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THE JOINT REPLENISHMENT PROBLEM:  
NEW HEURISTICS  
AND WORST CASE PERFORMANCE BOUNDS

by

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<sup>†</sup>This research was supported in part by NSF Grant DMC-8451984, and by AT&T Information Systems and DuPont Corporation

# The Joint Replenishment Problem: New Heuristics and Worst Case Performance Bounds

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## *Abstract*

The joint replenishment problem involves the lot sizing of several items with non-stationary demand in discrete time. The items have individual ordering costs and linear inventory holding costs. Besides, a joint ordering cost is incurred whenever one or more items is ordered together. This problem is known to be NP-complete. In this paper, we analyze the worst case performance of an existing multi-pass heuristic for the problem. Then a new single-pass heuristic is proposed, and it is proved that it has uniformly bounded worst case performance. Further, a lower bound on the cost of the optimal solution is obtained once the heuristic has been used. We then discuss a number of related heuristic algorithms and their worst case performance. The behavior of our heuristics for a randomly generated set of problems is also studied.

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# 1 INTRODUCTION

The joint replenishment problem concerns the lot sizing of several items over a given time horizon. We are considering systems where demand of each product in each time period over the time horizon is assumed to be known. The demand is not required to be constant over time, but may vary from one time period to another. This accounts for seasonality and other non-stationary demand characteristics. The external demand for each item must be met in every time period, and backorders are not permitted.

Each item has a fixed ordering cost or setup cost which is incurred whenever it is ordered, and a linear inventory holding cost. Besides, a joint setup or ordering cost is incurred whenever any one or more items are ordered together in the same time period. The underlying idea behind this joint cost is as follows. An important aspect of the model we are considering is relationships between the various items. It is often the case that families of products exist which, if ordered or produced together, result in lower costs. For example, items supplied to a retailer from the same wholesaler can constitute one family, incurring lower transportation costs if they are ordered together than if they are ordered in separate time periods. Thus it is natural to model these families using a joint setup (ordering) cost which is incurred whenever any member of the family is ordered. In the joint replenishment problem, all items belong to one family, with an associated joint setup (ordering) cost. We are focusing on decisions about when to order or produce each item, and in what lot sizes, in order to minimize the total ordering and inventory holding cost over the planning horizon.

This problem arises in a number of settings, both in the context of manufacturing, with individual and joint setup costs, and in distribution, with ordering costs. Certain multi-echelon inventory systems with a central warehouse supplying several retailers also reduce to the joint replenishment problem. Besides, it arises as a subproblem in more general lot sizing problems with assembly and distribution structures. Certain examples of these are discussed in [Jon87].

Problems with joint costs have been studied by a number of authors. The infinite horizon version with constant demands, with the objective of minimizing the average cost per time period, has been studied by [JMM85]. They develop a heuristic algorithm which is always within 6% of the optimal solution. The infinite horizon case with constant demands, but in a more general production and distribution model, has been considered by [Ro84]. Enumerative procedures are suggested by [Go74]. In the finite horizon case with non-constant demands, [Ve69] and [Za66] propose dynamic programming algorithms. Both of these algorithms take exponential time, and can only be used with very small problems due to a very large state space. Another dynamic programming formulation is provided by [Si79]. [Kao79] also has a dynamic programming formulation, which has a state space smaller than Zangwill's. He also suggests a multi-pass heuristic procedure for the problem, but

with no bounds on its performance.

The joint replenishment problem can be formulated as an integer programming problem, and has been proven to be NP-complete [Jon87]. Thus the emphasis in this paper is on the development of heuristic algorithms which are both fast and effective. To this end, we will define the relative cost of the heuristic as the ratio of the cost of the solution generated by heuristic to the cost of an optimal solution. In this paper, in the next section we analyze the heuristic proposed by [Kao79]. In addition to being multi-pass, we will show that its worst case performance can be arbitrarily bad. In section 3, we then propose a new heuristic for the problem. In section 4, we prove that in the worst case its relative cost is 3. We also show that this bound is tight for our heuristic. Further, a lower bound on the cost of the optimum solution is developed. In section 6, we present and discuss a number of related heuristics and their worst case performance. The behavior of our heuristics for a randomly generated set of problems is discussed in section 6. The last section presents our conclusions.

## 2 ANALYSIS OF AN EXISTING HEURISTIC

The following notation will be used in this paper:

$K_0$  = joint ordering (setup) cost,

$K_i$  = ordering (setup) cost of item  $i$ ,

$H_i$  = inventory holding cost of item  $i$  per unit per unit time,

$d_{it}$  = demand of item  $i$  in time period  $t$ ,

$N$  = number of items,

$T$  = the time horizon.

[Kao79] proposes a multi-pass heuristic for the joint replenishment problem. In each iteration, he solves a single item problem with adjusted costs, and uses the results to adjust the ordering costs of the next item. To be specific, a two item problem is first considered. The heuristic starts by assigning all the joint ordering cost to item 1. Thus the ordering cost of item 1 in every period  $t$  is set equal to  $K_1 + K_0$ , and the resulting single item problem is solved. This can be done efficiently using the well known Wagner-Whitin algorithm [WW58]. Let  $t_1, t_2, \dots$  be the time periods in which item 1 is ordered in the solution to this single item problem. If item 2 places an order in any of these time periods, it will not incur the joint cost again. Using this idea, the heuristic then assigns the joint cost  $K_0$  to item 2 in all time periods except  $t_1, t_2, \dots$ . It then solves the resulting one item problem for item 2. The solution for this item is then used to assign the joint cost to item 1 in the same manner, and the procedure iterates until the solutions do not change any more.

For  $N$  items, the heuristic moves from one item to the next in a round robin fashion, and continues to iterate until the ordering policy of each item converges. Though it can be shown that at each iteration the resulting ordering policy is no worse than the previous one and that the procedure converges in a finite number of steps, no a priori bound on the running time is guaranteed. We further show that the worst case performance of this procedure can be arbitrarily bad. For this, we define the *relative cost* of the heuristic as the ratio of the cost of the heuristic solution to the cost of an optimal solution to the problem. Thus the relative cost of any heuristic will be no less than 1, and the aim is to get a relative cost as close to 1 as possible.

Consider an  $N$  item problem in which all the items are identical. Thus let  $K$  represent their identical individual ordering cost, and let  $H$  be the inventory holding cost. Also, let the demand for any item in any time period be constant,  $d$ . In the first step,  $K_0$  is assigned to item 1 in every time period, and the resulting one item problem solved. For  $T$  sufficiently large, item 1 will order every  $(2(K_0 + K)/Hd)^{1/2}$  time periods (actually, the order time has to be integer, but if the data is selected so that the order intervals are large this will have a negligible effect on our calculation.) This is the standard Wilson lot sizing formula. Now,  $K_0$  is assigned to item 2 in every time period except those in which item 1 placed an order. When the resulting single item problem is solved, since the items are identical, item 2 will place orders in exactly the same time periods as item 1. This happens for every item and the heuristic converges in a single iteration. The average cost of the heuristic policy in per time period is then easily calculated to be  $\sqrt{Hd/2(K_0 + K)}[(N + 1)K_0 + 2NK]$ .

However, for  $N$  identical products, it is reasonable that each product shares the joint cost equally. Thus consider an alternate policy which orders each item every  $(2(K_0/N + K)/Hd)^{1/2}$  time periods. This is got by using the Wilson lot sizing formula with the joint cost equally divided among the items. The average cost of this policy per unit time is then calculated to be  $(2NHd(K_0 + NK))^{1/2}$ . This is then an upper bound on the cost of any optimal policy for this problem. Taking the ratio of the two average costs, the relative cost of the heuristic is thus at least:

$$\frac{1}{2} \left[ \sqrt{\frac{K + K_0/N}{K + K_0}} + \sqrt{\frac{K + K_0}{K + K_0/N}} \right]$$

Clearly, as  $N$  and  $K_0$  tend to  $\infty$ , and  $K_0/N$  tends to 0, this ratio will go to infinity, and so the heuristic can perform arbitrarily badly.

### 3 A NEW HEURISTIC FOR THE JOINT REPLENISHMENT PROBLEM

In this section we will discuss a new heuristic for the joint replenishment problem, which will be referred to as the *cost covering heuristic*. To get some intuition

regarding the structure of the solution, it is first noted that the basic tradeoff in lot-sizing is balancing the inventory holding cost against setup or ordering cost. Consider the simple part period balancing heuristic for a single item problem with no joint order cost [DM68]. In each time period it calculates the total inventory holding costs incurred since the last time an order was placed. If this total holding cost exceeds the ordering cost, an order is placed in the current time period. It then proceeds to the next time period, and repeats the procedure.

In the joint replenishment problem, each item incurs inventory holding costs, and has an individual ordering cost. The basic idea then is that the individual inventory holding cost of an item is balanced against the item's ordering cost. Thus when the total inventory holding cost of an item incurred since its last order exceeds its individual ordering cost, it becomes a candidate for reorder. However, we would like the holding costs of the candidate items to also cover the joint order cost. Thus a joint order is placed when the total holding cost of all the candidate items exceeds their total ordering cost and the joint cost.

To be specific, at time  $t$  the ordering policy in the first  $t - 1$  time period has been decided, and we want to decide whether an order is to be placed in the current time period. Let  $t_i$  denote the last time period in which item  $i$  placed an order. Also, let  $T_{it}$  denote the total inventory holding costs of item  $i$  since  $t_i$ , incurred in time periods  $t_i, t_{i+1}, \dots, t$ , assuming that no order is placed in time period  $t$ . Thus  $T_{it} = H_i \sum_{j=1}^{t-t_i} j d_{i,t_i+j}$ . At each time period  $t$  the heuristic calculates the quantity  $T_{it}$  for each item. If  $T_{it}$  exceeds  $K_i$ , then item  $i$  becomes a candidate for placing an order. However, an order is not placed until these candidate items can cover the joint cost. Thus in time period  $t$  apply the following rule:

*Rule 1: Order all candidate items (those for which  $T_{it} \geq K_i$ ) at time  $t$  if  $\sum_{i=1}^N (T_{it} - K_i)^+ \geq K_0$ .*

Thus all candidate items are ordered when a joint order is placed according to Rule 1. This is referred to as a *regular order* of the candidate items. The heuristic then proceeds to the next time period.

Let a joint order be placed at time  $t$  according to Rule 1. It is possible that an item which was not a candidate item may force an order very soon after time  $t$ . In this case it may be advantageous to order this item at time  $t$  and thus avoid the extra joint cost that would be incurred if it were ordered separately. To illustrate this point, consider two items with  $H_1 = H_2 = 1$  and demand as follows:

$$\begin{aligned} d_{1,t} &= 0 \text{ for } t = 1 \dots (m-1), & d_{1,m} &= (K_0 + K_1)/(m-1) \\ d_{2,t} &= 0 \text{ for } t = 1 \dots m, & d_{2,m+1} &= (K_0 + K_2)/m \end{aligned}$$

According to Rule 1, item 1 will place an order at time  $m$ , and item 2 is not a candidate for an order at this time. Item 2 will place an order at time  $(m+1)$ . However, if both items were ordered at time  $m$ , then an extra holding cost of

$(K_0 + K_2)/m$  is incurred for item 2, with a savings of  $K_0$ . Clearly, if  $m$  is large, the savings will be much larger than the extra holding cost. Besides, if the number of items  $N$  is large, then in a similar example the heuristic would order each item in a separate time period, so that the joint cost is incurred  $N$  times. By ordering all the items at the same time as the first one, we could save  $(N-1)K_0$  in order costs while incurring a small inventory holding cost. Thus the heuristic in its present form will have arbitrarily bad worst case performance if the number of items is large.

Let the last joint order be placed at time  $t_0$ , and let item  $i$  not be a candidate for order in this time period since  $T_{it_0} < K_i$ . The discussion above suggests that under certain conditions it may be advantageous to order item  $i$  at time  $t_0$ . Specifically, for such an item let  $t_i$  be the last time that it was ordered ( $t_i < t_0$ ). We maintain a quantity  $C_{it}$  defined by:

$$C_{it} = H_i(t_0 - t_i) \sum_{j=t_0}^t d_{ij}.$$

$C_{it}$  is the saving in the inventory holding cost of item  $i$  by placing an order for it at time  $t_0$ . In other words, we know that item  $i$  was last ordered at  $t_i$ , and that its inventory holding costs in periods  $t_i, \dots, t$  is  $T_{it}$ . Then  $C_{it}$  is the reduction in the holding cost that can be achieved by inserting an extra order for this item at time  $t_0$ . Clearly, this should be done if this savings is greater than the ordering cost  $K_i$  of item  $i$ . Thus we have:

*Rule 2: If at time  $t$  the last joint order was at time  $t_0$ , item  $i$  was not ordered at  $t_0$ , and  $C_{it} \geq K_i$ , then item  $i$  is also ordered at  $t_0$ .*

$T_{it}$  is then recalculated to reflect this extra order (i.e. it is now calculated from  $t_0$  instead of  $t_i$ ). This is referred to as *pulling* the item back, and the order is referred to as a *pull order* of item  $i$ .

To see why pull orders prevent the heuristic from performing arbitrarily badly, consider again the example above. At time  $m+1$ , for item 2 the quantity  $C_{2,m+1} = (m-1)(K_0 + K_2)/m$ . Then  $C_{2,m+1} \leq K_2$  whenever  $K_0 \leq K_2/(m-1)$ . Thus if the heuristic does not pull item 2 back to order with item 1, then either  $K_0$  or  $m$  is small. In either case it means that only a small saving can be obtained by ordering the two items together, and so the heuristic can not perform very badly. On the other hand, if  $K_0$  or  $m$  were large, then item 2 would be pulled back to order with item 1, and the heuristic would perform optimally.

The description below is simplified by considering a dummy time period  $T+1$  at the end of the time horizon in which the demand for every item is 0. With these ideas, we can now formulate the cost covering heuristic as:

### THE COST COVERING HEURISTIC

1.  $t = 1$ ,  $T_{it} = C_{it} = 0 \quad \forall i$ ;  $t_0 = t_i = 1 \quad \forall i$  (each item is ordered in the first time period.)

2.  $t = t + 1$ ; For each item  $i$ :
  - (a) Calculate  $C_{it}$ . If  $C_{it} \geq K_i$ , (Rule 2: pull order of item  $i$ ), place order for item  $i$  at  $t_0$  and set  $t_i = t_0$ .
  - (b) Calculate  $T_{it}$
3. If  $t = T + 1$ , stop. Otherwise (Rule 1: regular order) if  $\sum_{i=1}^N (T_{it} - K_i)^+ \geq K_0$  then place an order for all candidate items at time  $t$ . Set  $t_0 = t$ , and for the candidate items set  $t_i = t$ .
4. Go to step 2.

It is easily seen that  $T_{it}$  and  $C_{it}$  can be calculated in constant time from  $T_{i,t-1}$  and  $C_{i,t-1}$ . Thus the time complexity of the algorithm is  $O(NT)$ .

## 4 PERFORMANCE BOUND FOR THE COST COVERING HEURISTIC

In this section we will discuss the worst case performance of the cost covering heuristic. The following notation will be used:

$n_{ij}$  = number of pull orders of item  $i$  between the  $(j - 1)^{st}$  and  $j^{th}$  regular order,  $j = 2, 3, \dots$

$K_0^{it} = (T_{it} - K_i)^+ =$  portion of  $K_0$  covered by candidate item  $i$  in period  $t$ .

$K^{it} =$  Portion of  $K_0$  allocated to item  $i$ .

We will use the convention that if the demand of any item in a given time period is satisfied by holding it in inventory, then the holding cost is incurred in that time period. This is only for convenience in accounting for all costs in our analysis, and in the absence of discounting makes no difference to the total cost of a policy. Ordering cost is counted in the time period in which it is incurred.

The proof of the worst case performance of the heuristic is based on allocating the joint order cost incurred by any policy in each time period to the various items. After allocation the items have total allocated ordering costs that vary over time, and there is no joint order cost. The resulting problem thus decomposes into  $N$  independent single-item problems, one for each item. Let the single item problem for item  $i$  after cost allocation be denoted by  $P(i)$ . Let  $C_p^i$  denote the total cost of an optimal policy for  $P(i)$ . We show in lemma 2 that  $\sum_i C_p^i$  is a lower bound on the cost of any policy for the joint replenishment problem. Proofs of this form were first developed in continuous time in [Ro85]. At the same time, the orders placed by item  $i$  according to the cost covering heuristic constitute a feasible solution to



$P(i)$ . Let the cost of this solution to  $P(i)$  be denoted by  $C_h^i$ . In lemma 1 we show that  $\sum_i C_h^i$  is equal to the total cost incurred by the heuristic policy for the the original joint replenishment problem. Lemmas 4, 5 and 6 allow us to compare  $C_p^i$  and  $C_h^i$ , thus providing a uniform bound on the worst case relative cost of the heuristic policy.

To this end we reallocate the joint cost  $K_0$  in each time period in the following manner. Let two successive joint orders be placed by the heuristic in time periods  $t'$  and  $t$ , and let the  $I_t$  be the set of candidate items which place a regular order at  $t$ . For an item  $i \in I_t$ , let  $T_{it} = K_i + K_0^{it}$ . Then in each time period  $(t' + 1), \dots, t$ , an ordering cost of  $K^{it}$  is allocated to item  $i$ , where  $K^{it}$  are any numbers that satisfy the following two conditions:

- $K_0^{i,t-1} \leq K^{it} \leq K_0^{it} \quad \forall i \in I_t$
- $\sum_{i \in I_t} K^{it} = K_0.$

One possible choice is  $K^{it} = K_0^{i,t-1} + (K_0^{it} - K_0^{i,t-1})(K_0 - \sum_{i \in I_t} K_0^{i,t-1}) / \sum_{i \in I_t} (K_0^{it} - K_0^{i,t-1})$ . Thus in each of the time periods  $(t' + 1), \dots, t$  each item  $i \in I_t$  has a total ordering cost of  $K_i + K^{it}$ , and each item  $i \notin I_t$  has a total ordering cost of  $K_i$ .

**Lemma 1:** The total cost incurred by the policy generated by the cost covering heuristic is  $\sum_i C_p^i$ .

*Proof:* Clearly if the individual item problem  $P(i)$  follows the ordering pattern generated by the cost covering heuristic, then it incurs the same inventory holding and individual item ordering costs as the heuristic policy. Besides, at every time period  $t$  in which a joint order is placed, the candidate items incur an extra (allocated) ordering cost of  $\sum_{i \in I_t} K^{it} = K_0$ . Thus the total cost incurred by the individual items after the joint cost reallocation equals the cost incurred by the ordering policy generated by the heuristic.  $\square$

**Lemma 2: Lower bound:** Let  $C_p^i$  be the optimal holding and ordering cost for problem  $P(i)$ . Then  $\sum_i C_p^i$  is a lower bound on the total cost of any policy for the joint replenishment problem.

*Proof:* Consider any ordering policy  $\mathbf{P}$  for the joint replenishment problem, and let  $\mathbf{P}_i$  be the corresponding policy for the single item problem  $P(i)$ . The cost incurred by  $\mathbf{P}$  is made of the inventory holding costs of the items, the individual ordering costs of the items, and the joint ordering costs. Now if the joint costs are allocated to the items as described above, the inventory holding costs and the individual ordering costs incurred by  $\mathbf{P}$  are equal to the sum over all items of the corresponding costs incurred by  $\mathbf{P}_i$ . Besides, at time  $t$  let  $I_t$  be the set of items to which the joint cost is allocated. Let  $I_p$  be the set of items that place an order at  $t$  according to  $\mathbf{P}$ . Then the total allocated cost incurred by the the policies  $\mathbf{P}_i$  will be  $\sum_{i \in I_t \cap I_p} K^{it} \leq K_0$ . Thus the total cost of the  $\mathbf{P}$  is no less than the sum of the costs of  $\mathbf{P}_i$ ,  $1 \leq i \leq N$ . Now if  $P(i)$  are separately solved to optimality, and the

resulting optimal cost is  $C_p^i$ , clearly  $C_p^i$  is no larger than the cost of  $P_i$ , and thus the sum of these optimal costs will be a lower bound on the cost of  $P$ .  $\square$

Now we can concentrate on an individual item  $i$  with the reallocated costs. Let  $C_p^i$  be the minimum cost incurred by item  $i$ , and let  $C_h^i$  be the cost incurred by item  $i$  following the policy generated by the cost covering heuristic. We shall prove the  $C_h^i/C_p^i \leq 3$ , Then by lemmas 1 and 2 the worst case relative cost of the cost covering heuristic is 3.

First consider a single item problem over two adjacent intervals of time  $[t_1, t_2]$  and  $[t_2 + 1, t_3]$ . Let  $C$  be a lower bound on the cost of any solution to the problem over the first interval. Let  $C'$  be a lower bound on the cost of any solution over the second interval assuming a free order was placed at  $t_2$ . Then clearly a lower bound on the cost of any solution over the combined interval  $[t_1, t_3]$  is given by  $C + C'$ . Notice that the optimal over the combined interval may not have an order at  $t_2$ , so in  $C'$  we must assume a free order at  $t_2$  to get a lower bound. Formally, we can state the following lemma:

**Lemma 3:** Let  $C$  be a lower bound on the cost incurred by item  $i$  from time  $t = t_1$  to  $t = t_2$ . Let  $C'$  be a lower bound on the cost incurred by item  $i$  from  $t = t_2 + 1$  to  $t = t_3$ . assuming a free order at  $t_2$ . Then a lower bound on the cost of any policy from  $t = t_1$  to  $t = t_3$  is given by  $C + C'$ .  $\square$

We will break the problem of item  $i$  over the entire time horizon into three types of subproblems. Lower bounds are developed for each subproblem, and are combined over the whole time horizon using lemma 3. The cost covering heuristic may cause item  $i$  to place several pull orders between successive regular orders. The three types of subproblems this generates are:

- a regular order,
- two successive pull orders, and
- a pull order followed by a regular order.

The following three lemmas consider these three cases.

**Lemma 4:** According to the cost covering heuristic, let item  $i$  place an order at time  $t'$  followed by a regular order at time  $t$ . Let  $C_p$  be the minimum cost incurred by item  $i$  in the time periods  $(t' + 1), \dots, t$ . Let  $C_h$  be the cost incurred by item  $i$  in the same time periods by following the heuristic policy. Then  $C_h/C_p \leq 2$ .

*Proof:* Let the last joint order generated by the heuristic before  $t$  be at  $\bar{t}$ ,  $\bar{t} \geq t'$ . Item  $i$  has reallocated ordering cost of  $K_i$  in time periods  $(t' + 1), \dots, \bar{t}$ , and  $K_i + K^{it}$  in time periods  $(\bar{t} + 1), \dots, t$ . The heuristic will incur in the time interval  $[t' + 1, t]$  a holding cost of  $T_{i,t-1} \leq K_i + K_0^{i,t-1} \leq K_i + K^{it}$ , and an ordering cost (at  $t$ ) of  $K_i + K^{it}$ . Thus  $C_h = T_{i,t-1} + K_i + K^{it} \leq 2(K_i + K^{it})$ .

Now consider an optimal policy  $P$  over this time interval. If  $P$  places no order in this interval, then it incurs a holding cost of at least  $T_{it} \geq K_i + K_0^{it} \geq K_i + K^{it}$ . If  $P$  places at least one order in the interval  $[\bar{t} + 1, t]$ , then it will incur an ordering

cost of  $K_i + K^{it}$ . Finally, suppose that  $\mathbf{P}$  places no order in  $[\bar{t} + 1, t]$ , and at least one order in  $[t' + 1, \bar{t}]$ . Note that the heuristic placed a regular order at  $t$  instead of pulling back to  $\bar{t}$ . Hence the cost incurred by  $\mathbf{P}$  in this case must be greater than  $T_{it} \geq K_i + K^{it}$ . In every case, we get  $C_{\mathbf{P}} \geq K_i + K^{it}$ , so  $C_{\mathbf{h}}/C_{\mathbf{P}} \leq 2$ .  $\square$

**Lemma 5:** According to the heuristic, let item  $i$  place an order at time  $t'$ , followed by two pull orders at time  $t^1$  and  $t^2$ . Let  $C_{\mathbf{P}}$  be the minimum cost incurred by item  $i$  in the time periods  $(t' + 1), \dots, t^2$ . Let  $C_{\mathbf{h}}$  be the cost incurred by item  $i$  in the same time periods by following the heuristic policy. Then  $C_{\mathbf{h}}/C_{\mathbf{P}} \leq 3$ .

*Proof:* Consider the interval  $[t' + 1, t^2]$ . Let  $h_1$  and  $h_2$  be the holding costs incurred according to the heuristic by item  $i$  in intervals  $[t' + 1, t^1]$  and  $[t^1 + 1, t^2]$  respectively. Note that since we are considering pull orders,  $h_1, h_2 \leq K_i$ . Also, since the orders at  $t^1$  and  $t^2$  are pull orders, the reallocated ordering cost of item  $i$  in the entire interval is just  $K_i$ . The cost incurred by the heuristic in  $[t' + 1, t^2]$  is then  $C_{\mathbf{h}} = 2K_i + h_1 + h_2$ . Consider any ordering policy  $\mathbf{P}$ , and let  $C_{\mathbf{P}}$  be the cost it incurs in the same interval. The four possible cases are:

1.  $\mathbf{P}$  places no order in  $[t' + 1, t^2]$ , in which case  $C_{\mathbf{P}} \geq T_{i\bar{t}} \geq K_i$  (where  $\bar{t}$  is the time period in which Rule 2 becomes effective, causing the pull order in time  $t^1$ ), and  $C_{\mathbf{P}} \geq h_1 + h_2$ . Thus  $C_{\mathbf{h}}/C_{\mathbf{P}} = (2K_i + h_1 + h_2)/C_{\mathbf{P}} \leq 2K_i/K_i + (h_1 + h_2)/(h_1 + h_2) = 3$ .
2.  $\mathbf{P}$  places at least one order in  $[t' + 1, t^1]$  and none in  $[t^1 + 1, t^2]$ , in which case  $C_{\mathbf{h}}/C_{\mathbf{P}} \leq (2K_i + h_1 + h_2)/(K_i + h_2) \leq 1 + (K_i + h_1)/(K_i + h_2) \leq 3$ .
3.  $\mathbf{P}$  places an order in  $[t^1 + 1, t^2]$  and none in  $[t' + 1, t^1]$ , which is similar to case (2).
4.  $\mathbf{P}$  places more than one order in  $[t' + 1, t^2]$ , in which case  $C_{\mathbf{P}} \geq 2K_i$ , and  $C_{\mathbf{h}}/C_{\mathbf{P}} \leq 2 \leq 3$ .

In every case we get  $C_{\mathbf{h}}/C_{\mathbf{P}} \leq 3$ .  $\square$

**Lemma 6:** According to the heuristic, let item  $i$  place an order at time  $t'$ , followed by a pull order at time  $t^1$ , followed by a regular order at time  $t$ . Let  $C_{\mathbf{P}}$  be the minimum cost incurred by item  $i$  in the time periods  $(t' + 1), \dots, t$ . Let  $C_{\mathbf{h}}$  be the cost incurred by item  $i$  in the same time periods by following the heuristic policy. Then  $C_{\mathbf{h}}/C_{\mathbf{P}} \leq 3$ .

*Proof:* According to the heuristic, let  $h_1$  be the holding cost incurred by item  $i$  in time periods  $(t' + 1), \dots, t^1$ . Since the order at  $t^1$  is a pull order,  $h_1 \leq K_i$ . Let  $\mathbf{P}$  be an optimal policy, and let it place no order of item  $i$  in the interval  $[t' + 1, t^1]$ . Then it will incur a holding cost of at least  $h_1$  in this interval. On the other hand, if  $\mathbf{P}$  places at least one order in this interval, then it incurs an ordering cost of at least  $K_i$ . In either case, it incurs a cost of at least  $h_1$  in this interval.

In the interval  $[t^1 + 1, t]$ ,  $\mathbf{P}$  will incur a cost of at least  $K_i + K^{it}$  exactly as in lemma 4. Thus by lemma 3, in the combined interval  $[t' + 1, t]$ , we get

$C_p \geq h_1 + K_i + K^{it}$ . The costs incurred by item  $i$  according to the heuristic in this same interval are  $h_1$  (holding cost in  $[t' + 1, t^1]$ ),  $K_i$  (order at  $t^1$ ),  $T_{i,t-1} \leq K_i + K^{it}$  (holding in  $[t^1 + 1, t]$ ), and  $K_i + K^{it}$  (ordering at  $t$ ). Thus  $C_h \leq h_1 + 3K_i + 2K^{it}$ . Taking the ratio,  $C_h/C_p \leq 3$ .  $\square$

We are now ready to combine these results and prove our main theorem.

**Theorem 7:** The worst case relative cost of the cost covering heuristic is 3.

Consider the ordering policy generated by the heuristic for any item  $i$  over the time horizon. Clearly the time horizon can be broken into intervals

*Proof:* For any item  $i$  consider the ordering policy for  $P(i)$  generated by the cost covering heuristic. The time horizon for  $P(i)$  can be broken up into time intervals of the form  $[t' + 1, t]$ , where the heuristic place two successive regular orders of item  $i$  in time periods  $t'$  and  $t$ , with  $n_{ij}$  pull orders in between. If  $n_{ij} = 0$ , the time interval is of the form considered in lemma 4. If  $n_{ij}$  is an even number, then this time interval can be further divided into subintervals, where every two successive pull orders constitute a subinterval of the form considered in lemma 5, and the last regular order constitutes a time interval of the form considered in lemma 4. If  $n_{ij}$  is an odd number, the first  $n_{ij} - 1$  pull orders constitute subintervals of the form considered in lemma 5, and the last pull order followed by the regular order constitute a subinterval of the form considered in lemma 6.

For a subinterval  $f$ , let  $C_h^{if}$  be the cost incurred by  $P(i)$  over  $f$  by following an ordering policy as generated by the cost covering heuristic, and let  $C_p^{if}$  be the minimum cost incurred by  $P(i)$  over  $f$ . For each of the three types of subintervals, lemmas 4, 5, and 6 show that  $C_h^{if}/C_p^{if} \leq 3$ . Let  $C_h^i$  be the total cost incurred by  $P(i)$  following the heuristic policy, and let  $C_p^i$  be the minimum cost incurred by  $P(i)$ , both over the entire time horizon. Then  $C_h^i = \sum_f C_h^{if}$ , and by lemma 3  $C_p^i \geq \sum_f C_p^{if}$ . Therefore  $C_h^i/C_p^i \leq 3$ , which at once gives  $C_h^i/C_p^i \leq 3$ . Finally, lemma 1 shows that the total cost of the cost covering heuristic is given by  $\sum_i C_h^i$ , while lemma 2 shows that a lower bound on the cost of any policy for the joint replenishment problem is given by  $\sum_i C_p^i$ . This proves the theorem.  $\square$

We next show that this bound is tight by providing a suitable example. Consider a two item case with  $K_1 = 0$ . The demand structure is as follows:

$$\begin{aligned} d_{1,3j+1} &= K_0 + \epsilon, & d_{1,3j+2} &= K_0 - 2\epsilon, & d_{1,3j} &= \epsilon \\ d_{2,3j+1} &= 0, & d_{2,3j+2} &= K_2 + \epsilon, & d_{2,3j} &= 0 \end{aligned}$$

It is easy to check that the heuristic will place orders both items at time periods  $1, 3j$ , and  $3j + 1$ , for  $j = 1, 2, \dots$ . The policy cycles every three time periods, and in each cycle a cost of  $3K_0 + 3K_2 - \epsilon$  is incurred. The optimal policy will order item 1 at time periods  $1, 3j + 1$ , and  $3j + 2$ , and item 2 at time periods  $1$ , and  $3j + 2$ ,  $i = 1, 2, \dots$ . The optimal cost then is found to be  $2K_0 + K_2 + \epsilon$  in each three period cycle. The relative cost of the heuristic policy then is  $(3K_0 + 3K_2 - \epsilon)/(2K_0 + K_2 + \epsilon)$ . We only need  $K_0 \geq 2\epsilon$ , and if we let both  $K_0$  and  $\epsilon$  go to zero while satisfying this condition, the relative cost will go to 3.

## 5 RELATED ALGORITHMS AND WORST CASE PERFORMANCE BOUNDS

In this section we will discuss certain heuristic algorithms that are related to the cost covering heuristic. Clearly the cost covering heuristic reflects ideas from the single product dynamic lot sizing problem. A very well known, and in practice a very effective heuristic for the single item problem is the Silver-Meal heuristic. This motivates Variant 1 of the cost covering heuristic, and in the process illustrates how ideas from the single item problem can be adapted to the cost covering heuristic in general. Next, recall that for the constant demand, infinite horizon version of the joint replenishment problem, [JMM85] have proposed a heuristic which in the worst case performs 6% worse than the optimal. If a constant demand for each item were presented to our heuristic, we would like to achieve a similar performance. This however is not the case. In this section we propose related heuristics and show that they achieve similar worst case performance bounds if the demands are constant over time. This motivates Variants 2 and 3.

### 5.1 Variant 1

The first variant we consider is similar to the Silver-Meal heuristic for the single item problem [SM73]. The Silver-Meal heuristic for the single item  $i$  starts by placing an order in the first time period. In general, let the last order be placed at time  $t_i$ . The heuristic next wants to decide whether to place an order in time period  $t$ . Then the average cost per time period is  $T'_{it} = (K_i + H_i \sum_{j=t_i}^t (j - t_i) d_{i,j}) / (t - t_i + 1) = (K_i + T_{it}) / (t - t_i + 1)$ . The heuristic finds the first local minimum of  $T'_{it}$  as a function of  $t$ . If the local minimum occurs at  $t'$ , an order is placed at time  $t' + 1$ , and the procedure is repeated. For the single item problem, this is easily done by calculating  $T'_{it}$  for increasing values of  $t$ , and placing an order when  $T'_{it}$  first starts increasing.

In the joint replenishment problem, the  $T'_{it}$  are calculated for each item  $i$ . When a  $T'_{it}$  starts increasing, that item becomes a candidate for an order. Just as in the cost covering heuristic, we now have the joint cost  $K_0$  to consider. Since the candidate items must cover the joint cost, this is done by assigning a fraction  $K_0^{it}$  of the joint cost to candidate item  $i$  such that its  $T'_{it}$  does not increase. In other words, let  $T'_{it}$  start increasing at time  $t$ , so that  $T'_{i,t-1} < T'_{it}$ . Then  $K_i$  is replaced by  $K_i + K_0^{it}$  in the formula for  $T'_{i,t-1}$  and  $T'_{it}$  to maintain the equality  $T'_{i,t-1} = T'_{it}$ . In successive time periods this fraction of  $K_0$  assigned to item  $i$  may have to be increased to prevent  $T'_{it}$  from increasing. This is done for all candidate items until all of  $K_0$  has been assigned. Thus the variant of Rule 1 is *place a joint order at time  $t$  if  $\sum_i K_0^{it} \geq K_0$* . Once that happens, all candidate items are ordered at time  $t$ . Pull orders are placed by maintaining the test quantity  $C_{it}$  exactly as in the cost covering heuristic and using the same Rule 2. To summarise, Rules 1 and 2 of the cost covering heuristic are replaced by Rules 1\* and 2\* below.

1. *Rule 1\**: Order all candidate items at time  $t$  if  $\sum_i K_o^{it} \geq K_0$ .
2. *Rule 2\**: Same as Rule 2.

It is well known that the worst case performance of the Silver-Meal heuristic can be arbitrarily bad [Ax82]. The same example shows that this variant for the joint replenishment problem can perform arbitrarily badly. However, in computational tests the Silver-Meal heuristic has been found to outperform the part-period balancing heuristic, and this fact motivates this variant. It is also clear that other heuristics for the single item dynamic lot sizing problem can be similarly adapted to the joint replenishment problem by suitable assignment of  $K_o^{it}$ .

## 5.2 Variant 2

A second variant of the cost covering heuristic is obtained by modifying the computation of the joint order points. In the original heuristic, we used the quantity  $T_{it}$  which is the holding cost of item  $i$  since *its* last order. We now define  $\bar{T}_{it}$  as the holding cost of item  $i$  since the last joint order point. Thus if the last joint order was placed at time  $t_0$ , then  $\bar{T}_{it}$  is got by using  $t_0$  in place of  $t_i$  in the formula for  $T_{it}$ . In this variant of the heuristic, an item becomes a candidate for reorder when  $\bar{T}_{it} \geq K_i$ , and (Rule 1') all candidate items are ordered when:

$$\sum_{i=1}^N (\bar{T}_{it} - K_i)^+ \geq K_0.$$

This replaces rule 1 of the original heuristic. The quantity  $T_{it}$  is also maintained and is used to determine pull orders. If item  $i$  was not ordered at the last joint order point  $t_0$ , then a pull order for it is placed at  $t_0$  if  $T_{it} \geq K_i$ . This is Rule 2' and replaces Rule 2 of the original heuristic. Variant 2 thus operates by replacing Rules 1 and 2 of the cost covering heuristic by these new rules which are summarised below.

1. *Rule 1'*: Order all candidate items (those which have  $\bar{T}_{it} \geq K_i$ ) at time  $t$  if  $\sum_{i=1}^N (\bar{T}_{it} - K_i)^+ \geq K_0$ .
2. *Rule 2'*: If at time  $t$  the last joint order was at time  $t_0$ , item  $i$  was not ordered at  $t_0$ , and  $T_{it} \geq K_i$ , then order item  $i$  at  $t_0$  (pull order).

The following theorem establishes the worst case performance of this variant for both constant and non-constant demand. The heuristic by [JMM85] which has a worst case relative cost of 1.06 for joint replenishment systems with constant demand was formulated and analyzed in continuous time. For the purpose of comparison, the constant demand analysis in the following theorem is also done in continuous time. It is obvious how the analogous version of the heuristic can be run in continuous time.

**Theorem 8:** The worst case relative cost of variant 2 of the cost covering heuristic with non-constant demands is 3. If the demand for each item is constant over time, the worst case relative cost of the continuous time version of this variant is 1.25.

*Proof:* The proof that the worst case relative cost is 3 is very similar to the proof of Theorem 7, and is omitted. The example after Theorem 7 again achieves this worst case relative cost.

In the constant demand case, let  $d_i$  be the constant demand for item  $i$ . For convenience, let the items be indexed by increasing  $K_i/H_i d_i$ . Clearly then item 1 is the first to become a candidate for reorder, then item 2, and so on. If items  $1, \dots, p$  are the candidate items when the joint order is placed, then Rule 1' and simple algebra give:

$$t_0 = \sqrt{\frac{2(K_0 + \sum_{i=1}^p K_i)}{\sum_{i=1}^p H_i d_i}}.$$

Here  $p$  is the lowest index for which  $\sqrt{2K_{p+1}/H_{p+1}d_{p+1}} \geq t_0$ , and  $t_0$  is the time at which the first joint order is placed. Since  $\bar{T}_{it}$  is calculated for each item from the last joint order point, a joint order is placed every  $t_0$  time periods, and the same first  $p$  items are the candidate items each time. We can easily calculate that if items  $1, \dots, p$  order every  $t_0$  time periods, then this policy generated by our algorithm has an average cost per time period of  $\sqrt{2(\sum_{i=0}^p K_i)(\sum_{i=1}^p H_i d_i)}$  for the subsystem made of the first  $p$  items and the joint cost.

Further, all other items  $(p+1), \dots, N$  place orders every  $m_i t_0$  time periods, where  $m_i$  is an integer which depends on when Rule 2' triggers a pull order of item  $i$ . Thus by simple algebra  $m_i$  is the integer satisfying  $m_i t_0 \leq \sqrt{2K_i/H_i d_i} < (m_i + 1)t_0$ . These items thus never cause the joint cost to be incurred. Thus  $m_i t_0 \leq \sqrt{2K_i/H_i d_i} < 2m_i t_0$ . The average cost per time period for item  $i$  is then  $K_i/m_i t_0 + H_i d_i m_i t_0/2 \leq 1.25\sqrt{2K_i H_i d_i}$ .

Thus the total average cost per time period of the heuristic policy is at most  $\sqrt{2(\sum_{i=0}^p K_i)(\sum_{i=1}^p H_i d_i)} + 1.25 \sum_{i=p+1}^N \sqrt{2K_i H_i d_i}$ . [JMM85] prove that a lower bound on the average cost of any policy for this problem is  $\sqrt{2(\sum_{i=0}^p K_i)(\sum_{i=1}^p H_i d_i)} + \sum_{i=p+1}^N \sqrt{2K_i H_i d_i}$ . The worst case relative cost of this variant of the heuristic is thus 1.25.  $\square$

### 5.3 Variant 3

Thus variant 2 of the cost covering heuristic can perform at worst 25% worse than the optimal. Notice that this happens essentially because an item  $i$ ,  $i > p$ , which would have ordered every  $\sqrt{2K_i/H_i d_i} = 2t_0 - \epsilon$  periods if left to itself is pulled back to order every  $t_0$  time periods. It would be better if instead of pulling this order back, it could be *pushed* ahead, so that the item would order every  $2t_0$  time periods.

This can be done by modifying Rule 2'. Rule 2' pulls back to the last joint order point if  $T_{it} \geq K_i$ . Let item  $i$  not be ordered at joint order point  $t_0$ , and let the next joint order be at  $t'_0$ . Let  $T_{it} \geq K_i$  at some  $t$  between  $t_0$  and  $t'_0$ . Rule 2' would pull item  $i$  to place an order at  $t_0$ . The modification to this rule, called Rule 2'', will compare  $T_{it_0}$  and  $T_{it'_0}$  to  $K_i$ . It pulls to  $t_0$  or pushes to  $t'_0$  depending on whether  $K_i - T_{it_0}$  is smaller than  $T_{it'_0} - K_i$ . The rest of the algorithm is the same as variant 2. To summarise, we use the following two rules to replace the ones in the cost covering heuristic:

1. *Rule 1'': Same as Rule 1'.*
2. *Rule 2'': Let two successive joint orders be at time  $t_0$  and  $t'_0$ , and for item  $i$  let  $T_{it} > K_i$  at time  $t$ ,  $t_0 < t < t'_0$ . Then order item  $i$  at  $t_0$  (pull) if  $K_i - T_{it_0}$  is smaller than  $T_{it'_0} - K_i$ , and at  $t'_0$  (push) otherwise.*

By an argument similar to Theorem 7, we can show that in the worst case this variant can have a relative cost of 4. If demand for each item is constant over time, then by an argument similar to Theorem 8 we can show that items  $(p + 1), \dots, N$  have a relative cost no worse than 1.10. Thus a similar proof shows that in the constant demand case its worst case relative cost is 1.10. Thus in the constant demand case it will do no worse than 10% from the optimal.

## 6 COMPUTATIONAL RESULTS

The cost covering heuristic and its three variants were coded in FORTRAN and tested on a selection of randomly generated problems. The problems had 4 items and a planning horizon of 20 time periods. The test problems had the following characteristics:

- Two joint costs (35 and 55) were used.
- Individual setup costs for the items were generated from a uniform (10,20) distribution.
- The demands for each item in each time period were generated from uniform distributions with specified mean and variance. The demand patterns used in the various problems varied from constant demand to cases with high variance (up to a uniform (1,12) distribution).

A total of 156 test problems were generated and solved on an AT&T 6300 personal computer.

The solution obtained from the cost covering heuristic was compared to a lower bound for the optimal solution obtained as follows. The joint replenishment problem is formulated as a linear integer programming problem in [Jon87]. The linear



	Cost Covering	Variant 1	Variant 2	Variant 3
Average deviation from lower bound	1.80 %	1.71	1.87	1.80
Maximum deviation from lower bound	7.39 %	10.8	7.52	7.52
Cases where heuristic soln. is optimal	31 %	30.5	29.2	30.5
Cases with effectiveness $\geq 1.05$	4.8 %	4.8	6.2	4.2

Table 1: Computational results

relaxation of this problem was solved to yield a lower bound on the cost of the optimal solution.

The results show that on an average the heuristic had an solution that was 1.80% higher than the lower bound. In at least 31% of the cases the heuristic yielded the optimal solution, since the linear program had an integer solution which matched the heuristic solution. Only in 4.8% of the cases was the heuristic solution more than 5% larger than the lower bound. The worst divergence obtained was 7.39%. Similar values for the three variants are given in Table 1. All values are in percentage.

The results indicate that all four of the heuristic performs very well compared to the lower bound on the optimal solution. The performance was relatively good even in the worst case that was generated. In about 95% of all cases, the heuristics generate a solution which is within 5% of the lower bound. For the single item dynamic lot sizing problem, the Silver-Meal heuristic has been found to outperform the part-period balancing heuristic in computational tests, even though the former can perform arbitrarily badly in the worst case. In our tests too the Variant 1 performs better than the cost covering heuristic on the average, though the difference is not large. The running time of each heuristic was within 1.5 minutes for all 156 problems on an AT&T 6300 personal computer.

## 7 CONCLUSIONS

In this paper we proved that an existing heuristic for the joint replenishment problem can have arbitrarily bad relative cost in the worst case. On the other hand, the proposed cost covering heuristic not only provides us with a lower bound on the optimal solution, but also has uniformly bounded worst case relative cost. Besides, it is a single pass heuristic, and thus has the dual advantages of speed and low relative cost.

The cost covering heuristic is based on the idea of the joint cost being covered by the candidate items. A portion  $K_0^{it}$  of the joint cost is passed to candidate item  $i$ .

This value can be related to Lagrangean dual variables in the integer programming formulation of the joint replenishment problem [Jon87], and can be used in effective solution of the Lagrangean dual problem. The idea of the joint cost being passed to the individual items has also been found useful in developing similar algorithms for dynamic lot sizing under other product structures, such as linear or assembly structures. This will be an intermediate step towards designing fast heuristics with bounded worst case performance for general product flow structures.

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