THE EXTENDED ECONOMIC LOT SCHEDULING PROBLEM

by

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Abstract

We consider the problem of scheduling $m$ products on one facility to minimize the long run average cost associated with set-ups, stockouts, and holding inventory. A deterministic model that assumes all relevant parameters are known constants is formulated. Demand during stockouts is assumed to be backordered. Holding and backorder costs are linear. This model is an extension of the Economic Lot Scheduling Problem (ELSP), where demand is assumed to be met from inventories. We refer to this model as the Extended Economic Lot Scheduling Problem (EELSP). We extend to the EELSP recent work by Dobson and Zipkin on computing cyclic schedules for the ELSP. The combined procedure yields optimal or near optimal cyclic schedules. Finally we present a lower bound on the long run-average cost of cyclic schedules. We characterize instances of the EELSP for which cyclic schedules based on a common cycle length achieve these lower bounds.

1. Introduction

We address the following scheduling problem which is frequently encountered in practice.

There is a facility that can produce any of a number of products, but can only produce one product at a time. Expenditures in the form of time and money are incurred when production is stopped and the facility is set up, i.e., prepared, for a different product. All stockouts are backordered. The problem is to schedule production to minimize long run average cost associated with set-ups, stockouts and carrying inventories. Frequent production changeover is costly and time consuming;
it reduces the proportion of time spent on production lowering average inventory and possibly incurring stockouts. On the other hand, long production runs result in less frequent set-ups, higher average inventories, and again possible stockouts.

Our ultimate goal and motivation for the study of this problem is to find a practical solution to it. A first step is to formulate a mathematical model that captures the essence of the problem. The second step is to find a reasonable solution to the model in the form of a cyclic schedule. The third step is to find an efficient way to schedule the facility after an unforeseen disruption. This paper deals with the first two steps. In Gallego [87b] and [87c] we deal with the problem of scheduling a facility after an unforeseen disruption.

The model we have in mind charges linear time-weighted holding and backorder costs and assumes that all relevant parameters are known constants, i.e., production and demand rates, set-up times and costs, holding and backorder costs. In this paper we restrict attention to cyclic schedules, i.e., schedules that are periodic. The criterion is of course long run average cost.

The model we have just described is a generalization of a well known problem in the scheduling literature, the Economic Lot Scheduling Problem (ELSP), where demand is assumed to be met from inventories. The ELSP corresponds to the instance of the EELSP where the backorder costs are set to infinity. We refer to our model as the Extended Economic Lot Scheduling Problem (EELSP).

We were motivated to study the EELSP by two limitations of the ELSP. The first limitation is purely economical; the second has to do with problems that arise when one attempts to manage the facility in the presence of unforeseen disruptions. As in all optimization problems where average cost is minimized, one seeks an optimal balance between the different sources of cost. For the EELSP we seek a balance between holding, backorder, and set-up costs. The ELSP avoids stockouts.
This is clearly suboptimal in the presence of finite backorder costs. It forgoes the opportunity of achieving considerable savings, even when stockouts are expensive. For the EELSP, for example, it can be shown that if the backorder parameters are $k$ times larger than the holding parameters, then the optimal average cost of a schedule that allows backorders is up to $100(k+1)^{-1}\%$ cheaper than the optimal schedule that does not allow backorders.

Let us briefly touch on the problem of scheduling a facility that is subject to unforeseen disruptions. Assume that a cyclic schedule for the ELSP has been obtained. This cyclic schedule cannot be followed in practice because of unforeseen random disruptions. However, the cyclic schedule should still be valuable as a production guide. One could monitor the system disruptions in some way and take corrective measures to dampen their effect. In general, this cannot be done without incurring occasional stockouts. In fact, what one would like to do is to find corrective measures that are optimal in some economic sense. To do so we must be able to model the cost associated with stockouts.

In a sequel to this paper we formulate the problem of scheduling the facility after a single disruption as a control problem. There we show that under reasonable conditions the optimal policy effects corrective measures that are linear in the measure of disruption. In particular, the optimal policy reduces to a simple produce up to policy when the backorder costs are proportional to processing time.

Although we are dissatisfied with the limitations of the ELSP we find the literature very rich in insights and solution techniques. Maxwell [64] suggested two distinct stages in finding a cyclic schedule. In the first stage a cyclic production sequence is selected. This production sequence will then be repeated indefinitely. All products in consideration must have at least one position in the sequence; some products, however, may be produced more than once in the cycle. The second stage
involves finding the exact timing and quantities of the production lots. To form a cyclic schedule the inventory positions should agree at the start and at the end of the cycle.

Most research done for the ELSP restricts the class of schedules to those having equally spaced production lots and equal lot sizes for products with more than one position in the sequence. Maxwell [64] says that "this assumption rules out both the possibility of similar, but not identical, triangles (unequal lot sizes within a particular product) and the possibility of ordinate offset triangles (within any particular product the values of inventory at the times of production starts do not have to coincide), both of which may be either desirable or necessary to obtain schedulability and/or optimality."

In fact one of the most popular approaches to the ELSP is to choose a basic period and to assume that the cycle times for each item are integer multiples of that basic period. This approach assumes equal lot spacing and the zero switch rule (ZSR) which states that for any particular product the inventory level at the time a production run begins is zero. These assumptions were made to simplify the problem but lead to the subsidiary issue of whether there is a feasible schedule that satisfies them. See Elmaghraby [78] for an excellent review and, for recent developments, Fujita [78], Hassler [79], Hsu [83], Jones and Inman [87], Lee and Denardo [85], and Park and Yun [84].

On a different vein is the formulation originally suggested by Maxwell [64] that allows different lot sizes and unequally spaced production. Here the issue of feasibility is not present as long as the proportion of facility time available for setups is positive. This fact is pointed out by Maxwell [64] and proved formally by Dobson [85]. There are obvious advantages for this formulation. In particular any production sequence can be scheduled over a sufficiently large cycle length.
Moreover, as is shown by Gunther and Swansson [85], any schedule obtained by the basic period approach can also be obtained from a cycle length and a vector of production frequencies, but not vice versa. This means that this formulation is at least as general as the basic period approach.

Dobson [85] develops a heuristic that determines a production sequence. At the core of his algorithm is the determination of production frequencies. His procedure is similar in nature to the one developed by Jackson, Maxwell and Muckstadt [84]. The production frequencies are then rounded to integer powers of two, and a bin-packing algorithm is used to determine a production sequence. Roundy [85] shows that the cost penalty for rounding to powers of two is at most 6%. More recently Zipkin [87] took up the problem of finding an optimal cyclic schedule for a given production sequence. He succeeded by the use of a parametric quadratic programming algorithm. The combined approach of Dobson's determination of a production sequence and Zipkin's determination of an optimal cyclic schedule yields an efficient, if not optimal, solution to the ELSP.

Recently Jones and Inman [87] showed some conditions under which a rotation schedule (i.e., a cyclic schedule in which each product appears only once in the sequence) is optimal. These conditions are automatically recognized by Dobson's heuristic.

In this paper we formulate the EELSP allowing unequal lot sizes and unequally spaced production. We impose a rule that states that the inventory level of a product at the start of its production run is non-positive and its inventory level at the end of its production run is non-negative. We call this rule the extended zero switch rule (EZSR). Our formulation then splits the vector of production times into two vectors; one specifies the production time spent satisfying backorders and the other specifies the production time spent building inventory. The constraints on
cycle length and machine capacity are linear in the production time vectors just described and in the vector of idle times. We extend Dobson's feasibility result and his heuristic to determine a production sequence. Given a production sequence we extend Zipkin's parametric quadratic programming algorithm to determine an optimal cyclic schedule that follows the given sequence. Finally we obtain lower bounds on the long-run average cost of cyclic schedules and extend Jones and Inman's [87] results on when a rotation schedule is optimal.

The rest of this chapter is organized as follows: Section 2 introduces notation and formulates the EELSP under the EZSR. Section 3 deals with machine capacity and feasibility. In section 4 we extend Dobson's heuristic to determine a production sequence. Section 5 assumes a given production sequence and shows, as in Zipkin [87], how a quadratic programming algorithm can be used to obtain an optimal cyclic schedule that follows that sequence. Finally section 6 introduces bounds and extends the result obtained by Jones and Inman [87] on when a rotation schedule is optimal.

2. Notation and Formulation

To simplify the notation choose, without loss of generality, the unit of each product so that demand per unit time is equal to one. Part of the notation follows Dobson [87]. We start by describing the product parameters:

\[ m = \text{the number of products and} \]
\[ i = \text{a product index.} \]

We form the following vectors in \( \mathbb{R}^m \):

\[ e' = (1)^m_{i=1} : \text{vector of ones in } m \text{ space,} \]
\[ d' \equiv e' : \text{demand rates,} \]
\[ p' = (p'_i)_{i=1}^m \]: production rates,

\[ a' = (a'_i)_{i=1}^m \]: setup costs,

\[ s' = (s'_i)_{i=1}^m \]: setup times,

\[ h' = (h'_i)_{i=1}^m \]: holding costs, and

\[ b' = (b'_i)_{i=1}^m \]: backorder costs.

Let \( \kappa = 1 - \Sigma_{i=1}^m (p'_i)^{-1} \). \( \kappa \) is the proportion of facility time available for setups and idle time. We assume \( \kappa > 0 \).

The products are assumed to be produced in a fixed sequence; let

\[ n \] = number of positions in the sequence,

\[ j \] = position index,

\[ f_j \] = index of product produced in position \( j \),

\[ f = (f_j)_{j=1}^n \] sequence vector,

\[ J_i = \{ j : f_j = i \} \],

\[ S = \{ f \in \mathbb{R}^n : m \leq n < \infty, |J_i| \geq 1, \ i = 1,...,m \} \] the set of valid sequence vectors,

\[ F_j = \begin{bmatrix} 1 & f_j = i \\ 0 & f_j \neq i \end{bmatrix} \],

\[ F = (F_j)_{n \times m} \] an \( n \times m \) matrix, and

\[ e = (1)_{i=1}^n \] a vector of ones in \( n \)-space.

Let \( d = Fd' = Fe' = e \). Similarly define \( p, a, s, h, \) and \( b \). Note that all of these vectors are in \( n \)-space. Clearly \( p_j \) is the production rate \( p'_{f_j} \) of the product produced in the \( j^{th} \) position. The vectors \( a, s, h, \) and \( b \) have similar interpretations.
Let $j^+$ be the next position in the sequence after position $j$ where product $f_j$ is produced. The word "next" is interpreted in the circular sense; that is, if a product is not produced again in the cycle, then $j^+$ is the first position in the next cycle where product $f_j$ is produced. Clearly, if $|J_j| = 1$ then $j^+ = j$.

**Example 1.** Let $m = 3$, $f = (1,2,1,3)$ then

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$p = (p_1, p_2, p_1, p_3)$$ etc.

We now describe a cyclic schedule. Let

- $u_j =$ idle time at position $j$,
- $t_j =$ production time at position $j$,
- $y_j =$ inventory of product $f_j$ after the production run at position $j$,
- $u = (u_j)_{j=1}^n$, $t = (t_j)_{j=1}^n$, $y = (y_j)_{j=1}^n$ and $T =$ duration of a cycle.

We refer to a cyclic schedule that follows sequence $f$ as an $f$–cyclic schedule. The sequence of events in an $f$–cyclic schedule is: first idle the facility for $u_1 \geq 0$ units of time, then set up the product assigned to the first position of the sequence and produce it for $t_1$ units of time. Idle for $u_2$ units of time, then set up and produce the product assigned to the second position of the sequence. Continue in this fashion until the sequence is exhausted. Note that no idle time can be inserted in the middle of a production run and that any or all of the idle times can be zero.
Given an f–cyclic schedule we want to determine the total cost incurred during a cycle. Consider the inventory position of a product over a cycle (see Figure 1). We see that the production times \( t_j \) can be split into two parts \( t_j^- \) and \( t_j^+ \) by letting \( t_j^+ = \frac{y_j(p_j-1)^{-1}}{p_j} \) and \( t_j^- = t_j - t_j^+ \). Note that \( t_j^- \) and \( t_j^+ \) are not restricted in sign, although \( t_j \geq 0 \). Let \( t^- = (t_j^-)_{j=1}^n \) and \( t^+ = (t_j^+)_{j=1}^n \). The EZSR is equivalent to the requirement that \( t^- \geq 0 \) and \( t^+ \geq 0 \).

![Inventory Diagram](image)

**Figure 1.** An f–cyclic schedule

Let us now compute the holding and backorder cost incurred over the cycle. We associate with position \( j \) the cost corresponding to the shaded areas in Figure 2.
Figure 2. Production at position \( j \)

Clearly if \( t_j^- \geq 0 \) and \( t_j^+ \geq 0 \) then the cost associated with the shaded areas of Figure 2 is given by

\[
\frac{1}{2} p_j (p_j - 1) [b_j (t_j^-)^2 + h_j (t_j^+)^2].
\]

If \( t_j^- \geq 0 \) and \( t_j^+ \geq 0 \) for all positions \( j = 1, 2, ..., n \), then the total holding, backorder and set-up costs associated with the shaded areas for positions \( j = 1, ..., n \) is

\[
c(f, t^-, t^+, u) = \sum_{j=1}^{n} \{ a_j + \frac{1}{2} p_j (p_j - 1) [b_j (t_j^-)^2 + h_j (t_j^+)^2] \}. \tag{1}
\]

If \( t^- \) and \( t^+ \) are unrestricted in sign the reader can verify that the cost is given by the more elaborate expression
\[
\sum_{j=1}^{N} \{a_j + \frac{1}{2} p_j (p_j-1)[b_j(t_j^-)^2 + h_j(t_j^+)^2 - b_j(t_j^+)^2 - h_j(t_j^-)]\} \tag{2}
\]

where \( x_+ \) and \( x_- \) are respectively the negative and positive part of \( x \).

In what follows we assume the Extended Zero Switch Rule (EZSR). This rule restricts attention to f-cycle schedules that satisfy \( t^- \geq 0 \) and \( t^+ \geq 0 \). The interpretation should be clear. Never start producing a product that has positive net inventory, and never stop producing a product with a negative net inventory. The justification for the name comes from the Zero Switch Rule (ZSR) for the ELSP which states that a production run starts when inventories hit zero.

We now study the constraints that must be satisfied by the decision variables \((t^-, t^+, u \text{ and } T)\) to form a cyclic schedule. Clearly the total cycle length \( T \) satisfies

\[
\sum_{j=1}^{n} (u_j + s_j + t_j^- + t_j^+) = T. \tag{3}
\]

Between two consecutive production lots of a product, say at positions \( j \) and \( j^+ \), some other products are produced. The following equation balances the inventory levels of product \( f_j \) at positions \( j \) and \( j^+ \) with the demand accrued while products at positions \( j+1, \ldots, j^+-1 \) are produced

\[
p_j(t_{j^+}^- + t_{j^+}^+) - \sum_{k=j}^{j^+-1} (u_{k+1} + s_k + t_{k+1}^- + t_k^+) = 0 \tag{4}
\]

for all \( j = 1, 2, \ldots, n \). See Figure 3.
Figure 3.

Note that by the cyclic nature of the schedule demand over a cycle equals production on a cycle. Thus adding equation (4) over the set of positions where a particular product is produced, i.e. over, say $J_i$, yields

$$p_i \sum_{j \in J_i} (t_{j+}^- + t_{j+}^+) = p_i \sum_{j \in J_i} t_j = T, \quad i = 1, \ldots, m. \quad (5)$$

Consequently

$$\sum_{j=1}^{n} t_j = \sum_{i=1}^{m} \sum_{j \in J_i} t_j = \sum_{i=1}^{m} \left( \frac{1}{p_i} T \right) = (1-\kappa)T. \quad (6)$$

Substituting (6) into (3) we find
\[ \sum_{j=1}^{n} u_j = \kappa T - \sum_{j=1}^{n} s_j \]  

By recalling the definition of \( c(f,t^-,t^+,u) \) we can formulate the problem of finding a cyclic schedule that minimizes long-run average cost as

\[ \inf_{f \in \mathcal{F}} \min \frac{1}{T} c(f,t^-,t^+,u) \]

subject to (4), (7), \( t^- \geq 0, \ t^+ \geq 0, \ u \geq 0 \).  

Note that the outer optimization is over the set of valid sequences, while the inner optimization is over the choices of cycle length and idle and production times.

3. Capacity and Feasibility

In this section we assume that a sequence \( f \in \mathcal{F} \) is given. We will show that a feasible cyclic schedule that satisfies the EZSR can be constructed for any sequence provided that the cycle length is long enough. We introduce additional notation to handle the formulation more efficiently. Let \( N \) and \( L \) be \( n \times n \) matrices with entries defined by

\[
N_{jk} = \begin{cases} 
1 & \text{if } k = j^+ \\
0 & \text{else} 
\end{cases}
\]

and

\[
L_{jk} = \begin{cases} 
1 & \text{if } k = \{j, j+1, \ldots, j^+ - 1\} \\
0 & \text{else} 
\end{cases}
\]
Let

\[ P = \text{diag}(p_j^{-1})_{j=1}^n, \]

\[ A = PL, \]

\[ B = \text{diag}(b_j p_j(p_j-1)), \]

\[ H = \text{diag}(h_j p_j(p_j-1)), \]

\[ R = I + (I-P)(N-I), \]

\[ \hat{u}_j = u_{j+1}, \quad \text{and} \]

\[ \hat{u} = (\hat{u}_j)_{j=1}^n. \]

We can reformulate the inner optimization as

\[ \mathcal{P}(f) = \min \frac{1}{T} \left\{ \frac{1}{2} t^{-T} B t^- + \frac{1}{2} t^{+T} H t^+ - e^T a \right\} \]

(9a)

subject to

\[ (R-A) t^- + (I-A) t^+ - A \hat{u} = As \]

(9b)

\[ e^T \hat{u} = \kappa T - e^T s \]

(9c)

\[ t^- \geq 0, \ t^+ \geq 0, \ \hat{u} \geq 0. \]

(9d)

Equation (8b) may need some justification. Note that (4) may be written as

\[
\begin{bmatrix}
  t^{-}_{j+} - \frac{1}{p_j} \sum_{k=j+1}^{j+1} t^{-}_k \\
  t^{+}_{j} - \frac{1}{p_j} \sum_{k=j}^{j+1} t^{+}_k \\
\end{bmatrix} + \begin{bmatrix}
  \frac{1}{p_j} \sum_{k=j}^{j+1} t^{-}_k \\
  \frac{1}{p_j} \sum_{k=j}^{j+1} t^{+}_k \\
\end{bmatrix}
= \begin{bmatrix}
  \frac{1}{p_j} \sum_{k=j}^{j+1} s_k \\
  \frac{1}{p_j} \sum_{k=j}^{j+1} s_k \\
\end{bmatrix}
\]

after dividing by \( p_j \) and rearranging terms. Note that \( e_j^T A = \frac{1}{p_j} e_j^T L = \frac{1}{p_j} \sum_{k=j}^{j+1} e_k^T \).

Consequently we can write (10) as
\[ e_j^T[R-A]t^- + e_j^T[I-A]t^+ - e_j^T \hat{u} = e_j^T A \hat{s} \]

where

\[ e_j^T R t^- = t_{j+}^- + \frac{1}{p_j}(t_{j+}^- - t_{j+}). \]

Equation (9b) should be evident from (11). Dobson [86] shows that \( A \geq 0 \) and \( e^T A = (1-\kappa)e^T \). Consequently the spectral radius of \( A \), \( \rho(A) \leq \|A\|_\infty = 1-\kappa < 1 \).
An elementary theorem on non-negative matrices guarantees that \( (I-A)^{-1} \geq 0 \).
Dobson uses this fact to show that if the idle times are non-negative then there exists a feasible \( f \)-cyclic schedule for the ELSP. Conversely he shows that if \( \kappa = 0 \) and some setup times are positive then there is no feasible \( f \)-cyclic schedule. The reason is clear. If \( \kappa = 0 \) then there is no time available for setups. We now show

**Lemma 1.** \( (R-A) \) has a non-negative inverse.

**Proof.** \( N^T(R-A) = I - N^TP(L+N-I), N^TP(L+N-I) \geq 0 \) and \( e^T N^TP(L+N-I) = e^T A + e^T P(N-I) = e^T A + 0 = (1-\kappa)e^T \). \( 1-\kappa < 1 \) implies that the spectral radius \( \rho(N^TP(L+N-I)) < 1 \). By an elementary theorem on non-negative matrices we have \( (N^T(R-A))^{-1} = (R-A)^{-1} N \geq 0 \). Consequently \( (R-A)^{-1} \geq 0 \). \( \square \)

**Proposition 2.** System (9a–d) has a feasible solution if and only if \( \kappa T \geq e^T s \).

**Proof.** Assume \( (t^-,t^+,_\hat{u}) \) is a feasible solution of (8a–d). Then (9d) and (9c) imply that \( \kappa T \geq e^T s \geq 0 \). Conversely if \( \kappa T \geq e^T s \), then select any non-negative \( \hat{u} \) that satisfies (8c) and let

\[ t^- = (R-A)^{-1}H(B+H)^{-1}A(s+\hat{u}) \quad \text{and} \quad (12) \]

\[ t^+ = (I-A)^{-1}B(B+H)^{-1}A(s+\hat{u}). \quad (13) \]
Lemma 1 and the facts that \((I-A)^{-1} \geq 0\), \(A \geq 0\), and \(B\) and \(H\) are diagonal with positive elements guarantee that (12) and (13) satisfy (9b).

Note that as the backorder costs tend to infinity \(t^-\) tends to the zero vector and \(t^+\) tends to \((I-A)^{-1}A(s+\hat{u})\). Similarly, if holding costs tend to infinity \(t^+\) tends to zero and \(t^-\) tends to \((R-A)^{-1}A(s+\hat{u})\).

4. Sequence Determination

We briefly review the procedure used by Dobson [86] to determine a production sequence and show how this procedure can be used for the EELSP. The idea is to first determine approximate production frequencies by suppressing the sequencing aspects of the problem while retaining the capacity constraint on the time available for setups. Jackson, Maxwell and Muckstadt [84] applied this idea to multi-stage production systems. The next step involves rounding these frequencies to integer numbers. Roundy [86] shows how these frequencies can be rounded to powers of two with a cost penalty of at most 6%. The third step is to use these integer frequencies to create a production sequence. A bin-packing heuristic is used for this step.

The extension of this heuristic to the EELSP is very natural. The only algebraic change required is to replace \(h'_i\) by \(b'_i h'_i (b'_i + h'_i)^{-1}\) in Dobson's computation of the production frequencies. Note that as \(b'_i\) approaches infinity the quantity \(b'_i h'_i (b'_i + h'_i)^{-1}\) approaches \(h'_i\). This is consistent with the fact that \(b'_i = \infty\) for the ELSP. The rationale for this substitution is that if product \(i\) is produced independently over a cycle of length \(T\) the optimal average holding and backorder cost is \(h'_i b'_i (b'_i + h'_i)^{-1} (1-p'_i^{-1})T\) in contrast with \(h'_i (1-p'_i^{-1})T\) for the ELSP.

Given a production sequence \(f\) Dobson solves for production times assuming zero idle time. He gives bounds on the errors incurred when setup costs are zero.
5. **Obtaining an Optimal $f$–cycle Schedule**

In this section we assume a sequence $f \in \mathcal{F}$ is given. We want to obtain an optimal $f$–cyclic schedule, i.e. an optimal solution to problem (9a–d). We follow Zipkin [87] and fix the cycle length $T$. For a fixed cycle length $T$ the remaining problem becomes

$$c(T) = \min \frac{1}{2} t^{-\top} B t^- + \frac{1}{2} t^{+\top} H t^+ \quad (14a)$$

subject to

$$(R-A)t^- + (I-A)t^+ - A\hat{u} = As \quad (14b)$$

$$e^\top \hat{u} = \kappa T - e^\top s \quad (14c)$$

$$t^- \geq 0, \ t^+ \geq 0, \ \hat{u} \geq 0. \quad (14d)$$

We set $c(T) = \infty$ if (14a–d) is infeasible. Given $c(T)$ we can obtain the optimal cycle length by minimizing the long run average cost

$$C(T) = [c(T) + e^\top a] / T; \ T > 0.$$ 

Thus the inner optimization is equivalent to

$$\mathcal{P}(f) = \min_{T > 0} C(T).$$

The Karush–Kuhn Tucker conditions are necessary and sufficient for (14a–d). For each cycle length $T$ they can be conveniently expressed as
\[
\begin{bmatrix}
B & 0 & 0 & (R-A)^T & 0 & -I & 0 & 0 \\
0 & H & 0 & (I-A)^T & 0 & 0 & -I & 0 \\
0 & 0 & 0 & -A^T & e & 0 & 0 & -I \\
(R-A) & (I-A) & -A & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & e^T & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
t^- \\
t^+ \\
\hat{u} \\
\mu \\
\lambda \\
\phi^- \\
\phi^+ \\
\phi
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
As \\
-e^T s
\end{bmatrix} + T
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix},
\tag{15a}
\]

\[
\begin{bmatrix}
t^- \\
t^+ \\
\hat{u}
\end{bmatrix} \geq 0 \quad \text{complementary slack with} \quad \begin{bmatrix}
\phi^- \\
\phi^+ \\
\phi
\end{bmatrix} \geq 0, \tag{15b}
\]

and

\[\mu \text{ and } \lambda \text{ are unrestricted in sign.}\]

By proposition 2, (15a,b) has a solution for all cycle lengths \(T\) with the property \(T \geq e^T s\). Since \(B\) and \(H\) are positive definite, (15a–b) has a unique solution in \(t^-\) and \(t^+\). Optimal idle times \(\hat{u}\), however, need not be unique.

Equations (15a,b) correspond to the model developed by Markovitz [56] and are a special case of the model developed by Perold [84] where (15a,b) is solved parametrically in \(T\).
The parametric quadratic programming algorithm developed by Markovitz[56] computes \( c(T) \) for every \( T \geq \kappa^{-1} e^T \)’s. It adjusts the value of \( T \) while simultaneously maintaining the constraints (15a) and the complementarity conditions (15b). The algorithm yields a finite number of breakpoints

\[
\kappa^{-1} e^T s = T_0 < T_1 < T_2 < \ldots < T_K < T_{K+1} = \infty
\]

at which (15a,b) has a change of basis. Within each interval \([T_k, T_{k+1}]\) a solution to (15a,b) is linear in \( T \). Perold [84] shows how to compute vectors \( t^-_k \) and \( t^+_k \) and directions \( \theta_k \) and \( \delta_k \) so that

\[
t^-(T) = t^-_k + (T-T_k) \theta_k
\]

\[
t^+(T) = t^+_k + (T-T_k) \delta_k
\]

for all \( T \in [T_k, T_{k+1}] \). Within this interval \( c(T) = \frac{1}{2} [t^-(T)^T B t^-(T) + t^+(T)^T H t^+(T)] \) can be written as

\[
c(T) = \eta_k T^2 + \gamma_k T = \alpha_k
\]

where

\[
\eta_k = \frac{1}{2} [\theta_k^T B \theta_k + \delta_k^T H \delta_k],
\]

\[
\gamma_k = \theta_k^T B (t_k - T_k \theta_k) + \delta_k^T H (t^+_k - T_k \delta_k),
\]

and

\[
\alpha_k = \frac{1}{2} (t^-_k - T_k \theta_k)^T B (t^-_k - T_k \theta_k) + \frac{1}{2} (t^+_k - T_k \delta_k)^T H (t^+_k - T_k \delta_k).
\]
Clearly \( \eta_k > 0 \) for all \( k = 1,...,K \) so \( c(T) \) is piecewise convex. Zipkin [87] show that \( c(T) \) is actually convex for the ELSP.

Once (14a–d) is solved we can compute

\[
C_k(T) = (\alpha_k + e^T a)/T + \eta_k T + \gamma_k, \quad T > 0.
\]

Evidently

\[
C(T) = C_k(T) \quad \text{for} \quad T \in [T_k, T_{k+1}]
\]

for \( k = 1,...,K \). The optimal cycle length can then be found by first finding and recording the cycle length \( T^*_k \) which minimizes \( C_k(T) \) over the interval \([T_k, T_{k+1}]\) for each \( k = 1,...,K \), and then selecting from among these values the cycle length with minimal average cost. Let \( S_k = \sqrt{(\alpha_k + e^T a)/\eta_k} \). Clearly \( T^*_k = S_k \) if \( S_k \in [T_k, T_{k+1}] \), and if \( S_k \notin [T_k, T_{k+1}] \) then \( T^*_k \) is the endpoint of \([T_k, T_{k+1}]\) that is closest to \( S_k \).

6. Lower Bounds and Rotation Schedules

A crude lower bound for the long–run average cost of scheduling \( m \) products on a single facility can be obtained by independently finding the optimal cycle length for each product, ignoring capacity and synchronization constraints. Indeed, let

\[
\mathcal{J}_i = \frac{1}{2} b_i h_i (b_i + h_i)^{-1} (1-p_i^{-1}), \quad i = 1,...,n.
\]

Then the minimal average cost of scheduling product \( i \) over a cycle of length \( T_i \) is

\[
a_i T_i + \mathcal{J}_i T_i.
\]

(16)
This cost corresponds to setting \( t_i^- = \frac{h_i^+}{(b_i^++h_i^+)p_i^-}T_i \) and 
\( t_i^+ = \frac{b_i^+}{(b_i^++h_i^+)p_i^-}T_i. \)

The optimal cycle length for product \( i \) is obtained by minimizing (16) and is given by

\[
T_i^* = \sqrt{a_i^+T_i}. 
\] (17)

The set of cycle lengths \( \{T_i^* : i = 1, ..., n\} \) is known in the literature as the independent solution. The cost of the independent solution is a lower bound on the long-run average cost of any cyclic schedule.

A tighter lower bound can be obtained by explicitly considering the amount of time that is available for set-ups. Let \( T_i \) be the average interval of time between successive setups for product \( i \) in a cyclic schedule. A lower bound on the average cost of the cyclic schedule is \( \sum \left[ \frac{a_i^+}{T_i} + \frac{\mathcal{J}_iT_i}{T_i} \right] \). This bound is achieved if every product is produced at equal intervals of time. The proportion of time consumed in setups is \( \sum s_i^+/T_i \), which cannot exceed the proportion of time \( \kappa \) available for setups. Thus a stronger lower bound is given by

\[
\min \sum_{i=1}^{m} \frac{a_i^+}{T_i} + \frac{\mathcal{J}_iT_i}{T_i} \quad \text{(18a)}
\]

subject to 
\[
\sum_{i=1}^{m} \frac{s_i^+}{T_i} \leq \kappa. \quad \text{(18b)}
\]

The solution to (18a,b) is given by

...
\[
\hat{T}_i^* = \frac{\sum_{i=1}^{n} (a_i' + \lambda s_i')}{\sum_{i=1}^{n} J_i'}
\]  

(19)

where \( \lambda \) is the non-negative dual variable corresponding to the capacity constraint (19) and is complementary slack with it.

Jones and Inman [87] note that a common cycle solution is always feasible. For the ELSP they showed that if \( a_i' = \tau J_i \) for \( i = 1, \ldots, m \) for some constant \( \tau \), and the capacity constraint (18b) is not binding, i.e., \( \lambda = 0 \), then the lower bound given by the independent solution is achieved by scheduling all products on a common cycle length \( T_i^* = \hat{T}_i^* = \tau \). Thus every product is produced only once in the cycle giving rise to a rotation schedule. Clearly this result also holds for the EELSP.

In fact, a common cycle solution is still optimal among cyclic schedules when it agrees with the tighter lower bound obtained when the capacity constraint (18b) is binding, i.e., \( \lambda > 0 \). This happens if \( s_i' = \psi a_i' = \psi \tau J_i \) for \( i = 1, \ldots, m \) for some constant \( \psi \). To see this note that the optimal cycle length (19) for product \( i \) is given by

\[
\hat{T}_i^* = \sqrt{(1+\psi \lambda)} \sqrt{a_i'/J_i'}
\]

\[
= \sqrt{(1+\psi \lambda)} \tau,
\]

which is independent of \( i \). Thus a rotation cycle again achieves a lower bound and is therefore optimal.
We have proved

**Proposition 3.** If \( a'_i = \tau^2 J'_i \) for all \( i \) for some constant \( \tau \) and the optimal common cycle length is not at its lower bound then a rotation schedule is optimal (extension of Jones and Inman to EELSP). Moreover, if in addition \( s'_i = \psi a'_i \) for \( i \) and for some constant \( \psi \) then a rotation schedule is optimal regardless of the capacity constraint.

7. **Conclusions**

The ELSP has been extended to allow backorders in an effort to model effectively a frequently encountered scheduling problem. We have extended earlier methods for the ELSP to the EELSP. The resulting algorithm finds a good production sequence \( f \) and an optimal \( f \)-cyclic schedule. This research is a first step in an effort to obtain an algorithm that schedules several products on a single machine in the presence of unforeseen disruptions.
REFERENCES


