarkar's projective algorithm that handles standard-form linear programming problems nicely. We suggest modifications to his method that we suspect will lead to better search directions and a more useful algorithm. Much of the analysis depends on a two-constraint linear programming problem that is a relaxation of the scaled original problem.

Abbreviated title: On Anstreicher's Phase I–Phase II Projective Algorithm

Keywords: Linear programming, interior-point method, projective algorithm, combining phase I–phase II.
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Anstreicher has proposed a variant of Karmarkar's projective algorithm that handles standard-form linear programming problems nicely. We suggest modifications to his method that we suspect will lead to better search directions and a more useful algorithm. Much of the analysis depends on a two-constraint linear programming problem that is a relaxation of the scaled original problem.
1. INTRODUCTION

Consider the standard form linear programming problem

\[(\bar{P}) \quad \min \bar{c}^T\bar{x} \]
\[\bar{A}\bar{x} = \bar{b} \]
\[\bar{x} \geq 0.\]

A number of authors (Anstreicher [1], de Ghellinck and Vial [3], Gay [5], Gonzaga [6], Jensen and Steger [11] and Ye and Kojima [13]) proposed a variant of Karmarkar's polynomial-time projective algorithm [8] for dealing with the reformulation

\[\min(\bar{c}^T,0)(\bar{x},\bar{x})\]
\[\begin{bmatrix} \bar{A} & -\bar{b} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{x} \end{bmatrix} = 0 \]
\[\begin{bmatrix} 0^T & 1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{x} \end{bmatrix} = 1 \]
\[\begin{bmatrix} \bar{x} \\ \bar{x} \end{bmatrix} \geq 0.\]

However, these methods (except [3]) still required a strictly positive feasible solution. The usual remedy is to use an artificial variable with a high cost, but the choice of this coefficient is not easy and its use inelegant.

Anstreicher [2] proposed a method that deals with the artificial variable directly, by imposing a constraint that it be zero, leading to
\[
\min (\tilde{c}^T, 0, 0) \begin{pmatrix}
\tilde{x} \\
\sigma \\
\tau
\end{pmatrix}
\]
\[
(\tilde{A}, -\tilde{b}, \tilde{b} - \tilde{A} e) \begin{pmatrix}
\tilde{x} \\
\sigma \\
\tau
\end{pmatrix} = 0
\]
\[
(\hat{P}) 
\begin{pmatrix}
0^T, 1, 0 \\
0^T, 0, 1
\end{pmatrix}
\begin{pmatrix}
\tilde{x} \\
\sigma \\
\tau
\end{pmatrix} = 1
\]
\[
\begin{pmatrix}
\tilde{x} \\
\sigma \\
\tau
\end{pmatrix} \geq 0,
\]

where \( e \) is a vector of ones of appropriate dimension. His algorithm generates a sequence of strictly positive points satisfying all constraints except \( \tau = 0 \), starting with \( e \). Anstreicher showed that (if \( \hat{P} \) has a nonempty bounded set of optimal solutions) his method simultaneously approaches feasibility and optimality, and provided strong convergence results.

In this paper we discuss modifications to his algorithm. In particular, we develop new search directions that we believe are likely to be better than Anstreicher's; preliminary computational experience [12] bears this out. Our analysis is based on a single crucial two-constraint linear programming problem, which provides bounds and shows the existence of certain directions. The directions we propose are then generated by a norm-minimization subproblem. These two subsidiary problems are closely linked and have a very similar form whether a lower bound has been generated or not. This contrasts somewhat with Anstreicher's approach. Another difference is that, when a lower bound is found, we continue the algorithm from the current iterate, whereas Anstreicher reverts to the initial solution.
De Ghellinck and Vial [3] have also proposed an algorithm that does not require feasibility. Their lower-bounding procedure requires the solution of $n$ quadratic equations, whereas ours (and Anstreicher's) needs the analysis of a piecewise-linear concave function with $n$ breakpoints. It is therefore unlikely that these methods are too closely related, although the precise relationship is unclear. The search directions of Anstreicher [2] and de Ghellinck and Vial [3], as well as those proposed here and those arising from a "big M" formulation, are linear combinations of three directions that are the projections of vectors associated with the scaled problem, so that only the weights are different. (See also Gonzaga [7] and Mitchell and Todd [10], who show that several algorithms that maintain feasibility throughout use linear combinations of two directions.) However, the choice of weights is likely to be crucial in the efficiency of a method.

It is important to point out that Anstreicher's method and ours solve a problem $(P)$ where a strictly positive vector is known that satisfies all but one constraint. This subsumes problem $(\bar{P})$ above, but also includes situations where one or more constraints are added cutting off a previously feasible strictly positive solution. A simple manipulation incorporates all the infeasibilities into a single constraint, with the other added equations satisfied by the current point.

The paper is organized as follows. Section 2 describes the algorithm and introduces the two important subproblems: the relaxed scaled problem $(RP)$ and the direction finding subproblem $(DFSP)$. In section 3 we prove the convergence result, following Anstreicher's analysis. Sections 4 and 5 discuss the solution of $(RP)$ and $(DFSP)$ respectively, and section 5 concludes with further discussion of the stepsize selection.

I would like to thank Kurt Anstreicher for very helpful discussions related to this work.
2. THE ALGORITHM

We consider the problem

\[
\min \ c^T x \\
Ax = 0 \\
(P) \quad d^T x = 1 \\
\xi^T x = 0 \\
x \geq 0,
\]

following Anstreicher's notation. This includes the standard form problem with artificial variable as in the introduction. (Actually, Anstreicher considers the fractional linear programming problem of minimizing \( c^T x / d^T x \), with the normalizing constraint \( d^T x = 1 \) replaced by \( e^T x = n \). We prefer the formulation above, which facilitates working with the dual and is closer to the original problem.)

We assume (as holds for the problem derived from the standard form problem) that:

(A1) \( Ae = 0, \ d^T e = 1, \) and \( \xi^T e = 1 \);

(A2) \( d^T x \geq 0 \) for all \( x \) satisfying \( Ax = 0, \ x \geq 0 \); and

(A3) \( Ax = 0, \ d^T x = 0, \ \xi^T x = 0 \) and \( x \geq 0 \) imply \( x = 0 \) or \( c^T x > 0 \).

We do not need to require that \( \xi^T x \geq 0 \) for all \( x \) with \( Ax = 0, \ d^T x = 1 \) and \( x \geq 0 \). If \( \xi^T e \) were 0, we could incorporate \( \xi \) in \( A \) and proceed in a standard way. Given that we may deal with infeasible points, we can collect all the infeasibilities into one constraint, and then assume without loss that \( \xi^T e > 0 \) at the initial solution \( e \). Scaling
then gives assumption (A1). Assumption (A2) is convenient, and without loss of
generality; if it fails, the constraint \( d^T x \geq 0 \) can be added, and then incorporated with a
surplus variable into the homogeneous constraints \( Ax = 0 \). The resulting problem then
satisfies (A1)–(A3) if the original satisfies (A1) and (A3). With (A2), any strictly positive
solution to \( Ax = 0 \) has \( d^T x > 0 \) and hence can be scaled to satisfy \( d^T x = 1 \) also. When
(P) is feasible, (A3) is equivalent to the existence of a bounded nonempty set of optimal
solutions.

The algorithm generates a sequence \( \{x^k\} \) satisfying

\[
Ax^k = 0, \quad d^T x^k = 1, \quad \xi^T x^k \geq 0, \quad \text{and} \quad x^k > 0,
\]

starting with \( x^0 := e \). At each iteration, we have a valid lower bound \( z^k \) on the optimal
value of (P), but we allow \( z^k = -\infty \).

Given \( x^k \), let \( \Delta \) denote the diagonal matrix with \( \Delta e = x^k \), and define

\[
\bar{A} := A \Delta, \quad \bar{c} := \Delta c, \quad \bar{d} := \Delta d, \quad \text{and} \quad \bar{\xi} := \Delta \xi.
\]

Then (P) is equivalent to the scaled problem

\[
\begin{align*}
\min \quad & \bar{c}^T \bar{x} \\
(\bar{P}) \quad & \bar{A} \bar{x} = 0 \\
& \bar{d}^T \bar{x} = 1 \\
& \bar{\xi}^T \bar{x} = 0 \\
& \bar{x} \geq 0
\end{align*}
\]
in the rescaled variables \( \bar{x} = \Delta^{-1} x \), and \( \bar{x} = e \) corresponds to \( x^k \) and satisfies all the constraints of (P) except possibly \( \zeta^T \bar{x} = 0 \). Its objective value is denoted

\[
v = v^k := c^T e = c^T x^k.
\]

The dual to (P) is

\[
\begin{align*}
\max z \\
(\mathcal{D}) \\
\tilde{A}^T u + \bar{d} z + \xi \lambda \leq \bar{c},
\end{align*}
\]

which is equivalent to the dual (D) to (P) with its constraints scaled by \( \Delta \).

For any vector \( \bar{a} \), let \( \bar{a}_q \) and \( \bar{a}_p \) denote its projection into \( \{ \bar{x} \colon \tilde{A} \bar{x} = 0, \ e^T \bar{x} = 0 \} \) respectively. Note that, if \( \bar{x} \) satisfies \( \tilde{A} \bar{x} = 0 \), then \( \bar{a}^T \bar{x} = \bar{a}_q^T \bar{x} = \bar{a}^T_q \bar{x} \) for any \( \bar{a} \), and similarly with subscript \( p \) if also \( e^T \bar{x} = 0 \). Also, if \( e^T \bar{a}_q = 0 \) (or equivalently \( e^T \bar{a} = 0 \)), then \( \bar{a}_p = \bar{a}_q \). Finally, \( \bar{a}_q = \bar{a} - \tilde{A}^T u_a \) for some \( u_a \) that is often a byproduct of the computation of \( \bar{a}_q \). We will usually assume that such \( u_a \)'s are available henceforth without further comment.

Let us calculate \( \bar{c}_q, \bar{d}_q \) and \( \bar{\xi}_q \) and form the problem

\[
\begin{align*}
\min \bar{c}_q^T \bar{x} \\
(\mathcal{RP}) \\
\bar{d}_q^T \bar{x} = 1 \\
\bar{\xi}_q^T \bar{x} = 0 \\
\bar{x} \geq 0;
\end{align*}
\]

by our remarks above, this is a relaxation of (P). Its dual is
\[
\max z \\
(\text{RD}) \\
\bar{d}_q z + \bar{z}_q \lambda \leq \bar{c}_q.
\]

(If (RD) is feasible, it can be viewed as the problem of finding the maximum \( z \) such that Anstreicher's \( \theta_2(z) \) is nonnegative, the basis for updating lower bounds in [2].) We can also write (RD) as

\[
\max z \\
(\text{RD}) \\
\bar{A}^T u + \bar{d} z + \bar{z}_q \lambda \leq \bar{c} \\
u = u_c - z u_d - \lambda u\xi
\]

which is clearly a restriction of (D). Hence any feasible solution \((z, \lambda)\) to (RD) can be extended to a feasible solution \((u, z, \lambda)\) to (D) and (D), which then certifies \( z \) as a lower bound on the value of (P) or (P) by duality.

The analysis of subproblem \((RP)\) is a key ingredient of our algorithm. In section 4, we discuss how to solve it. Here we show how to use its solution. Primarily, its value (if finite) is used to improve the lower bound \( z^k \), but its solution is also useful to demonstrate the existence of "good" directions of motion.

First we deal with the case that \((RP)\) is infeasible. We have:

**Lemma 2.1.** \((RP)\) is infeasible iff \( \bar{z}_q \geq \mu \bar{d}_q \) for some \( \mu > 0 \); in this case, for \( u = u_\xi - \mu u_d \) we have

\[
\xi \geq A^T u + \mu d
\]

so that \( Ax = 0, \ d^T x = 1 \) and \( x \geq 0 \) imply \( \xi^T x \geq \mu > 0 \), and (P) is infeasible.
Proof: Note that the linear programming problem \( \min \{ \xi^T \bar{x}; \; \bar{d}^T \bar{x} = 1, \; \bar{x} \geq 0 \} \) is feasible, since \( \bar{x} = e \) satisfies the constraints. Hence (RP) is infeasible iff the optimal value of this problem is positive, which holds iff there is a dual solution \( \mu > 0 \), giving \( \xi_q \geq \mu \bar{d}_q \). Since \( \xi_q = \xi - A^T u \xi \) and similarly for \( \bar{d}_q \), it then follows that

\[
\xi \geq A^T u + \mu \bar{d},
\]

and hence the inequality in the lemma holds by scaling by \( \Delta^{-1} \).

For the rest of the analysis we assume that (RP) is feasible, and let \( z \) be its optimal value, finite or \(-\infty\). Since (RP) is a relaxation of (P), \( z \) is a valid lower bound on the value of (P) or (P), and hence so is

\[
z^{k+1} := \bar{z} := \max \{ z, z^k \}. \tag{2.3}
\]

This shows how the lower bound is updated. The algorithm then proceeds by moving from \( e \) in the direction \( g \) which is the solution to the following direction-finding subproblem:

\[
\min \| g \|
\]

(DFSP) \hspace{1cm} \( \bar{A} g = 0, \; e^T g = 0 \); \hspace{1cm} \tag{2.4}

\[\xi^T g \leq -\xi^T e; \tag{2.5}\]

if \( v \geq \bar{z} \),

\[ (\bar{c} - v \bar{d})^T g \leq 0; \tag{2.6}\]

and

\[ (\bar{c} - \bar{z} \bar{d})^T g \leq -(v - \bar{z}). \tag{2.7}\]
Here we replace (2.5) by an equation if \( \xi^T e = 0 \), so that \( x^k \) is feasible. Also, (2.7) (which is to hold regardless of the values of \( v \) and \( \bar{z} \)) should be read as \( d^T g \leq -1 \) if \( \bar{z} = -\infty \). Note that \( 0 = -(\bar{c} - v\bar{d})^T e \) and \( -(v - \bar{z}) = -(\bar{c} - \bar{z}\bar{d})^T e \) (or \( -1 = -d^T e \)), so that (2.5)-(2.7) can be expressed as

\[
\xi^T(e + g) \leq 0;
\]

if \( v \geq \bar{z} \),

\[
(\bar{c} - v\bar{d})^T(e + g) \leq 0;
\]

and

\[
(\bar{c} - \bar{z}\bar{d})^T(e + g) \leq 0 \quad (\text{or} \quad d^T(e + g) \leq 0).
\]

The motivation for (DFSP) is as follows. Roughly, (2.5) assures progress toward feasibility, (2.6) monotonicity if possible, and (2.7) progress toward optimality. Suppose first that \( \bar{z} > -\infty \). Assume also that it is possible to move from \( e \) to \( e + g \) without violating nonnegativity, and that \( \bar{d}^T(e + g) > 0 \). Consider \( \bar{x} = (e + g)/\bar{d}^T(e + g) \). Then (2.4) and (2.5') (with equality) would show that \( \bar{x} \) is feasible, and (2.7') would imply that \( \bar{x} \) achieves the lower bound \( \bar{z} \) and hence is optimal. This is perhaps a little too much to hope for. However, the closer we can approach \( e + g \), the more nearly we satisfy our goals. We know that we can move from \( e \) a distance \( r := \sqrt{n/(n-1)} \) in the direction \( g/\|g\| \), since \( r \) is the radius of the sphere around \( e \) inscribing the simplex \( S := \{x : e^T x = n, \ x \geq 0\} \). Hence it is very natural to ask for the direction \( g \) of minimum norm satisfying these conditions. A similar motivation prompts our use of an inequality in (2.5). If the optimal \( g \) satisfies this inequality strictly, then an even shorter step will lead to feasibility, and our stepsize selection will ensure that we do not overshoot. If feasibility is ever attained, we replace (2.5) by an equation so that it is preserved from then on.

This argument justifies constraints (2.5) and (2.7). To explain (2.6), we remark that, when \( v \geq \bar{z} \), (2.7) implies that \( (\bar{c} - \bar{z}\bar{d})^T(e + \alpha g) \leq (\bar{c} - \bar{d})^T e \) for \( \alpha > 0 \). However,
it does not follow that $\bar{c}^T\bar{x} \leq v = \bar{c}^Te$ for $\bar{x} = (e + \alpha g)/\bar{d}^T(e + \alpha g)$. Imposing (2.6) ensures this monotonicity when $v \geq \bar{z}$. (We cannot hope for monotonicity when $v < \bar{z}$, since then the current value is less than the optimal value of (P).) The idea of imposing constraint (2.6) is due to Anstreicher [1,2].

If $\bar{z} = -\alpha$, the argument above seems rather strange, since (2.7) now states $\bar{d}^T(e + g) \leq 0$. Why should we seek to decrease $\bar{d}^T\bar{x}$? One answer is that Anstreicher's convergence result shows that, by doing so, we will obtain a finite lower bound in a finite number of iterations. The intuition for this result is as follows. If

$\bar{x} = (e + \alpha g)/\bar{d}^T(e + \alpha g)$ for some $0 < \alpha < 1$, then $\bar{d}^T(e + \alpha g) < \bar{d}^Te = 1$, so $e^T\bar{x} > e^Te$. But by (A3) we cannot keep moving away from the origin while maintaining $Ax = 0$, $d^T \bar{x} = 1$, $\xi^T \bar{x} \leq 1$, $c^T \bar{x} \leq v$, and $x \geq 0$. So eventually this case cannot occur, i.e., we will have $\bar{z} > -\alpha$. Also, moving away from the origin without compromising the objective function or the level of infeasibility provides "more room" for future steps.

The constraint $e^Tg = 0$ implies that $e^T(e+g) = n$, so that $e+g$ lies in the affine hull of the simplex $S$. There is no obvious reason to impose this restriction, besides the fact that the analysis then proceeds in a standard way. Suppose instead that we minimize the angle between the rays $e$ and $e+g$, subject to $\bar{A}(e+g) = \bar{A}g = 0$, and (2.5')-(2.7'). Then the objective would be to maximize $e^T(e+g)/\|e\|\|e+g\|$. It follows from the results below that this objective can be made positive, and it is therefore equivalent to minimize its reciprocal, $\|e\|\|e+g\|/e^T(e+g)$. This objective (and the constraints) are homogeneous of degree zero in $(e+g)$, so we may normalize and add the constraint that the denominator is, say, $n$. But then $e^Tg = e^T(e+g) - e^Te = n-n = 0$, and $\|e+g\| = (e^Te + e^Tg + g^Tg)^{1/2} = (n + \|g\|^2)^{1/2}$. Hence this problem is equivalent to (DFSP), which can be thus viewed as that of finding the ray closest (in angle) to $e$ satisfying the constraints above.
We now show how the solution of (RP) provides a feasible solution to (DFSP) with objective function value not too large. We distinguish two cases.

**Lemma 2.2.** Suppose (RP) is feasible but unbounded. In that case, let \( \overline{w} \) satisfy

\[
\begin{align*}
\overline{c}_q \overline{w} < 0, & \quad \overline{d}_q \overline{w} = 0, & \quad \overline{\xi}_q \overline{w} = 0, & \quad e^T \overline{w} = n, & \quad \overline{w} \geq 0.
\end{align*}
\] (2.8)

Hence \( \overline{w} \) is the direction of a ray of (RP) along which the objective function is unbounded below, scaled so that \( e^T \overline{w} = n \). Then

\[
g := \overline{w} - e
\] (2.9)

is feasible in (DFSP) with \( ||g|| \leq R := \sqrt{n(n-1)} \).

**Proof:** The existence of \( \overline{w} \) satisfying (2.8) follows from the unboundedness of (RP). Now \( \overline{w} \) lies in the simplex \( S = \{\overline{x}: e^T \overline{x} = n, \overline{x} \geq 0\} \), and so \( ||\overline{w} - e|| \leq R \), the radius of the sphere around \( e \) circumscribing \( S \). It follows that \( g = (\overline{w} - e)_q \) has norm at most \( R \).

Now \( e^T \overline{w} = e^T e = n \), so \( e^T(\overline{w} - e) = 0 \), and thus \( A g = 0 \) and \( e^T g = 0 \). From \( \overline{\xi}_q \overline{w} = 0 \) we get \( \overline{\xi}^T g = \overline{\xi}_q (\overline{w} - e) = -\overline{\xi}_q e = -\overline{\xi}^T e \). Finally, \( \overline{c}^T g = \overline{c}_q (\overline{w} - e) < -\overline{c}^T e \) and \( \overline{d}^T g = \overline{d}_q (\overline{w} - e) = -\overline{d}^T e = -1 \) show that (2.6) and (2.7) hold (whether or not \( \overline{z} = -\overline{w} \)).

If the hypotheses of the lemma hold, and we move in the direction \( g \) given by (2.9), then we find that \( \overline{d}(e + \alpha g) = 1 - \alpha \) so that, for \( 0 < \alpha < 1 \),

\[
\overline{x} = \frac{e + \alpha g}{\overline{d}^T(e + \alpha g)} = e + \frac{\alpha}{1 - \alpha} \overline{w}_q.
\]

Thus, after renormalizing, we can suppose that we are moving in the direction \( \overline{w}_q \)

satisfying \( \overline{c}_q \overline{w}_q < 0, \overline{d}_q \overline{w}_q = 0 \) and \( \overline{\xi}_q \overline{w}_q = 0 \). This seems quite reasonable.
Now suppose (RP) has a finite optimal value. We then have

**Lemma 2.3.** Suppose (RP) has optimal solution $\overline{w}$ with value $z$. In that case, since $\overline{z} \geq z$ by (2.3), we have

$$
(c - zd)^Tq \overline{w} \leq 0, \quad d_q^Tw = 1, \quad \xi_q^T\overline{w} = 0, \quad w \geq 0.
$$

Then

$$
g := n\overline{w}_q/e^T\overline{w} - e
$$

is feasible in (DFSP) with $\|g\| \leq R$.

**Proof:** Again, $n\overline{w}/e^T\overline{w}$ lies in the simplex $S$ so that $\|g\| \leq R$. As in the proof of lemma 2.2, $g$ satisfies (2.4), (2.5) and (2.7) since $\overline{w}$ satisfies (2.10). Finally, if $v \geq \overline{z}$ then (2.10) implies $(c - vd)^Tq \overline{w} \leq 0$, whence (2.6) holds.

Here, if we choose $g$ by (2.11), the result of a step in direction $g$ followed by renormalizing is

$$
e + \alpha g
d^T(e + \alpha g) = (1 - \beta)e + \beta\overline{w}_q
$$

for some $0 \leq \beta \leq 1$. Thus we move toward the point $\overline{w}_q$ which satisfies all constraints except possibly nonnegativity.

The algorithm is then as follows. Given $x^k$, calculate $\overline{x}, \overline{c}, \overline{d}$ and $\overline{\xi}$ by (2.2) and thence $\overline{c}_q, \overline{d}_q$ and $\overline{\xi}_q$, and form (RP). If (RP) is infeasible, then so is (P) and we stop. Otherwise we set $z^{k+1} := \overline{z} := \max\{z, z^k\}$ as in (2.3), where $z$ is the optimal value of
(RP). We then solve (DFSP) for g. If the optimal solution is \( g = 0 \), then \( x^k \) is optimal in (P) and we stop. Otherwise, set

\[
\hat{x} = e + \alpha g \tag{2.12}
\]

and

\[
x^{k+1} = \Delta \hat{x} / \Delta^T \hat{x} \tag{2.13}
\]

for a suitable \( \alpha > 0 \) and proceed to iteration \( k+1 \).

We will discuss the choice of \( \alpha \) further below. For our convergence results, we assume

\[
\alpha = \min\{-\xi^T e / \xi^T g, \ (n-1)r / ((2n-3)\|g\|)\}, \tag{2.14}
\]

where the first term is omitted if \( \xi^T e = 0 \). This choice ensures that \( \hat{x} \) is strictly positive, and then assumption (A2) implies that \( \Delta^T \hat{x} > 0 \) so that \( x^{k+1} \) is defined.

Note that, if \( v = \bar{z} \), then (2.6) and (2.7) coincide, so that the monotonicity constraint (2.6) can be omitted. Moreover, if \( v^k \leq \bar{z} \), then (2.7) implies \( v^{k+1} \leq z^{k+1} \leq z^{k+2} \), so that the monotonicity constraint should be omitted in all future iterations. This will certainly happen eventually if \( c^T x^0 \) is smaller than the optimal value of (P).

3. CONVERGENCE RESULTS

The results of Anstreicher [2] extend naturally to the present setting, since we have assured in the constraints of (DFSP) that \( g \) satisfies the appropriate conditions. The only significant difference from Anstreicher's approach is that, when a finite lower bound \( \bar{z}^k \) is generated, we continue from \( x^k \) rather than restarting with \( x^0 = e \) and \( z^0 = z^k \); the
reason is that progress has already been made toward feasibility and optimality. We have the following

**Theorem 3.1.**

a) Unless infeasibility is demonstrated by an infeasible subproblem (RP), a finite lower bound $z^{k+1}$ will be generated at some iteration, say the Kth.

b) Unless or until infeasibility is demonstrated, the iterates $\{x^k\}$ satisfy:

(i) $Ax^k = 0$, $d^T x^k = 1$, $x^k \geq 0$;

(ii) $\xi^T x^k \leq (e^T x^k/n) \exp(-.3k/n)$;

(iii) For $k < K$, $c^T x^{k+1} \leq c^T x^k$ and

$$c^T x^{k+1} - z \leq (e^T x^{k+1}/n) \exp(-.3k/n) (c^T e - z),$$

where $z = \min\{z^{K+1}, v, K\}$.

(iv) For $k \geq K$ either

$$c^T x^k \leq z^{k+1}, \text{ which implies } c^T x^{k+1} \leq z^{k+1} \leq z^{k+2}; \text{ or}$$

$c^T x^{k+1} \leq c^T x^k$ and

$$c^T x^{k+1} - z^{k+1} \leq (e^T x^{k+1}/n) \exp(-.3k/n) (c^T e - z^{K+1}).$$

(Note that the theorem does not claim that infeasibility will always be detected; it is possible that $z^k \to \infty$.)

**Proof:** This result follows from arguments of Anstreicher [2] using reduction in the potential functions
\[ f(x; h) = n \ln h^\top x - \sum_j \ell_j n x_j \]

for \( h = d \), \( h = \xi \) and \( h = c - zd \). (Since \( \sum_j \ell_j n x_j \leq n \ln(e^\top x/n) \), \( f(x; h) \leq \beta \) implies that \( h^\top x \leq (e^\top x/n) \exp(\beta/n) \).) We give a sketch; details are as in sections 4 and 6 of [2].

First, the constraint \( \xi^\top g \leq -\xi^\top e \) in (DFSP) and the choice of \( \alpha \) shows that

\[
either \quad \xi^\top x^{k+1} = 0 \quad or \\
\quad f(x^{k+1};\xi) \leq f(x^k;\xi) - .3
\]

(3.1)

for each \( k \); the first case can occur if \( \alpha = -\xi^\top e / \xi^\top g \), and then \( \xi^\top x^\ell = 0 \) for all \( \ell > k \), by the resulting equality constraint in (DFSP). This implies (b)(ii); (i) follows since \( g = g_p \), the step size \( \alpha \) in (2.14) is chosen so that \( \dot{x} \) and \( x^{k+1} \) are nonnegative, and the normalization (2.13) ensures that \( d^\top x^{k+1} = 1 \).

Now suppose that no finite lower bound is ever generated. Then all directions \( g \) satisfy \( d^\top g \leq -1 \). It follows that

\[
f(x^{k+1};d) \leq f(x^k;d) - .3, \quad (3.2)
\]

except possibly for an iteration where \( \xi^\top x^{k+1} = 0 \), in which case \( f(\cdot; d) \) does not increase. Since \( d^\top x^{k+1} = 1 \), (3.2) implies that

\[
e^\top x^k \to \infty. \quad (3.3)
\]

The constraint \( (c - vd)^\top g \leq 0 \) in (DFSP) implies that

\[
c^\top x^{k+1} \leq c^\top x^k \leq ... \leq c^\top e. \quad (3.4)
\]
Now let $x$ be any limit point of $x^k/e^Tx^k$. Then $c^Tx \leq 0$, $d^Tx = 0$, and (b) (ii) implies $ξ^Tx = 0$. Since $Ax = 0$ and $x ≥ 0$, this contradicts assumption (A3). Hence a finite lower bound is generated, say at iteration $K$, and (a) is proved.

Condition (2.6) in (DFSP) ensures that, for $k < K$, $(c - νd)^T \hat{x} ≤ 0$, so that $c^Tx^{k+1} ≤ ν = c^Tx^k$. Hence $ν ≥ ν^K ≥ z$, where $z$ is as in (iii). Together with (2.6) and (2.7), which gives $d^Tg ≤ -1$, this shows that $(c - zd)(e + g) ≤ 0$. Thus

$$f(x^{k+1}; c - zd) ≤ f(x^k; c - zd) - .3, \quad (3.5)$$

except possibly for an iteration where $ξ^Tx^{k+1} = 0$, in which case $f(·; c - zd)$ does not increase. Then (iii) follows by a standard argument.

Now consider iteration $k$, where $k ≥ K$. If $c^Tx^k ≤ z^{k+1}$, so that $ν ≤ \bar{z}$, then (DFSP) includes the constraint $(c - zd)^Tg ≤ -(ν - \bar{z})$, which implies that $(c - zd)^T \hat{x} ≤ 0$ (since $(c - zd)^Te ≤ 0$ and $(c - zd)^T(e + g) ≤ 0$), so that $c^Tx^{k+1} ≤ \bar{z}$ as required. Otherwise, constraint (2.6) implies $c^Tx^{k+1} ≤ c^Tx^k$ as above, and $(c - zd)^Tg ≤ -(ν - \bar{z})$ yields

$$f(x^{k+1}; c - z^{k+1}d) ≤ f(x^k; c - z^{k+1}d) - .3, \quad (3.6)$$

except possibly for an iteration where $ξ^Tx^{k+1} = 0$, in which case $f(·; c - z^{k+1}d)$ does not increase. Since $z^{k+1} ≥ z^K+1 ≥ z$, (3.5) and (3.6) by a standard argument lead to (iv).

The inequalities on the infeasibility and the duality gap in (ii)–(iv) include the term $e^Tx^k/n$ (or $e^Tx^{k+1}/n$). Thus it appears that these terms can tend to infinity, rendering the inequalities useless. However, suppose (P) has optimal value $z^*$. Then the offending terms can be explicitly bounded as follows.
We know from (A3) that

\[ S_{\alpha} := \{ x : Ax = 0, e^T x = n, x \geq 0, c^T x \leq \alpha \max \{ 0, c^T e, z^* \}, \]
\[ d^T x \leq \alpha, 0 \leq \xi^T x \leq \alpha \} \]

is empty for \( \alpha = 0 \). Hence there is some \( 0 < \alpha^* < 1 \) such that \( S_{\alpha} \) is empty for all \( 0 \leq \alpha \leq \alpha^* \); otherwise, if we take \( x^k \in S_{\alpha_k} \) for a sequence \( \alpha_k \to 0 \), then all \( x^k \)'s lie in the compact set \( \{ x : e^T x = n, x \geq 0 \} \), and any limit point lies in \( S_0 \), a contradiction.

We claim that \( e^T x^k/n \leq 1/\alpha^* \) for all sufficiently large \( k \). Indeed, suppose \( k \geq n |\ell n \alpha^*|/3 \), and assume that \( e^T x^k/n =: 1/\alpha > 1/\alpha^* \). Then let \( x = x^k/(e^T x^k/n) = \alpha x^k \). By (b)(i), \( Ax = 0 \), \( d^T x = \alpha \leq \alpha^* \), and \( x \geq 0 \). By definition, \( e^T x = n \), and from (b)(ii) \( 0 \leq \xi^T x \leq \exp(-3k/n) \leq \alpha^* \). Finally, (b)(iii) and (iv) imply that \( c^T x^k \leq \max \{ 0, c^T e, z^* \} \), so that \( c^T x \leq \alpha \max \{ 0, c^T e, z^* \} \). Hence \( x \in S_{\alpha^*} \), a contradiction.

4. SOLVING THE TWO–CONSTRAINT LINEAR PROGRAMMING PROBLEM

Here we discuss problem (RP), which we rewrite below:

\[
\begin{align*}
\min_{\mathbf{q}} & \quad c^T \mathbf{x}^\mathbf{q} \\
\text{(RP)} & \quad d^T \mathbf{x}^\mathbf{q} = 1 \\
& \quad \xi^T \mathbf{x}^\mathbf{q} = 0 \\
& \quad \mathbf{x} \geq 0.
\end{align*}
\]
In order to perform an iteration, we need to be able to determine whether (RP) is infeasible, and, if not, determine that it is unbounded or find its optimal value $z$.

To check infeasibility, according to lemma 2.1, we need to see whether $\xi_q \geq \mu \lambda_q$ for some $\mu > 0$. This requires computing at most $n$ ratios of the form $(\xi_q)_j/(\lambda_q)_j$; and is thus trivially performed. Henceforth we assume that (RP) is feasible.

In this case, we can determine unboundedness of (RP) or find its optimal value by solving its dual

$$(\text{RD}) \quad \begin{align*}
\max z \\
\bar{d}_q z + \lambda \xi_q \leq \bar{c}_q.
\end{align*}$$

This linear programming problem has only two variables, and can therefore be solved in $O(n)$ time using the method of Dyer [4] and Megiddo [9]. This method shows (RD) infeasible (and hence (RP) unbounded) or obtains an optimal solution by systematically cutting down the number of inequalities. In order to obtain an $O(n)$ bound, use of a linear—time median—finder is necessary; such algorithms are complicated, and Megiddo recommends instead a simple randomized algorithm. For details, see [4,9].

Alternatively, a simple method requiring at most $O(n^2)$ time can be derived from Anstreicher's analysis of his $\bar{c}_q(z)$. Let $J_+, J_0$, and $J_-$ denote the set of indices $j$ with $(\xi_q)_j$ positive, zero, and negative, respectively, and let $\lambda_j(z) = (\bar{c}_q - z\lambda_q)_j/(\xi_q)_j$ for $j \in J_+ \cup J_-$. The constraints of (RD) indexed by $J_0$ define an interval $Z$ for $z$, possibly empty, while the other constraints require

$$\lambda_-(z) \leq \lambda \leq \lambda_+(z), \quad (4.1)$$
where
\[ \lambda_+(z) = \min \{ \lambda_j(z) : j \in J_+ \} , \]  
(4.2)
\[ \lambda_-(z) = \max \{ \lambda_j(z) : j \in J_- \} . \]  
(4.3)

If \( Z \) is empty we stop ((RD) is infeasible); otherwise we choose some \( z \in Z \) and check (4.1). By examining the slopes of \( \lambda_+ \) and \( \lambda_- \), we determine whether \( z \) should be increased or decreased to reduce \( \lambda_-(z) - \lambda_+(z) \) if (4.1) is infeasible; otherwise we wish to increase \( z \). We can move to the next breakpoint of \( \lambda_- - \lambda_+ \) in \( O(n) \) time, and evaluate it similarly. After examining at most \( O(n) \) breakpoints we either solve (RD) or show it infeasible. (The latter occurs if the piecewise-linear convex function \( \lambda_- - \lambda_+ \) is minimized and still positive at some \( z \), or if it is still positive when we encounter an endpoint of \( Z \), and these can all be checked by inspecting slopes.) We omit the details.

5. SOLVING THE DIRECTION-FINDING SUBPROBLEM

This section shows how to solve the problem (DFSP) of section 2. To simplify notation, we write \( \beta = \bar{c} - v\bar{d} \) and \( \zeta = \bar{c} - \bar{z}\bar{d} \) if \( \bar{z} > -\omega \), \( \zeta = \bar{d} \) if \( \bar{z} = -\omega \). We also write \( \theta = \zeta^T e \) and \( \kappa = v - \bar{z} \) for \( \bar{z} > -\omega \), \( \kappa = 1 \) if \( \bar{z} = -\omega \). Then our problem can be written as

\[
\begin{align*}
\min & \quad \frac{1}{2} g^T g \\
\text{(DFSP)} & \quad g = g_p; \\
& \quad \zeta_p^T g \leq -\theta; \\
& \quad \beta_p^T g \leq 0; \\
& \quad \zeta_p^T g \leq -\kappa,
\end{align*}
\]  
(5.1)  
(5.2)  
(5.3)  
(5.4)
where (5.2) is replaced by an equation if \( \theta = 0 \). Now the Karush–Kuhn–Tucker conditions for the problem with (5.1) removed imply that its optimal solution \( g^* \) satisfies

\[
g^* = -\lambda \bar{c}_p - \mu \bar{\beta}_p - \nu \bar{\zeta}_p \quad (5.5)
\]

for some \( \lambda, \mu, \nu \). Since \( g^* = g^*_p \), (5.1) is redundant. Note that \( g \) is thus a linear combination of \( \bar{c}_p, \bar{d}_p \) and \( \bar{\zeta}_p \).

Suppose first that \( \theta = 0 \) so that (5.2) reads \( \bar{\zeta}_p^T g = 0 \). Then we project \( \bar{\beta}_p \) and \( \bar{\zeta}_p \) orthogonal to \( \bar{\zeta}_p \), to get

\[
\bar{\beta}_o = \left( I - \frac{\xi_p \xi_p^T}{\xi_p^T \xi_p} \right) \bar{\beta}_p \quad \text{and} \quad \bar{\zeta}_o = \left( I - \frac{\xi_p \xi_p^T}{\xi_p^T \xi_p} \right) \bar{\zeta}_p.
\]

Note that we cannot have \( \bar{\zeta}_p = 0 \), since then \( \theta = 0 \) implies \( \bar{\zeta}_q = 0 \) so that \( \xi^T x = 0 \) whenever \( Ax = 0 \), contradicting (A1). If \( \kappa \leq 0 \), \( g^* = 0 \) solves (DFSP), and otherwise the optimal \( g^* \) is a linear combination of \( \bar{\beta}_o \) and \( \bar{\zeta}_o \). Using the Karush–Kuhn–Tucker conditions yields

**Lemma 5.1.** If \( \theta = 0 \) and \( \kappa \leq 0 \) then \( g^* = 0 \). If \( \theta = 0 \) and \( \kappa > 0 \), then if \( \bar{\beta}_o^T \bar{\zeta}_o \geq 0 \),

\[
g^* = -\kappa \bar{\zeta}_o / \bar{\zeta}_o^T \bar{\zeta}_o \quad (5.6)
\]
while if \( \beta_o^T \zeta_o < 0 \),

\[
g^* = -\mu \beta_o - \nu \zeta_o \quad \text{where}
\]

\[
\begin{pmatrix} \mu \\ \nu \end{pmatrix} = \left[ \frac{-\kappa \beta_o^T \zeta_o}{\kappa \beta_o^T \beta_o} \right] / \left[ (\beta_o^T \beta_o)(\zeta_o^T \zeta_o) - (\beta_o^T \zeta_o)^2 \right].
\]

Note that feasibility of (DFSP) implies that the denominators of the fractions above are positive.

Now assume that \( \theta > 0 \). In this case too, \( \bar{\xi}_p \) cannot vanish, since otherwise (5.2) cannot hold, whereas we know (DFSP) is feasible.

First we suppose that (5.3) is not present or not binding.

**Lemma 5.2.** If \( \theta > 0 \), the minimum norm solution to (5.2) and (5.4) is as follows.

If \( \kappa \bar{\xi}_p^T \xi_p \leq \sigma \bar{\xi}_p^T \xi_p \), then

\[
g^* = -\sigma \bar{\xi}_p / \xi_p^T \bar{\xi}_p.
\]  

(5.8)

Otherwise, if \( \sigma \bar{\xi}_p^T \xi_p \leq \kappa \bar{\xi}_p^T \xi_p \), then

\[
g^* = -\kappa \bar{\xi}_p / \xi_p^T \bar{\xi}_p.
\]  

(5.9)

If neither of these cases holds, then
\[ g^* = -\lambda \xi_p - \nu \zeta_p, \text{ where} \] (5.10)

\[
\begin{pmatrix}
\lambda \\
\nu
\end{pmatrix} = \begin{pmatrix}
\theta \xi_p \xi_p - \kappa \xi_p \zeta_p \\
\kappa \zeta_p \xi_p - \theta \zeta_p \zeta_p
\end{pmatrix} \bigg/ \left[ (\xi_p \xi_p)(\zeta_p \zeta_p) - (\xi_p \zeta_p)^2 \right].
\]

**Proof:** If \( \zeta_p = 0 \), then \( \kappa \leq 0 \) by the feasibility of (DFSP), so that the first case holds. Moreover, if \( \xi_p \) and \( \zeta_p \) were collinear, the feasibility of (DFSP) would imply that the first case held. Hence all denominators are positive. In all cases, we use the Karush–Kuhn–Tucker conditions. In the first case, \( \lambda = \theta / (\xi_p \xi_p) \geq 0 \), (5.2) holds with equality, and (5.4) holds by the condition. In the second case, we must have \( \kappa > 0 \), for otherwise \( \xi_p \zeta_p < 0 \) and one can deduce \( \kappa < 0 \) and \( \kappa[(\xi_p \xi_p)(\zeta_p \zeta_p) - (\xi_p \zeta_p)^2] > 0 \), a contradiction. So \( \nu = \kappa / (\xi_p \zeta_p) \geq 0 \), (5.4) holds with equality, and (5.2) holds by the condition. In the final case, \( \lambda \) and \( \nu \) are well defined and positive by the condition, and (5.2) and (5.4) hold with equality. The lemma is proved.

If \( \theta > 0 \) and \( \kappa < 0 \), lemma 5.2 shows how to compute the solution \( g^* \) to (DFSP). If \( \theta > 0 \) and \( \kappa \geq 0 \), we first compute \( g^* \) by lemma 5.2; if \( \beta_p^T g^* \leq 0 \), \( g^* \) solves (DFSP). If not, we know that constraint (5.3) is tight. Hence we compute

\[
\xi'_o = \begin{pmatrix}
I - \frac{\beta_p \beta_p^T}{\beta_p^T \beta_p}
\end{pmatrix} \xi_p \quad \text{and}
\]

\[
\zeta'_o = \begin{pmatrix}
I - \frac{\beta_p \beta_p^T}{\beta_p^T \beta_p}
\end{pmatrix} \zeta_p
\]

and apply lemma 5.2 with \( \xi'_o \) and \( \zeta'_o \) replacing \( \xi_p \) and \( \zeta_p \).
Anstreicher's [2] choice of direction \( \mathbf{g} \) is derived by considering a problem defined over the circumscribing ball, \( \{ \mathbf{x} : \mathbf{e}^T \mathbf{x} = n, \| \mathbf{x} - \mathbf{e} \| \leq R \} \). We suspect that, in most cases, this ball does not approximate the simplex particularly closely in the important directions. In any case, the direction \( \mathbf{g} \) discussed mainly in [2] satisfies (5.2) with equality and \( \| \mathbf{g} \| = R; \) thus feasibility can only be nearly achieved if a step of length nearly \( R \) is feasible. At the end of section 4 of [2], an improved \( \mathbf{g} \) is proposed, which satisfies (5.2) and (5.4) with equality and satisfies \( \| \mathbf{g} \| < R; \) this is exactly the \( \mathbf{g}^* \) of (5.10), except that the coefficient \( \lambda \) there may be negative. This improved \( \mathbf{g} \) is likely to be much more effective in practice. The direction \( \mathbf{g} \) that solves (DFSP) covers all eventualities and seems to us to be a natural and useful choice. Indeed, our preliminary experience [12] indicates that Anstreicher's original direction performs very poorly, while using the modified direction when appropriate gives an algorithm that is usually comparable to that proposed here.

Finally, we return to our discussion of the choice of \( \alpha \). The convergence analysis implies that we only need to make a sufficient reduction in \( f(\cdot ; \xi) \) and \( f(\cdot ; \mathbf{c} - \bar{z} \mathbf{d}) \) at each step. (If \( \bar{z} = -\mathbf{w} \), replace \( \mathbf{c} - \bar{z} \mathbf{d} \) by \( \mathbf{d} \) here and henceforth.) If we can achieve \( \xi^T \mathbf{x} = 0 \), while maintaining nonnegativity, this is generally worthwhile, if it does not increase \( f(\cdot ; \mathbf{c} - \bar{z} \mathbf{d}) \). However, the latter is not even defined at the new iterate if \( \xi^T \mathbf{x} \) is some component of \( \mathbf{x} \), say \( x_n \). In this case, we recommend making the full step if \( \mathbf{v} \leq \bar{z} \) or if

\[
f'(x^{k+1}; \mathbf{c} - \bar{z} \mathbf{d}) \leq f'(x^k; \mathbf{c} - \bar{z} \mathbf{d}),
\]

where

\[
f'(x; h) := (n-1)\ell_n h^T x - \sum_{j=1}^{n-1} \ell_n x_j,
\]

and then dropping the final component of \( \mathbf{x} \) from then on.
Now suppose we cannot drive $\xi^T x$ to zero. Then we must balance reductions in $f(\cdot; \xi)$ and $f(\cdot; c - \tilde{z}d)$. If both (5.2) and (5.4) are binding, as in (5.10), then the reduction will be the same for both, and we search on $f(\cdot; \xi)$. If only the first is binding, as in (5.8), we search on the more restrictive objective, i.e. $f(\cdot; \zeta)$. Finally, if only the second is binding, as in (5.6), (5.7), or (5.9), we search on $f(\cdot; c - \tilde{z}d)$, which is now more restrictive. In any case, the analysis of section 3 remains valid.
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