ORIENTED MATROIDS AND THE LINEAR COMPLEMENTARITY PROBLEM

By

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ABSTRACT

The linear complementarity problem (LCP) is given a matrix $A \in \mathbb{R}^{n \times n}$ and a vector $b \in \mathbb{R}^n$, find nonnegative $x, y \in \mathbb{R}^n$ such that $y - Ax = b$ and $y^T x = 0$. It is natural to study the problem in the context of oriented matroids, which are combinatorial generalizations of the collection of sign patterns of vectors in subspaces of $\mathbb{R}^n$.

The thesis is divided into four chapters, the first being introductory. The second chapter deals with sets of matrices and oriented matroids generalizing two classes important in linear complementarity theory: the class $P$ of matrices with positive principal minors, and the class of symmetric matrices.

In the third chapter, a systematic procedure based on lexicographic extensions of oriented matroids is given for perturbing a matrix so that it has all nonzero minors or all nonzero principal minors. This corresponds in the matrix case to adding a certain VanderMonde matrix to the original matrix. Relationships among different classes of perturbed oriented matroids are given, and implications for using these perturbations to resolve degeneracy in algorithms for the linear complementarity problem are discussed.

The last chapter is concerned with the $Q$-matrix problem, the problem of characterizing the class of matrices $A$ for which the LCP has a solution for all $b$. It is shown that this is not a property of the oriented matroid associated with the matrix $(I,-A)$, in contrast to the $Q$ and $P$-matrix problems. The $Q$-matrix problem is then studied in the context of oriented adjoints of Bachem and Kern.
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CHAPTER I

INTRODUCTION

The linear complementarity problem is to find \( x \) and \( y \) satisfying

\[
(1) \quad y - Ax = b \\
(2) \quad y \geq 0, x \geq 0 \\
(3) \quad x^T y = 0
\]

for a given \( n \times n \) matrix \( A \) and \( n \)-vector \( b \). This problem has applications in the areas of linear programming, quadratic programming [31] [38], computation of economic equilibria [34], and other fields. The study of the linear complementarity problem in the context of oriented matroids was begun by Todd [43]. In this thesis, we extend his work to cover several topics of current interest to researchers in complementarity theory.

Oriented matroids, introduced independently by Bland and Las Vergnas [7], Folkman and Lawrence [13], and Novoa [21], give a framework for studying the combinatorial properties of the sign patterns of vectors in a vector space. There are two major motivations for studying the linear complementarity problem in the context of oriented matroids. The first is that oriented matroid theory has been shown to give insights into the simplex method for linear programming [6], [44]. Following [43], we attempt to obtain similar insights into Lemke's algorithm for
the linear complementarity problem. The second motivation is that many classes of matrices associated with the linear complementarity problem are defined by properties of signs of subdeterminants, properties that have analogs in an equivalent [28] axiomatization of oriented matroids as abstract assignments of signs to square submatrices. In this context we study the interrelationships among these classes of matrices, without taking other properties of the matrices into consideration.

We will call the system (1)-(3) for a given A and b the LCP (A,b). Several classes of matrices A arising in linear complementarity theory can be defined in terms of signs of subdeterminants. These include P-matrices, which have positive principal minors, and P₀-matrices, which have nonnegative principal minors. On the other hand, some classes, such as symmetric matrices, positive (semi)definite matrices, and copositive plus matrices [31] cannot be characterized by signs of subdeterminants. Based on earlier work on the simplex method for linear programming [6], one might hope that all of the properties of matrices relevant to Lemke's pivoting algorithm are expressable in terms of signs of subdeterminants. In [43], Todd defined the class of P-oriented matroids, which capture the combinatorial properties of sign patterns of vectors in the nullspaces of matrices (I,-A) for P-matrices A. Here we define P₀-oriented matroids and study how this and related classes relate to the convergence properties of Lemke's algorithm. We extend the work of [36], [37] on symmetric oriented matroids, Fiedler and Ptak's [11], [12] results on sign patterns and principal minor signs, Aganagic and Cottle's [1], [2] work on the matrix classes Q n P₀ and Q₀ n P₀, and others. Some of our work gives alternate proofs of known results.
that do not rely on operations such as addition of matrices. On the other hand, this gives our results a wider validity, because there are oriented matroids that are not representable as sign patterns of vectors in a vector space [7], [33]. We give an example of such a nonrepresentable symmetric P-oriented matroid in section II.3. In section II.4, we give an oriented matroid framework for studying factorizations of square matrices, such as LU-factorization and eigenvalue decomposition.

Lemke's algorithm for the linear complementarity problem relies heavily on the use of lexicographic pivoting rules to resolve degeneracy [10], [31]. These techniques can be generalized to oriented matroids, as shown by Las Vergnas [27] and Todd [44]. Chapter III is concerned with applying these techniques to perturbations of the left-hand side of (1) as well as to the right-hand side. The framework of lexicographic extensions of oriented matroids is natural for this study. We study different kinds of perturbations of the left-hand side of (1), and show that various classes of matrices are related by these perturbations. We apply a particular perturbation of the left-hand side to the linear complementarity problem and show that the class of LCP's that Lemke's algorithm will solve can be extended by implicitly solving the problem on the perturbed matrix.

The class of matrices $A$ for which the LCP $(A,b)$ has a solution for all $b$ is the class of $Q$-matrices. It has proved very difficult to obtain a reasonably efficient characterization of these matrices, in spite of the large amount of work devoted to this problem [1], [9], [25], [40], [41]. A class of matrices related to $Q$ is the class $Q_0$
of matrices $A$ for which the LCP $(A,b)$ has a solution for all $b$ such that (1) and (2) have a solution. Doverspike and Lemke [9] conjectured that the class $Q_0$ could be described in terms of signs of subdeterminants of matrices. We use perturbations of a matrix from Kelly and Watson [25] to show that neither $Q$ nor $Q_0$ can be described in terms of signs of subdeterminants of matrices. This suggests that oriented matroids are of limited usefulness in studying the $Q$-matrix problem. However, the sign patterns of vectors in the row space of a matrix related to $A$ determine if a matrix is in $Q$, according to a characterization of $Q$ by Gale [2]. We show that this characterization can be put into the framework of the oriented adjoints of Bachem and Kern [3]. Finally, we study the problem of $Q$-oriented matroids, which is related to the problem of characterizing the matrices $A$ that can be shown to be in $Q$ by considering only the signs of subdeterminants, and give some results for low dimensions.

1. Oriented Matroids

Let $E$ be a finite set. A signed set in $E$ is a pair $X = (X^+, X^-)$, with $X^+ \subseteq E$, $X^- \subseteq E$, and $X^+ \cap X^- = \emptyset$. The opposite of $X$ is the signed set $-X = (X^-, X^+)$, and the set underlying $X$ is $X = X^+ \cup X^-$. A signed set $X$ contains a signed set $Y$, $Y \subseteq X$, if $Y^+ \subseteq X^+$, $Y^- \subseteq X^-$, and $X$ contains an unsigned set $Z$, $Z \subseteq X$, if $Z \subseteq X$.

An oriented matroid $M$ is a pair $(E, C)$, where $E$ is a finite set and $C$ is a collection of signed sets in $E$, called circuits, that satisfies
(C1) \[
\begin{cases}
(\emptyset, \emptyset) \notin C \\
C \in C \Rightarrow -C \in C \\
C_1, C_2 \in C \text{ and } C_1 \subseteq C_2 \Rightarrow C_1 = C_2 \text{ or } C_1 = -C_2
\end{cases}
\]

(C2) \[
\begin{cases}
C_1, C_2 \in C \text{ and } e \in (C_1^+ \cap C_2^-) \cup (C_1^- \cap C_2^+) \text{ imply that there exists } C_3 \in C \text{ with } C_3^+ \subseteq (C_1^+ \cup C_2^+) \setminus \{e\}, \ C_3^- \subseteq (C_1^- \cup C_2^-) \setminus \{e\}.
\end{cases}
\]

We write \( C = C(M) \).

It will be useful for the reader to keep the following example in mind. Let \( V \) be the nullspace of an \( m \times n \) real matrix \( R \). Let \( E = \{e_1, \ldots, e_n\} \) index the columns of \( R \), and for every vector \( x \) of \( V \), define \( S_x = \{e : x_e \neq 0\} \), called the support of \( x \). Let \( C \) be the collection of setwise minimal supports of nonzero vectors in \( V \), i.e. \( C \in C \) iff \( C = S_x \) for some \( x \neq 0 \) in \( V \), and \( S_z \notin C \) for any \( z \neq 0 \) in \( V \). Let \( C \) be the collection of signed sets \( C = (C^+, C^-) \) such that \( C = C^+ \cup C^- = S_x \) for some \( x \in V \), \( C \in C \), and \( C^+ = \{e : x_e > 0\} \), \( C^- = \{e : x_e < 0\} \). Then \( C \) is the collection of circuits of an oriented matroid \( M(R) \), called the oriented matroid represented by \( R \). The vectors \( x \) for \( S_x \in C \) are called elementary vectors of \( V \), and the collection of sets \( C \) is the set of circuits (unsigned) of the matroid underlying \( M \).

Oriented matroids were axiomatized independently by Bland and Las Vergnas [7], Folkman [13], and Novoa [21]. Bland [6] showed that the simplex method of linear programming could be studied in the context of oriented matroids. Todd [43] showed that complementary pivoting
algorithms could be studied in this context. Other concepts from geometry, linear algebra, and topology have been related to oriented matroids, see [3], [5], [29], [33].

Several concepts from matroid theory (see [46]) are important for oriented matroids. Let $C = C(M)$ be the set of circuits of an oriented matroid on $E$. Let $\mathcal{C}$ be the collection of sets underlying the circuits of $C$. Then $\mathcal{C}$ is the set of circuits of $M$, the matroid underlying $M$. A subset $I$ of $E$ is independent in $M$ if there is no $C \in \mathcal{C}$ with $C \subseteq I$. A base of $M$ is a maximal independent subset of $E$. The rank in $M$ of a subset $J$ of $E$ is the cardinality of a maximal independent subset of $J$. The rank of $M$ is the rank of $E$ in $M$. A closed set of $M$ is a maximal subset $F$ of $E$ with a given rank. A hyperplane of $M$ is a closed set $H$ of $E$ with rank one less than the rank of $M$. A loop (coloop) of $M$ is an element $e \in E$ that is a one-element circuit (is in no circuit) of $M$.

We say that oriented matroids $M_1$ and $M_2$ on sets $E_1$ and $E_2$, respectively, are isomorphic if there is a bijection $\phi: E_1 \to E_2$ such that $C(M_2) = \{(\phi(C^+), \phi(C^-)) : C \in C(M_1)\}$.

A signed set $X$ in $E$ has a conformal decomposition into circuits of an oriented matroid $M$ on $E$ if $X^+ = C_1^+ \cup \ldots \cup C_m^+$, and $X^- = C_1^- \cup \ldots \cup C_m^-$ for circuits $C_1, \ldots, C_m$ of $C(M)$. We call such signed sets cycles of $M$. We also say that $(\emptyset, \emptyset)$ is a cycle of $M$. The set of cycles of $M$ is denoted by $K(M)$, and is also called the signed span of the circuits of $M$. The collection $K(M)$ satisfies the property:
\( \begin{align*} K_1, K_2 \in K(M) \text{ and } e \in (K_1^+ \cap K_2^-) \cup (K_1^- \cap K_2^+) \text{ imply that there exists } K_3 \subseteq K(M) \text{ with } e \not\subseteq K_3, \\
( K_1^- \setminus K_2^- ) \cup ( K_2^+ \setminus K_1^+ ) \subseteq K_3^- \\
c( K_1 \setminus K_2 ) \cup ( K_2 \setminus K_1 ) \subseteq K_3^+ \subseteq K_3 \subseteq K_1 \cup K_2. \end{align*} \)

If \( K_1, K_2, K_3 \) are as above, we say that \( K_3 \) is obtained from \( K_1 \) and \( K_2 \) by eliminating \( e \) between \( K_1 \) and \( K_2 \). Given such a \( K_1 \) and \( K_2 \), the \( K_3 \) satisfying (K1) will in general not be unique. The set \( K(M) \) is also closed under Bland's operation of composition:

(K2) If \( K_1, K_2 \subseteq K(M) \), then \( K_1 \circ K_2 \equiv (K_1^+ \cup (K_2^+ \setminus K_1^-), K_1^- \cup (K_2^- \setminus K_1^+)) \subseteq K(M) \).

The operation \( \circ \) is called the composition operator. Note that the circuits of \( M \) are cycles that have minimal underlying sets, so that \( M \) can be recovered from \( K(M) \).

Let \( B \subseteq E \) be a base of \( M \). For every element \( e \in E \setminus B \), there is a circuit \( C \) of \( C \) with \( e \in C^+ \) and \( C \subseteq B \cup e \). This is denoted by \( C = C(B,e) \), called a fundamental circuit with respect to \( B \).

For any oriented matroid \( M \) on \( E \) we can define a family of related oriented matroids, called minors, on subsets of \( E \). For any signed set \( X \) on \( E \) and any subset \( F \) of \( E \), let the signed set \( X \setminus F \) be \( (X^+ \setminus F, X^- \setminus F) \). Now let \( F, G \subseteq E \) satisfy \( F \cap G = \emptyset \). Let \( K \) be the family of cycles of \( M \). Let \( K \setminus F \setminus G \) denote the collection of signed sets \( \{ K \setminus G : K \in K, K \cap F = \emptyset \} \). Then \( K \setminus F \setminus G \) is the set of cycles of an oriented matroid \( M \setminus F \setminus G \) on \( E \setminus (F \cup G) \). For the sake of brevity, \( M \setminus F (M/G) \) is written for \( M \setminus F / \emptyset \) \( (M/ \emptyset /G) \). We say that \( M \setminus F /G \) is
obtained from $M$ by deleting $F$ and contracting $G$. Deletions and contractions may be performed all at once, or sequentially, in any order, always yielding the same result.

Corresponding to every oriented matroid $M$ on a set $E$, there is a uniquely defined dual oriented matroid on $E$, denoted $M^*$. The circuits of $M^*$, called cocircuits of $M$, are signed sets $D$ that satisfy the following orthogonality property

\[
\forall C \in C, \text{ either } \overline{C} \cap D = \emptyset, \text{ or } (C^+ \cap D^+) \cup (C^- \cap D^-) \neq \emptyset \text{ and } (C^+ \cap D^-) \cup (C^- \cap D^+) \neq \emptyset
\]

and have minimal nonempty underlying sets. Further, $(M^*)^* = M$. If $M$ is represented by an $m \times n$ matrix $R$, then the cocircuits of $M$ correspond to elementary vectors of the rowspace of $R$. If $M = M(I, -A)$, then $M^* = M(A^T, I)$. We write $D(M) = C(M^*)$.

The Minty Coloring Property [7] is satisfied by the oriented matroids $M$ and $M^*$. Distinguish an element $e \in E$ and partition $E$ into subsets $e \in R, B,$ and $W$. Then exactly one of the following alternatives holds:

(i) There is a circuit $X$ of $M$ with $e \in \overline{X} \subseteq R \cup B$, $X^- \cap R = \emptyset$

(ii) There is a cocircuit $Y$ of $M$ with $e \in \overline{Y} \subseteq R \cup W$ and $Y^- \cap R = \emptyset$.

Let $M$ be an oriented matroid on $E$, and let $F$ be a subset of $E$. Then $C(F) = \{(C^+ \setminus F) \cup (C^- \cap F), (C^- \setminus F) \cup (C^+ \cap F) : C \in C\}$ is
the set of circuits of an oriented matroid $M_F^-$, obtained from $M$ by reversing signs on $F$.

Up to this point, all of the material of this section can be found in Bland and Las Vergnas [7]. We finish this section with the concept of oriented bases [28], [30],[43]. In the representable case, where $M$ is represented by an $m \times n$ matrix $R$ of full row rank, each ordered subset of $m$ independent columns of $R$ can be given the sign of its determinant. These signs relate to the sign patterns of elementary vectors of the nullspace of $R$ by Cramer's rule. Las Vergnas showed that there are precisely two assignments of signs (one being the opposite of the other) to ordered bases of an oriented matroid satisfying an analogous rule. If $M = M(I,-A)$, the determinants of $n \times n$ submatrices of $(I,-A)$ are determined by the subdeterminants of $A$. Thus the study of oriented matroids represented by matrices $(I,-A)$ is equivalent to the study of collections of signs of subdeterminants of square matrices.

2. Extensions

Let $M$ be an oriented matroid on $E$, and let $p \notin E$. Let $D$ be the set of cocircuits of $M$. If $\hat{M}$ on $E \cup p$ satisfies $\hat{M} \setminus p = M$, then $\hat{M}$ is called a point extension of $M$. We confine ourselves to point extensions of $M$ that have rank equal to that of $M$. Las Vergnas [27] showed that every such extension is determined by a partition $(y^+, z, y^-)$ of $D$, such that $D \in y^+$ iff $-D \in y^-$, and
(i) $Z$ is line-closed, i.e. if $D_1, D_2 \in Z$, $D_1 \neq \pm D_2$, and 
\[ \text{rank}(E \setminus (D_1 \cup D_2)) = \text{rank}(E) - 2, \] 
then every $D$ with $D \cap \overline{(D_1 \cup D_2)}$ is in $Z$.

(ii) If $D_1, D_2 \in V^+$, and $D_3$ is obtained from $D_1, D_2$ by axiom (C2) 
then $D_3 \in V^+$.

Las Vergnas showed that such a partition $(V^+, Z, V^-)$ corresponds 
to a unique oriented matroid $\hat{M}$ on $E \cup p$ with cocircuits $\hat{D}$, so 
that there is a $\hat{D} = (D^+ \cup p, D^-)$ for every $D \in V^+$, a $\hat{D} = (D^+, D^-)$ 
for every $D \in Z$, and a $\hat{D} = (D^+, D^- \cup p)$ for every $D \in V^-$. 

Lexicographic extensions were also introduced by Las Vergnas.

Suppose we are given an independent set $\{e_1, \ldots, e_k\}$ in $E$, and 
$p \notin E$. Then there is a unique extension $\hat{M}$ on $\hat{E} = E \cup p$ with 
cocircuits $\hat{D}$ satisfying 

(i) $\hat{D} \subseteq \hat{D}$ if $\hat{D} \in \hat{D}$ and $\{e_1, \ldots, e_k\} \cap \hat{D} = \emptyset$, and 

(ii) $\hat{D} \subseteq \hat{D}$ if $\hat{D}\setminus\{p\} = D \in \hat{D}$ and $\{e_1, \ldots, e_k\} \cap D \neq \emptyset$, with $p$ 

appearing in $\hat{D}$ with the same sign as the first $e_i$ in $D$.

Moreover, each cocircuit $\hat{D}$ of $\hat{M}$ containing $p$ is of the latter form.

The following definition is due to Todd [43].

Definition 2.1. Suppose $\hat{M}$ arises as above from $M$. Then, if $k > 0$, 
we say $p = \text{lex}(e_1, \ldots, e_k)$ extends $M$ to $\hat{M}$ and call $\hat{M}$ a lexicographic extension of $M$. If $k > 0$, and $\hat{M}$ is obtained from $\hat{M}$ by 
reversing the sign of $p$, we say $p = -\text{lex}(e_1, \ldots, e_k)$ extends $M$ to 
$\hat{M}$. Similarly, if we reverse the sign of $e_1$ in $M$ to get $\hat{M}$, then 
extend $\hat{M}$ to $\hat{M}$ as above, then reverse the sign of $e_1$ to get $\hat{M}$, 
we say $p = \text{lex}(-e_1, e_2, \ldots, e_k)$ extends $M$ to $\hat{M}$, and so on.
In [44], Todd gave a characterization of the circuits of a lexicographic extension of an oriented matroid. We will restate his characterization with new notation.

**Definition 2.2.** Let $M$ be an oriented matroid on $E$. Let $B$ be a base of $M$, and let $e \in E$. Then

$$ \mathcal{G}(B,e) = \begin{cases} C(B,e) \setminus e & \text{if } e \notin B \\ (\emptyset, e) & \text{if } e \in B. \end{cases} $$

**Theorem 2.3 [44].** Let $M$ be an oriented matroid on $E$, and let $p = \text{lex}(e_1, \ldots, e_k)$ extend $M$ to $\hat{M}$. For every base $B$ of $M$, we have $\hat{C}(B,p) = (p, \emptyset) \circ \mathcal{G}(B,e_1) \circ \cdots \circ \mathcal{G}(B,e_k)$.

The next theorems come from [43].

**Theorem 2.4.** If $p = \text{lex}(e_1, \ldots, e_k)$ extends $M$ to $\hat{M}$, then every circuit $\hat{C}$ of $\hat{M}$ containing $p$ contains at least $k+1$ elements.

**Theorem 2.5.** If $p = -\text{lex}(e_1, \ldots, e_k)$ extends $M$ to $\hat{M}$, then there is a positive circuit $\hat{C}$ of $\hat{M}$ with $\hat{C} = \{p, e_1, \ldots, e_k\}$.

In the representable case, when $e_1, \ldots, e_k$ index columns $r_1, \ldots, r_k$ of a matrix $R$ representing $M$, a lexicographic extension $\text{lex}(e_1, \ldots, e_k)$ corresponds to adding a column $\varepsilon r_1 + \varepsilon^2 r_2 + \cdots + \varepsilon^k r_k$ to the matrix, for sufficiently small $\varepsilon > 0$. 
3. Sphere Systems and the Point Picture

Suppose $M$ on $E$ is represented by an $m \times n$ matrix $R = [r_1, \ldots, r_n]$. The cycles of $M^*$ are signed sets $K_x = \{e: x_e > 0\}, \{e: x_e < 0\}$ for vectors $x$ in the rowspace of $R$. For simplicity, suppose that $M$ has no loops. Let $S^{m-1} = \{z \in \mathbb{R}^m: z_1^2 + \ldots + z_m^2 = 1\}$. For every $i = 1, \ldots, n$, the set $\{z \in S^{m-1}: z^T r_i = 0\}$ is an image of $S^{m-2}$ on $S^{m-1}$ that divides $S^{m-1}$ into three regions:

$H_i = \{z \in S^{m-1}: z^T r_i = 0\}$, $N_i = \{z \in S^{m-1}: z^T r_i < 0\}$, and $P_i = \{z \in S^{m-1}: z^T r_i > 0\}$. We call these regions respectively the $i$ hyperplane, the negative side of $i$, and the positive side of $i$.

The collection of hyperplanes corresponding to elements of $E$ thus partitions $S^{m-1}$ into a collection of regions that are determined by their positions relative to these hyperplanes. For every such region $U$, we associate the signed set $\{(i: U \subseteq P_i), (i: U \subseteq N_i)\}$. This collection of signed sets is the family of cycles of $M^*$. In [13] and [33] it is proved that every oriented matroid can be thought of as a set of regions on a sphere $S^{m-1}$ that is divided up by homeomorphic images of $S^{m-2}$, provided that certain intersection properties hold among the images of $S^{m-2}$. For non-representable oriented matroids, these "hyperspheres" need not be intersections of $S^{m-1}$ with hyperplanes of $\mathbb{R}^m$. We will refer to this interpretation as the sphere system picture. We give an example in Figure 1.1 of a rank three oriented matroid $M$ on $E = \{e_1, \ldots, e_4\}$, where the elements $(e_1, \ldots, e_4)$ correspond to the great circles $(r_1, \ldots, r_4)$ on the sphere $S^2$ in Figure 1.1. The cocircuits of $M$ correspond to the maximal intersections of the circles $(r_1, \ldots, r_4)$. 
For example, points A, B, and C represent the cocircuits \((\{e_2, e_4\}, \emptyset)\), \((\{e_1, e_3\}, \emptyset)\), \((\{e_3\}, \{e_4\})\). For each element \(r_i\), the arrow points to the positive side of \(i\). The negatives of these cocircuits are represented by the intersections of the corresponding circles on the opposite side of \(S^2\).

Another geometrical interpretation of representable oriented matroids is more prevalent in the linear complementarity literature. Here we have an oriented matroid \(M\) (of rank \(m\)) on \(E\), represented by an \(m \times n\) matrix \(R = [r_1, \ldots, r_n]\), and we associate to each column of \(R\) the corresponding point in \(\mathbb{R}^m\). Every hyperplane of \(M\) defines a hyperplane of \(\mathbb{R}^m\) containing the points corresponding to
the hyperplane of \( M \). For every hyperplane of \( M \), there is a pair of cocircuits \( D_1, D_2 (D_1 = -D_2) \) of \( M \) such that the points of \( D_1^+ \) are separated from those of \( D_1^- \) by this hyperplane.

If we scale the columns of \( R \) by positive numbers and we assume that \( M \) has no loops (\( R \) has no zero columns), the points representing elements of \( E \) can be taken to be on \( S^{m-1} \). The collection of hyperplanes of \( \mathbb{R}^m \) corresponding to hyperplanes of \( M \), together with an assignment to each hyperplane of a negative side and a positive side, defines a sphere system picture of another oriented matroid, called an adjoint of \( M \). This concept first arose in [5]. The definition of oriented adjoints was given by Bachem and Kern [3].

**Definition 3.1 [3].** Let \( M \) and \( \tilde{M} \) be two oriented matroids on \( E \) and \( \tilde{E} \) respectively, without loops and of the same rank. Then \( \tilde{M} \) is called an adjoint of \( M \) if there are maps

\[
\phi: \tilde{E} \to D(M) \quad \text{and} \quad \psi: E \to D(\tilde{M})
\]

such that for all elements \( e \in E \) and \( \tilde{e} \in \tilde{E} \)

\[(1) \quad e \in \phi(\tilde{e})^\pm \quad \text{if and only if} \quad \tilde{e} \in \psi(e)^\pm \]

holds and the induced map \( \phi \) mapping points (sets of rank one) of \( \tilde{M} \) onto hyperplanes \( H = E \backslash \phi(\tilde{e}) \) (\( \tilde{e} \in \tilde{E} \)) is bijective.
We illustrate this by a rank three example in Figure 1.2. Let $R$ be a matrix representing the oriented matroid $M$ on $E = \{e_1, \ldots, e_4\}$, such that $\|r_i\| = 1$ for $i = 1, \ldots, 4$. Let $\tilde{r}_1, \ldots, \tilde{r}_6 \in \mathbb{R}^3$ give equations defining hyperplanes of $\mathbb{R}^3$ that contain pairs $(r_i, r_j)$, and let $\tilde{\mathcal{M}}$ be the oriented matroid on $\tilde{E} = \{\tilde{e}_1, \ldots, \tilde{e}_6\}$ represented by the matrix $\tilde{R}$ with columns $\tilde{r}_i$, $i = 1, \ldots, 6$. For $i = 1, \ldots, 6$, define $\phi(\tilde{e}_i) = D \in \mathcal{D}(M)$ such that $e_j \in D^+$ iff $\tilde{r}_i^T r_j > 0$, $e_j \in D^-$ iff $\tilde{r}_i^T r_j = 0$. Also, for $i = 1, \ldots, 6$ define $\psi(e_i) = \tilde{D} \in \mathcal{D}(\tilde{M})$ such that $\tilde{e}_j \in \tilde{D}^+$ iff $\tilde{r}_j^T \tilde{r}_i > 0$, $\tilde{e}_j \in \tilde{D}^-$ iff $\tilde{r}_j^T \tilde{r}_i < 0$. Then (1) is trivially satisfied, and it is easy to check that $\psi$ is bijective. Thus $\tilde{\mathcal{M}}$ is an adjoint of $M$.

![Figure 1.2. Point picture with adjoint](image)
Certain non-representable oriented matroids, such as orientations of the Vamos matroid [7], do not have oriented adjoints. Concepts such as the line joining two elements of $M$, often used in linear complementarity theory, have little meaning unless an adjoint of $M$ is specified. On the other hand, if an oriented matroid has an adjoint it may have many different adjoints.

4. Oriented Matroids and Linear Complementarity

In the study of the linear complementarity problem, we deal with matrices of the form $[I,-A]$, where $A$ is an $n \times n$ matrix. The following definition gives an oriented matroid framework for studying such matrices. We follow the notation of Todd [43].

**Definition 4.1.** An oriented matroid on $E$ is called **square** (with rank $n$) if $E = S \cup T$, $S \cap T = \emptyset$, $S = \{s_1, ..., s_n\}$, $T = \{t_1, ..., t_n\}$, and $S$ is a base of $M$.

In the representable case, if $M$ is represented by $[I,-A]$, then $S$ indexes the columns of $I$, while $T$ indexes the columns of $A$.

**Definition 4.2.** Let $M$ be a square oriented matroid on $E = S \cup T$. A subset $X$ of $E$ is called **complementary** if $|\{s_i, t_i\} \cap X| \leq 1$ for all $i = 1, ..., n$. A circuit $C \in C(M)$ is called complementary if its underlying set is complementary.

**Definition 4.3.** Let $M$ be a square oriented matroid on $E$, and let $\hat{M}$ be a point extension of $M$ on $\hat{E} = E \cup p$. We say that a circuit $\hat{C} \in C(\hat{M})$ solves the linear complementarity problem of $\hat{M}$ if $\hat{C}$ is positive, complementary, and contains $p$. 
We now give Todd's extension of Lemke's algorithm for oriented matroids. Let \( M \) be a square, rank \( n \), oriented matroid on \( E = S \cup T \), let \( p \) extend \( M \) to \( \hat{M} \) on \( \hat{E} = E \cup p \), and let \( q \) extend \( \hat{M} \) to \( \hat{\hat{M}} \) on \( \hat{\hat{E}} = E \cup p \cup q \), such that every circuit of \( \hat{\hat{M}} \) containing \( q \) has \( n+1 \) elements and \( C(S,q) \) is positive. Call a circuit \( \hat{C} \) of \( \hat{\hat{M}} \) special if it is positive, complementary, and includes \( p \).

**Definition 4.4 [43].** Two special circuits are **adjacent** if the union of their underlying sets is complementary. A special circuit \( \hat{C}_k \) is the endpoint of a \( q \)-ray \( \hat{C} \) if both are positive circuits of \( \hat{\hat{M}} \), \( p \in \hat{C}_k \setminus \hat{C} \), \( \hat{C} \cap T \neq \emptyset \), and \( \hat{C}_k \cup \hat{C} \) is complementary.

Let \( \hat{C}_p \) and \( \hat{C}_q \) be the fundamental circuits of \( \hat{\hat{M}} \) associated with the base \( S \) and \( p \) and \( q \) respectively. Note that \( \hat{C}_p \) solves the LCP \( \hat{M} \) if \( \hat{C}_p^- = \emptyset \). If not, let \( \hat{C}_1 \) be the unique (see [43]) positive circuit of \( \hat{\hat{M}} \) in \( S \cup p \cup q \) that is not \( \hat{C}_q \).

**Theorem 4.5 [43].** Suppose \( \hat{C}_p^- \neq \emptyset \). Then there is a chain \( \hat{C}_1, \hat{C}_2, \ldots, \hat{C}_k \) of distinct special circuits of \( \hat{\hat{M}} \) with each consecutive pair adjacent and \( \hat{C}_k \) the endpoint of a \( q \)-ray or a positive complementary circuit of \( \hat{\hat{M}} \) containing \( p \).

Thus Lemke's algorithm traces through this sequence of circuits. Todd [43] showed that in the special case of SSM-oriented matroids (see [43] or ch. III), no \( q \)-rays existed, and thus \( \hat{C}_k \) solves the LCP associated with \( M \).

In the representable case, adjacent special circuits correspond to adjacent basic feasible solutions and \( q \)-rays correspond to secondary rays (see [31]).
5. Overview of the Thesis

The rest of this thesis is divided into three chapters. Chapter II is concerned with classes of oriented matroids related to matrices that arise in quadratic programming problems. The chapter is divided into four sections.

The first section is concerned with symmetric oriented matroids. This extends the work of [36] and [37], in which these oriented matroids were shown to capture many important properties of symmetric matrices. An equivalent definition to the one given in [36] is derived, using the orientations of bases of oriented matroids [28]. Symmetric oriented matroids related to the weakly oriented matroids of Bland and Jensen [23] are given, and two problems on representations of symmetric oriented matroids are discussed.

Section 2 introduces the class of $P_0$-oriented matroids. This is a generalization of the class of $P$-oriented matroids of [43]. We show the equivalence between a characterization of $P_0$-oriented matroids in terms of their circuits and a characterization in terms of the orientations of their bases. An analogous theorem for $P_0$-matrices was proved by Fiedler and Ptak [12], but their proof uses addition of matrices, an operation which has no analog in oriented matroids. It is known [10] that Lemke's algorithm will not solve all LCP's $(A,b)$ with $A$ a $P_0$-matrix. We study several classes of oriented matroids contained in the class $P_0$. In particular, we note that the work of Aganagic and Cottle on the matrix classes $Q \cap P_0$ and $Q_0 \cap P_0$ extends to oriented matroids. We define the classes of $P_1$-oriented matroids from [36] and
[37], which attempt to capture properties of positive semidefiniteness, and adequate oriented matroids, analogous to the adequate matrices of Ingleton [22].

Section 3 uses Mandel's [33] surgery technique to create an example of a nonrepresentable symmetric P-oriented matroid. Nonrepresentability is proved as in [15] and [33], by setting up a "linear program" on the oriented matroid, and showing that it produces a monotonic cycle of pivots, some of which are nondegenerate.

Factorization theorems from linear algebra are introduced in section II.4. A definition of oriented matroid products is given, and in this framework we derive analogs of LU-factorizations and eigen-decompositions [18].

Part of section 4 is devoted to investigating a very natural question for oriented matroid theory, which comes from the framework of oriented matroid products. The problem is: Given $M$ on $B \cup G \cup H$, where $B$ is a base of $M$, $C = G \cup H$, $C^- = \emptyset$, do there exist $K_1, K_2 \in K(M)$ such that $K_1 \subseteq B \cup G$, $K_2 \subseteq B \cup H$, $G \subseteq K_1^+$, $H \subseteq K_2^+$, $K_1 \setminus G = -K_2 \setminus H$? We only have partial answers to this "reverse elimination" property.

Chapter III is concerned with lexicographic perturbations of oriented matroids. Such a perturbation is defined by extending an oriented matroid on $S \cup T$, where $|S| = m$, $|T| = n-m$, $S$ is a base of $M$, by $t_i = \text{lex}(t_i, -s_{i_1}, \ldots, -s_{i_k})$ for some $i \in \{1, \ldots, n-m\}$, \(\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}\), and then deleting $t_i$. We show that it is not necessary to assume that $t_i$ is independent of $\{s_{i_1}, \ldots, s_{i_k}\}$,
as in the definition of lexicographic extensions [27]. We sequentially perturb \( t_i \) by \( \hat{t}_i = \text{lex}(t_i, -s_1, \ldots, -s_n) \) for \( i = 1, \ldots, n-m \), and call this a Vandermonde perturbation. We show how the circuits of such a perturbation can be calculated. Vandermonde perturbations are then applied to the linear complementarity problem, and it is shown that they can be used to solve some LCP's \((A, b)\) for \( A \) semimonotone (see [24]) for which Lemke's algorithm applied to \((A, b)\) will fail. Interrelationships among several classes of oriented matroids based on perturbation properties are shown.

In chapter IV we study the Q-matrix problem. A matrix \( A \) is in the class \( Q \) if the LCP \((A, b)\) has a solution for all \( b \). The class \( Q_0 \) is the class of matrices \( A \) such that the LCP \((A, b)\) has a solution whenever the system \( y - Ax = b, \ y \geq 0, \ x \geq 0 \) has a solution. Much work has gone into trying to obtain efficient characterizations of these matrix classes [1], [9], [25], [40], [41]. The characterization due to Gale [2] is highly inefficient to check. In [9], Doverspike and Lemke introduced the superclass \( M^0 \) of classes of square matrices. A class \( X \) is in \( M^0 \) if and only if two matrices \( A_1 \) and \( A_2 \) having common signs of corresponding subdeterminants are either both in \( X \) or neither in \( X \). They conjectured that \( Q_0 \in M^0 \). We show that \( Q \notin M^0, \ Q_0 \notin M^0 \), based on a perturbation of a matrix of Kelly and Watson [25]. We then derive an oriented matroid analog of Gale's characterization of the matrix classes \( Q \) and \( Q_0 \), based on the oriented matroid adjoints of Bachem and Kern [3]. The final section of chapter IV is concerned with the Q-oriented matroid problem. Here we
define an extension $\hat{M}$ of a square oriented matroid $M$ to be feasible if it has a positive circuit containing the new element, and to have a complementary solution if it has such a circuit which is complementary. Then we call $M$ a Q-oriented matroid if every extension has a complementary solution and a $Q_0$-oriented matroid if this holds for every feasible extension. From the work of Todd [43], we know that the class of Q-oriented matroids includes the $SSM$-oriented matroids, for which there is a complementary base $B$ such that when $t_i$'s in $B$ are renamed $s_i$'s, the result is an SSM (strictly semi-monotone)-oriented matroid. It is shown that this characterizes rank two Q-oriented matroids, and that the class of $SSM$-oriented matroids is a proper subset of the rank three Q-oriented matroids.

The sections within each chapter are labelled 1, 2, etc., and the results within them 1.1, 2.3, etc. A result in another chapter is denoted II.3.4 for the fourth result in section 3 of chapter II.
CHAPTER II
SYMMETRY AND POSITIVE DEFINITENESS

This chapter is concerned with several classes of matrices and oriented matroids related to symmetry and positive definiteness, building on the work done in [36]. These matrices and oriented matroids arise in linear complementarity problems associated with quadratic programming.

Given \( G \in \mathbb{R}^{n \times n}, D \in \mathbb{R}^{m \times n}, c \in \mathbb{R}^n, b \in \mathbb{R}^m \), we want \( u, y \in \mathbb{R}_+^m, v, x \in \mathbb{R}_+^n \) such that

\[
\begin{pmatrix}
    u \\
    y
\end{pmatrix}
- \begin{pmatrix}
    O & -D \\
    D^T & G
\end{pmatrix}
\begin{pmatrix}
    v \\
    x
\end{pmatrix}
= \begin{pmatrix}
    b \\
    -c
\end{pmatrix},
\quad u^T y = 0, v^T x = 0.
\]

The usual hypothesis on \( G \) is that it is symmetric and positive semidefinite. This implies that \( \begin{pmatrix}
    O & -D \\
    D^T & G
\end{pmatrix} \) is positive semidefinite.

The chapter is divided into four sections. In section 1, symmetric oriented matroids are defined. Theorem 1.5 characterizes symmetric oriented matroids in terms of the orientations of their bases. Properties of matrices \( A \) for which \( M(I, -A) \) is symmetric are discussed. Section 2 surveys the class of \( P(P_0) \)-oriented matroids. A matrix \( A \) is a \( P(P_0) \) matrix if its principal minors are positive (non-negative). Several classes of matrices within the class \( P_0 \) have been studied (see [10]). Oriented matroid analogs for these classes, and the relationships among them, are studied. In particular, their relevance to the solvability of associated linear complementarity problems is considered, related to the work of Aganagic and Cottle [1], [2].

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Mandel's [33] surgery technique is used in section 3 to create an example of a symmetric P-oriented matroid that is non-representable. This emphasizes the fact that the results of this chapter are proper extensions of the theory of P-matrices.

Section 4 examines the extent to which decomposition of square matrices (e.g. LU, Cholesky) is possible in the oriented matroid setting.

1. Symmetry

Assume that \( M \) is a square (see chapter I) oriented matroid of rank \( n \) on \( E = S \cup T \).

**Definition 1.1.** The switch, \( swX \), of a signed set \( X \) in \( E \), is obtained by changing signs on \( X \cap T \), followed by interchanging occurrences of \( t_i \)'s and \( s_i \)'s in their appearances in the signed set. The switch \( swY \) of an unsigned set \( Y \) in \( E \) is obtained by interchanging occurrences of \( t_i \)'s and \( s_i \)'s in the set.

**Definition 1.2** [36]. A square oriented matroid \( M \) is called symmetric if the oriented matroid \( swM \), obtained from \( M \) by replacing each circuit of \( M \) with its switch, is the dual of \( M \).

In the case where \( M \) is representable, e.g. \( M = M(I,-A) \) for some square matrix \( A \), we have \( swM = M(A,I) \). It is well known ([7]) that \( M(A^T,I) \) is the dual of \( M \). Thus \( M(I,-A) \) will be a symmetric oriented matroid when \( A \) is symmetric.
1.1 Symmetry and the orientation of the bases of $M$

The first goal of this section is to characterize symmetric oriented matroids in terms of the orientations of their bases.

**Theorem 1.3** [28]. Let $M$ be an oriented matroid. There exists a mapping $\varepsilon$ assigning values 1 or -1 to orderings of bases of $M$ such that given bases $B, B'$ of $M$ and orderings $\beta, \beta'$ of $B, B'$ then

(i) if $B = B'$, then $\varepsilon(\beta) = \varepsilon(\beta')$ iff $\beta$ and $\beta'$ can be obtained from each other by an even permutation.

(ii) if $\beta$ and $\beta'$ agree in all but one position, then $\varepsilon(\beta) = \varepsilon(\beta')$ iff the two elements of $B \Delta B'$ appear in the circuit $C$ with $C \subseteq B \cup B'$ with opposite signs.

Furthermore, a mapping $\varepsilon$ with this property is unique up to multiplication by -1 and determines uniquely the orientation of $M$.

Such an $\varepsilon$ is called an **orientation** of the bases of $M$. In the following, it will be useful to extend $\varepsilon$ by defining $\varepsilon(\beta) = 0$ for all orderings $\beta$ of $n$-sets $B$ that are not bases of $M$ ($n$ is the rank of $M$). Then, for every $n$-set $B$, define $\bar{\varepsilon}(B) = \varepsilon(\beta)$, where $\beta$ is $B$ ordered as in a given natural ordering of $E$. For $E = S \cup T$, the natural ordering is taken as $(s_1, \ldots, s_n, t_1, \ldots, t_n)$.

**Definition 1.4.** Let $E = S \cup T$, where $|S| = |T| = n$, and $S, T$ are disjoint. For any $n$-set $B = S_J \cup T_K$ contained in $E$, let $n(B) = \sum_{i=1}^{j} (J_i - i)$, where $j = |J|$, $J = \{J_1, \ldots, J_j\}$. 
Theorem 1.5. \( M \) is symmetric iff 
\[
\varepsilon(B)(-1)^{\eta(B)} = \varepsilon(swB)(-1)^{\eta(swB)}
\]
for all \( n \)-sets \( B \).

Consider the representable case. Given a square submatrix 
\(-A_{N\setminus J,K}\) of \(-A\), \( \det(-A_{N\setminus J,K}) \) is equal to the determinant of the \( n \times n \) 
submatrix of \( (I,-A) \) indexed by \( B = S_J \cup T_K \), multiplied by \( (-1)^{\eta(B)} \).
In the symmetric case, we would like \( \det(-A_{N\setminus J,K}) \) to have the same 
sign as \( \det(-A_{K,N\setminus J}) \), which is related to the determinant of the \( n \times n \) 
submatrix of \( (I,-A) \) indexed by \( S_{N\setminus K} \cup T_{N\setminus J} = swB \).

Proof. The "only if" part of the theorem is proved first. The next 
result shows that the conclusion of the theorem holds if \( B \) is not a 
base.

Lemma 1.6. Let \( M \) be symmetric. Then an \( n \)-set \( B \) is a base of \( M \) 
iff \( swB \) is a base of \( M \).

Proof. If \( B \) is a base of \( M \), then \( swB \) is a cobase, by symmetry.
Thus \( swB \) is a base of \( M \). The converse is the same since 
\( B = sw(swB) \). \( \square \)

Now let \( B = S_J \cup T_K \) be a base of \( M \). We use induction on \( |B \cap T| \).
For \( |B \cap T| = 0 \), we have \( B = swB = S \), and the implication is trivially 
true. If \( |B \cap T| > 1 \), there is an \( m \) such that \( s_m \notin B \). Let \( \ell \) be 
such that \( B' = (B \setminus t_\ell) \cup s_m \) is a base of \( M \). Then 
\( swB' = ((swB) \setminus t_m) \cup s_\ell \).
\( C(B,s_m) \) and \( sw(C(swB),s_\ell) \) meet on \( t_\ell \) and \( s_m \) only.

Define \( \sigma(B,B') = 1 \) if \( s_m \) and \( t_\ell \) agree in sign in \( C(B,s_m) \) 
and \( \sigma(B,B') = 0 \) otherwise. Define \( \sigma(swB,swB') \) similarly. By
symmetry of \( M \), which implies orthogonality of \( C(B, s_m) \) and 
\( sw(C(swB, s_x)) \), we get \( \sigma(B, B') = \sigma(swB, swB^T) \).

Define \( \rho(B, B') \) to be the number of interchanges it takes to 
move \( t_x \) from its natural position in \( B \) so that the resulting 
ordering of \( B \) differs from the natural ordering of \( B' \) in exactly 
the one position where \( B \) has \( t_x \). Define \( \rho(swB, swB^T) \) similarly.

Then

\[
\rho(B, B') = |J| + |\{k \in K: k < x\}| - |\{j \in J: j < m\}|.
\]

\[
\rho(swB, swB^T) = |N \setminus K| + |\{i \in N \setminus J: i < m\}| - |\{i \in N \setminus K: i < x\}|.
\]

Noting that \( |J| = |N \setminus K| \), we get that \( \rho(B, B') - \rho(swB, swB^T) = 
(x-1) - (m-1) = x-m. \)

For \( J' = J \cup m \), inspection of the formula for \( \eta(B) \) gives
\[
\eta(B') = \sum_{i=1}^{J'} (J'_i - i) = \sum_{i=1}^{J} (J_i - i) + m - |J| - 1. \quad \text{Thus} \quad \eta(B') - \eta(swB^T) =
\eta(B) + m - |J| - 1 = \eta(swB) + m - x + 1 + |N \setminus K| = \eta(B) - \eta(swB) + m - x.
\]

The induction hypothesis gives that \( \varepsilon(B')(-1)\eta(B') = 
\varepsilon(swB^T)(-1)^{\eta(swB^T)}. \) Las Vergnas' theorem 1.3 tells us that
\( (-1)\sigma(B, B') = \varepsilon(B)\varepsilon(B')(-1)\rho(B, B'). \)

Now we combine the last few results.

\[
\varepsilon(B)(-1)\eta(B) = \varepsilon(B')(-1)\rho(B, B')(-1)\sigma(B, B')(-1)\eta(B') + \eta(swB) - \eta(swB^T) - (m-x)
= \varepsilon(swB^T)(-1)\rho(swB, swB^T)(-1)\sigma(swB, swB^T)(-1)\eta(swB)
= \varepsilon(swB)(-1)\eta(swB).
\]

Thus the "only if" part of theorem 1.5 is proved.
For the converse, suppose that \( M \) is square and for every \( n \)-set \( B \) we have \( \varepsilon(B)(-1)^n(B) = \varepsilon(swB)(-1)^n(swB) \), where \( \varepsilon \) is an orientation of the bases of \( M \) that determines \( M \), as in theorem 1.3. Then since \( \varepsilon(B) \) is nonzero iff \( \varepsilon(swB) \) is, the underlying matroids of \( swM \) and \( M^* \) are the same. Let \( \varepsilon \perp \) be an orientation of the bases of \( M^* \), the dual of \( M \), that determines \( M^* \). Let \( \varepsilon \perp \) be the corresponding orientation of \( n \)-sets of \( E \) ordered naturally in \( E \). Las Vergnas [28] showed that \( \varepsilon \) and \( \varepsilon \perp \) must satisfy the following property.

**Theorem 1.7 [28].** Given two bases \( B_1, B_2 \) of \( M \), \( \varepsilon(B_1)\varepsilon(B_2) = \varepsilon \perp(B_1)\varepsilon \perp(B_2)(-1)^a \), where \( a \) is the number of elements of \( B_1 \Delta B_2 \) with odd rank in the natural ordering of \( E \).

**Lemma 1.8.** The following rule gives an orientation of the bases of \( swM \), ordered naturally in \( E \): \( \varepsilon_{sw}(swB) = \varepsilon(B)(-1)^{|B \cap S|} \) if \( n \) is odd, \( \varepsilon_{sw}(swB) = \varepsilon(B) \) if \( n \) is even.

**Proof.** An orientation of the bases of \( M \) is clearly given by the rule: \( \varepsilon(S)(B) = \varepsilon(B)(-1)^{|B \cap S|} \). For \( B \) a base of \( M \), we need to exchange occurrences of \( s_i \)'s with \( t_i \)'s and vice versa. This gives \( \varepsilon_{sw}(swB) = \varepsilon(S)(B) \) ordered as in the vector \((t_1, \ldots, t_n, s_1, \ldots, s_n)\). It takes \(|B \cap T|(|B \cap S|)\) transpositions to order this as in \((s_1, \ldots, s_n, t_1, \ldots, t_n)\). Note that \(|B \cap T| + |B \cap S| = n\), so \(|B \cap T|(|B \cap S|)\) agrees with \(|B \cap S|\) in parity if \( n \) is even, and is always even when \( n \) is odd. In the first case, this cancels the \((-1)^{|B \cap S|}\) obtained from switching signs on \( S \), in the second case it does not, so lemma 1.8 is proved. \( \square \)
Theorem 1.5 will now be proved if we can show that $\overline{\epsilon}$ and $\overline{\epsilon}_{SW}$ satisfy Las Vergnas' property of theorem 1.7.

Let $B_1 = S_{J_1} \cup T_{K_1}$ and $B_2 = S_{J_2} \cup T_{K_2}$ be bases of sw$M$. Assume that $n$ is odd. Then

$$
\overline{\epsilon}_{SW}(swB_1) \overline{\epsilon}_{SW}(swB_2) \\
= \overline{\epsilon}(B_1)(-1) \left| B_1^n \right| \overline{\epsilon}(B_2)(-1) \left| B_2^n \right| \\
= \overline{\epsilon}(B_1) \overline{\epsilon}(B_2)(-1) \left| (B_1 \Delta B_2)^n \right| \\
= \overline{\epsilon}(swB_1) \overline{\epsilon}(swB_2)(-1) \left[ \eta(swB_1) + \eta(swB_2) - \eta(B_1) - \eta(B_2) + \left| (B_1 \Delta B_2)^n \right| \right] \\
= \overline{\epsilon}(swB_1) \overline{\epsilon}(swB_2)(-1) \eta(swB_1) + \eta(swB_2) - \eta(B_1) - \eta(B_2) + \left| (B_1 \Delta B_2)^n \right| \left| B_1 \right|
$$

(1)

by the hypothesis. We analyze the exponent from (1).

$$
\eta(swB_1) + \eta(swB_2) - \eta(B_1) - \eta(B_2) + \left| (B_1 \Delta B_2)^n \right| \\
= \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left[ (N \setminus K_1)_i - i \right] + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left[ (N \setminus K_2)_i - i \right] - \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (J_{1i} - i) - \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (J_{2i} - i) \\
+ \left| (B_1 \Delta B_2)^n \right| \\
= \left[ \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (N \setminus K_1)_i + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (N \setminus K_2)_i \right] - \left[ \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} J_{1i} + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} J_{2i} \right] + \left| (B_1 \Delta B_2)^n \right|.
$$

The term $a_1 = [\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (N \setminus K_1)_i + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (N \setminus K_2)_i]$ has the same parity as the number of elements of $(N \setminus K_1) \Delta (N \setminus K_2) = K_1 \Delta K_2$ with odd rank in the natural ordering of $N = (1, \ldots, n)$. The term $a_2 = [\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} J_{1i} + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} J_{2i}]$ has the same parity as the number of elements of $J_1 \Delta J_2$ with odd rank in the natural ordering of $N$. Then $\left| (B_1 \Delta B_2)^n \right| = \left| (swB_1 \Delta swB_2)^n \right| T_1$,
and \(|(swB_1 \Delta swB_2) \cap T| - a_2\) has the same parity as the number of elements of \(J_1 \Delta J_2\) with even rank in the natural ordering of \(N\). Since \(n\) is odd, the exponent in (1) is \(a_1 + (|(swB_1 \Delta swB_2) \cap T| - a_2)\) which has the same parity as the number of elements of \(swB_1 \Delta swB_2\) with odd rank in the natural ordering of \(E\).

When \(n\) is even, the proof is simpler. In that case, the term \(|(B_1 \Delta B_2) \cap S|\) does not occur in the exponent from (1). Then an element \(k\) of \(K\) has odd rank in the natural ordering of \(N\) iff \(t_k\) has odd rank in the natural ordering of \(E\), so it is easy to see that \(a_1 - a_2\) must agree in parity with the number of elements of \((swB_1 \Delta swB_2)\) with odd rank in the natural ordering of \(E\).

Thus \(e_{sw}\) is an orientation of the bases of \(swM\) (which are the bases of \(M^*\)) with \(e_{sw} = \pm e_M\), since it satisfies Las Vergnas' property. This shows that \(swM = M^*\), so that \(M\) is symmetric. \(\square\)

**Corollary 1.9.** If \(M\) is represented by \((I, -A)\), then \(M\) is symmetric iff for every \(J, K \subseteq N, |J| = |K|\), we have \(\text{det}(A_{JK})\) agreeing in sign with \(\text{det}(A_{KJ})\).

Matrices satisfying the property that \(\text{det}(A_{JK})\text{det}(A_{KJ}) \geq 0\), for \(J, K \subseteq N, |J| = |K|\), are called sign symmetric. These matrices have been studied by several authors [26], [35]. Their research centers around the inequality

\[
\text{det}(A_{IJ}, InJ)\text{det}(A_{IJ}, InJ) \leq \text{det}(A_{II})\text{det}(A_{JJ}),
\]

which is known to be true for sign symmetric matrices. However, since
this relation involves the magnitudes of the determinants rather than just their signs, it is not relevant to our study of the corresponding oriented matroids.

1.2 **Symmetric weakly oriented matroids**

The "if" part of the proof of the characterization of symmetric oriented matroids in terms of their bases required that the assignment of signs to the bases of \( M \) determined an oriented matroid. This leads one to ask what sort of structure one might get from an assignment of signs \( \varepsilon \) to the bases of a square matroid \( M \) that did not necessarily determine an oriented matroid, but that satisfied the property 
\[
\varepsilon(B)(-1)^{\eta(B)} = \varepsilon(\text{swB})(-1)^{\eta(\text{swB})}.
\]
One can show that the orientation of the circuits of \( M \) determined by such an assignment \( \varepsilon \) and the correspondence of theorem 1.3 gives a weakly oriented matroid, in the sense of Bland and Jensen [23].

One family of matroids that admit such an assignment of signs to their bases, but are not orientable, is the class of matroids \( \{M_n\} \) from Bland and Las Vergnas [7]. For a given \( n \geq 4 \), \( M_n \) is a matroid on \( E = \{s_1, \ldots, s_n\} \cup \{t_1, \ldots, t_n\} \), with circuits \( \{t_1, \ldots, t_n\}, \{s_i, t_i, s_j, t_j\} \) for \( 1 \leq i < j \leq n \), \( \{s_1, \ldots, s_{i-1}, t_i, s_{i+1}, \ldots, s_n\} \) for \( 1 \leq i \leq n \), and all \((n+1)\)-subsets of \( E \) not containing any of the preceding \( 1 + \frac{n(n-1)}{2} + n \) sets. An \( n \)-set \( B \) contained in \( E \) contains one of the circuits of \( M_n \) iff \( \text{swB} \) contains one. Thus one can take pairs \((B, \text{swB})\) of bases of \( M_n \) and define \( \varepsilon(B) \) and \( \varepsilon(\text{swB}) \) such that \( \varepsilon \) satisfies 
\[
\varepsilon(B)(-1)^{\eta(B)} = \varepsilon(\text{swB})(-1)^{\eta(\text{swB})}
\]
for each pair. This yields a family of weak orientations of \( M_n \). Bland and Las Vergnas
showed that the matroids $M_n$ are not orientable, so none of these weak orientations gives an orientation of $M_n$.

1.3 Representations of symmetric oriented matroids

Recall from corollary 1.9 that $M(I, -A)$ is symmetric iff for every $J, K \subseteq N$, $|J| = |K|$, we have $\text{det}(A_{JK})$ agreeing in sign with $\text{det}(A_{KJ})$. Here we investigate two properties of such matrices. We say $A \in S$ if $M(I, -A)$ is symmetric.

Every square symmetric matrix is diagonalizable. Every $2\times2$ matrix in $S$ is diagonalizable. For $n \geq 3$, there are $n \times n$ matrices in $S$ that are not diagonalizable. One $3\times3$ example is based on the matrix $A = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{pmatrix}$. $A$ is symmetric and therefore in $S$. Furthermore, the determinant of every square submatrix of $A$ is nonzero, so matrices $A' = A + \varepsilon B$ will satisfy $M(I, -A) = M(I, -A')$ for sufficiently small $\varepsilon > 0$ and any $3\times3$ matrix $B$. Consider $A' = A + \varepsilon \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$. The characteristic polynomial of $A$, given by $\text{det}(A - \lambda I)$, has a double root at $\lambda = 4$ and a simple root at $\lambda = 7$. The value of $\text{det}(A' - \lambda I)$ at its stationary point close to $\lambda = 4$ ($\lambda = 4$ is a stationary point of $\text{det}(A - \lambda I)$) is greater than zero. Thus $\text{det}(A' - \lambda I)$ has only one root, near $\lambda = 7$. Therefore $A'$ is not diagonalizable, even though it is in $S$.

We conclude this section with an unsolved problem. Given $A \in S$, does there exist a symmetric matrix $A'$ such that $M(I, -A) = M(I, -A')$? The answer is yes for $2\times2$ matrices, since every $2\times2$ matrix $A \in S$ can be made symmetric by positive scaling of one of its columns. This operation does not change $M(I, -A)$. This technique does not extend to
the case of 3x3 matrices. A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} is an example of a matrix in S that cannot be rescaled by positive numbers so that the result is symmetric. However \( M(I,-A) \) coincides with \( M(I,-A') \) for \( A' = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \). Thus the question remains unanswered for \( n \geq 3 \).

If \( A \in S \), then \( A^T \in S \). \( A + A^T \) will be a symmetric matrix, but it will not necessarily be true that \( M(I,-A) = M(I,-(A+A^T)) \). For example, \( A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \in S \), \( A + A^T = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \). Clearly \( T \) is a base of \( M(I,-A) \) but not a base of \( M(I,-(A+A^T)) \).

The isotopy conjecture of Goodman and Pollack [20] states that for two \( n \times m \) matrices \( R \) and \( R' \) representing the same oriented matroid, there is a path in \( R^{n \times m} \) from \( R \) to \( R' \) such that the oriented matroid represented by every matrix on the path is the same. The example shows that the straight line from \( R = [I,-A] \) to \( R' = [I,-A^T] \) will not be such a path. If the conjecture were proved, it would not necessarily imply that the path would go through a matrix \([I,-A']\) with \( A' \) symmetric. Thus this problem is, in a sense, more difficult than the isotopy conjecture.

2. Positive Definiteness

2.1 P-Oriented matroids

A matrix \( A \) (not necessarily symmetric) is positive definite if \( x^T A x > 0 \) for \( x \neq 0 \), and \( A \) is a P-matrix if \( x_i (A x)_i > 0 \) for some \( i \), for all \( x \neq 0 \). A positive definite matrix is a P-matrix, and the converse is true if the matrix is symmetric. The property P captures the features of positive definiteness that are relevant for the linear complementarity problem. Todd [43] studied an oriented matroid analog of the class P, and we briefly summarize his results.
Definition 2.1. A cycle $K$ of a square oriented matroid is called **sign reversing** (sr) if there is no $i$ such that $s_i \in K^+$, $t_i \in K^+$ or $s_i \in K^-$, $t_i \in K^-$. $K$ is called **strictly sign reversing** (ssr) if, in addition, there is an $i$ such that $s_i \in K^+$, $t_i \in K^-$ or $s_i \in K^-$, $t_i \in K^+$. (Strictly) sign preserving cycles are defined analogously.

A **violator** of a strictly sign reversing cycle $K$ is an $s_i$ such that $s_i \in K$, $t_i \notin K$ or a $t_i$ such that $t_i \in K$, $s_i \notin K$. A subset $X$ of $E$ is called **complementary** if $|\{s_i, t_i\} \cap X| \leq 1$ for all $i$. A cycle $K$ of $M$ is complementary if $|\{s_i, t_i\} \cap K| \leq 1$ for all $i$.

Definition 2.2. A square oriented matroid is a $P$-oriented matroid if it has no sign reversing circuits.

The following theorem was proved in [43].

Theorem 2.3 [43]. The following are equivalent for a square oriented matroid $M$:

a) $M$ is a $P$-oriented matroid.

b) Every complementary subset $B$ of $E$ with cardinality $n$ is a base of $M$. Furthermore, there is an orientation $\varepsilon$ of the bases of $M$ with $\varepsilon(\beta) = (-1)^{|BnT|}$, where $\beta$ is the ordering $(b_1, \ldots, b_n)$ of the complementary base $B$ with $b_i \in \{s_i, t_i\}$ for each $i$.

c) Every point extension $\hat{M}$ of $M$ to $\hat{E}$ contains precisely one positive complementary circuit.

Condition b generalizes the property that $P$-matrices have positive principal minors, and condition c generalizes the property that the LCP with $A$ a $P$-matrix has a unique solution for every right-hand side.
The concept of eigencycles was introduced in [36].

Definition 2.4. A cycle \( K \) of \( K(M) \), where \( M \) is a square oriented matroid, is called a **positive eigencycle** if for every \( i \) either 
\[ s_i \in K^+, \ t_i \in K^+, \text{ or } s_i \in K^-, \ t_i \in K^- \], or \( s_i \notin K, \ t_i \notin K \). \( K \) is a **negative eigencycle** if, for all \( i \), we have \( s_i \in K^+ \iff t_i \in K^- \), \( s_i \in K^- \iff t_i \in K^+ \).

A vector \((Ax, x)\) in the nullspace of a matrix \((I, -A)\), such that \( x \) is an eigenvector corresponding to a positive eigenvalue, must give rise to a positive eigencycle of \( M(I, -A) \). It was shown in [36] that every symmetric P-oriented matroid has a positive eigencycle. The following theorem, based on a result of Fiedler and Ptak [12], greatly improves on that result.

**Theorem 2.5.** Let \( M \) be a P-oriented matroid and let \( \sigma \) be any pattern of +'s and -'s on \( n \) elements. Then \( M \) has a positive eigencycle that agrees with \( \sigma \) on both \( S \) and \( T \).

**Proof.** (Analogous to Fiedler and Ptak's proof.) Todd [43] showed that \( M \) is a P-oriented matroid iff for every cocircuit \( D \) there is an \( i \) such that \( s_i \in K^-, \ t_i \in K^+ \), or \( s_i \in K^+, \ t_i \in K^- \). This implies that \( M \) has no positive cocircuit. This implies, by duality (see [7]), that there is a \( K \in K(M) \) such that \( E = K^+ \). \( K \) is a positive eigencycle. The theorem now follows from the observation that changing signs on pairs \( \{s_i, t_i\} \) preserves the property that \( M \) is a P-oriented matroid. \( \square \)
The P-oriented matroid class seems well understood. However, in quadratic or linear programming, we usually deal with a larger class of oriented matroids, P_0.

2.2 The class P_0

A square matrix A is positive semidefinite if x^TAx \geq 0 for all x. A is a P_0-matrix if for every nonzero x there is an i such that x_i \neq 0 and x_i(Ax)_i \geq 0. Positive semidefinite matrices are in P_0. Unfortunately, the class P_0 does not capture all the features of positive semidefiniteness that are relevant to the linear complementarity problem. We study a class of oriented matroids analogous to the class of P_0-matrices.

Definition 2.6. A square oriented matroid is a P_0-oriented matroid if for every (\emptyset,\emptyset) \neq K \subset \mathcal{K}(M) there is an i such that t_i \in K and s_i does not have the opposite sign in K as t_i.

A cycle K failing to satisfy the requirements of definition 2.6 is a strictly sign reversing (ssr) cycle with all its violators in S. The following lemma was proved in [36].

Lemma 2.7. If a square oriented matroid M has an ssr cycle K with all its violators in S, then there is a negative eigencycle K' of M with K conforming to K'.

Proof. The following algorithm generates K'. Suppose we have an ssr cycle K with all its violators in S. If there is an i such that s_i \in K^+, t_i \notin K, replace K by K \cdot C(S,t_i). If there is an i such
that \( s_i \in K^-, t_i \notin K \), replace \( K \) by \( K \circ C(S, t_i) \). Repeat this step until there is no \( i \) with \( s_i \in K, t_i \notin K \). At that stage we have \( \overline{K} = K \), a negative eigencycle. \( \square \)

Thus \( P_0 \)-oriented matroids can equivalently be defined as those square oriented matroids that have no negative eigencycles.

**Theorem 2.8.** Let \( M \) on \( S \cup T \) be a \( P_0 \)-oriented matroid and \( B = S_{N \setminus J} \cup T_J \) a complementary base. Then \( M' \), obtained from \( M \) by interchanging occurrences of \( s_j \) and \( t_j \) for all \( j \in J \) in cycles of \( M \), is a \( P_0 \)-oriented matroid.

**Proof.** Clearly, \( S \) is a base of \( M' \), since it corresponds to the base \( B \) of \( M \). If we have an ssr cycle \( K \) of \( M' \) with all its violators in \( S \), we can use lemma 2.7 to construct a negative eigencycle \( \overline{K} \) of \( M' \). This cycle \( \overline{K} \) after relabelling will also be a negative eigencycle of \( M \), implying that \( M \) is not \( P_0 \). \( \square \)

The main theorem of this section proves an analog to Todd's equivalence of a) and b) in theorem 2.3. Fiedler and Ptak [12] proved this for the matrix case. Their proof does not carry over into the oriented matroid setting, because it involves addition of matrices. The oriented matroid proof is much longer, but it may offer new insights.

**Theorem 2.9.** A square oriented matroid is a \( P_0 \)-oriented matroid iff

(1) There is an orientation \( \varepsilon \) of its bases with \( \varepsilon(\beta) = -(1)|BnT| \), where \( \beta \) is the ordering \( (b_1, \ldots, b_n) \) of a complementary base \( B \) with \( b_i \in \{s_i, t_i\} \) for each \( i \), for all complementary bases \( B \).
The ordering \((b_1, \ldots, b_n)\) of a complementary base \(B\) will be called the \(N\)-ordering of \(B\) from now on. The proof of theorem 2.9 is by induction on \(|S|\). This requires that we establish some properties of minors \(M \setminus T_j/S_j\) for subsets \(S_j \subseteq S, T_j \subseteq T\).

Recall from matroid theory that the bases of \(M \setminus t_j/s_j\) are exactly the sets \(B \setminus s_j\) for bases \(B\) of \(M\) with \(s_j \in B, t_j \not\in B\). The assignment of signs \(\varepsilon_j\) to ordered bases of \(M \setminus t_j/s_j\) given by \(\varepsilon_j(\beta) = \varepsilon(\beta')\), where \(\beta = (b_1, \ldots, b_{n-1})\), \(\beta' = (b_1, \ldots, b_{j-1}, s_j, b_j, \ldots, b_{n-1})\), and \(\varepsilon\) is the orientation of the bases of \(M\) such that \(\varepsilon(s_1, \ldots, s_n) = 1\), can easily be shown (see [30]) to be an orientation of the bases of \(M \setminus t_j/s_j\). Call \(\varepsilon_j\) the orientation induced by \(\varepsilon\). For arbitrary \(J \subseteq N\), get the orientation \(\varepsilon_j\) of bases of \(M \setminus T_j/S_j\) by taking the orientations of the corresponding \(n\)-sets of \(E\) with the element \(s_{j_1}\) added in the \(j_1\)th position for each \(J_1 \in J\).

**Lemma 2.10.** Let \(M\) be a square oriented matroid of rank \(n\), and let \(J \subseteq N\). Then the orientation \(\varepsilon\) of the bases of \(M\) satisfies \(\varepsilon(\beta) = (-1)^{|B \cap T|}\) for the \(N\)-orderings of complementary bases \(B\) containing \(S_J\) iff the orientation \(\varepsilon_J\) of the oriented matroid \(M \setminus T_J/S_J\) satisfies \(\varepsilon_J(\beta) = (-1)^{|B \cap T_N \setminus J|}\) for \((N \setminus J)\)-orderings of complementary bases \(B\) of \(M \setminus T_J/S_J\).

**Proof.** If \(\beta\) is an \((N \setminus J)\)-ordering of the complementary base \(B\) of \(M \setminus T_J/S_J\), then the ordered \(n\)-set of \(E\) that one gets from adding the \(s_{j_1}\)'s in the \(j_1\)th positions of \(1, \ldots, n\) is the \(N\)-ordering of
$B \cup S_J$. Then $|B \cap T_{N \setminus J}| = |(B \cup S_J) \cap T|$, so we must have $\varepsilon_J(\beta) = (-1)^{|BnT_{N \setminus J}|}$ iff $\varepsilon(\beta') = (-1)^{|(BuS_J) \cap T|}$, with $\beta'$ the ordering of $B \cup S_J$ obtained from $\beta$ by placing each $S_{J_i}$ in the $J_i$th position of $(1, \ldots, n)$. \qed

**Lemma 2.11.** A square oriented matroid $M$ is a $P_0$-oriented matroid iff $M \backslash T_J/S_J$ is a $P_0$-oriented matroid for all $J \subseteq N$.

The proof is trivial.

We now outline the proof of theorem 2.9. The theorem is trivially true for square oriented matroids of rank one. Now assume that it is true for square oriented matroids of rank less than $n$. Let $K$ be an ssr cycle of a square oriented matroid $M$ of rank $n$, with all its violators in $S$. Suppose that $t_i \notin K$ for some $i$. Then $K \backslash s_i$ is an ssr cycle of $M \backslash t_i/s_i$ with all its violators in $S$. By the induction hypothesis this implies that there is no orientation $\varepsilon_i$ satisfying (1) with respect to the $(N \setminus i)$-orderings of complementary bases $B$ of $M \backslash t_i/s_i$. By lemma 2.10, this implies that there is no $\varepsilon$ satisfying (1). Hence the equivalence is established if $M$ contains such a cycle.

Now suppose that all ssr cycles of $M$ with all their violators in $S$ contain $T$, and let $K$ be such a cycle of $M$. Lemma 2.12 shows that this implies that $T$ is a base of $M$, and that for every complementary base $B = S_{N \setminus J} \cup T_J$, the oriented matroid $M \backslash S_J/T_J$ has a negative eigencycle. Lemmas 2.13-2.15 show that when $T$ is a base of $M$ and $B = S_{N \setminus J} \cup T_J$ is a complementary base where $J$ is maximal
with this property and $|J| < n$, then $M \setminus S_J / T_J$ has a negative eigencycle iff $\varepsilon(\beta) \neq (-1)^{|J|}$ for the $N$-orderings $\beta$ of $B$ in the orientation of bases of $M$ in which $\varepsilon(\tau) = (-1)^n$ for $\tau = (t_1, \ldots, t_n)$. Thus the existence of a negative eigencycle of $M \setminus S_J / T_J$, proved in lemma 2.12, shows that (1) cannot hold.

Suppose next that $\varepsilon$ is an orientation of the bases of $M$ such that $\varepsilon(s_1, \ldots, s_n) = 1$, but (1) is not satisfied. Let $B = S_{N \setminus J} \cup T_J$ be a complementary base with $J \neq N$ such that $\varepsilon(\beta) \neq (-1)^{|J|}$. Then by lemma 2.10, property (1) is not satisfied for $M \setminus t_i / s_i$, for $i \in N \setminus J$. By the induction hypothesis there is an ssr cycle of $M \setminus t_i / s_i$ with its violators in $S$, so by lemma 2.11, $M$ is not $P_0$. Suppose, then, that $\varepsilon(\beta) = (-1)^{|B \cap T|}$ for complementary bases $B$ with $|B \cap T| < n$, and that since (1) is not satisfied, $T$ is a base and $\varepsilon(\tau) \neq (-1)^n$ for $\tau = (t_1, \ldots, t_n)$. By lemmas 2.13-2.15 we show that this implies that $M \setminus S_J / T_J$, for a maximal set $J$ with $|J| < n$ such that $S_{N \setminus J} \cup T_J$ is a complementary base, has a negative eigencycle. This implies that $M$ has an ssr cycle with its violators in $T_J$. However, since $T$ is a base of $M$, theorem 2.8 tells us that $M$ is not in $P_0$.

Thus theorem 2.9 is proved, once we establish lemmas 2.12-2.15.

**Lemma 2.12.** Let $M$ be a $P_0$-oriented matroid, and let $K$ be an ssr cycle of $M$ that has all its violators in $S$. Further assume all such cycles $K$ have $\overline{K} = E$. Then $T$ is a base of $M$, and for every complementary base $B = S_{N \setminus J} \cup T_J$ there is a negative eigencycle of $M \setminus S_J / T_J$. 

Proof. Let \( T \) contain a circuit \( C \) of \( M \), and without loss of
generality assume that \( C \) does not conform to \( K \). Then \( C \subseteq K \), so
we can find (see [33]) a \( C \)-approximation \( K' \) of \( K \), a cycle \( K' \neq K \)
that conforms to \( K \) and agrees with \( K \) for all \( e \in (K^+ \setminus C^+) \cup (K^- \setminus C^-) \).
This \( K' \) will contain \( S \) and thus be an ssr cycle of \( M \) with its
violators in \( S \), and with some \( t_i \in T \setminus K' \). This contradicts the
hypothesis that \( T \setminus K = \emptyset \) for such cycles. Thus \( T \) is a base of \( M \).

For \( i = 1, \ldots, n \), let \( C_i = C(T, s_i) \) if \( s_i \in K^+ \), \( C_i = -C(T, s_i) \)
if \( s_i \in K^- \). If \( C_i \) does not conform to \( K \), for some \( i \), then we
can construct a \( C_i \)-approximation \( K' \) of \( K \) as above, such that
\( K' \) is ssr with its violators in \( S \), and \( T \setminus K' \neq \emptyset \). Thus the
circuits \( C_i \) conform to \( K \). Let \( B = S_{N \setminus J} \cup T_J \) be a complementary
base of \( M \). From matroid theory, we have a bijection \( \pi: T \rightarrow B \) such
that \( (T \setminus t_i) \cup \pi(t_i) \) is a base of \( M \) for all \( i \). Furthermore,
\( \pi(t_k) \) can be taken to be \( t_k \) for \( k \in J \). For every \( j \in N \setminus J \)
there is an \( s_j = \pi(t_j) \in S_{N \setminus J} \) such that \( (T \setminus t_j) \cup s_j \) is a base,
which implies that \( t_j \in C(T, s_j) \). This implies that
\( K_J = C_{j_1} \circ \ldots \circ C_{j_n} \) for \( \{j_1, \ldots, j_n\} = N \setminus J \) contains \( T_{N \setminus J} \), and
since it conforms to \( K \) it is an ssr cycle of \( M \). Since \( S_J \subseteq E \setminus K_J \),
\( K_J \setminus T_J \) is a negative eigencycle of \( M \setminus S_J / T_J \). \( \square \)

If \( T \) is a base of \( M \) and \( J \) is a maximal subset of \( N \) such
that \( |J| < n \) and \( S_{N \setminus J} \cup T_J \) is a base of \( M \) then \( M \setminus S_J / T_J \)
satisfies the conditions of the next lemma.
Lemma 2.13. A square oriented matroid $M$ has $S$ and $T$ as its only complementary bases if and only if $M = M(I,-PD)$, where $P$ is a permutation matrix, with a single cycle when $\text{rank } M > 1$, and $D$ is a diagonal matrix with diagonal elements in $\{+1,-1\}$.

Proof. The results are trivial if $\text{rank } M = 1$, so suppose it is at least 2. For $M = M(I,-PD)$, define $\pi : N \to N$ by $\pi(i) = j$ iff $Pu_i = u_j$, where $u_i$ and $u_j$ are the $i$th and $j$th unit vectors. Then the circuits of the underlying matroid $M(I,-PD)$ are the sets $\{t_i, s_{\pi(i)}\}$, for $i = 1, \ldots, n$. Suppose that $B$ is a complementary base containing $t_1$. Then it does not contain $s_{\pi(1)}$, so it contains $t_{\pi(1)}$. Since $\pi$ has a single cycle, continuing this way shows that for all $i$, $B$ cannot contain $s_{\pi(i)}$, so $B = T$. If $B$ contains $s_1$, a similar analysis shows that $B = S$.

Now suppose that $S$ and $T$ are the only complementary bases of $M$. Consider $C(S, t_1)$; then $s_1 \not\in C(S, t_1)$, for otherwise $(S \cup t_1) \setminus s_1$ would be a base of $M$. There exists $s_i \in C(S, t_1)$, so say that $s_2 \in C(S, t_1)$. Continuing this way, suppose that $s_i \not\in C(S, t_k)$ for all $1 \leq i < k < j < n$, but that $s_{k+1} \in C(S, t_k)$ for $1 \leq k < j$. Suppose then that $s_k \in C(S, t_j)$ for some $k < j$, and let $k'$ be the maximum such $k$. Let $L = \{\ell : k' < \ell < j\}$, and let $C_L$ be a circuit contained in $S_{N\setminus L} \cup T_L$. We can use the fundamental circuits $C(S, t_\ell)$ for $\ell \in L$ to eliminate members of $T_L$ from $C_L$. Call the resulting cycle $K_L$. If $t_j \notin C_L$, then $s_k \in K_L$, since $s_k \in C(S, t_j)$ and $s_k \not\in C(S, t_\ell)$ for $\ell \in L \setminus j$. If $t_j \notin C_L$, let
\( \ell' = \min(\{ \ell : t_\ell \in C_L \}) \). Then \( s_{\ell'+1} \in C(S,t_{\ell'}) \) and \( s_{\ell'+1} \notin C(S,t_\ell) \) for \( \ell \in L \setminus \ell' \), so \( s_{\ell'+1} \in K_L \). In both cases \( K_L \) will be a nonempty cycle of \( M \) with \( K_L \subseteq S \), contradicting the assumption that \( S \) is a base of \( M \). Thus, for \( j < n \), we have \( s_k \notin C(S,t_j) \) for \( k \leq j \); and we can renumber so that \( s_{j+1} \in C(S,t_j) \) and continue. For \( C(S,t_n) \) we will have \( s_i \notin C(S,t_n) \) for \( i > 1 \) by the argument above, but \( s_1 \in C(S,t_n) \), since \( t_n \) is not a loop.

Note that from the above, \( C(S,t_n) = \{s_1,t_n\} \), and \( C(S,t_j) \) contains \( s_{j+1} \) but not \( s_k \), \( k \leq j \). Suppose now we applied the same argument to the \( t \)’s starting with \( t_{j+1} \). Since \( C(S,t_{j+1}) \) contains \( s_{j+2} \), we could choose \( j+2 \) as our next index, and continue thus through \( t_{j+2}, \ldots, t_n, t_1, \ldots, t_j \). Then \( C(S,t_j) \) would play the role of \( C(S,t_n) \), and hence \( C(S,t_j) = \{s_{j+1},t_j\} \). Thus all the sets \( \{s_{j+1},t_j\}, j = 1, \ldots, n-1 \) and \( \{s_1,t_n\} \) are circuits.

There can be no other circuits, or else we could produce a circuit entirely in \( S \) by eliminations. It follows that \( M = M(I,-PD) \) for \( P \) a permutation matrix with a single cycle, and \( D \) a diagonal matrix with entries in \( \{+1,-1\} \). \( \square \)

**Lemma 2.14.** \( M(I,-PD) \), for \( P \) a permutation matrix of an \( n \)-cycle \( \pi \) has a negative eigencycle iff the orientation of the bases of \( M \) with \( \varepsilon(s_1, \ldots, s_n) = +1 \) has \( \varepsilon(t_1, \ldots, t_n) = (-1)^n \).

**Proof.** Consider the sequence of ordered bases \( (s_1, \ldots, s_n), (t_\pi(1), s_2, s_3, \ldots, s_n), \ldots, (t_\pi(n), \ldots, t_\pi(1), s_n), (t_\pi(1), \ldots, t_\pi(n-1), s_n), (t_\pi(1), \ldots, t_\pi(n)) \). The orientations of these \( n+1 \) ordered bases are determined by the circuits
in the sets \( \{s_i, t_{\pi(i)}\} \), for \( i = 1, \ldots, n \). This implies that 
\[ \varepsilon(s_1, \ldots, s_n) = \varepsilon(t_{\pi(1)}, \ldots, t_{\pi(n)}) \]
if and only if there is an even number of the circuits contained in the sets \( \{s_i, t_{\pi(i)}\} \), in which 
\( s_i \) and \( t_{\pi(i)} \) agree in sign. Note 
\[ \varepsilon(t_{\pi(1)}, \ldots, t_{\pi(n)}) = (-1)^n \varepsilon(t_1, \ldots, t_n) \]. Construct the cycle \( K \) of \( M \) as follows. First, 
let \( K \) be \( C(S, t_{\pi(1)}) \). At a general step, let \( K \rightarrow (K \circ C) \), where 
\( C \) contains \( \{s_i, t_{\pi(i)}\} \) for \( i \) such that \( t_i \in K \), \( s_i \notin K \), and 
\( s_i \) appears in \( C \) with the opposite sign of \( t_i \)'s sign in \( K \). 
After the \( n \)th step we have a cycle \( K \) with \( K = E \) such that \( s_i \) and \( t_i \) disagree in \( K \) for all \( i \) except possibly for the pair 
\( \{s_1, t_1\} \). If there is an even number of the circuits contained in 
the pairs \( \{s_i, t_{\pi(i)}\} \) with \( s_i, t_{\pi(i)} \) agreeing in sign, then \( K \) 
will be a negative eigencycle. Conversely, suppose \( K \) is a 
negative eigencycle of \( K \). Then it must be conformally composed from 
the circuits contained in the sets \( \{s_i, t_{\pi(i)}\} \). If \( K \) is a negative 
eigencycle, then 
\[ |S \cap K^+| = |T \cap K^-|, |S \cap K^-| = |T \cap K^+| \]. An 
odd number of circuits of the form \( \{s_i, t_{\pi(i)}, 0\} \), in which \( s_i \) and 
t_{\pi(i)} \ agree, would prohibit these equalities by making the parities 
unequal for both equations. So there must be an even number of such 
circuits. \( \square \)

**Lemma 2.15.** Let \( M \) be a square oriented matroid of rank \( n \), and let 
\( J \) be a maximal subset with \( |J| < n \) such that \( B = S_{N \setminus J} \cup T_J \) is a 
basis of \( M \). Suppose also that \( T \) is a basis of \( M \). Then \( M \setminus S_J / T_J \) 
has a negative eigencycle iff the orientation \( \varepsilon \) of the bases of \( M \) 
that satisfies \( \varepsilon(\tau) = (-1)^n \) also satisfies \( \varepsilon(B) \neq (-1)^{|J|} \), where 
\( B \) is the \( N \)-ordering of \( B \).
Proof. Suppose first that \(|J| = n-1\). Then \(B = (T \setminus t_i) \cup s_i\) for some \(i\), and we get from the definition of orientations of bases that \(C(T,s_i)\) is strictly sign reversing iff the orientation of the bases of \(M\) that satisfies \(\varepsilon(\tau) = (-1)^n\) also satisfies \(\varepsilon(\beta) \neq (-1)^{n-1}\), for \(\beta\) the N-ordering of \(B\). Also, \(C(T,s_i) \setminus (T \setminus t_i)\) is a negative eigencycle of \(M \setminus (S \setminus s_i)/(T \setminus t_i)\) iff \(C(T,s_i)\) is ssr. Next suppose that \(|J| \leq n-2\). Then lemmas 2.13 and 2.14 apply to \(M \setminus S_J / T_J\). We need to show that the orientation \(\varepsilon'_J\) of the bases of \(M \setminus S_J / T_J\) with \(\varepsilon'_J(\tau_{N \setminus J}) = (-1)^{|N \setminus J|}\) for \(\tau_{N \setminus J}\) the \((N \setminus J)\)-ordering of \(T_{N \setminus J}\) satisfies \(\varepsilon'_J(S_{N \setminus J}) = -1\) for \(S_{N \setminus J}\) the N-ordering of \(S_{N \setminus J}\) iff the orientation \(\varepsilon\) of bases of \(M\) satisfying \(\varepsilon(\tau) = (-1)^n\) satisfies \(\varepsilon(\beta) \neq (-1)^{|J|}\). But this is exactly analogous to lemma 2.10, with \(\varepsilon'_J\) taking the place of \(\varepsilon_J\) and \(T_J\) taking the place of \(S_J\). \(\square\)

2.3 Oriented matroid classes between \(P\) and \(P_0\) of interest for

the linear complementarity problem

Suppose we have an oriented matroid \(\hat{M}\) on \(E \cup p \cup q\), and oriented matroids \(\hat{M} = \hat{M} \setminus q\), \(M = \hat{M} \setminus p\) of the same rank as \(\hat{M}\), with \(M\) a square oriented matroid. Furthermore, assume \(S \cup q\) is the set underlying a positive circuit of \(\hat{M}\), and every circuit of \(\hat{M}\) involving \(q\) has \(n+1\) elements. We want to study how well Lemke's algorithm [see chapter I] works when \(M\) belongs to certain classes of \(P_0\)-oriented matroids. Recall from chapter I that a circuit \(\hat{C}\) of \(\hat{M}\) is called special if it is positive and complementary and contains \(p\). A circuit \(\hat{C}_k\) of \(\hat{M}\) is the endpoint of a q-ray \(\hat{C}\) if both are positive circuits of \(\hat{M}\), \(p \in \hat{C}_k \setminus \hat{C}\), \(\hat{C} \cap T \neq \emptyset\) and \(\hat{C}_k \cup \hat{C}\) is complementary.
Theorem 2.16. If $M$ is a $P_0$-oriented matroid and $\hat{\mathcal{C}}_k$ is the endpoint of a q-ray $\hat{\mathcal{C}}$, then $q \notin \hat{\mathcal{C}}$.

Proof. Let $\hat{\mathcal{C}}_q$ be the positive circuit of $\hat{\mathcal{M}}$ supported by $S \cup q$. Suppose $q \in \hat{\mathcal{C}}$. Then $\hat{\mathcal{C}}$ has $n+1$ elements. Let $K$ be the result of eliminating $q$ between $\hat{\mathcal{C}}_q$ and $-\hat{\mathcal{C}}$. Then $p \notin K$, $q \notin K$, so $K$ is a cycle of $M$. Furthermore, $K$ is an ssr cycle of $M$ with its violators in $S$. This contradicts the assumption that $M$ is $P_0$. □

For an $n \times n$ matrix $A$, membership in $P_0$ does not insure that Lemke's algorithm will solve the LCP for all right-hand sides that have a solution. For example, when $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, the system $y - Ax = b$ has a positive complementary solution $y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, but application of Lemke's algorithm to the system $y - Ax = b + z_0 d$ for any positive $d$ leads to the discovery of a secondary ray (see ch. I) with $z_0 = 0$. Theorem 2.16 shows that when $A$ is a $P_0$-matrix (when $M(I,-A)$ is a $P_0$-oriented matroid), all secondary rays will have $z_0 = 0$ ($q \notin \hat{\mathcal{C}}$). For some classes of matrices in $P_0$, such as positive semidefinite matrices, termination on a secondary ray always implies infeasibility of the LCP [31]. This has led to the study of other classes of matrices in $P_0$ [10], some of which can be characterized by properties of the oriented matroids that they represent.

2.3.1 The class $P_1$

This class was introduced in [36] to generalize the class of oriented matroids representable by positive semidefinite matrices.
Definition 2.17. A square oriented matroid is a $P_1$-oriented matroid if it has no strictly sign reversing cycles.

For the matrix $A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$, we have $M(I,-A)$ in $P_0$ but not in $P_1$, so $P_1$ is a proper restriction of the class $P_0$. The class $P_1$ includes the oriented matroids $M(I,-A)$ for positive semidefinite matrices $A$, since the existence of a strictly sign reversing cycle of $M(I,-A)$ gives a vector $(Ax, x)$ in the nullspace of $(I,-A)$ such that $x^TAx < 0$.

The following theorem was proved in [36].

Theorem 2.18. A symmetric $P_0$-oriented matroid is a $P_1$-oriented matroid.

The class $P_1$ includes some matrices for which Lemke's algorithm will not always yield a solution when one exists. For example, consider $M(I,-A)$, where $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. $M(I,-A)$ is a $P_1$-oriented matroid. Let $p = \text{lex}(s_2,s_1)$ extend $M$ to $\hat{M}$, and then let $q = -\text{lex}(s_2,s_1)$ extend $\hat{M}$ to $\bar{M}$. Apply Lemke's algorithm to the oriented matroid $\bar{M}$. In the first step, the element $s_1$ leaves the basis and $q$ enters. Then $\chi_k = (\{s_2,p,q\}, \emptyset)$ is the endpoint of the q-ray $\chi = (\{t_1\}, \emptyset)$. The LCP does, however, have a solution, namely $(\{t_2,s_1,p\}, \emptyset)$. This is an example of an LCP for which the choice of the extension by $q$ to $\bar{M}$ affects the performance of the algorithm. The choice of $q = -\text{lex}(s_1,s_2)$ would have led to solution of the LCP.
Thus the knowledge that an oriented matroid is in $P_1$ (unless it is also symmetric) fails to give information about whether or not Lemke's algorithm will be successful in solving an LCP arising from the oriented matroid. The next section considers classes of oriented matroids that generalize classes of matrices previously studied in linear complementarity theory.

2.3.2 The classes $Q \cap P_0$ and $Q_0 \cap P_0$

As we shall see in chapter IV, the classes $Q$ and $Q_0$, defined for oriented matroids in chapter I, do not have the property that a matrix $A$ is a $Q(Q_0)$-matrix if and only if $M(I,-A)$ is a $Q(Q_0)$-oriented matroid. This section is inspired by the work of Aganagic and Cottle [1], [2], which showed that the above statement is true if $A$ and $M(I,-A)$ are in $P_0$. The following theorem is a generalization of Aganagic and Cottle's theorem [2].

**Theorem 2.19.** Let $M$ be a $P_0$-oriented matroid, and let $K \in K(M)$ satisfy: $K \subseteq S \cup T_I$ for some $I \subseteq N$, $T_I \subseteq K^+$, $S_I \cap K^+ = \emptyset$, $S_N \setminus I \cap K^- = \emptyset$. Then for any extension $\hat{M}$ of $M$ on $E \cup p$, such that $\hat{C}(S,p)$ satisfies $s_i \in \hat{C}(S,p)^-$ if $s_i \in S_I \setminus K$, $s_i \in \hat{C}(S,p)^+$ if $s_i \in S_N \setminus I \setminus K$, then there is no positive complementary circuit of $\hat{M}$ including $p$.

**Proof.** (Analogous to Aganagic and Cottle's proof.) Suppose we have a $K$ and $\hat{C}(S,p)$ as in the theorem, and let $\tilde{C}$ be a positive complementary circuit involving $p$. Note that $S_I \setminus K$ is not empty, for
otherwise \( K \) would be a strictly sign reversing cycle with all its violators in \( S \). Thus \( \hat{C}(S,p) \) is not positive, and \( \hat{C} \) is not \( \hat{C}(S,p) \). Let \( \overline{K} \) be a result of eliminating \( p \) between \( \hat{C}(S,p) \) and \( -\overline{C} \). Then consider \( \hat{k} = K \circ \overline{K} \). We have \( S_I \subseteq \hat{k}^-, T_I \subseteq \hat{k}^+, \) and if \( t_i \in \hat{k} \) for \( i \in N \setminus I \), then \( t_i \in \overline{C}^+ \) so that \( t_i \in \hat{k}^- \), and hence \( s_i \not\in \overline{C} \), so \( s_i \in \hat{k}^+ \). Thus \( \hat{k} \) is a strictly sign reversing cycle with its violators in \( S \), contradicting \( P_0 \). □

**Corollary 2.20.** A \( P_0 \)-oriented matroid is \( Q \) iff it has no positive complementary circuits.

**Proof.** Suppose \( M \) is in \( P_0 \), and let \( C \) be a positive complementary circuit. Then \( \overline{C} \) fulfills the hypotheses of the \( K \) of the theorem, so there is an extension, e.g. \( p = \text{lex}(s_{i_1}, ..., s_{i_k}, -s_{i_{k+1}}, ..., -s_{i_n}) \), where \( \{i_1, ..., i_k\} = \{i: t_i \in C\} \), that has no solution. Note that if \( M \) is representable, there is a representable extension that has no solution. Conversely, suppose \( M \) has no positive complementary circuits, so it has no q-rays \( \overline{C} \) with \( q \not\in \overline{C} \) when we solve the LCP's of extensions of \( M \). From theorem 2.16, since \( M \) is \( P_0 \), there are no q-rays \( \overline{C} \) with \( q \in \overline{C} \), so Lemke's algorithm will yield a solution. □

**Definition 2.21.** A square oriented matroid is said to have property \((T)\) if the existence of \( K \in K(M) \) satisfying the conditions of the previous theorem implies the existence of \( D \in D(M) \), a positive cocircuit contained in \( (S_I \setminus K \cup T_{N \setminus I}) \).

**Lemma 2.22.** If a \( P_0 \)-oriented matroid \( M \) is a \( Q_0 \)-oriented matroid, then it satisfies property \( (T) \).
Proof. (In this case, Aganagic and Cottle's proof does not work for oriented matroids.) Suppose we have a $K$ fulfilling the conditions of the last theorem. Let $p = \text{lex}(-s_{i_1}, \ldots, -s_{i_k}, s_{i_{k+1}}, \ldots, s_{i_n})$ extend $M$ to $\hat{M}$, where $j > k$ implies that $s_{i_j} \in S_1 \setminus K$. Then by that theorem, $\hat{M}$ has no complementary solution. Then, since $M$ is $Q_0$, $\hat{M}$ is infeasible. Thus, by Minty's lemma, there is a positive cocircuit $D$ containing $p$. $D$ intersects $S$, since $T \cup p$ is a cobase, and since the sign of $p$ is determined by the first $s_{i_j}$ in $D$, we must have $D \cap (S_1 \setminus K) = \emptyset$. Then by orthogonality with $K$, we have $D \cap T_1 = \emptyset$, so $M$ satisfies property $(T)$. □

This leads us to Aganagic and Cottle's characterization of $Q_0 \cap P_0$, which they prove entirely using arguments that are valid for oriented matroids.

Theorem 2.23. A $P_0$-oriented matroid is a $Q_0$-oriented matroid iff $M$ satisfies property $(T)$ with respect to every complementary base $B$.

The proof of the "if" part involves applying Lemke's algorithm to an LCP and showing that the appearance of a q-ray $\hat{C}$ with $q \notin \hat{C}$ either indicates a positive cocircuit containing the new element or points to a violation of property $(T)$ for some complementary base $B$. See [2].

We have seen, then, that $P_0$ forms a class of matrices/oriented matroids in which the characterizations of $Q_0$ and $Q$ are the same for the matrices as for the oriented matroids that represent them.
2.4 Adequate and symmetric $P_0$-oriented matroids

Adequate matrices were introduced by Ingleton [22], and have a natural analog in oriented matroids.

**Definition 2.24.** A $P_0$-matrix $A$ is adequate if for every $I \subseteq N$, $\det A_{II} = 0$ implies that the rows of $A_I$ and the columns of $A_I$ are linearly dependent.

**Definition 2.25.** A $P_0$-oriented matroid is adequate if every complementary circuit is in $T$ and every complementary cocircuit is in $S$.

**Theorem 2.26.** A symmetric $P_0$-oriented matroid is adequate.

**Proof.** By induction on $|C \cap T|$, where $C$ is a complementary circuit. For $|C \cap T| = 1$, $C = C(S,t_i)$ for some $i$. If $s_j \in C(S,t_i)$ for some $j \neq i$, then by symmetry $s_i$ appears in $C(S,t_i)$ with the same sign. If this sign is $+$, then $C(S,t_i) \circ -C(S,t_j)$ is ssr with its violators in $S$, and if it is $-$, then $C(S,t_i) \circ C(S,t_j)$ is ssr with its violators in $S$. Now suppose $C \subseteq T$ for complementary circuits $C$ with $|C \cap T| = k-1$. Suppose $C$ is a complementary circuit with $|C \cap T| = k$. Suppose $C \cap S \neq \emptyset$ and let $t_j \in C \cap T$. Then there is no complementary circuit in $S_{N\setminus I \cup T_I}$, where $T_I = (C \cap T) \setminus t_j$. Since $S_{N\setminus I \cup T_I}$ is a base, $M \setminus S_I / T_I$ is a symmetric $P_0$-oriented matroid. In this matroid, then, $|C \setminus T_I| = |t_j| = 1$, so the argument above shows that $t_j$ is a loop of $M \setminus S_I / T_I$, and that therefore $C \subseteq T$. This argument also shows that the complementary cocircuits are in $S$, since they are switches of complementary circuits. $\Box$
The converse to this theorem is false, since P-oriented matroids are adequate, and there are many nonsymmetric P-oriented matroids.

Theorem 2.27. If $M$ is an adequate oriented matroid, then $M$ is $Q_0$.

Proof. Suppose that Lemke's algorithm is applied to an oriented matroid $\hat{M}$ on $E \cup p \cup q$, with $\hat{M}\{p,q\} = M$ and $\text{rank}(\hat{M}) = \text{rank}(M)$. Suppose that a q-ray $\hat{\mathcal{C}}$ is reached. Then $q \notin \hat{\mathcal{C}}$, by theorem 2.15. Let $\hat{\mathcal{C}}'$ be an endpoint of the q-ray $\hat{\mathcal{C}}$. $(\hat{\mathcal{C}}' \cup \hat{\mathcal{C}}) \setminus \{p, q\}$ is a complementary n-set. Let $T_J = T \setminus \hat{\mathcal{C}}$, and consider $M \setminus T_J / S_J$. If $M \setminus T_J / S_J$ had no positive cocircuit in $S \setminus S_J$, then there would be a cycle $K \in K(M \setminus T_J / S_J)$ with $S \setminus S_J \subseteq \mathcal{K}^+$, by Minty's lemma. Then $\hat{\mathcal{C}} \setminus S_J$ is a circuit of $M \setminus T_J / S_J$, and $(\hat{\mathcal{C}} \setminus S_J) \circ K$ is a negative eigencycle of $M \setminus T_J / S_J$, contradicting the property $P_0$. Therefore there is a positive cocircuit of $M \setminus T_J / S_J$ in $S \setminus S_J$. This implies that there is a complementary cocircuit of $M$ in $(S \setminus S_J) \cup T_J$ that is positive on $S \setminus S_J$. By the adequate property, this cocircuit is in $S \setminus S_J$. This implies that there is a cocircuit $\hat{\mathcal{D}}$ of $\hat{M}$ contained in $(S \setminus S_J) \cup p \cup q$ that is positive on $S \setminus S_J$. $\hat{\mathcal{D}}$ must be orthogonal to the positive circuit of $\hat{M}$ supported by $S \cup q$, so $q \in \hat{\mathcal{D}}^-$. $\hat{\mathcal{D}}$ must also be orthogonal to the endpoint $\hat{\mathcal{C}}'$ of the q-ray, and $(S \setminus S_J) \cap \hat{\mathcal{C}}' = \emptyset$, so $p \in \hat{\mathcal{D}}^+$. Thus $\hat{\mathcal{D}} \setminus q$ is a positive cocircuit of $\hat{M}$ including $p$. This implies that there are no positive circuits of $\hat{M}$ involving $p$. □

This last proof followed the lines of arguments of Eaves [10]. His proof must be changed considerably, because there is no oriented matroid analog to taking the symmetric part $A + A^T$ of a matrix.
Corollary 2.27. A symmetric \( P_0 \)-oriented matroid is a \( Q_0 \)-oriented matroid.

We have established the following relationships for classes of oriented matroids:

\[
\begin{align*}
\text{(symmetric } &\cap \text{ } P_0) \\
\uparrow &\\
P_0 &\iff P_1 \\
Q_0 \cap P_0 &\iff Q \cap P_0 \\
&\iff \text{adequate} \\
&\iff P
\end{align*}
\]

3. A Nonrepresentable Symmetric \( P \)-Oriented Matroid

Mandel's surgery technique [33] is used here to transform an oriented matroid \( M(I,-A) \), for a matrix \( A \) that is a symmetric \( P \)-matrix, into a nonrepresentable oriented matroid that is still a symmetric \( P \)-oriented matroid.

Consider the symmetric \( P \)-matrix

\[
\begin{pmatrix}
4 & -1 & -1 & -1 \\
-1 & 4 & -1 & -1 \\
-1 & -1 & 4 & -1 \\
-1 & -1 & -1 & 4
\end{pmatrix} = A.
\]

The circuits of the oriented matroid \( M(I,-A) \) are given below, in
an abbreviated form. The circuit \( \{s_2, t_2, \{s_1, t_1\} \} \), for example, is written as \( \overline{s_1} s_2 \overline{t_1} t_2 \). A complete list of the circuits of \( M(I,-A) \) would include the negatives of those given below.

\[
\begin{array}{cccccc}
\overline{s_1} s_2 \overline{t_1} t_2 & s_1 \overline{s_2} s_3 \overline{s_4} t_1 & s_1 \overline{s_2} s_3 t_1 t_4 & s_1 \overline{s_2} s_4 t_2 t_4 & s_1 \overline{s_4} t_1 t_2 t_3 \\
\overline{s_1} s_3 \overline{t_1} t_3 & s_1 \overline{s_2} s_3 \overline{s_4} t_2 & s_1 \overline{s_2} s_3 t_2 t_4 & s_2 \overline{s_4} t_1 t_2 t_4 & s_1 \overline{s_4} t_2 t_3 t_4 \\
\overline{s_1} s_4 \overline{t_1} t_4 & s_1 \overline{s_2} s_3 \overline{s_4} t_3 & s_1 \overline{s_2} s_3 t_3 t_4 & s_2 \overline{s_3} s_4 t_1 t_3 & s_2 \overline{s_3} t_1 t_2 t_4 \\
\overline{s_2} s_3 \overline{t_2} t_3 & s_1 \overline{s_2} s_3 \overline{s_4} t_4 & s_1 \overline{s_2} s_4 t_1 t_3 & s_2 \overline{s_3} s_4 t_1 t_4 & s_2 \overline{s_3} t_1 t_3 t_4 \\
\overline{s_2} s_4 \overline{t_2} t_4 & s_1 t_1 t_2 t_3 t_4 & s_1 \overline{s_2} s_4 t_2 t_3 & s_1 \overline{s_2} t_1 t_3 t_4 & s_2 \overline{s_4} t_1 t_2 t_3 \\
\overline{s_3} s_4 \overline{t_3} t_4 & s_2 t_1 t_2 t_3 t_4 & s_1 \overline{s_2} s_4 t_3 t_4 & s_1 \overline{s_2} t_2 t_3 t_4 & s_2 \overline{s_4} t_1 t_3 t_4 \\
\overline{s_3} s_4 \overline{t_3} t_4 & s_2 t_1 t_2 t_3 t_4 & s_1 \overline{s_2} s_4 t_3 t_4 & s_1 \overline{s_2} t_2 t_3 t_4 & s_3 \overline{s_4} t_1 t_2 t_3 \\
\overline{s_4} t_1 t_2 t_3 t_4 & \overline{s_1} s_3 \overline{s_4} t_2 t_3 & \overline{s_1} s_3 t_2 t_3 t_4 & \overline{s_1} s_3 t_2 t_3 t_4 & \overline{s_3} s_4 t_1 t_3 t_4 \\
\end{array}
\]

**Theorem 2.28** (Mandel). Let \( C \) be a circuit of an oriented matroid \( M \) on \( E \). Let \( H = E \setminus C \). Suppose that we are given an oriented matroid \( \overline{M} \) on \( H \cup p \), where every circuit of \( \overline{M} \) involving \( p \) has rank(\( \overline{M} \)) + 1 elements, and \( \overline{M} \setminus p = M/C \). Then the following set \( \overline{K} \) is the set of cycles of an oriented matroid: \( \overline{K} = K(M) \setminus (\pm C) \cup \{ C \circ \overline{K} : \overline{K} = \overline{K} \setminus p \ \text{for some} \ \overline{K} \in K(\overline{M}) \ \text{with} \ p \in \overline{K}^+ \} \).

For \( M = M(I,-A) \), let \( C_1 = \overline{s_1} s_2 \overline{t_1} t_2, \ C_2 = \overline{s_3} s_4 \overline{t_3} t_4 \).

Define \( M_1 \) on \( (E \setminus C_1) \cup p_1 \) and \( M_2 \) on \( (E \setminus C_2) \cup p_2 \) by

\[
C(M_1) = \pm \{ s_3 s_4, t_3 t_4, s_3 t_3, s_3 t_4, t_3 t_4, s_3 p_1, t_3 p_1, t_4 p_1 \},
\]

\[
C(M_2) = \pm \{ s_1 s_2, t_1 t_2, s_1 t_1, s_2 t_2, t_1 t_1, t_2 t_1, s_1 p_2, t_1 p_2, t_2 p_2 \}.
\]
Both $M_1$ and $M_2$ are oriented matroids of rank 1, so we see that Mandel's condition that every circuit of $M_1$ ($M_2$) containing $p_1$ ($p_2$) have two elements is satisfied for $M_1$ and $M_2$. Furthermore, one can check that $C(M_i \setminus p_i) = M \setminus C_i$ for $i = 1, 2$, satisfying Mandel's requirement. We therefore replace $\pm C_1$ in $C(M)$ by the set $\pm \{s_1^1 s_2^2 s_3^3 s_4^4 t_1^1 t_2^2, s_1^1 s_2^2 s_3^3 s_4^4 t_1^3 t_2^3, s_1^1 s_2^2 s_3^3 s_4^4 t_1^1 t_2^4, s_1^1 s_2^2 s_3^3 s_4^4 t_1^3 t_2^4\}$ to get $\hat{M}$. It is easy to see that $\hat{M} \setminus C_2 = M \setminus C_2$, and thus we replace $\pm C_2$ in $C(\hat{M})$ by the set $\pm \{s_1^1 s_2^2 s_3^3 s_4^4 t_1^4 t_2^4, s_1^1 s_2^2 s_3^3 s_4^4 t_1^3 t_2^3, s_1^1 s_2^2 s_3^3 s_4^4 t_1^1 t_2^3, s_1^1 s_2^2 s_3^3 s_4^4 t_1^3 t_2^4\}$.

From Mandel's theorem we get that the resulting set of circuits is the set of circuits of an oriented matroid, and we call this oriented matroid $\hat{M}$.

First we show that $\hat{M}$ is a symmetric P-oriented matroid. The almost-complementary circuits of $\hat{M}$, circuits $C$ for which $|\{i: \{s_i^i, t_i^i\} \subseteq C\}| = 1$, are also circuits of $M$. Therefore $\hat{M}$ is a P-oriented matroid, by theorem A of Todd [43]. In order for $\hat{M}$ to be symmetric, every pair $(\hat{C}_1, \hat{C}_2)$ of circuits of $\hat{M}$ must have $(\text{swC}_1)$ orthogonal to $\hat{C}_2$. Note first that $|\text{swC}_1 \cap C| \geq 2$ for all circuits $C$ of $M$ except $C_2$, and $|\text{swC}_2 \cap C| \geq 2$ for all circuits $C$ of $M$ except $C_1$. For every new circuit $\hat{C}_1$ of $\hat{M}$ that replaces $\pm C_1$, we also have $\pm C_1$ contained in $\hat{C}_1$. Thus $\text{swC}_1$ will be orthogonal to every circuit of $\hat{M}$ except possibly the ones replacing $\pm C_1$ and $\pm C_2$. The same rule applies to circuits $\hat{C}_2$ replacing $\pm C_2$. Orthogonality among pairs $(\text{swC}_i, \hat{C}_j)$ for new circuits $\hat{C}_i$, $\hat{C}_j$ is quickly checked.

We still need to show that $\hat{M}$ is not representable. Suppose that $\hat{M}$ is represented by a matrix $(I, \hat{A})$. For every circuit $C$ of $M$
with $s_4 \in \mathbb{C}^+$, define $x(c)$ to be the elementary vector in $V$, the null space of $(I, -\tilde{A})$, with signed support equal to $C$, and with $x_{s_4}(c) = 1$. The submatrices of $(I, -\tilde{A})$ obtained by deleting two or three columns all have rank four since all of the oriented matroids $M \setminus \{e_i, e_j, e_k\}$ have rank four. For any pair $\{e_i, e_j\}$, we can order the vectors $x$ in $V$ with $x_{e_i} = x_{e_j} = 0$, $x_{s_4} = 1$ as follows: If $x_{t_4} \neq 0$ in some vector $x$ in $V$ with $x_{e_i} = x_{e_j} = x_{s_4} = 0$, then order the vectors $x$ with $x_{e_i} = x_{e_j} = x_{s_4} = 1$ in order of increasing $x_{t_4}$. Define $x_{e_i e_j}$, the endpoint of the line $e_i e_j$, to be the unique vector $x$ in $V$ with $x_{e_i} = x_{e_j} = x_{s_4} = 0$, $x_{t_4} = 1$. If $x_{t_4} = 0$ for all vectors $x$ in $V$ with $x_{e_i} = x_{e_j} = x_{s_4} = 0$, then $x_{t_4}$ is constant for all $x$ in $V$ with $x_{e_i} = x_{e_j} = 0$, $x_{s_4} = 1$. In that case, we let $x_{e_i e_j}$ be an arbitrary nonzero element of $V$ with $x_{e_i} = x_{e_j} = x_{s_4} = 0$. In both cases, for $x \neq y$ with $x_{s_4} = y_{s_4} = 1$, define $x < y$ if and only if $x_{e_i e_j} < y_{e_i e_j}$ when $x_{e_i e_j} > 0$, $x_{e_i e_j} = y_{e_i e_j}$ when $x_{e_i e_j} = 0$. Consider the following sequence of circuits of $\tilde{M}$: $(s_1 s_4 t_1 t_2, s_1 s_3 s_4 t_1 t_2, s_2 s_3 s_4 t_1 t_2, s_2 s_4 t_1 t_2)$. We take the vector $x_{s_2 t_2}$ to be a vector supported by the signed set $s_1 t_1 t_2$. This implies that $x(s_1 s_4 t_1 t_2) < x(s_1 s_3 s_4 t_1 t_2)$, and $x_{t_4}(s_1 s_4 t_1 t_2) = x_{t_4}(s_1 s_3 s_4 t_1 t_2)$. The vector $x_{s_1 t_1 t_2}$ is the vector supported by $s_1 s_2 s_3 t_1 t_2$, so $x(s_1 s_2 s_3 s_4 t_1 t_2) < x(s_2 s_3 s_4 t_1 t_2)$, and $x_{t_4}(s_1 s_2 s_3 s_4 t_1 t_2) = x_{t_4}(s_2 s_3 s_4 t_1 t_2)$. We take $s_1 t_1$ to be the vector supported by $s_2 s_3 t_2$, so $x(s_2 s_3 t_2) < x(s_2 s_3 s_4 t_2)$ and $x_{t_4}(s_2 s_3 s_4 t_2) = x_{t_4}(s_2 s_3 s_4 t_2)$. Finally we see that $s_3 t_3$ is the vector supported by $s_1 s_2 t_1 t_2$, so $x(s_2 s_3 t_2) < x(s_1 s_4 t_2)$ and $x_{t_4}(s_2 s_3 s_4 t_2) < x_{t_4}(s_1 s_4 t_2)$. This is a contradiction, so $\tilde{M}$ is not representable.
In figure 2.1 we give a part of the sphere system picture for $\hat{M}$, restricting ourselves to the relevant circuits. The intersection of two hyperplanes is represented by a line, directed toward the circuit $C$ on this line with $s_4 \notin C$, $t_4 \notin C^-$. Two of these circuits $C$ have $t_4 \in C^+$, so we have a monotonic cycle of pivots. In Mandel's terminology, the sequence \((s_1 s_4 t_1 t_4, s_1 s_3 s_4 t_3 t_4, s_2 s_3 s_4 t_3 t_4, s_2 s_4 t_2 t_4)\) is a directed $t_4$-walk with $s_4$ as the "infinity element". The discovery of such a walk is the most common technique for showing that an oriented matroid is non-representable.

Figure 2.1. A monotonic cycle of pivots
4. Factorizations

4.1 Definitions

There are several factorization theorems in the theory of positive (semi) definite matrices [18]. In this section, we study the extent to which these theorems remain true in the context of $P(P_0)$-oriented matroids. We start by defining the general framework for this investigation.

Let $\hat{M}$ be an oriented matroid on $E = S \cup T \cup V$, where $|S| = |T| = |V| = n$, and $S$, $T$, and $V$ are disjoint bases of $\hat{M}$. If $\hat{M}$ is representable, we can write the representation as $\hat{M} = M(I,-AB,-A)$, for $n \times n$ matrices $A$ and $B$, where the columns of $I$, $-AB$, and $-A$ are indexed by $S$, $T$, and $V$ respectively. Then $\hat{M} \setminus T = M(I,-A)$ and $\hat{M} \setminus S = M(B,I) = swM(I,-B)$. This motivates the following definition.

**Definition 4.1.** Let $M$ be a square oriented matroid on $E = S \cup T$, and let $M^1$ and $M^2$ be square oriented matroids on $E^1 = S^1 \cup T^1$ and $E^2 = S^2 \cup T^2$, respectively of the same rank as $M$. We say that $M$ is a product of $M^1$ and $M^2$ if there exists an extension $\hat{M}$ of $M$ on $E = S \cup T \cup V$, where $S$, $T$, and $V$ are bases of $M$, and $M^1$ and $M^2$ are isomorphic to $\hat{M} \setminus T$ and $sw(\hat{M} \setminus S)$ respectively, via the bijections

$$
\phi^1: S^1 \cup T^1 \to S \cup V \quad \text{defined by} \quad \phi^1(s^1_i) = s_i, \quad \phi^1(t^1_i) = v_i, \quad i = 1, \ldots, n
$$

$$
\phi^2: S^2 \cup T^2 \to T \cup V \quad \text{defined by} \quad \phi^2(s^2_i) = t_i, \quad \phi^2(t^2_i) = v_i, \quad i = 1, \ldots, n.
$$

Note first that for nonsingular square matrices $A$ and $B$, the oriented matroid $M(I,-AB)$ will be a product of $M(I,-A)$ and $M(I,-B)$. 
On the other hand, one can easily show that oriented matroids $M_1$ and $M_2$ can have many different products, in the framework of definition 4.1. However, this framework is sufficient to derive analogs of several theorems from matrix theory.

4.2 A P-oriented matroid problem

The product of a nonsingular matrix $A$ with its transpose is positive definite. For nonsingular matrices $A$ and $B$ such that $M(I,-B) = M(I,-A^T)$, the product $AB$ is a P-matrix. This follows from the formula

$$AB\left(\begin{array}{c} i_1, \ldots, i_k \\ j_1, \ldots, j_k \end{array}\right) = \sum_{\alpha_1 < \ldots < \alpha_k} \left[ A\left(\begin{array}{c} i_1, \ldots, i_k \\ \alpha_1, \ldots, \alpha_k \end{array}\right) \right] \left[ B\left(\begin{array}{c} \alpha_1, \ldots, \alpha_k \\ j_1, \ldots, j_k \end{array}\right) \right].$$

For the case $(i_1, \ldots, i_k) = (j_1, \ldots, j_k)$, the property $M(I,-B) = M(I,-A^T)$ implies that $A\left(\begin{array}{c} i_1, \ldots, i_k \\ \alpha_1, \ldots, \alpha_k \end{array}\right)$ and $B\left(\begin{array}{c} \alpha_1, \ldots, \alpha_k \\ j_1, \ldots, j_k \end{array}\right)$ have the same sign (see section 1). The nonsingularity of $A$ and $B$ implies that at least one term in the sum will be nonzero, so the sum will be positive. We try to see if this property holds for oriented matroids.

Let $M$ be a square oriented matroid of rank $n$, and suppose that $M$ is a product of square oriented matroids $M^1$ and $M^2$, when $M^1$ is isomorphic to $(swM^2)^*$ via the bijection $\phi^3: S^1 \cup T^1 \rightarrow S^2 \cup T^2$ with $\phi^3(s^1_i) = s^2_i$, $\phi^3(t^1_i) = t^2_i$. Let $\hat{M}$ be as in definition 4.1. We would like to show that $M = M \setminus V$ is a P-oriented matroid.

This problem in general remains unsolved, but theorems 4.2 and 4.4 prove special cases.
Theorem 4.2. Suppose $M$ and $\hat{M}$ are as above. If $\hat{M}$ has an adjoint, then $M$ is a P-oriented matroid.

Proof. We want to show that $\hat{M}\setminus V$ has no sign reversing circuits. Let $C \in C(\hat{M}\setminus V)$. Suppose that we could produce $K_1 \in K(\hat{M}\setminus T)$ and $K_2 \in K(\hat{M}\setminus S)$ such that $K_1 \setminus S = -(K_2 \setminus T)$, and $C = (K_1 \circ K_2) \setminus V$. Then $K_1$ and $K_2$ would be nonzero on $V$, and strictly disagree in sign on $V$. We hypothesized that $\hat{M}\setminus T$ is isomorphic to $(\hat{M}\setminus S)^*$. Thus $(\Phi(K_1^+), \Phi(K_1^-))$ and $K_2$ must agree in sign somewhere on $T$. This implies that $C$ is not sign reversing. In the representable case, it is always possible to produce such a $K_1$ and $K_2$. Let $M = (I, -AB, -A)$ represent $\hat{M}$, and let $(x, y, 0)$ be a vector of the nullspace of $M$. Then the nonsingularity of $AB$ and $A$ give $z$ such that $(0, y, z)$ and $(y, 0, -z)$ are vectors of the nullspace of $M$ corresponding to $K_1$ and $K_2$.

Lemma 4.3. If $\hat{M}$ has an adjoint and $C$ is a circuit of $\hat{M}\setminus V$, then there exist $K_1 \in K(\hat{M}\setminus T)$ and $K_2 \in K(\hat{M}\setminus S)$ such that $K_2 \setminus S = -(K_2 \setminus T)$ and $C = (K_1 \circ K_2) \setminus V$.

Proof. Let $F_1$ be the closure in $\hat{F} = S \cup T \cup V$ of $C \cap S$, and let $F$ be the closure in $\hat{E}$ of $C \cap T$. If $r(F_1 \cap F_2) \neq 0$, we can extend $\hat{M}$ to $\hat{M}$ on $\hat{E} \cup p$, with $p = \text{lex}(f)$ for some $f \in F_1 \cap F_2$ not a loop of $\hat{M}$. If $r(F_1 \cap F_2) = 0$, then $F_1$ and $F_2$ are nonmodular, i.e. they satisfy $r(F_1 \cap F_2) + r(F_1 \cup F_2) < r(F_1) + r(F_2)$. Then by Bachem and Kern's [3] theorem there is an extension of $\hat{M}$ to $\hat{M}$ on $\hat{E} \cup p$, where $p$ is in the closure of $F_1$ and of $F_2$ in $\hat{M}$, and $p$ is not a loop of $\hat{M}$. In both cases, this gives us circuits $C_1, C_2$ of
$\hat{M}$ with $C_1 \subseteq (C \cap S) \cup p$, and $C_2 \subseteq (C \cap T) \cup p$. Assume that $p \in C_1^-$, $p \in C_2^+$, without loss of generality. Then $(C_1 \circ C_2) \setminus p = \pm C$, since eliminating $p$ between $C_1$ and $C_2$ yields a cycle of $\hat{M}$ in $C$. Assume $(C_1 \circ C_2) \setminus p = C$. The cycles $K_1$ and $K_2$ as required by the lemma are then obtained by eliminating $p$ between $C_1$ and $C(V,p)$, and between $C_2$ and $-C(V,p)$, respectively. \[\] The assumption that there is an extension of $\hat{M}$ by an element in the closure of $F_1$ and $F_2$ for nonmodular flats $F_1, F_2$ of $\hat{M}$ was crucial to the previous theorem. This property is called the intersection property by Bachem and Wanka [4]. There are many nonrepresentable oriented matroids, for example the non-euclidean oriented matroids of Fukuda [15] and Mandel [33], for which this is not always possible.

The next result gives another situation in which we can prove that $\hat{M}/V$ is a $P$-oriented matroid.

**Theorem 4.4.** If all circuits $C$ of $\hat{M}/V$ satisfy the property 
$$\min(|C \cap S|, |C \cap T|) \leq 2,$$
then $\hat{M}/V$ is a $P$-oriented matroid.

**Proof.** As in the proof of theorem 4.2, we let $C \in C(\hat{M}/V)$, and we look for $K_1 \in K(\hat{M}/T)$ and $K_2 \in K(\hat{M}/S)$ such that $K_1 \setminus S = -(K_2 \setminus T)$, and $C = (K_1 \circ K_2) \setminus V$. Assume that $|C \cap S| \leq |C \cap T|$, $C$ is positive, and let $G = C \cap S$. Since $T$ is a base of $\hat{M}$, $|G| > 0$. Suppose $|G| = 1$, say $G = \{g\}$. We let $K_1 = C(V,g)$, and let $K_2$ be the result of eliminating $g$ between $-K_1$ and $C$. Then $K_1$ and $K_2$ will satisfy $K_1 \setminus S = -(K_2 \setminus T)$, and $C = (K \circ K_2) \setminus V$. 

Next, assume \(|G| = 2\), and let \(G = \{g_1, g_2\}\). For any cycle \(K\) with \(K \subseteq V \cup G\), \(G \subseteq K^+\), any cycle \(K'\) obtained from eliminating \(g\) between \(-K\) and \(C\) will either have \(g_2 \in K'^+\), \(g_2 \in K'^-\), or \(g_2 \notin K'\). We would like to show that the third case, \(g_2 \notin K'\), holds for some \(K, K'\).

The oriented matroid \(\hat{M} \setminus \{T \cup (S \setminus G)\}\) is a corank two oriented matroid [33]. This implies [33] that there is a unique ordering \((K_1, \ldots, K_r)\) of the cycles \(K\) of \(\hat{M} \setminus \{T \cup (S \setminus G)\}\) such that \(G \subseteq K^+\), such that \(K_1 = C(V, g_1) \circ C(V, g_2)\), \(K_r = C(V, g_2) \circ C(V, g_1)\), and for all \(1 \leq j < r\), \(K_j\) and \(K_{j+1}\) never strictly disagree in sign on any element. Let \(\hat{K}\) be the result of eliminating \(g_1\) between \(-C(V, g_1)\) and \(c\). If \(r = 1\), define \(K_1'\) to be the result of eliminating \(g_2\) between \(\hat{K}_1 = \hat{K} \circ -C(V, g_2)\) and \(\hat{K}_2 = -C(V, g_2) \circ \hat{K}\). Then, since \(C(V, g_1) \circ C(V, g_2) = C(V, g_2) \circ C(V, g_1)\) when \(r = 1\), we will have \(-\left(K_1' \setminus T\right) = K_1' \setminus G\), and \(G \cap K_1' = \emptyset\). If \(r > 1\), define \(K_1'\) to be as above, and define \(K_r'\) to be \(\hat{K}_2\) as above. For \(1 < j < r\), let \(K_j'\) be a result of eliminating \(g_1\) between \(-K_j\) and \(C\). We want \(j \in \{1, \ldots, r\}\) such that \(g_2 \notin K_j'\). If there is no such \(j\), we know that there is a \(k \in \{1, \ldots, r\}\) such that \(g_2 \in K_k'^+\), \(g_2 \in K_k'^-\), since \(g_2 \in K_1'^+\) and \(g_2 \in K_r'^-\). Let \(\bar{K}\) be \(K_k \circ K_{k+1}\), and let \(\bar{K}'\) be the result of eliminating \(g_2\) between \(K_k'\) and \(K_k'^+\). Then we will have \((\bar{K}' \setminus H) = K \setminus G\), since \(K_k'\) and \(K_k'^+\) never disagree in sign.

Thus we can always produce \(K, K'\) such that \(K \setminus G = -K' \setminus H\), and \(G \cap K' = \emptyset\), so the theorem is proved. \(\Box\)

This theorem shows that an example \(\hat{M}\) so that \(\hat{M} \setminus V\) is not a \(P\)-oriented matroid, must have rank five or more, since \(C\) would have at least six elements.
4.3 LU Factorizations

Definition 4.5. Let a square oriented matroid $M$ be a product of $M^1$ and $M^2$, with $\hat{M}$ as defined in definition 4.1. $\hat{M}$ is called an LU-factorization of $M$ if $s_j \notin C(S,v_i)$ for $j < i$, $t_j \notin C(T,v_i)$ for $j > i$.

A well known theorem in numerical analysis (see [18]) states that a matrix $A$ has an LU-decomposition with lower triangular $L$ and upper triangular $U$ both positive on the main diagonal iff the leading principal minors of $A$ are positive. Theorems 4.6 and 4.8 generalize part of this result in our framework.

Theorem 4.6. Let $\hat{M}$ be an LU factorization of $M = \hat{M}\setminus V$, and suppose that $s_i \in C(S,v_i)^+$, $t_i \in C(T,v_i)^-$ for all $i$. Then there are circuits $C_1, \ldots, C_n$ of $M$ such that $C_i \subseteq \{t_1, \ldots, t_i\} \cup \{s_1, \ldots, s_n\}$ and $C_i$ is sign preserving for every $i$.

Proof. For $i = 1$, $C(T,v_1)$ is $(v_1, t_1)$ by the LU property. Eliminate $v_1$ between $C(S,v_1)$ and $-C(T,v_1)$ to get $C(S,t_1)$ which will be sign preserving, since $s_1 \in C(S,v_1)^+$. For a general $i > 1$, suppose $B_i = \{t_1, \ldots, t_{i-1}\} \cup \{s_1, \ldots, s_n\}$ is a base of $M$. The result $K$ of eliminating $v_i$ between $C(S,v_i)$ and $-C(T,v_i)$ is in $\{s_1, \ldots, s_n\} \cup \{t_1, \ldots, t_i\}$. The cycle $K$ has $s_i \in K^+$, $t_i \in K^+$ and is contained in $B_i \cup t_i$. Hence it is a fundamental circuit, say $C_i$. Since $s_i \in C_i$, the set $(B_i \cup s_i) \setminus t_i$ is a base of $M$. Thus the theorem is true by induction. ⊓⊔
The converse of this theorem is not true, however. Consider the Vamos Matroid, a rank four matroid on \( E = \{e_1, \ldots, e_8\} \) such that every subset of four elements of \( E \) is a cobase (base of the dual), except for the sets \( \{e_1e_2e_3e_4\}, \{e_1e_2e_5e_6\}, \{e_1e_2e_7e_8\}, \{e_3e_4e_5e_6\}, \) and \( \{e_3e_4e_7e_8\} \). We relabel the set \( \{e_1, \ldots, e_8\} \) as \( (s_2, s_4, s_3, t_3, t_1, t_2, s_1, t_4) \). Let \( M \) be any orientation of the Vamos matroid, such as in [7]. A sphere system picture of \( M \) would look like figure 2.2 below.

![Diagram](image)

**Figure 2.2.** An orientation of the Vamos matroid

The lines defined by the intersections of planes \( t_1, t_2 \) and \( s_1, t_4 \), respectively, do not meet. However, the lines defined by the intersections
of planes $s_3,t_3$ and $s_2,s_4$ meet each of the previous two lines and intersect at the point $C$, which gives the cocircuit contained in \(\{t_1,t_2,s_1,t_4\}\). The points $A$, $B$, $D$, and $E$ give the other four degenerate cocircuits, contained in the sets $\{s_1,s_3,t_3,t_4\}$, $\{s_1,s_2,s_4,t_4\}$, $\{s_3,t_1,t_2,t_3\}$, and $\{s_2,s_4,t_1,t_2\}$. Note that none of these is a complementary 4-set, so every complementary 4-set is a cobase. This implies that the circuits $C(B_i,t_i)$ for complementary bases $B_i = \{t_1,\ldots,t_{i-1}\} \cup \{s_1,\ldots,s_n\}$ for $i = 1,\ldots,4$ contain $s_i$. We can therefore change signs on any orientation of this matroid to get circuits $C_i = C(B_i,t_i)$ with $s_i \in C_i^+$ for $i = 1,\ldots,4$. Call such an oriented matroid $\hat{M}$.

We would like to extend this oriented matroid by $n$ elements $v_1,\ldots,v_n$ to get $\hat{\hat{M}}$ such that $\hat{\hat{M}}$ is an LU factorization of $\hat{M} = \hat{\hat{M}} \setminus \iset{V}$ Consider the element $v_2$ of such an $\hat{M}$. The oriented matroid $\hat{\hat{M}} \setminus (V \setminus v_2) = \hat{\hat{M}}$ is a point extension of $\hat{M}$ satisfying $s_1 \notin C(S,v_2)$, $\{t_3,t_4\} \subset E \setminus C(T,v_2)$. The cocircuits of $\hat{\hat{M}}$ contained in $T \cup s_1 \cup v_2$ and in $S \cup \{t_3,t_4\} \cup v_2$ cannot contain $v_2$, then, by orthogonality. Therefore the cocircuits of $\hat{M}$ on the line through $A$ and $B$ and the cocircuit on $C$ are cocircuits of $\hat{\hat{M}}$.

From Las Vergnas' theorem ([27], see section I.2), all cocircuits in the line-closure of $A$, $B$, and $C$ are cocircuits of $\hat{\hat{M}}$. This line-closure contains the cocircuits on $D$ and $E$, and thus all of the vertices of the tetrahedron, which are fundamental cocircuits with respect to the cobase $\{s_2,s_3,s_4,t_3\}$. Thus all the cocircuits of $\hat{M}$ are cocircuits of $\hat{\hat{M}}$. But $v_2$ is not a loop of $\hat{\hat{M}}$ since $V$ is a base of $\hat{\hat{M}}$, so it is impossible to extend $\hat{M}$ to $\hat{\hat{M}}$ this way.
Theorem 4.7. If a square oriented matroid $M$ on $E = S \cup T$ has an adjoint, and there are circuits $C_1, \ldots, C_n$ such that $C_i \subseteq \{t_1, \ldots, t_i\} \cup \{s_i, \ldots, s_n\}$ and $\{s_i, t_i\} \subseteq C_i$ for all $i$, then for any $i$ there exists an extension by $v_i$ such that $s_j \not\in C(S, v_i)$ for $j < i$ and $t_j \not\in C(T, v_i)$ for $j > i$.

Proof. Let $F_1$ be the closure in $E$ of $\{t_1, \ldots, t_i\}$, and let $F_2$ be the closure in $E$ of $\{s_i, \ldots, s_n\}$. Then if there is an element $f \in F_1 \cap F_2$, with $r(f) \neq 0$, extend $M$ to $\hat{M}$ by $v_i = \text{lex}(f)$. If there is no such $f$, then by Bachem and Kern's [3] theorem we can extend $M$ to $\hat{M}$ by $v_i$, which is in the closure of $\{t_1, \ldots, t_i\}$ and of $\{s_i, \ldots, s_n\}$, and is not a loop. Thus $s_j \not\in C(S, v_i)$ for $j < i$, and $t_j \not\in C(T, v_i)$ for $j > i$. □

Unfortunately, it is not clear that all of these extensions could be done together, to get the desired $\hat{M}$.

This section concludes with some relationships between LULU-factorizations and symmetry.

Theorem 4.8. If $M$ on $E = S \cup T$ is symmetric and there exist $C_1, \ldots, C_n$ such that $C_i \subseteq \{s_i, \ldots, s_n\} \cup \{t_1, \ldots, t_i\}$, $\{s_i, t_i\} \subseteq C_i^+$ for each $i$, then $M$ is a P-oriented matroid.

Proof. By induction on $n$. For $n = 1$, $M$ has one circuit, and it is not sign reversing. Suppose $n > 1$. The $n$-sets $B_i = \{t_1, \ldots, t_{i-1}\} \cup \{s_i, \ldots, s_n\}$, for $i = 1, \ldots, n$ are bases of $M$, by an argument in the proof of theorem 4.6. Therefore the sets $B_i \setminus s_n$, for $i = 1, \ldots, n-1$
are bases of \( M \setminus t_n/s_n \). For each \( i < n \), the signed set \( C_i/s_n \) is a cycle of \( M \setminus t_n/s_n \) contained in \( (B_i/s_n) \cup t_i \), and therefore each \( C_i/s_n \) is a fundamental circuit of \( M \setminus t_n/s_n \) with respect to \( B_i/s_n \). These give us circuits \( C_i' = C_i/s_n \), \( i = 1, \ldots, n-1 \) with \( \{s_i, t_i\} \subseteq C_i'^+ \). Thus \( M \setminus t_n/s_n \) satisfies the hypotheses of the theorem, and the inductive hypothesis implies that \( M \setminus t_n/s_n \) is a P-oriented matroid.

We need to show by theorem 2.3 that every complementary n-set \( B \) of \( E \) is a base of \( M \), and for each \( i \), \( C(B, B_i) \) is strictly sign preserving (ssp). We do this by induction on \( |B \cap S| \). For \( |B \cap S| = 0 \), \( B = T \), which is a base of \( M \), and we also have that \( C(T, s_n) \) is ssp. For \( i \neq n \), \( C(T, s_i) \) contains \( t_n \) with the same sign that \( t_i \) has in \( C(T, s_n) \), by symmetry. If \( t_n \notin C(T, s_i) \), \( t_i \notin C(T, s_n) \), then \( C(T, s_i) \setminus s_n \) is a cycle of \( M \setminus t_n/s_n \), and thus ssp. If \( t_n \in C(T, s_i)^- \), \( t_i \in C(T, s_n)^- \), eliminate \( t_n \) between \( C(T, s_n) \) and \( C(T, s_i) \). The result, with \( s_n \) deleted, is a cycle of \( M \setminus t_n/s_n \), so is ssp, which implies that \( t_i \in C(T, s_i) \). If \( t_n \in C(T, s_i)^+ \), \( t_i \in C(T, s_n)^+ \), eliminate \( t_n \) between \( -C(T, s_n) \) and \( C(T, s_i) \) to get the same result. Thus \( C(T, s_i) \) is ssp for all \( i \).

Now suppose that every complementary n-set \( B \) with \( |B \cap S| < k \) is a base, and \( C(B, B_i) \) is ssp for each \( i \). Let \( \hat{B} \) be a complementary n-set with \( |\hat{B} \cap S| = k > 1 \). There is an \( s_i \in \hat{B} \), so by the fact that \( \hat{B} \setminus s_i \cup t_i = \hat{B}' \) has \( |\hat{B}' \cap S| < k \), we have that \( C(\hat{B}', s_i) \) is ssp, and thus \( \hat{B} \) is a base, with \( C(\hat{B}, t_i) = C(\hat{B}', s_i) \). If \( s_n \in \hat{B} \), then for \( i \neq n \), \( C(\hat{B}, B_i) \setminus s_n \) is a cycle of \( M \setminus t_n/s_n \), and
thus \( C(\hat{B}, t_n) = C((\hat{B} \setminus s_n) \cup t_n, s_n) \) which is also ssp since
\[ |((\hat{B} \setminus s_n) \cup t_n) \cap S| < k. \]
So assume that \( t_n \in \hat{B}. \)

**Lemma 4.9.** \( C(\hat{B}, s_n) \) is ssp.

**Proof.** If \( t_n \not\in \overline{C(\hat{B}, s_n)} \), then \( C(\hat{B}, s_n) \setminus s_n \) is a sign reversing cycle of \( M \setminus t_n/s_n \), a contradiction. So \( t_n \in \overline{C(\hat{B}, s_n)} \). Suppose \( t_n \in C(\hat{B}, s_n)^- \). \( \hat{B} \) contains an \( s_i \), so let \( \hat{B}' = (\hat{B} \setminus s_i) \cup t_i \).

Consider \( C(\hat{B}', s_n) \), which is ssp, so \( t_n \in C(\hat{B}', s_n)^+ \). The signed sets \( \text{sw}C(\hat{B}, s_n) \) and \( C(\hat{B}', s_n) \) are both nonzero on at most three elements, \( s_n, t_n, \) and \( t_i \). Now \( t_n \in [\text{sw}C(\hat{B}, s_n)]^+ \), and \( \{s_n, t_n\} \subseteq C(\hat{B}', s_n)^+ \). Since \( t_n \in C(\hat{B}', s_n)^- \), we have \( s_n \in [\text{sw}C(\hat{B}, s_n)]^+ \).

By duality of \( M \) and \( \text{sw}M \), \( t_i \) must appear in \( \text{sw}C(\hat{B}, s_n) \) and \( C(\hat{B}', s_n) \) with opposite signs, and thus \( s_i \) appears in \( C(\hat{B}, s_n) \) with the opposite sign of \( t_i \) in \( C(\hat{B}', s_n) \). Since \( t_n \in C(\hat{B}', s_n)^- \), \( t_n \in C(\hat{B}', s_n)^+ \)
we can eliminate \( t_n \) between them to get a cycle \( K \) of \( M \setminus t_n/s_n \) when we remove \( s_n \). Then \( s_i \) and \( t_i \) appear in \( K \) with opposite signs, and no other pair appears in \( K \). This contradicts the hypothesis that \( M \setminus t_n/s_n \) is P. \( \square \)

Now we show that \( C(\hat{B}, \overline{B}_j) \) is ssp for \( j \neq n \). If \( \overline{B}_j = t_j \), then
\[ C(\hat{B}, t_j) = C((\hat{B} \setminus s_j) \cup t_j, s_j) \], which is ssp since \( |((\hat{B} \setminus s_j) \cup t_j) \cap S| < k. \)
If \( \overline{B}_j = s_j \), then the proof that \( C(\hat{B}, s_j) \) is ssp mimics the proof that \( C(T, s_i) \) is ssp for \( i \neq n \), so \( C(\hat{B}, s_j) \) is ssp. Thus we have completed the induction. \( \square \)

**Definition 4.10.** If \( \hat{M} \) is an LU-factorization of \( \hat{M} \setminus V \), such that
\[ \hat{M} \setminus S = (\hat{M} \setminus T)^* \], \( \hat{M} \) is called an LL\(^T\) factorization of \( \hat{M} \setminus V \).
Recall that section 4.2 was concerned with investigating when such a property implied that \( \hat{\mathcal{M}} \setminus \mathcal{V} \) was a P-oriented matroid. The following example shows that the existence of such a factorization does not imply symmetry of \( \hat{\mathcal{M}} \setminus \mathcal{V} \). Consider

\[
\hat{\mathcal{M}} = \begin{pmatrix}
1 & 0 & 0 & -1 & -1 & -3 & -1 & 0 & 0 \\
0 & 1 & 0 & -1 & -2 & -4 & -1 & -1 & 0 \\
0 & 0 & 1 & -2 & -3 & -8 & -2 & -1 & -1
\end{pmatrix}.
\]

Then

\[
\hat{\mathcal{M}} \setminus \mathcal{S} = \begin{pmatrix}
1 & 1 & 3 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1
\end{pmatrix},
\]

and one can check that \( \hat{\mathcal{M}} \setminus \mathcal{S} = (\hat{\mathcal{M}} \setminus \mathcal{T})^* \). But \( \hat{\mathcal{M}} \setminus \mathcal{V} \) is not symmetric, since for \( A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 2 & 3 & 8 \end{pmatrix} \), we have \( 0 = A \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} \neq A \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix} = 1. \)

### 4.4 Eigendecompositions

**Definition 4.11.** A square oriented matroid is **orthogonal** if \( M^* = M_S \).

The motivation for this definition is clear in the linear case, where \( M(I,-A) \) has \( M^* = M(A^T, I) \), \( M_S = M(A^{-1}, I) \).

A symmetric matrix \( A \) has a decomposition into \( QDQ^T \), where \( Q \) is orthogonal and \( D \) is diagonal. In that case, \( M(I,-A) \) is the product of \( M_1 = M(I,-Q) \) and \( M_2 = M(I,-DQ^T) \). The inertia of a matrix \( A \) is the triple \( (r^+, r^0, r^-) \), where \( r^+ \), \( r^0 \), and \( r^- \) are the numbers of positive, zero, and negative diagonal elements of \( D \).
If $A$ is nonsingular, $r^0 = 0$. Motivated by this we make the following definition.

**Definition 4.12.** $\hat{M}$ on $S \cup T \cup V$, where $|S| = |T| = |V|$, and $S, T, V$ are disjoint bases of $\hat{M}$ is an eigendecomposition of $\hat{M} \setminus V$ if $\hat{M} \setminus T$ is orthogonal, and $\hat{M}_R \setminus S$ and $(\hat{M} \setminus T)$ are isomorphic, for some subset $R$ of $V$, via the bijection $\phi: T \cup V \rightarrow S \cup V$ given by $\phi(t_i) = s_i$, $\phi(v_i) = v_i$ for $i = 1, ..., n$. The triple $(|R|, 0, n-|R|)$ is called the inertia of the decomposition.

**Theorem 4.13.** If $\hat{M}$ is an eigendecomposition of $\hat{M} \setminus V$, then any $\hat{M}'$ obtained from $\hat{M}$ by switching signs on $V$ and reordering elements of $V$ is an eigendecomposition of $\hat{M}' \setminus V = \hat{M} \setminus V$.

The proof is straightforward. Furthermore, these operations preserve the inertia of the decomposition. It is not known if two oriented matroids $\hat{M}, \hat{M}'$ on $S \cup T \cup V$, as above, such that $\hat{M} \setminus V = \hat{M}' \setminus V$ and $\hat{M}, \hat{M}'$ are both eigendecompositions of $\hat{M} \setminus V$, always have the same inertia.

**Theorem 4.14.** If $\hat{M}$ is an eigendecomposition of $\hat{M} \setminus V$ with $|R| = n$, then $[D_{\hat{M}}(S \cup T, v_i)]v_i$ is a negative eigencycle of $(\hat{M} \setminus V)^*$, for all $i$. Also, eliminating $v_i$ between $C(S, v_i)$ and $-C(T, v_i)$ gives a positive eigencycle of $\hat{M} \setminus V$, for all $i$.

**Proof.** $(\hat{M} \setminus V)^* = \hat{M}^*/V$, so $[D_{\hat{M}}(S \cup T, v_i)]v_i \in K(\hat{M} \setminus V)^*$. It is a negative eigencycle, since $[D_{\hat{M}}(S \cup T, v_i)](S \cup v_i) = [D_{\hat{M} \setminus S}(T, v_i)]v_i$ which corresponds to $[-D_{\hat{M} \setminus T}(S, v_i)]v_i$ in the isomorphism of $\hat{M} \setminus S$ and
\( \hat{M} \backslash T \), since \( |R| = n \) and similarly \(-D_{\hat{M} \backslash T}(S,v_i)\)\backslash v_i = \\
[D_M(S \cup T, v_i)]\backslash (T \cup v_i)\). Also, \([C(S,v_i)]\backslash v_i = [-C(T,v_i)]\backslash v_i\), since \(|R| = n\). □

**Corollary 4.15.** If \( \hat{M} \) is an eigendecomposition of \( \hat{M} \backslash V \), with \(|R| = n\), then the positive eigencycles of \( \hat{M} \backslash V \) given by theorem 4.14 form a set of \( n \) different eigencycles.

**Proof.** Let \( C(S,v_i) \) and \( C(S,v_j) \) be two different fundamental circuits of \( \hat{M} = \hat{M} \backslash V \) with respect to \( S \). By the duality of \( M \) and \( M_S \), \( C(S,v_i) \) and \( C(S,v_j) \) are different on \( S \). □

We make the following observation for the representable case. For an \( n \times n \) matrix \( A \), existence of an eigendecomposition of \( M(I,-A) \) does not imply diagonalizability of \( A \). Recall that in section II.1 we produced matrices \( A \) and \( A' \) such that \( M(I,-A) = M(I,-A') \), \( A \) was symmetric, and \( A' \) was not diagonalizable. Then the matrix obtained by adjoining the negatives of the eigenvectors of \( A \) to \( (I,-A) \) represents an eigendecomposition of \( M(I,-A) = M(I,-A') \).

This area of factorizations appears worthy of further study.
CHAPTER III
LEXICOGRAPHIC PERTURBATIONS

Lexicographic extensions of oriented matroids were introduced by Las Vergnas [27]. In the representable case, such an extension corresponds to adding a column that is a weighted sum of certain other columns. This allows one to apply degeneracy-resolving techniques of linear programming that involve implicit perturbation of the right-hand-side to oriented matroids. This chapter extends this technique to sequential perturbation of several of the elements of an oriented matroid. The aim is to find a reasonable characterization of the circuits of the resulting oriented matroid, so that an implementable pivoting algorithm can operate implicitly on the perturbed oriented matroid.

The first section of this chapter extends the definition of lexicographic extensions (see ch. I) to include extensions determined by sets of elements that are not necessarily independent. In the second section, we introduce the sequential perturbation of all the elements \( T \) of an oriented matroid \( M \) on \( S \cup T \), where \( S \) is a base of \( M \). It is shown that a circuit of the perturbed oriented matroid can be obtained from a single tableau of \( M \). Applications of the perturbation of section 2 to the linear complementarity problem are discussed in section 3.

There we show that for linear complementarity problems \((A, b)\) with \( A \) semi-monotone and \( b \) nondegenerate, the implicit use of
Lemke's algorithm on the perturbed oriented matroid will solve the problem in some cases in which Lemke's algorithm fails. Thus we extend the class of matrices for which the LCP may be solved by pivoting algorithms. The last section generalizes the concept of sequential perturbations of columns of a square matrix in which the most significant perturbations occur on the main diagonal. Some results relating classes of matrices with this perturbation scheme are given.

1. Preliminaries

A lexicographic extension is usually given in terms of an independent set. The first theorem of this chapter shows that this can be generalized to arbitrary sets. Recall (see ch. I) that for a base $B$ of $M$ and element $e$

$$\mathcal{G}(B,e) = \begin{cases} (\emptyset, e) & \text{if } e \in B \\ C(B, e) \setminus e & \text{if } e \notin B. \end{cases}$$

**Theorem 1.1.** Let $M$ be an oriented matroid on $E$, and let $R = \{r_1, \ldots, r_t\}$, $t < n$ be a subset of $E$. Let $I = (i_1, \ldots, i_s)$ be a maximal independent subset of $R$, found by applying the greedy algorithm to $(r_1, \ldots, r_t)$. Then the set of circuits $\hat{C}$ on $\hat{E} = E \cup p$ obtained by the rules

(a) for all circuits $C \in C(M)$, $C \in \hat{C}$

(b) for every base $B$ of $M$, $C(B, p) = (p, \emptyset) \circ G_M(B, r_1) \circ G_M(B, r_2) \circ \cdots \circ G_M(B, r_t)$

is the set of circuits of $\hat{M}$, the lexicographic extension of $M$ by $p = \text{lex}(i_1, \ldots, i_s)$ as defined in chapter I.
Proof. If \( \{r_1, \ldots, r_t\} \) is independent, rules (a) and (b) simply give Todd's [44] characterization of the circuits of \( \hat{M} \). We would like to show that, for a given \( B \) and \( b_k \in B \), the first \( C_M(B, r_j) \) in which \( b_k \) appears is always \( C_M(B, i_k) \) for some \( i_k \). Let \( B \) be a base of \( M \), \( b_k \) an element of \( C(B, p) \). Let \( r_f \) be the first \( r_j \) such that \( b_k \in C(B, r_j) \). Suppose \( C \in C(M) \), \( r_f \in C \subseteq \{r_1, \ldots, r_f\} \). Use the circuits \( C(B, r_i) \), for \( r_i \) in \( (C \cap \{r_1, \ldots, r_f-1\}) \setminus B \) to eliminate \( C \cap \{r_1, \ldots, r_f-1\} \) from \( C \). The result is a cycle \( K \) of \( M \) with \( K \subseteq B \cup r_f \). If \( r_f \notin B \), then \( K = \pm C(B, r_f) \). However, \( b_k \notin K \) and \( b_k \in C(B, r_f) \), so this is a contradiction. If \( r_f \in B \), i.e. \( r_f = b_k \), then \( K \subseteq B \), which is also impossible. So no such circuit exists, and \( r_f \) is independent of \( \{r_1, \ldots, r_{f-1}\} \). Since the sign of \( b_k \) in \( C(B, p) \) is determined by the first \( r_i \) such that \( b_k \in C_M(B, r_i) \), it is clear that we only need to consider \( r_i \)'s picked by the greedy algorithm. \( \square \)

In the following, if \( M \) and \( \hat{M} \) are as in the theorem, we write \( p = \text{lex}(r_1, \ldots, r_t) \) extends \( M \) to \( \hat{M} \), without requiring that \( \{r_1, \ldots, r_t\} \) be an independent set.

Definition 1.2. Let \( M \) be an oriented matroid on \( E \), and let \( e \) be an element of \( E \). Let \( \hat{e} = \text{lex}(e, r_1, \ldots, r_t) \) extend \( M \) to \( \hat{M} \), where \( \{r_1, \ldots, r_t\} \) is a subset of \( E \) not containing \( e \). Then \( M_{\hat{e}} = \hat{M} \setminus e \) is called a perturbation of \( M \), with \( \hat{e} \) the perturbed element.

Proposition 1.3 [43]. Let \( M \) be an oriented matroid, and let \( p = \text{lex}(r_1, \ldots, r_t) \) extend \( M \) to \( \hat{M} \), where \( \text{rank}(r_1, \ldots, r_t) = m \). Then every circuit of \( \hat{M} \) containing \( p \) has at least \( m+1 \) elements.
2. VanderMonde Perturbations

Let $M$ be an oriented matroid on $E = S \cup T$, where $S = \{s_1, \ldots, s_m\}$ and $T = \{t_1, \ldots, t_{n-m}\}$ are disjoint, and $S$ is a base of $M$. Define $M_0 = M$, and for $i = 1, \ldots, n-m$ let $\xi_i = \text{lex}(t_i, -s_1, \ldots, -s_m)$ extend $M_{i-1}$ to $\hat{M}_i$, and then let $M_i = \hat{M}_i \setminus t_i$.

**Definition 2.1.** $\hat{M} = M_{n-m}$ is called a VanderMonde perturbation of $M$.

The reason for this name becomes clear in the representable case. When $M = M(I, -A)$, where $T$ indexes the columns of $-A$, then $\hat{M} = M(I, -\hat{A})$ for $A = \hat{A} + \epsilon V$, where $\epsilon > 0$ is sufficiently small and

$$V = \begin{pmatrix}
  v_1 & v_2 & \cdots & v_{n-m} \\
  v_1 & v_2 & \cdots & v_{n-m} \\
  \vdots & \vdots & \ddots & \vdots \\
  v_1 & \cdots & \cdots & v_{n-m}
\end{pmatrix}, \quad \epsilon \gg v_1 \gg v_2 \gg \ldots \gg v_{n-m} > 0,$$

a VanderMonde matrix.

We would like a reasonable characterization of the circuits of $\hat{M}$ in terms of the circuits of $M$. Note first that by proposition 1.3, every circuit of $M_i$ containing $\xi_j$ for some $j \leq i$ will have $m+1$ elements. Thus $\hat{M}$ will be a uniform oriented matroid of rank $m$ on $n$ elements. Every $(m+1)$-set will support a circuit of $\hat{M}$. Suppose we have an $(m+1)$-set $S_Q \cup T_R$ contained in $E$. The following independent sets of $M$ will be useful in determining the circuit of $\hat{M}$ in $S_Q \cup T_R$. 

Definition 2.2. For an \((m+1)\)-set \(S_Q \cup T_R = \{s_q(1), \ldots, s_q(k)\} \cup \{t_r(1), \ldots, t_r(\lambda)\}\) with \(\{s_q(1), \ldots, s_q(k)\}\) and \(\{t_r(1), \ldots, t_r(\lambda)\}\) naturally ordered in \(E\), define \(I_0, I_1, \ldots, I_{\lambda-1}\) as follows. Let \(I_0 = S_Q\), and for \(i = 1, \ldots, \lambda-1\), let \(I_i = I_{i-1} \cup \{e\}\), where \(e\) is the first element of \((t_r(\lambda-i+1), s_1, \ldots, s_m)\setminus I_{i-1}\) such that \(I_i\) is independent. Let \(BQR = I_{\lambda-1}\).

Lemma 2.3. For \(i = 1, \ldots, \lambda-1\) the sets \(B_i = I_{i-1} \cup \{t_{r(j)}: j \leq \lambda-i\}\) are bases of \(\mathcal{M}_{\lambda-i}\).

Proof. For each \(i\), \(I_{i-1}\) is independent in \(\mathcal{M}\), so a circuit of \(\mathcal{M}_{\lambda-i}\) in \(B_i\) must contain a \(t_{r(j)}\) for \(j \leq \lambda-i\). However, such a circuit must have \(m+1\) elements, and \(B_i = I_{i-1} \cup \{t_{r(j)}: j \leq \lambda-i\}\) is constructed so that it has \(m\) elements. □

The goal is to show that the orientation of the circuit of \(\mathcal{M}\) in \(S_Q \cup T_R\) can be determined from the tableau associated with \(BQR\).

For \(i = 1, \ldots, \lambda-1\) define

\[
K_i = \begin{cases} 
(\emptyset, \emptyset) & \text{if } t_r(\lambda-i+1) \not\in I_i \\
\mathcal{G}(BQR, t_r(\lambda-i+1)) \circ \mathcal{G}(BQR, s_1) \circ \ldots \circ \mathcal{G}(BQR, f_1) & \text{if } t_r(\lambda-i+1) \not\in I_i,
\end{cases}
\]

where \(f_1\) is the last member of \((t_r(\lambda-i+1), s_1, \ldots, s_m)\) before \(I_i \setminus I_{i-1}\).

Theorem 2.4. The circuit of \(\mathcal{M}\) in \(S_Q \cup T_R\) is given by the following algorithm
(0) \( \bar{K} + (\hat{t}_{r(1)}, \emptyset) \circ G(B_{QR}, t_{r(1)}) \circ -G(B_{QR}, s_1) \circ \ldots \circ -G(B_{QR}, s_m). \)

(1) For \( i = \ell - 1 \) down to 1 do

(2) \( \bar{K} + [((\hat{t}_{r(\ell-i+1)}, \emptyset) \circ K_i \circ \pm\bar{K}] \setminus (I_i \setminus I_{i-1}) \)

where we have \( \pm\bar{K} \) Depending on whether \( I_i \setminus I_{i-1} \in \bar{K}^c \).

Proof. From theorem 1.1 and the construction of \( I_i \), we have

\[
C_{M_{r(\ell-i)}}(B_i, \hat{t}_{r(\ell-i+1)}) = (\hat{t}_{r(\ell-i+1)}, \emptyset) \circ G_{M_{r(\ell-i)}}(B_i, t_{r(\ell-i+1)}) \circ -G_{M_{r(\ell-i)}}(B_i, s_1) \circ \ldots \circ -G_{M_{r(\ell-i)}}(B_i, s_m).
\]

The first \( C_{M_{r(\ell-i)}}(B_i, e) \)

for \( e \in (t_{r(\ell-i+1)}, s_1, \ldots, s_m) \) that contains a \( \hat{t}_{r(j)} \) for \( j \leq \ell - i \)
will have \( m+1 \) elements and thus render the succeeding ones irrelevant.

This circuit is \( C_{M_{r(\ell-i)}}(B_i, I_i \setminus I_{i-1}) \). For elements \( f \) of \( (t_{r(\ell-i+1)}, s_1, \ldots, s_n) \) before \( I_i \setminus I_{i-1} \), \( C_{M_{r(\ell-i)}}(B_i, f) = G(B_{QR}, f) \).

This gives us, with \( f_i \) the last such \( f \) before \( I_i \setminus I_{i-1} \),

\[
C_{M_{r(\ell-i+1)}}(B_i, \hat{t}_{r(\ell-i+1)}) = (\hat{t}_{r(\ell-i+1)}, \emptyset) \circ G(B_{QR}, t_{r(\ell-i+1)}) \circ -G(B_{QR}, s_1) \circ \ldots \circ -G(B_{QR}, f_i) \circ -G_{M_{r(\ell-i)}}(B_i, I_i \setminus I_{i-1}).
\]

An application of step (2) of the algorithm changes \( \bar{K} \) from \( \bar{K} = C_{M_{r(\ell-i)}}(B_i, \hat{t}_{r(\ell-i+1)}) \)

\[
= \pm C_{M_{r(\ell-i)}}(B_i, I_i \setminus I_{i-1}) \rightarrow C_{M_{r(\ell-i+1)}}(B_i, \hat{t}_{r(\ell-i+1)}).
\]

When \( i = 1 \), we have \( C_{M_{r(\ell)}}(B_1, \hat{t}_{r(\ell)}) \), which is a circuit of \( \hat{M} \) in \( S_{Q \cup \hat{t}_{R}} \).

With the help of theorem 2.4 one can determine the circuit of \( \hat{M} \)
contained in an \( (m+1) \)-set \( S_{Q \cup \hat{t}_{R}} \) by finding the corresponding \( B_{QR} \).

Note that if, at some stage of the algorithm, \( I_i \setminus I_0 \) only contains \( t_i \)'s, we can change the \( t_i \)'s in \( C_{M_{r(\ell-i)}}(B_i, \hat{t}_{r(\ell-i+1)}) \) to \( \hat{t}_i \)'s and stop.
In the next section we will give a detailed description of an application of this procedure that is specially designed for the LCP. We end this section with a curious example of a VanderMonde perturbation.

Theorem 2.5. Let \( M \) be a square oriented matroid such that every element of \( T \) is a loop of \( M \). Let \( \tilde{M} \) be the VanderMonde perturbation of \( M \). Then \( \tilde{M} \) is a symmetric oriented matroid.

Proof. This can be proved directly from the characterization of the circuits of \( \tilde{M} \), but an easier way is to note that \( \tilde{M} \) can be represented by \(( I, -A)\), where

\[
A = \begin{pmatrix}
\varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_n \\
\varepsilon_2 & \vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\varepsilon_n & \cdots & \cdots & \varepsilon_n \\
\end{pmatrix}
\]

for \( 0 < \varepsilon_n < \cdots < \varepsilon_1 \). Thus we can choose \( \varepsilon_i = (\frac{1}{2})^i \), which will yield a symmetric matrix \( A \). \( \square \)

3. Application to the LCP

In this section we show how the perturbation procedure of section 2 can be applied to solve certain LCP's for which the usual Lemke's algorithm encounters \( q \)-rays \( C \) with \( q \notin C \).
Definition 3.1. A square oriented matroid is called semi-monotone (SM) if for every \((\emptyset, \emptyset) \neq K \in K(M)\) such that \(K^- \cap T = \emptyset\), there is an index \(i\) such that \(t_i \in K^+, s_i \notin K^-\).

Clearly, a \(P_0\)-oriented matroid is semi-monotone.

Theorem 3.2. Let \(M\) be a semi-monotone oriented matroid. For some \(j \in N\) and \(I \subseteq N\), let \(\hat{t}_j = \text{lex}(t_j, -s_i(1), \ldots, -s_i(r))\) extend \(M\) to \(M_j\), where \(\{i(1), \ldots, i(r)\} = I\). Let \(\hat{M} = M_j \backslash \hat{t}_j\). Then \(\hat{M}\) is semi-monotone.

Proof. Suppose \(\hat{M}\) has \((\emptyset, \emptyset) \neq K \in K(\hat{M})\) such that \(K^- \cap T = \emptyset\), and \(t_i \in K^+ \Rightarrow s_i \in K^-\) for all \(i\). Let \(C_1 \circ C_2 \circ \ldots \circ C_r = K\) be a conformal decomposition of \(K\) into circuits of \(\hat{M}\). If \(\hat{t}_j \notin K\), then \(K\) is a cycle of \(M\), contradicting the assumption that \(M\) was SM. Let \(C = C_p\) be a circuit containing \(\hat{t}_j\) in the conformal decomposition of \(K\). Extend \(C \backslash \hat{t}_j\) to a base \(B\) of \(\hat{M} \backslash \hat{t}_j\).

Then \(C_{\hat{M}}(B, \hat{t}_j) = (\hat{t}_j, \emptyset) \circ G(B, t_j) \circ -G(B, s_i(1)) \circ \ldots \circ -G(B, s_i(r))\).

Consider the subsequence \((s_{\hat{t}_j}(1), \ldots, s_{\hat{t}_j}(r))\) of elements \(s_i(k)\) such that \(s_i(k) \notin B\). Then let \(K_p = C(B, t_j) \circ -C(B, s_{\hat{t}_j}(1)) \circ \ldots \circ -C(B, s_{\hat{t}_j}(r))\).

Then \(K_p\) (with \(\hat{t}_j\) replacing \(t_j\)) will conform to \(C_p\), except possibly on elements \(s_i(k) \notin B\) which may be in \(C_p^+\) because of the term \((s_i(k), \emptyset) = -G(B, s_i(k))\) in the expansion of \(C_p\). Nevertheless, if \(s_i(k) \in C_p^-\), then \(s_i(k) \in K_p^-\). For every \(C_p\) containing \(\hat{t}_j\) in the conformal decomposition of \(K\), define the corresponding \(K_p\) of \(M\). For \(C_p\) with \(\hat{t}_j \notin C_p\), let \(K_p = C_p\). Then let \(\hat{K} = K_1 \circ \ldots \circ K_r\).
For each $k$ we have $t_k \in C^\pm$ iff $t_k \in \hat{K}^\pm$, and $s_k \in C^- \Rightarrow s_k \in \hat{K}^-$. Thus $\hat{K}^- \cap T = \emptyset$, and $t_i \in \hat{K}^+ \Rightarrow s_i \in \hat{K}^-$. This contradicts the hypothesis that $M$ was SM. □

These semi-monotone oriented matroids are related to the class of strictly semi-monotone oriented matroids, introduced by Todd [43].

**Definition 3.3.** A square oriented matroid is called strictly semi-monotone (SSM) if it has no nonzero sign reversing cycles that are positive on $T$.

**Theorem 3.4.** Let $M$ be a semi-monotone oriented matroid. If every complementary $n$-set of $E$ is a base, then $M$ is a strictly semi-monotone oriented matroid.

**Proof.** By induction on $n$. For $n = 1$, we have $E = \{s_1, t_1\}$. When $t_1$ is independent and $(s_1, t_1)$ is not a circuit of $M$, then $M$ must be $M\setminus \{1, -1\}$, which is SSM. Now let the rank of $M$ be $n$, and let $C$ be a sign reversing circuit of $M$, positive on $T$. It is easy to show that if $M$ is semi-monotone, then so is $M\setminus T_I/S_I$ for any $I \subseteq N$.

Thus, if $T \setminus C$ is nonempty, say $T \setminus C = T_I$ for $I \neq \emptyset$, then $C \setminus S_I$ becomes a cycle $K_C$ of $M\setminus T_I/S_I$, which has a sign reversing circuit positive on $T \setminus T_I$ in its conformal decomposition. Note also that every complementary $|N\setminus I|$-set of $M\setminus T_I/S_I$ is a base, since it corresponds to a complementary base of $M$ that contain $S_I$. So we can assume that $T \subseteq C$. Thus $C = -C(T, s_i)$ for some $s_i$. We can create a negative eigencycle of $M$ that is positive on $T$ by taking $-C(T, s_i) \circ -C(T, s_i+1) \circ \ldots \circ -C(T, s_i-1)$ (indices mod $n$). This contradicts the SM property of $M$. □
Now suppose that we have an oriented matroid $M$ on $E \cup p$ with $M \setminus p$ semimonotone. We want to solve the LCP of this oriented matroid. We know that if we perturb $t_i$ by $\hat{t}_i = \text{lex}(t_i, -s_1, \ldots, -s_n)$ for $i = 1, \ldots, n$, to get $\tilde{M}$ on $E \cup p$, every $n$-set of $\tilde{M} \setminus p$ will be a base. This implies that $\tilde{M} \setminus p$ is strictly semi-monotone. From [43] we know that Lemke's algorithm gives a solution to the problem for $\tilde{M}$. In many cases this will correspond to a solution for $M$.

Let $\hat{M}$ on $E \cup p \cup q$ be such that $\hat{M} \setminus q = M$, $S \cup q$ is the set underlying a positive circuit of $\hat{M}$, and every circuit of $\hat{M}$ containing $q$ contains $n+1$ elements. Perturb $p$ first so that $\hat{p} = \text{lex}(p, -s_1, \ldots, -s_n)$, and then perturb the $t_i$'s sequentially by $\hat{t}_i = \text{lex}(t_i, -s_1, \ldots, -s_n)$ for $i = 1, \ldots, n$, to get $\tilde{M}$ on $S \cup \hat{p} \cup \hat{p} \cup q$. Every $n$-set of $\tilde{M}$ is a base, and $\tilde{M} \setminus \{\hat{p}, q\}$ is strictly semi-monotone.

In the first step of Lemke's algorithm, we have $C_1 = C(S, q)$, which is a circuit of $\tilde{M}$ with $n+1$ elements. It is also a circuit of $\hat{M}$. Also, $s_i \in C_{\tilde{M}}(S, p)^-$ iff $s_i \in C_{\hat{M}}(S, \hat{p})^-$, which can be verified from the formula for $C_{\tilde{M}}(S, \hat{p})$. Thus $S$ is a solution to $\hat{M}$ iff $S$ is a solution to $\tilde{M}$. Otherwise, there is a unique $s_i \in C_{\hat{M}}(S, p)^-$ such that $(S \setminus s_i) \cup p \cup q$ supports a positive circuit with $n+1$ elements. Thus this circuit, with $\hat{p}$ replacing $p$, is a circuit of $\hat{M}$.

Suppose now that at some step of the algorithm we have $H$, a complementary ($n$-1)-set of $E$, which together with $p$ and $q$ supports a positive circuit $C$ of $\hat{M}$ that has $n+1$ elements. There is a corresponding circuit $\hat{C}$ of $\hat{M}$ with $\hat{p}$ replacing $p$, $\hat{t}_i$ replacing $t_i$ for $t_i \in \hat{C}$. Let $e$ be the incoming variable specified by the
algorithm applied to \( \hat{M} \). There is a unique positive circuit \( C' \) of \( \hat{M} \) in \( H u e u p u q \) other than \( C \). If \( C' \) contains \( q \), then \( C' \) has \( n+1 \) elements and there is a corresponding positive circuit \( \hat{C}' \) of \( \hat{M} \) that agrees with \( C' \). If \( q \not\in C' \), the two algorithms may diverge.

If \( p \in C', q \not\in C' \), then \( C' \) solves the LCP for \( \hat{M} \). If then \( \mid C' \mid < n+1 \), the positive circuit \( \hat{C}' \) of \( \hat{M} \) in \( H u e u p u q \) different from \( \hat{C} \) may contain \( q \), and thus the algorithm on \( \hat{M} \) may fail to terminate. However, we shall see that in determining \( \hat{C}' \), theorem 2.4 requires the examination of \( C' \), so that with a trivial modification the algorithm on \( \hat{M} \) halts. If neither \( p \) nor \( q \) is in \( C' \), the algorithm on \( \hat{M} \) terminates unsuccessfully, while the algorithm on \( \hat{M} \) continues searching for a solution.

In order to calculate the circuits of \( \hat{M} \) using theorem 2.4, we implicitly rename elements \( q \) and \( \check{p} \) in \( \hat{M} \) as \( t_{-1} \) and \( t_0 \), and rename \( q \) and \( p \) in \( \hat{M} \) as \( t_{-1} \) and \( t_0 \). At some step of the algorithm we have a complementary \((n-1)\)-set \( \check{H} \subseteq S u \bar{Y} \) such that \( \check{H} u \check{p} u q \) contains a positive circuit \( C \). Let \( e \) be the entering variable determined by the algorithm. There exists a unique positive circuit \( \check{C}' \) different from \( \check{C} \) in \( \check{H} u \check{p} u q u e \). We must find \( \check{C}' \).

This involves checking the orientations of circuits of \((n+1)\)-sets contained in \( (\check{H} u \check{p} u q u e) \). Note that we do not check \( \check{H} u \check{p} u q \), since it is \( \check{C} \), or \( \check{H} u q u e \), since if it supported a positive circuit, this would be a q-ray, contradicting the property that \( \hat{M}\backslash\{\check{p},q\} \) is SSM.
Associated with \( \mathcal{C} \) we have a base \( B \) of \( M \), which is the \( B_{QR} \) of \( \hat{\mathcal{C}} = S_Q \cup \mathcal{R} \) when \( \hat{p} \) and \( q \) are renamed \( \hat{\xi}_0 \) and \( \hat{\xi}_{-1} \). If \( H \cup p \) obtained from \( \hat{H} \cup \hat{p} \) by replacing \( \hat{\xi}_i \)'s by \( t_i \) and \( \hat{p} \) by \( p \) is independent in \( M \), then \( B_{QR} = H \cup p \).

Theorem 3.5. If the algorithm applied to \( \hat{M} \) finds a solution to the LCP, then this solution is also found by the algorithm applied to \( \hat{M} \).

Proof. We have already seen that this is true if the solution is \( C(S,p) \). Suppose that \( H \cup p \) for \( H \subseteq E \) is independent, and that \( H \cup p \cup q \) supports a positive circuit \( C \) of \( M \). We have seen that we have a corresponding positive \( \mathcal{C} \) supported by \( \hat{H} \cup \hat{p} \cup q \) of \( \hat{M} \). Let \( e \) be the incoming variable specified by the algorithm, and suppose that \( H \cup p \cup e \) contains a positive circuit containing \( p \). Then \( H \cup e \) is a base of \( M \). The base \( B_{QR} \) of \( M \) that is used to find the circuit of \( \hat{M} \) in \( \hat{H} \cup \hat{p} \cup e \) is \( H \cup e \), since \( \hat{p} = \hat{\xi}_0 \), which is \( \mathcal{C}_{r(1)} \) for \( S_Q \cup \mathcal{R} = \hat{H} \cup \hat{p} \cup e \). In the (0) step of the algorithm, we calculate \( C(B_{QR}, \hat{p}) = (\hat{p}, \emptyset) \circ \mathcal{G}(B_{QR}, p) \circ -\mathcal{G}(B_{QR}, s_1) \circ \ldots \circ -\mathcal{G}(B_{QR}, s_n) \). Thus the algorithm on \( \hat{M} \) needs to examine the circuit \( C(B_{QR}, p) \) and will recognize the solution. \( \square \)

Corollary 3.6. For semi-monotone oriented matroids, the algorithm applied to \( \hat{M} \) will solve every problem that the algorithm applied to \( \hat{M} \) will solve.

The algorithm applied to \( \hat{M} \) will also yield a solution in some cases where the algorithm applied to \( \hat{M} \) fails to give a solution.

We give an example.
Example. Let \( \hat{M} = M \begin{pmatrix} 1 & 0 & 0 & -3 & 2 & -1 \\ 0 & 1 & 0 & -1 & 1 & -1 \end{pmatrix} \). This matrix comes from the LCP \((A,b)\) with \( A = \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix} \), \( b = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \), and \( q = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) is the artificial vector. For the first step, we proceed as usual, yielding the tableau \( \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{3}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \). This is the tableau for \( B = \{s_2, p\} \), where \( \hat{H} = s_2 \). The entering variable is \( t_1 \), giving the q-ray \((t_1, 0)\), which implies that the algorithm applied to \( \hat{M} \) will fail to give a solution. To apply the algorithm to \( \hat{M} \), we need to calculate the circuits in \( \{s_2, \hat{t}_1, \hat{p}\} \) and \( \{\hat{t}_1, \hat{p}, q\} \). We show the calculation for \( \{\hat{t}_1, \hat{p}, q\} \). Here, for \( \hat{H} = \hat{t}_1 \), we have \( B_{QR} = \{s_1, p\} \) and the tableau is \( \begin{pmatrix} 0 & 1 & 0 & -1 & 1 & -1 \\ 1 & -2 & 0 & -1 & 0 & -1 \end{pmatrix} \). We apply the algorithm of theorem 2.4.

(i) \( \bar{K} \mapsto (q, 0) \circ G(B_{QR}, q) \circ -G(B_{QR}, s_1) \circ -G(B_{QR}, s_2) = \{(q, p), s_1\} \)

(ii) \( \bar{K} \mapsto (\hat{p}, 0) \circ \bar{K} \setminus p = \{(q, \hat{p}), s_1\} \)

(iii) \( \bar{K} \mapsto (\hat{t}_1, 0) \circ G(B_{QR}, t_1) \circ \bar{K} \setminus s_1 = \{(q, \hat{p}, \hat{t}_1), 0\} \).

This is a positive circuit. The entering variable now is \( \hat{t}_2 \). We need the circuits of \( \hat{M} \) in \( \{\hat{t}_1, \hat{t}_2, p\} \) and \( \{\hat{t}_2, \hat{p}, q\} \). We show the calculation for \( \{\hat{t}_2, \hat{p}, q\} \). For \( \hat{H} = \hat{t}_2 \), \( B_{QR} = \{t_2, p\} \), and the tableau is

\( \begin{pmatrix} -1 & 3 & 0 & 0 & 1 & -2 \\ -1 & 2 & 0 & 1 & 0 & -1 \end{pmatrix} \).

(i) \( \bar{K} \mapsto (q, 0) \circ G(B_{QR}, q) \circ -G(B_{QR}, s_1) \circ -G(B_{QR}, s_2) = \{(q, p, t_2), 0\} \)

(ii) \( \bar{K} \mapsto (p, 0) \circ \bar{K} \setminus p = \{(q, p, t_2), 0\} \)

(iii) \( \bar{K} \mapsto (t_2, 0) \circ \bar{K} \setminus t_2 = \{(q, p, \hat{t}_2), 0\} \).
The next entering variable is $s_1$. We need the circuits of $\hat{M}$ in $\{s_1, \hat{\xi}_2, \hat{\nu}\}$ and $\{\hat{s}_1, \hat{\nu}, q\}$. We show the calculation for $\{s_1, \hat{\xi}_2, \hat{\nu}\}$. Here $B_{QR} = \{s_1, t_2\}$, and the tableau is

$$
\begin{pmatrix}
1 & -3 & 0 & 0 & -1 & 2 \\
0 & -1 & 0 & 1 & -1 & 1
\end{pmatrix}.
$$

(i) $\bar{K} + (\hat{\nu}, 0) \circ G(B_{QR}, p) \circ -G(B_{QR}, s_1) \circ -G(B_{QR}, s_2) = (\{s_1, t_2, \hat{\nu}\}, 0)$

(ii) $\bar{K} + (\hat{\xi}_2, 0) \circ \bar{K} \setminus t_2 = (\{s_1, \hat{\xi}_2, \hat{\nu}\}, 0)$.

Thus the algorithm concludes with a solution to the LCP for $\hat{M}$. In this case, the solution with $t_2$ replacing $\hat{\xi}_2$ and $p$ replacing $\hat{\nu}$ is a solution to the problem for $\hat{M}$.

**Example.** Let $\hat{M} = M \begin{pmatrix} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$. In this LCP (from [10]), $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $b = (-1)$, and $q = (1)$ is the artificial vector. The first step gives the same tableau, with $p$ replacing $s_1$ in the basis. Then $t_1$ is the entering variable, again a q-ray. We need to calculate the circuits of $\hat{M}$ in $\{\hat{\xi}_1, s_2, \hat{\nu}\}$ and in $\{\hat{\xi}_1, \hat{\nu}, q\}$. We show the calculation for $\{t_1, s_2, \hat{\nu}\}$. Here $B_{QR} = \{s_1, s_2\}$, and the tableau is the original tableau.

(i) $\bar{K} + (\hat{\nu}, 0) \circ G(B_{QR}, p) \circ -G(B_{QR}, s_1) \circ -G(B_{QR}, s_2) = (\{\hat{\nu}, s_1, s_2\}, 0)$

(ii) $\bar{K} + (\hat{\xi}_1, 0) \circ G(B_{QR}, t_1) \circ \bar{K} \setminus s_1 = (\{\hat{\nu}, \hat{\xi}_1, s_2\}, 0)$.

Thus the solution given by the algorithm applied to $\hat{M}$ does not correspond to a solution for the LCP of $\hat{M}$. The algorithm failed to find the solution $(\{t_2, p\}, 0)$ of the LCP for $\hat{M}$. 
4. Diagonally Dominant Perturbations

Definition 4.1. Let $M = M_0$ be a square oriented matroid. For $i = 1, \ldots, n$, let $\hat{t}_i = \text{lex}(t_i, -s_{i1}, -s_{i2}, \ldots, -s_{in})$ extend $M_{i-1}$ to $\hat{M}_i$, and let $M_i = \hat{M}_i \setminus t_i$, where $\{s_{i1}, \ldots, s_{in}\} \subseteq S \setminus s_i$. Then $M_n = \hat{M}$ is called a diagonally dominant perturbation of $M$.

The point of studying these perturbations is to try to keep the perturbed oriented matroids within certain classes of oriented matroids. An example is provided by the next theorem, which concerns a particular diagonally dominant perturbation with $\{s_{i1}, \ldots, s_{in}\} = \emptyset$.

Theorem 4.2. Let $M = M_0$ be a $P_0$-oriented matroid. For $i = 1, \ldots, n$, let $\hat{t}_i = \text{lex}(t_i, -s_i)$ extend $M_{i-1}$ to $\hat{M}_i$, and then let $M_i = \hat{M}_i \setminus t_i$. Then $M_n = \hat{M}$ is a $P_i$-oriented matroid.

Proof. We show first that each $M_i$ is a $P_0$-oriented matroid. This is done by induction on $i$. Clearly $M_0$ is a $P_0$-oriented matroid.

For $M_k$, $k \geq 1$, suppose $K$ is an ssr cycle of $M_k$ with its violators in $S$. If $\hat{t}_k \notin K$, then $K$ is a cycle of $M_{k-1}$, contradicting the inductive hypothesis that $M_{k-1}$ is $P_0$. Suppose $\hat{t}_k \in K$, say $\hat{t}_k \in K^+$, which implies that $s_i \in K^-$. Let $K = C_1 \circ C_2 \circ \ldots \circ C_k$ be a conformal composition of $K$, where $1 \leq r \leq k$ is such that $s_k \in C_i$ for $i \leq r$, $s_k \notin C_i$ for $i > r$.

Let $C_p \in \{C_1, \ldots, C_k\}$ contain $\hat{t}_k$. Extend $C_p \setminus \hat{t}_k$ to a base $B$ of $M_k$. Then $C_p \setminus \hat{t}_k = (\hat{t}_k, \emptyset) \circ G_{M_{k-1}}^{-1} (B, t_k) \circ G_{M_{k-1}} (B, s_k)$. If $p \leq r$, then $s_k \in B$, so $s_k \in [-G_{M_{k-1}} (B, s_k)]^+$, which implies $s_k \in G_{M_{k-1}}^+ (B, t_k)$. In that case, let $C'_p = C_{M_{k-1}} (B, t_k)$. If $p > r$
and \( s_k \not\in B \), let \( C'_p = C_{M_{k-1}}'(B,t_k) \circ -C_{M_{k-1}}'(B,s_k) \). If \( p > r \) and \( s_k \in B \), let \( C'_p = C_{M_{k-1}}'(B,t_k) \). For all \( p \) with \( \xi_k \in C_p' \), replace \( C_p \) by \( C'_p \) according to the above rules. For \( p \) with \( \xi_k \not\in C_p' \), let \( C'_p = C_p \). Then \( K' = C_1' \circ \ldots \circ C_2' \) is a cycle of \( M_{k-1} \), and it agrees with \( K \) except on \( t_k \), where \( \xi_k \) is changed to \( t_k \), and possibly on \( s_k \), where \( s_k \in C_p' \) for some \( p > r \). However, \( s_k \in K' \) because there is a \( C_p' \) for \( p \leq r \) with \( s_k \in C_p' \). Thus \( K' \) is an ssr cycle of \( M_{k-1} \) with its violators in \( S \), contradicting the inductive hypothesis.

Next we show by induction that each \( M_k \setminus T_k/S_k \) is \( P \), where \( T_k = \{ t_i : i > k \} \), \( S_k = \{ s_i : i > k \} \). For \( k = 1 \), the circuits of \( M_k \setminus T_k/S_k \) come from \( C_{M_1}(S,\xi_1) = (\xi_1,0) \circ G_{M_0}(S,t_1) \circ -G_{M_0}(S,s_1) \). Here \( s_1 \not\in C_{M_0}(S,t_1) \) by the \( P_0 \)-property, and since \( s_1 \in C_{M_1}(S,\xi_1) \), we have \( M_k \setminus T_k/S_k \) in \( P \) for \( k = 1 \). Suppose \( C \) is a complementary circuit of \( M_k \setminus T_k/S_k \). If \( \xi_k \not\in C \), \( C \) is a complementary circuit of \( M_{k-1} \setminus T_{k-1}/S_{k-1} \), contradicting the inductive hypothesis that \( M_{k-1} \setminus T_{k-1}/S_{k-1} \) is \( P \). Every complementary \((k-1)\)-set of \( M_{k-1} \setminus T_{k-1}/S_{k-1} \) is a base, and thus every complementary \( k \)-set of \( M_k \setminus T_k/S_k \) containing \( s_k \) is a base. Thus, if \( \xi_k \not\in C \), then \( C \setminus \xi_k \) can be extended to a complementary base \( B_k \) of \( M_k \setminus T_k/S_k \) containing \( s_k \). Let \( B \) be \( B_k \cup S_k \), so that \( C = C_{M_k}(B,\xi_k) \setminus S_k \). Then \( C_{M_k}(B,\xi_k) = (\xi_k,0) \circ G_{M_k}(B,t_k) \circ -G_{M_{k-1}}(B,s_k) \), which must contain \( s_k \). Then, however, we have \( s_k \in C \), which is a contradiction. Thus every complementary \( k \)-set of \( M_k \setminus T_k/S_k \) is a base. By a theorem from Todd [43], we now need to show that all almost complementary circuits of \( M_k \setminus T_k/S_k \), those with
\(|\{s_i, \hat{t}_i\} \cap C| \leq 1 \) for all but one \( i \), are ssp. Let \( C \) be such a circuit and let \( i \) be such that \( \{s_i, \hat{t}_i\} \subseteq C \). Then \( C \setminus \hat{t}_i \) can be extended to a complementary base \( B_k \) of \( M_k \setminus T_k / S_k \). As above, we have \( B = B_k \cup S_k \), and \( C_{M_k}(B, \hat{t}_i) \) is a strictly sign reversing cycle with its violators in \( B \). But \( B \) is a complementary base, and thus by theorem II.2.8 we get a contradiction to the fact that \( M_k \) is \( P_0 \). \( \square \)

The perturbation of theorem 4.2 can be shown to turn SM-oriented matroids into SSM-oriented matroids, and also to preserve symmetry. Unfortunately, this type of perturbation seems much more difficult to implement implicitly to improve Lemke's algorithm. The bases of the original oriented matroid from which one can read off the circuits of the perturbed oriented matroid are much more difficult to characterize in this case. Thus the results of this last chapter are more of a theoretical nature than those of the rest of the chapter. One might ask if general diagonally dominant perturbations, in which the set \( \{s_{i_1}, \ldots, s_{i_r}\} \) is not empty, preserve the property \( P_0 \). The answer is no, based on the following counterexample.

If \( A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \), the oriented matroid \( M(I, -A) \) is \( P_0 \). Perturb \( t_1 \) by \( \hat{t}_1 = \text{lex}(t_1, -s_1, -s_2) \), and \( t_2 \) by \( \hat{t}_2 = \text{lex}(t_2, -s_2, -s_1) \). Then the circuits of the perturbed oriented matroid turn out to be \( (\{s_1, s_2, \hat{t}_1\}, \emptyset) \), \( (\{s_1, s_2, \hat{t}_2\}, \emptyset) \), \( (\{s_2, \hat{t}_1\}, \{\hat{t}_2\}) \), \( (\{s_1, \hat{t}_2\}, \{\hat{t}_1\}) \) and their negatives, which give an SSM-oriented matroid that is not in \( P \).

Finally, we prove a converse of theorem 4.2.
Theorem 4.3. Let \( M \) be a square oriented matroid that is not a \( P_0 \)-oriented matroid. Then let \( \hat{t}_i = \text{lex}(t_i, -s_i) \) extend \( M \) to \( \hat{M}_i \) for some \( i \), and let \( M_i = \hat{M}_i \setminus t_i \). Then \( M_i \) is not a \( P_0 \)-oriented matroid.

Proof. Let \( K \) be a strictly sign reversing cycle of \( M \), with all its violators in \( S \). If \( t_i \not\in K \), then \( K \) is a cycle of \( M_i \), and the result is proved. So say \( t_i \in K^+ \). This implies that \( s_i \in K^- \). Let \( K = C_1 \circ C_2 \circ ... \circ C_\ell \) be a conformal composition of \( K \) in which for some \( 1 \leq r \leq \ell \), we have \( s_i \in C_p \) for \( p \leq r \), \( s_i \not\in C_p \) for \( p > r \). Since \( s_i \in K^- \), we know that \( r > 1 \). Suppose \( C_p \) contains \( t_i \). Extend \( C_p \setminus t_i \) to a base \( B_p \) of \( M \setminus t_i \) such that \( B_p \) contains \( s_i \) if possible. Consider the circuit \( C_{M_i}(B_p, t_i) = (t_i, 0) \circ G(B_p, t_i) \circ G(B_p, s_i) \). If \( s_i \not\in B_p \), then \( C(B_p, s_i) \subseteq C(B_p, t_i) \) is \( C_p \), by our choice of \( B_p \). If \( s_i \in B_p \), then \( G(B_p, s_i) = (s_i, 0) \).

Thus in both cases we will have \( C_{M_i}(B_p, t_i) \) agreeing with \( C \) except on \( t_i \), where \( t_i \) replaces \( t_i \), and on \( s_i \), where we might have \( s_i \in C_{M_i}(B_p, t_i)^+ \) if \( s_i \in B_p \). Now let \( K' = C_1' \circ ... \circ C_\ell' \), where \( C_p' = C_p \) if \( t_i \not\in C_p \), and \( C_p' = C_{M_i}(B_p, t_i) \) if \( t_i \in C_p \). Then \( K' \) is an ssr cycle of \( M_i \), since it agrees with \( K \) except on \( t_i \), where \( t_i \) replaces \( t_i \). \( K' \) agrees with \( K \) on \( s_i \), since the \( C_p' \)'s with \( s_i \in C_p \) are earlier than the ones for which \( s_i \in C_p^+ \).

The violators of \( K' \) are also all in \( S \). \( \square \)

A similar proof shows that non-SM oriented matroids do not become SM after this type of perturbation. This leads us to the theorem:
Theorem 4.4. A square oriented matroid is $P_0$ (SM) if and only if the square oriented matroid obtained by successively replacing $\tau_i$ by $\tau'_i = \text{lex}(t_i, -s_i)$ for $i = 1, \ldots, n$, is $P$ (SSM).
CHAPTER IV
THE Q-MATRIX PROBLEM

Finding a characterization of the class of Q-matrices has been a major focus of LCP research [1], [2], [8], [9], [25], [41]. A matrix $A$ is a Q-matrix if the LCP $(A,b)$ has a solution for all $b$. A matrix $A$ is a $Q_0$-matrix if the LCP $(A,b)$ has a solution whenever the system $y - Ax = b$, $x \geq 0$, $y \geq 0$ has a solution. The only characterization of the class $Q$ is due to Gale [2]. This characterization is highly inefficient to check. The characterizations of P-matrices as those $A$ for which the LCP $(A,b)$ has a unique solution for all $b$ [39], and of SSM-matrices (also called $\bar{Q}$-matrices [8]) as the matrices $A$ for which the LCP $(A,b)$ and the LCP's $(A',b')$ for all principal submatrices $A'$ of $A$ have solutions for all $b$, $b'$ [8], lead one to hope for a similar characterization of the class $Q$.

Doverspike and Lemke [9] introduced the superclass $M^0$ of classes of square matrices. A class $X$ of square matrices is in $M^0$ if and only if any two square matrices that have all corresponding subdeterminants agreeing in sign are either both in $X$ or neither in $X$. The superclass $M^0$ relates to oriented matroids in that $X \in M^0$ if and only if $M(I,-A_1) = M(I,-A_2)$ implies that $A_1$ and $A_2$ are both in or neither in $X$. The classes $P$ and SSM are in $M^0$, and Doverspike and Lemke conjectured that $Q_0 \in M^0$.  

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The first section of this chapter introduces the work of Kelly and Watson [25], and uses a perturbation of a matrix from their paper to show that $Q \not\in M^0$ and $Q_0 \not\in M^0$. The second section shows that it is still possible to study the $Q$-matrix problem with oriented matroids, by making use of Bachem and Kern's [3] concept of oriented matroid adjoints. Every representation of an oriented matroid gives rise to an adjoint. Gale's characterization and Kelly and Watson's analysis are reproduced in the framework of oriented matroid-oriented adjoint pairs. The $Q$-oriented matroid problem is studied in the last section. The goal is to characterize square oriented matroids for which every point extension has a positive complementary circuit containing the new element. From Todd's paper [43], we know that $P$-oriented matroids and SSM-oriented matroids are $Q$-oriented matroids, but little more is known.

1. The Matrix Case

Let $A$ be an $n \times n$ matrix. A complementary submatrix $M$ of $(I, -A)$ is an $n \times n$ matrix such that the columns $(m_1, \ldots, m_n)$ of $M$ have $m_i \in \{u_i, -a_i\}$ for $i = 1, \ldots, n$, where $u_i$ is the $i$th unit vector and $-a_i$ is the $i$th column of $-A$. The LCP $(A, b)$ has a solution if there exists a complementary matrix $M$ such that $b \in \text{cone}(m_1, \ldots, m_n)$.

Lemma 1.1. If the LCP $(A, b)$ has no solution, then the LCP $(A, b_\epsilon)$ for $b_\epsilon = b + \sum_{i=1}^{n} \epsilon u_i$ has no solution, for sufficiently small $\epsilon > 0$.

Proof. Suppose that $(A, b_\epsilon)$ has a solution for arbitrarily small $\epsilon > 0$. Since there is a finite number of complementary submatrices,
there exists an $M$ with $b_\varepsilon \in \text{cone}\{m_1, \ldots, m_n\}$ for arbitrarily small $\varepsilon > 0$. But $\text{cone}\{m_1, \ldots, m_n\}$ is closed, so $b_\varepsilon \in \text{cone}\{m_1, \ldots, m_n\}$. \qed

**Lemma 1.2.** If $A$ has a zero column $a_j$, then the LCP $(A, b_\varepsilon)$ with 

$$b_\varepsilon = \sum_{i \neq j} \varepsilon^i u_i - u_j$$

has no solution, for sufficiently small $\varepsilon > 0$.

**Proof.** Suppose that $(A, b_\varepsilon)$ has a solution for arbitrarily small $\varepsilon > 0$. As above, there exists $M$ with $b_\varepsilon \in \text{cone}\{m_1, \ldots, m_n\}$ for arbitrarily small $\varepsilon > 0$. This implies that $M$ is nonsingular and that $x > 0$ when $Mx = b_\varepsilon$. Since $a_j$ is the zero column, $m_j = u_j$. Thus we have $x = -M^{-1}u_j + M^{-1}(\sum_{i \neq j} \varepsilon^i u_i)$. But $-M^{-1}u_j = -u_j$, so $x_j < 0$, since $M^{-1}(\sum_{i \neq j} \varepsilon^i u_i)$ is negligible for sufficiently small $\varepsilon > 0$. \qed

From lemma 1.1, we only need to show that right-hand sides $b$ in general position with respect to the columns of $(I, -A)$ have solutions to the LCP $(A, b)$, in order to show that $A$ is in $Q$. We can also assume, from lemma 1.2, that the columns $a_i$ can be scaled by positive numbers so that $\|a_i\| = 1$, $i = 1, \ldots, n$.

**Definition 1.3.** An element $b$ of $\mathbb{R}^n$ is visible from $-m_i \in \{-u_i, a_i\}$ if the line segment \( \{x \in \mathbb{R}^n : x = \lambda(-m_i) + (1-\lambda)b, \ 0 \leq \lambda < 1 \} \) does not intersect any of the cones: $\text{cone}\{m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_n\}$ for complementary submatrices $M$ containing $m_i$.

Clearly $\lambda(-m_i) + (1-\lambda)b \in \text{cone}\{m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_n\}$ for some $0 \leq \lambda < 1$ iff $b \in \text{cone}\{m_1, \ldots, m_n\}$. An element is therefore
visible from $-m_i$ iff there is no solution to the LCP involving $m_i$. Let $\text{Vis}(m_i)$ be the set of points visible from $-m_i$.

**Theorem 1.4 [25].** A matrix $A$ is a Q-matrix if and only if $\text{Vis}(-u_i) \cap \text{Vis}(a_i) = \emptyset$ for some $i = 1,...,n$.

Visibility sets $\text{Vis}(m_i)$ are useful as aids for picturing when a matrix is in $Q$. We now reproduce an example from [25]. This example will then be perturbed to form a pair of matrices that share corresponding subdeterminant signs, but one of these matrices will be in $Q$, while the other will not be. Consider the following matrix [25].

$$A = \begin{pmatrix} 21 & 25 & -27 & -36 \\ 7 & 3 & -9 & 36 \\ 12 & 12 & -20 & 0 \\ 4 & 4 & -4 & -8 \end{pmatrix}.$$ 

The matrix $(I,-A)$ can be transformed by elementary row operations that preserve signs of subdeterminants into

$$R = \begin{pmatrix} 2 & -2 & 0 & -3/4 & 1 & -1 & 0 & 3/4 \\ 2 & 2 & -2 & -1/3 & -2 & -2 & 2 & -1/3 \\ 0 & 0 & 2 & -1/2 & 0 & 0 & 2 & -1/2 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \end{pmatrix}.$$ 

We denote the columns of this matrix by $(s_1,s_2,s_3,s_4,t_1,t_2,t_3,t_4)$. Figure 4.1 gives the points $s_1,s_2,s_3,-s_4,t_1,t_2,t_3,-t_4$. They lie in
Figure 4.1. Kelly and Watson's configuration
the affine space \( x \in \mathbb{R}^4 : x_4 = 1 \), which we identify with \( \mathbb{R}^3 \). Then the cones associated with complementary submatrices are represented by convex hulls in \( \mathbb{R}^3 \).

The analysis is made possible by restricting the visibility sets of \(-s_4\) and \(-t_4\) by enclosing them in "boxes" formed by triangles of the form \( \Delta m_1 m_2 m_3 = \text{conv}(m_1, m_2, m_3) \), for affinely independent complementary sets \( m_1, m_2, m_3 \).

The coordinates of the points \( J, K, L, S, T \) in Figure 4.1 are:

\[
J = \begin{pmatrix} 0 \\ 2/5 \\ 4/5 \\ 1 \end{pmatrix}, \quad K = \begin{pmatrix} 6/7 \\ -2/7 \\ 8/7 \\ 1 \end{pmatrix}, \quad L = \begin{pmatrix} 0 \\ -2/3 \\ 0 \\ 1 \end{pmatrix}, \quad S = \begin{pmatrix} 3/5 \\ -2/5 \\ 4/5 \\ 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix}.
\]

The points \( K', S', T' \) are obtained from \( K, S, T \) by multiplying the first coordinate by \(-1\).

First, consider the point \(-s_4\). It is contained in the tetrahedron \( \text{tet}JKLs_1 = \text{conv}(J, K, L, S_1) \). The triangle \( \Delta KLS_1 \) is contained in the triangle \( \Delta S_1 t_2 s_3 \), \( \Delta JLs_1 \) is contained in \( \Delta S_1 t_2 t_3 \), and \( \Delta JKS_1 \) is contained in \( \Delta S_1 s_2 s_3 \). Points on and beyond these faces are not visible from \(-s_4\). The remaining face \( \Delta JKL \) is the union of \( \text{conv}(JLST) \), which is contained in \( \Delta t_1 s_2 t_3 \), and the triangle \( \Delta KST \), which is in no triangle \( \Delta m_1 m_2 m_3 \). The interior of \( \Delta KST \) is thus a "window" for \(-s_4\) to see through. The visibility set of \(-s_4\) is further constrained by the triangle \( \Delta S_1 Ls_3 \), which is \( \Delta S_1 t_2 s_3 \), and the triangle \( \Delta S_1 Js_3 \), which is in \( \Delta S_1 s_2 s_3 \). Also \(-s_4\) is coplanar with the points \( S, T, \)
and $s_3$. This gives us the visibility set of $-s_4$: $\text{Vis}(-s_4) = \text{int } \text{tet } JKLs_1 \cup \text{int } \text{tet } KSTs_3 \cup \text{relint } \Delta KST$, where $\text{int } \text{tet } JKLs_1$ is the interior of $\text{tet } JKLs_1$, and $\text{relint } \Delta KST$ is the relative interior of $\Delta KST$ in the plane containing $\{K,S,T\}$. The situation for $-t_4$ is similar, $\text{Vis}(-t_4) = \text{int } \text{tet } JK'Ls_2 \cup \text{int } \text{tet } K'S'T's_3 \cup \text{relint } \Delta K'S'T'$. These sets are disjoint (see [25] for details) and thus $A$ is a Q-matrix.

Kelly and Watson use this example to show that the set of Q-matrices is not open. A slight perturbation of the point $-t_4$ will make $\text{Vis}(-t_4)$ and $\text{Vis}(-s_4)$ intersect. If $-t_4$ is moved off of the plane formed by $S', T'$, and $s_3$, to the same side as $s_2$, the perturbed point $-t'_4$ will be able to "see" through the relative interiors of the triangles $\Delta S'T's_3$ and $\Delta STs_3$ into the interior of $\text{tet } STKs_3$, which is in $\text{Vis}(-s_4)$.

Unfortunately, this perturbation involves changing $M(I,-A)$. Note that the points $t_2$, $-t_4$, $t_3$, and $s_3$ are coplanar and thus correspond to a hyperplane of $M(I,-A)$. When $-t_4$ is moved off of the plane, this set no longer corresponds to a hyperplane. Therefore Kelly and Watson's example does not show that $Q \not\in M^0$, as noted by Doverspike and Lemke [9]. Kelly and Watson opine that it is unlikely that the class $Q$ can be described in terms of linear independence and nonzero minors, since modifications of their configuration produce a great variety of Q-matrices that can be perturbed to matrices not in $Q$.

The following example shows that there is a matrix $A$ in $Q$ such that a perturbation to $A'$ maintains $M(I,-A) = M(I,-A')$ but with $A' \not\in Q$. 
In the matrix $R$, make the following changes:

$$
t_2^\varepsilon = \frac{1}{1+\varepsilon} \begin{pmatrix} -1 \\ -2 + 2\varepsilon \\ 2\varepsilon \\ 1 + \varepsilon \end{pmatrix}, \quad s_4^\varepsilon = \begin{pmatrix} 3 \\ \frac{8\varepsilon}{3} - \frac{1}{3} \\ \frac{2\varepsilon}{3} - \frac{1}{2} \\ -1 \end{pmatrix}, \quad t_4^\varepsilon = \begin{pmatrix} 3 + \varepsilon \\ \frac{4\varepsilon}{3} - \frac{1}{3} \\ -\frac{1}{2} \\ -1 \end{pmatrix}.
$$

For $\varepsilon > 0$, replace $t_2$ by $t_2^\varepsilon$, $s_4$ by $s_4^\varepsilon$, and $-t_4$ by $-t_4^\varepsilon$, and call the resulting matrix $R^\varepsilon$. Note that $t_2^\varepsilon = \frac{1}{1+\varepsilon} t_2 + \frac{\varepsilon}{1+\varepsilon} t_3$. Let $x$ be the point $\frac{1}{1+\varepsilon} (s_3 + \varepsilon s_1)$. The perturbations from $-s_4$ to $-s_4^\varepsilon$ and $-t_4$ to $-t_4^\varepsilon$ are made so that the sets of points $\{x,S',T',-t_4^\varepsilon\}$ and $\{x,S,T,-s_4^\varepsilon\}$ will be coplanar.

**Lemma 1.4.** For sufficiently small $\varepsilon > 0$, every set of four columns of $R^\varepsilon$ is independent.

**Proof.** For $\varepsilon = 0$, we have $R^\varepsilon = R$. For sufficiently small $\varepsilon > 0$, every set of four independent columns of $R$ will give a corresponding set of four independent columns of $R^\varepsilon$. There are five sets of four dependent columns of $R$. These are $\{t_1,t_2,s_1,s_2\}$, $\{t_1,t_3,s_3,s_4\}$, $\{s_1,s_2,s_4,t_4\}$, $\{s_3,t_2,t_3,t_4\}$, and $\{t_1,t_2,t_4,s_4\}$. It can be checked that for small $\varepsilon > 0$, each of these sets of four columns is independent.

**Lemma 1.5.** $\text{Vis}(-t_4) \cap \text{Vis}(-s_4) = \emptyset$ when $t_2$ is changed to $t_2^\varepsilon$.

**Proof.** Let $L_1^\varepsilon$ be the point where the line from $s_1$ to $t_2^\varepsilon$ hits the triangle $\triangle t_1 s_2 t_3$, and let $L_2^\varepsilon$ be the point where it hits
$\Delta t_1 s_2 s_3$. Since $t^e_2$ is on the line segment from $t_2$ to $t_3$, the point $L^e_1$ will be on the segment from $L$ to $J$. Let $S^e$ be the point where the line from $L^e_1$ to $K$ hits the segment $ST$. The visibility set of $-s^e_4$ then becomes $\text{int tet}JKL^e_1 s_1 \cup \text{int tet}S^eTKs_3 \cup \text{relint} \Delta KS^eT$, which is contained in the original visibility set of $-s^e_4$. The visibility set of $-t^e_4$ stays the same, except that a second "hole" in the face $\Delta JK'L$ of the tetrahedron $\text{tet} JK'Ls_2$ is given by the triangle $\Delta L^e_1 L^e_2 L$. This hole does not let the visibility set of $-t^e_4$ intersect that of $-s^e_4$, however, since lines from $-t^e_4$ through the hole are blocked from $\text{Vis}(-s^e_4)$ by the triangle $\Delta t^e_1 s^e_2 t^e_3$.

Thus $\text{Vis}(-s^e_4)$ is strictly contained in what it was before, and the new "hole" in $\text{tet} JK'Ls_2$ does not allow $-t^e_4$ to "see" into $\text{Vis}(-s^e_4)$, so $\text{Vis}(-t^e_4) \cap \text{Vis}(-s^e_4) = \emptyset$. □

When $s^e_4$ is changed to $s^e_4$, the visibility set of $-s^e_4$ is that of $-s^e_4$, except that the term $\text{int tet} Ks^eT s_3$ becomes $\text{int tet} Ks^eTx$, since $x$ is on the plane containing $S^e, T$, and $-s^e_4$. Similarly, the set $\text{int tet} S'T'xs_3 \cup \text{relint} \Delta S'T's_3$ is added to the visibility set of $-t^e_4$, since $x$ is on the plane containing $S', T'$, and $-t^e_4$, and some modifications to $\text{Vis}(-t^e_4)$ are made around the "hole" near the point $L$, to obtain $\text{Vis}(-t^e_4)$. Note that the point $x$ is on the boundary of $\text{Vis}(-s^e_4)$ and of $\text{Vis}(-t^e_4)$. Now we do a perturbation of $t^e_4$ to $t^e_4$, analogous to Kelly and Watson's perturbation, to tilt the plane containing the points $\{t^e_4, S', T'\}$ so that $x$ is in the interior of $\text{Vis}(-t^e_4)$. Then $\text{Vis}(-t^e_4)$ and $\text{Vis}(-s^e_4)$ intersect.
Theorem 1.6. \( Q \not\in M^0 \).

Proof. Pivot on the columns \( s_1, \ldots, s_4 \) of the matrix \( R^e \) to get a matrix \((I, -A^e)\). The perturbation of \(-t_{4}^{e}\) in \( R^e \) will yield a new matrix \( R^{e,\delta} \). Pivot on the columns \( s_1, \ldots, s_4 \) of \( R^{e,\delta} \) to get a matrix \((I, -A^{e,\delta})\). The signs of all of the subdeterminants of \( A^e \) are nonzero by lemma 1.4. Thus, for small enough \( \delta \), the signs of the corresponding subdeterminants of \( A^e \) and \( A^{e,\delta} \) will all agree. But \( A^e \) is in \( Q \), while \( A^{e,\delta} \) is not. \( \square \)

Corollary 1.7. \( Q_0 \not\in M^0 \).

Proof. Recall that a matrix \( A \) is in \( Q_0 \) if the LCP \((A,b)\) has a solution whenever \( b \) is in \( \text{cone}(I,-A) \). Given \( A^e \) and \( A^{e,\delta} \) as above, clearly \( \text{cone}(I,-A^e) = \mathbb{R}^n \), since \( A^e \) is a Q-matrix. The property that \( \text{cone}(I,-A^e) = \mathbb{R}^n \) is a property of \( M(I,-A^e) \), since it is equivalent to the property that there is a strictly positive vector in the nullspace of \((I,-A^e)\), and since \( M(I,-A^{e,\delta}) = M(I,-A^e) \), we have that \( \text{cone}(I,-A^{e,\delta}) = \mathbb{R}^n \). However, there are vectors \( b \in \mathbb{R}^n \) for which the LCP \((A^{e,\delta},b)\) has no solution, since \( A^{e,\delta} \not\in Q \). \( \square \)

Remark: To construct \( A^{e,\delta} \), \( A^e \), from \( R \), so that \( A^e \) was \( Q \) but \( A^{e,\delta} \) was not, it was not necessary to perturb element \( t_2 \). This was only done to make the matrix \( A^e \) totally nondegenerate, in the sense of Doverspike and Lemke [9].

In light of theorem 1.6, the following theorem of Doverspike and Lemke [9] is surprising. Let \( S_0 \) be the class of matrices \( A \) for
which there is an $x > 0$, $x \neq 0$ such that $Ax > 0$, and let $V$ be the class of square matrices with all subdeterminants nonzero.

**Theorem 1.8** [9]. $Q_0 \cap S_0^C \cap V$ is in $M^0$.

Doverspike and Lemke were unable to drop the restriction that all subdeterminants be nonzero. It is possible that the degeneracy-resolving apparatus of chapter III can help to decide if $Q_0 \cap S_0^C \in M^0$.

2. **Oriented Adjoints and the Q-matrix Problem**

The results of section one of this chapter may lead to pessimism about the role of oriented matroids in the study of the Q-matrix problem. This section will show that the oriented adjoints of Bachem and Kern [3] give a fully adequate, if somewhat cumbersome, structure for studying the Q-matrix problem.

The point $x$ played a special role in the perturbed Kelly and Watson configuration $R^6$ of section 1. In $R^6$, the planes containing the triples $\{t_4^e,t_2^e,t_3^e\}$, $\{s_4^e,t_1^e,t_3^e\}$, $\{s_1^e,s_2^e,s_3^e\}$, and $\{s_1^e,t_2^e,s_3^e\}$ all intersected at $x$, whereas in $R^{6\delta}$, only the last three of these intersected at $x$. This change did not lead to a change in $M(R^6)$.

There therefore seems to be more information in the incidence relationships among the complementary hyperplanes (linearly independent $(n-1)$-sets $H$ of columns of $(I,-A)$ with $|\{u_i,-a_i\} \cap H| \leq 1$ for all $i$) than is present in the oriented matroid $M(I,-A)$.

The idea of studying the incidence relationships among the complementary hyperplanes is present in Gale's characterization of Q-matrices, stated below.
Let $M_1, \ldots, M_k$ be the set of nonsingular complementary submatrices of a matrix $(I, -A)$. Given an $M_i$, the rows $(M_i^{-1})_j$ of $M_i^{-1}$ define facets of the cone generated by the columns of $M_i$. Suppose that a vector $b$ is in general position in $\mathbb{R}^n$. If for every $M_i$ there exists a $j$ such that $u_j^T M_i^{-1} b < 0$, then there is no solution to the LCP $(A, b)$. If we then create $n^2$ systems of inequalities, each with $\ell$ rows of the type $u_j^T M_i^{-1} b < 0$, enumerating all the ways to pick one row from each matrix $M_i^{-1}$, we see that $A$ is a Q-matrix iff each of these systems has no feasible solution $b$. This is Gale's characterization. We recall the definition of adjoint of an oriented matroid from Chapter I. Given a matrix $(I, -A)$, for $A \in \mathbb{R}^{n \times n}$, we can form a matrix $F$, which we call an adjoint of $(I, -A)$, as follows. Let $H_1, \ldots, H_k$ be the set of independent $(n-1)$-subsets $\{h_1, \ldots, h_{n-1}\}$ of columns of $(I, -A)$. For each $H_i$, define a column $0 \neq f_i \in \mathbb{R}^n$ so that $f_i^T h_j = 0$ for all $h_j \in H_i$. If $H_i$ is contained in a complementary submatrix $M_j$ of $(I, -A)$, and the $m$th column of $M_j$ is not in $H_i$, then $f_i$ is proportional to $M_j^{-T} u_m$. Gale's characterization can then be stated as requiring the nonexistence of certain sign patterns of vectors in the rowspace of $F$. Thus the oriented matroid $M(F)$ determines whether or not $A$ is in $Q$. Note that $F$ is unique up to scalar multiplication of the columns, and rearrangement of the columns corresponding to a reindexing of the set $\{H_1, \ldots, H_k\}$.

In general, we have the concept of oriented adjoints of oriented matroids, defined by Bachem and Kern [3]. We recall the definition here.
Definition 2.1 [3]. Let \( M \) and \( \hat{M} \) be two oriented matroids on \( E \) and \( \hat{E} \) respectively, without loops and of the same rank. Then \( \hat{M} \) is called an adjoint of \( M \) if there are maps

\[
\phi: \hat{E} \rightarrow \mathcal{D}(M) \quad \text{and} \quad \psi: E \rightarrow \mathcal{D}(\hat{M})
\]

such that for all elements \( e \in E \) and \( \hat{e} \in \hat{E} \)

1. \( e \in \phi(\hat{e})^\pm \) if and only if \( \hat{e} \in \psi(e)^\pm \)

holds and the induced map \( \phi \) mapping points (sets of rank one) of \( \hat{M} \) onto hyperplanes \( H = E \setminus \phi(\hat{e}) \) (\( \hat{e} \in \hat{E} \)) is bijective.

Recall the example from chapter I of an oriented matroid \( M \) with adjoint \( \hat{M} \). Note that \( \hat{M} \) is the oriented matroid represented by \( \hat{R} \), and members of the signed span \( K(\hat{M}^*) \) of cocircuits of \( \hat{M} \) are given by the intersections of halfspaces determined by \( r_1, \ldots, r_6 \).

Theorem 2.2 [3]. Let \( \hat{M} \) be an adjoint of \( M \), with mappings \( \phi \) and \( \psi \) as in definition 2.1. For every \( K \in K(\hat{M}^*) \), the cocircuit span of \( \hat{M} \), there is a unique extension of \( M \) to \( \hat{M} \) by an element \( p \) such that

1. There is a cocircuit \( (D^+ \cup p, D^-) \) of \( \hat{M} \) for all \( D \in \phi(K^+) \cup -\phi(K^-) \),

2. There is a cocircuit \( (D^+, D^-) \) of \( \hat{M} \) for all \( D \in \mathcal{D}(M) \) such that \( D \notin [\phi(K^+) \cup -\phi(K^-)] \).
A given adjoint does not determine all of the extensions of $M$ in this way. As we saw in the Kelly and Watson example of section 1, different adjoints may determine different extensions.

**Definition 2.3.** An oriented matroid configuration is a four-tuple $(M, \hat{M}, \phi, \psi)$, where $\hat{M}$ is an adjoint of $M$, $\hat{M}$ has no parallel elements, and $\phi, \psi$ are as in definition 2.1. Two configurations $(M, \hat{M}, \phi, \psi)$ and $(M', \hat{M}', \phi', \psi')$ are equivalent if $M' = M$, $\hat{M}'$ (on $\hat{E}'$) is isomorphic to $\hat{M}_X$ (on $\hat{E}$) for some $X \subseteq \hat{E}$ via a bijection $\alpha: \hat{E} \to \hat{E}'$, $\phi'(\hat{e}) = \phi(\alpha(\hat{e}))$ for $\hat{e} \in \hat{E} \setminus X$, $\phi(\hat{e}) = -\phi(\alpha(\hat{e}))$ for $\hat{e} \in X$, and

$$\hat{e} \in \psi(e) \iff \begin{cases} \alpha(\hat{e}) \in \psi'(e)^+ & \text{for } \hat{e} \in \hat{E} \setminus X \\ \alpha(\hat{e}) \in \psi'(e)^- & \text{for } \hat{e} \in X. \end{cases}$$

The next lemma follows directly from theorem 2.2.

**Lemma 2.4.** The extensions of $M$ determined by members of the signed spans of the cocircuits of $\hat{M}_1$ and $\hat{M}_2$, for two equivalent configurations $(M, \hat{M}_1, \phi_1, \psi_1)$ and $(M, \hat{M}_2, \phi_2, \psi_2)$ are the same.

For a given rank, we will have a finite set of equivalence classes of configurations $(M, \hat{M}, \phi, \psi)$ in which $M$ is square and $M, \hat{M}$ are representable. We can say that an equivalence class is in $Q$, if all extensions associated with the cocircuit spans of adjoints $\hat{M}$ of these configurations yield solutions to the associated LCP's. We thus have
an oriented matroid framework for studying the Q-matrix problem, since all of the representable extensions of a square \( M \) correspond to members of the cocircuit span of \( \tilde{M} \), for some configuration \((M, \tilde{M}, \phi, \psi)\). Unfortunately, this framework is very unwieldy, even for rank four. For rank three oriented matroids, we can prove, in contrast to theorem 1.6:

**Theorem 2.5.** The class \( Q \cap \mathbb{R}^{3 \times 3} \) is in \( M^0 \).

**Proof.** Given a configuration \((M, \tilde{M}, \phi, \psi)\), where \( M \) is square of rank three, define a configuration \((M_3, \tilde{M}_3, \phi_3, \psi_3)\) by \( M_3 = M \setminus \{s_3, t_3\} \), \( \tilde{M}_3 = M \setminus H_3 \), where \( H_3 = \{ \phi^{-1}(H) : H \text{ a hyperplane of } M, \text{ rank}(H \setminus \{s_3, t_3\}) < 2 \} \). Let \( \phi_3 \) and \( \psi_3 \) be the restrictions of \( \phi \) and \( \psi \) to \( E \setminus H_3 \) and \( E \setminus \{s_3, t_3\} \) respectively. Then \( \tilde{M}_3, \phi_3, \psi_3 \) satisfy the conditions of definition 2.1. One can show, by case checking:

**Lemma 2.6.** Every configuration \((M_3, \tilde{M}_3, \phi_3, \psi_3)\) obtained this way is determined (up to equivalence) by \( M_3 \).

Note next that the oriented matroids \( M \setminus s_3 \) and \( M \setminus t_3 \) are point extensions of \( M_3 \). This leads us to the following lemma, essentially proved by Kelly and Watson by drawing all possible configurations \((M_3, \tilde{M}_3, \phi_3, \psi_3)\) up to equivalence and symmetries among \( s_i \)'s and \( t_i \)'s.

**Lemma 2.7.** To know if a configuration \((M, \tilde{M}, \phi, \psi)\) is in \( Q \) it is enough to know \((M_3, \tilde{M}_3, \phi_3, \psi_3)\) and the extensions \( M \setminus s_3 \) and \( M \setminus t_3 \).

Kelly and Watson's catalog of configurations \((M_3, \tilde{M}_3, \phi_3, \psi_3)\) misses one configuration, the one determined when \( M_3 \) is the rank three oriented
matroid on four elements with one circuit which is positive on every element. This configuration also fits into the general framework of their analysis. Putting together lemmas 2.6 and 2.7, we get that a configuration \((M, \hat{M}, \phi, \psi)\) is in \(Q\) depending only on \(M\). □

3. **The Q-Oriented Matroid Problem**

The results of section one show that the oriented matroid \(M(I,-A)\) does not give enough information to decide if a given matrix is in \(Q\). Section two shows that adjoints give an adequate but overly cumbersome combinatorial structure to study the Q-matrix problem. This leads us to investigation of the following two questions, which may be easier than using oriented matroids to describe the Q-matrix property.

1. What is the largest class of \(n \times n\) Q-matrices that is in \(M^0\)?
2. Characterize the class of Q-oriented matroids, the square oriented matroids for which every point extension has a positive complementary circuit containing the new element.

Problems 1 and 2 are not equivalent, since there exist many nonrepresentable square oriented matroids (see e.g. section II.3). We will concentrate on problem 2, since we feel that solution of problem 2 may give a solution to problem 1 for representable oriented Q-oriented matroids.

The following theorem was proved by Todd [43].

**Theorem 3.1.** Let \(M\) be a square oriented matroid on \(E = S \cup T\). The following are equivalent:

1. There is no sign reversing circuit of \(M\) that is positive on \(T\).
2. For all $I \subseteq \{1, \ldots, n\}$, every point extension of $M \setminus T_I / S_I$ has a positive complementary circuit containing the new element.

3. For every $I \subseteq \{1, \ldots, n\}$. If $\hat{M}_I$ is a point extension of $M \setminus T_I / S_I$ containing a positive complementary circuit $C$ with $C \cap T = \emptyset$, then $\hat{M}_I$ contains no other positive complementary circuit.

The class of oriented matroids satisfying 1-3 is the class of SSM-oriented matroids, defined in chapter III. P-oriented matroids are SSM-oriented matroids, so the example of chapter II, section 3, shows that this class is not limited to representable oriented matroids.

**Definition 3.2.** A square oriented matroid $M$ on $E = S \cup T$ is an SSM-oriented matroid if there is a complementary base $B$ such that there is no sign reversing circuit of $M$ that is positive on $E \setminus B$.

Matrices $A$ such that $M(I,-A)$ is an SSM-oriented matroid are those that are related to SSM matrices by principal pivot transforms. The square oriented matroids of rank two can be enumerated efficiently. There are 24 nondegenerate square oriented matroids of rank two, and a small number of degenerate ones. From enumeration of all of these, we find:

**Theorem 3.3.** A square oriented matroid of rank two is a $Q$-oriented matroid iff it is SSM.

As a consequence of theorem 2.5, we have the following.
Theorem 3.4. A rank three oriented matroid is a Q-oriented matroid iff A is a Q-matrix for some representation (I,-A) of M.

Proof. An extension of a square rank three oriented matroid M to \( \hat{M} \) is representable, by [19], since it has seven elements. \( \square \)

Unfortunately, the property \( \overline{SSM} \) does not characterize Q-oriented matroids, even for rank three. Consider the matrices

\[
A_1 = \begin{pmatrix} 2 & 24 & -7 \\ 24 & 2 & -7 \\ 9 & 9 & -5 \end{pmatrix} [45] \quad \text{and} \quad A_2 = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} [39].
\]

These are known to satisfy the property that the LCP's \((A_i,b)\) have positive even numbers of solutions for all b. Property 3 of theorem 3.1 implies that every positive right hand side must have exactly one solution. Thus these matrices and (by theorem 3.4) the oriented matroids that they represent are in Q but not \( \overline{SSM} \). These examples also show that oriented matroid analogs of Garcia's [17] class will fail to characterize Q-oriented matroids, since they also involve specifying extensions with unique solutions.
BIBLIOGRAPHY


