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AN ABSTRACT DUALITY

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Abstract

We present a notion of abstract duality that provides a common characterization of several combinatorial dualities, as well as the orthogonality relation on vector spaces coordinatized over fields having no nontrivial involutory automorphism.
Several combinatorial structures exhibit an interesting duality relation that yields interesting theorems, and, sometimes, useful explanations or interpretations of results that do not concern duality explicitly. We present a common characterization of the duality relations associated with matroids, Sperner families, oriented matroids, and weakly oriented matroids. The same conditions characterize the orthogonality relation on certain families of vector spaces. This leads to a notion of abstract duality. Antimatroids (convex geometries) have no abstract duality. Details of this joint work are presented in the Ph.D. dissertation [4] of the second author, and in a paper in preparation [1].

Let \( \mathcal{F} \) denote the family of all matroids \( F \) on a finite set of elements \( E(F) \). There are many different, but equivalent, axiomatizations of matroids, e.g. in terms of circuits (minimal dependent sets), independent sets, bases, rank functions, closure operators, etc. (see [14]). However, our interest here is confined to fundamental properties of the matroid duality relation \( D: \mathcal{F} \to \mathcal{F} \), whose descriptions do not require the notation of any one particular axiomatization.

The matroid duality relation \( D: \mathcal{F} \to \mathcal{F} \) is an involution,

\[
(I) \quad D(D(F)) = F \quad (\forall F \in \mathcal{F}),
\]

that preserves the ground set,

\[
(II) \quad E(D(F)) = E(F) \quad (\forall F \in \mathcal{F}).
\]

There are standard operations called contraction (\( / \)) and deletion (\( \setminus \)) that take each \( F \) and \( e \in E(F) \) to matroids \( F/e \) and \( F\setminus e \) having \( E(F/e) = E(F) = E(F)\setminus\{e\} \). It is well known that the duality relation interchanges contraction and deletion,
(III) a) \( D(F \setminus e) = D(F)/e \) \((\forall F \in \mathcal{F}, e \in E(F))\);

b) \( D(F/e) = D(F) \setminus e \) \((\forall F \in \mathcal{F}, e \in E(F))\).

In order to use \( \mathcal{F} \) later as a generic symbol, it will be useful at this point to let \( \mathcal{F}_m \) denote the family of all finite matroids.

Theorem 1a. Matroid duality is the unique function \( D : \mathcal{F}_m \rightarrow \mathcal{F}_m \) satisfying (I)-(III).

Properties (I)-(III) also characterize the blocking duality of Sperner families (clutters) [8]. That is, if we let \( \mathcal{F}_\alpha \) be the family of all Sperner families \( F \) on a finite ground set \( E(F) \), and take the operations of (III) to be the usual contraction and deletion in this context (see [9, section 3]), then we get

Theorem 1b. Sperner family duality is the unique \( D : \mathcal{F}_\alpha \rightarrow \mathcal{F}_\alpha \) satisfying (I)-(III).

G. Kalai pointed out to us that Theorem 1.a is a strengthening of a result of J.P.S. Kung [13]. Kung proved the version of Theorem 1.a in which one imposes the additional restriction that \( D \) preserves isomorphisms,

\[
(IV) \quad F_1 = \psi(F_2) \Rightarrow D(F_1) = \psi(D(F_2))
\]

\((\forall F_1, F_2 \in \mathcal{F} \text{ and isomorphisms } \psi \text{ from } F_2 \text{ to } F_1)\).

An isomorphism \( \psi \), as in (IV), is a bijection from \( E(F_2) \) to \( E(F_1) \) that takes \( F_2 \) to \( F_1 \). It is evident that (IV) holds, not only for the matroid duality relation on \( \mathcal{F}_m \), but also for the Sperner family duality relation on \( \mathcal{F}_\alpha \). Furthermore, if we let \( \mathcal{F}_\circ \) (respectively, \( \mathcal{F}_m \)) denote the family of all (weakly) oriented matroids on finite ground sets [3] ([2, 11]) then we get
Theorem 2. Conditions (I)-(IV) characterize: (a) matroid duality on $\mathcal{F} = \mathcal{F}_m$; (b) Sperner family duality on $\mathcal{J} = \mathcal{F}_o$; (c) oriented matroid duality on $\mathcal{G} = \mathcal{F}_o$; (d) weakly oriented matroid duality on $\mathcal{H} = \mathcal{F}_w$.

We emphasize that in each case (a) - (d) the operations of contraction ($\setminus$) and deletion ($\setminus$) in property (III) are the ordinary contraction and deletion operations on the pertinent family $\mathcal{F}$. For each family $\mathcal{J}$, the forms of the contraction and deletion operations depend on the chosen description of $\mathcal{J}$. Recall that there may be many equivalent descriptions (in terms of circuits, etc.). There is a choice of a description for each of $\mathcal{F}_m$, $\mathcal{F}_o$, $\mathcal{F}_o$, and $\mathcal{F}_w$, that results in the contraction and deletion operations, respectively, taking the same form for each family. The descriptions that we have in mind are closely related to the following linear algebraic example of families $\mathcal{J}$ having a unique $D$ satisfying (I)-(IV).

Let $K$ be a field. Let $\mathcal{F}_K$ be the family of ordered pairs $F = (E(F), V(F))$, where $E(F)$ is a finite set of coordinates, and $V(F)$ is a vector subspace of the space $K^E$ of all maps $x: E \to K$. For each $F \in \mathcal{F}_K$ and $e^* \in E(F)$ let $F/e^*$ and $F\setminus e^*$ be given by

(1) $E(F/e^*) = E(F/\setminus e^*) = E(F) \setminus \{e^*\}$, $\quad V(F/e^*) = \{x: E(F) \setminus \{e^*\} \to K \mid \exists y \in V(F) \text{ with } x(e) = y(e) \forall e \in E(F) \setminus \{e^*\}\},$ $\quad V(F\setminus e^*) = \{x: E(F) \setminus \{e^*\} \to K \mid \exists y \in V(F) \text{ with } y(e^*) = 0 \text{ and } x(e) = y(e) \forall e \in E(F) \setminus \{e^*\}\}.$

Note that $V(F/e^*)$ and $V(F/\setminus e^*)$, respectively, are the subspaces of $K^E \setminus \{e^*\}$ obtained by projecting $V(F)$ onto the hyperplane $H = \{x: E(F) \to K \mid x(e^*) = 0\},$

and intersecting $V(F)$ with $H$. 

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Theorem 2e. Let $K = \mathbb{R}$, or $Q$, or $GF(p^n)$ for $p$ prime and $n$ odd.

Then for $\mathcal{F} = \mathcal{F}_K$ conditions (I)-(IV) are satisfied uniquely by the orthogonality relation $D$ taking each $F = (E, V) \in \mathcal{F}_K$ to

$D(F) = (E, V^\perp)$, where $V^\perp = \{x : E \to K \mid x \cdot y = 0 \quad \forall y \in V\}$.

The isomorphism-preserving condition (IV) takes the following form here. Let $F_1, F_2 \in \mathcal{F}$ and let $\psi$ be a bijection from $E(F_2)$ to $E(F_1)$ such that $V(F_2) = \{x \circ \psi \mid x \in V(F_1)\}$. Then (IV) requires that $V(D(F_2)) = \{y \circ \psi \mid y \in V(D(F_1))\}$.

The earlier examples, those of cases (a)-(d) of Theorem 2, can be put in a form similar to that of $\mathcal{F}_K$, in the following way. In each case we have

(2) a set $T$ having $0 \in T$, and a family $\mathcal{F}$ of pairs $F = (E(F), V(F))$, where $E(F)$ is a finite set and $V(F)$ is a set of maps $x : E(F) \to T$.

Contraction ($/$) and deletion ($\setminus$) are as in (1). In each case $E(F)$ is the usual ground set. The target set $T$ is $\{0, 1\}$ for $\mathcal{F}_m$ and $\mathcal{F}_o$, and $\{-1, 0, 1\}$ for $\mathcal{F}_o$ and $\mathcal{F}_w$. The set $V(F)$ of maps from $E(F)$ to $T$ is the set of: (a) incidence vectors of unions of circuits of a matroid $F \in \mathcal{F}_m$; (b) incidence vectors of supersets of members of a Sperner family $F \in \mathcal{F}_o$; (c) signed incidence vectors of conformal unions of signed circuits of an oriented matroid $F \in \mathcal{F}_o$; (d) signed incidence vectors of consistent unions (see [1,4]) of signed circuits of a weakly oriented matroid $F \in \mathcal{F}_w$. 

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When the examples are described in this form, the corresponding form of the duality relation \( D \) is especially simple. For instance, for a set \( \mathcal{G} \) of pairwise noncomparable subsets of \( E \), the corresponding \((E, V) \in \mathcal{F}_\alpha\) has \( V \) the set of incidence vectors of
\[
\mathcal{I} = \{I \subseteq E \mid \exists S \in \mathcal{G}, S \subseteq I\},
\]
and \( V(D(F)) \) is the set of incidence vectors of
\[
\{R \subseteq E \mid R \cap I \neq \emptyset, \forall I \in \mathcal{I}\}.
\]
For a matroid \( F \in \mathcal{F}_m \) on \( E \) with \( V(F) \) the set of all incidence vectors of unions of circuits of \( F \), \( D(F) \) is the set of incidence vectors of subsets \( S \subseteq E \) that intersect no union of circuits on exactly one element.

Suppose that \( \mathcal{I} \) is of the form (2) and is closed under the contraction and deletion operations (1). We will say that a function \( D: \mathcal{I} \rightarrow \mathcal{I} \) that satisfies (I)-(III) is a weak abstract duality on \( \mathcal{I} \). A weak abstract duality that satisfies (IV) will be called an abstract duality. For each of the five examples, families \( \mathcal{F}_m, \mathcal{F}_\alpha, \mathcal{F}_C, \mathcal{F}_w, \mathcal{F}_K \), it is clear that there is at least one abstract duality - the standard duality, or orthogonality, relation. The proofs that there is at most one abstract duality take the same general form. We shall describe it now.

For each of the examples \( \mathcal{I} \), it is not difficult to show that for all \( F \in \mathcal{I} \) with \( E(F) \) sufficiently large, \( F \) is determined uniquely by its set of simple minors:

\[
(3) \quad \{F/e \mid e \in E(F)\} \cup \{F\e \mid e \in E(F)\}.
\]
Such an $F$ is called reconstructible. It is not difficult to see why one might expect reconstructibility. For $F \in \mathcal{F}$ having $|E(F)| \geq 2$, let $\mathcal{M}$ be the set (3) of simple minors of $F$. First note that $E(F)$ is just the union of $E(F')$ over $F' \in \mathcal{M}$. Also note that: (a) for any $e \in E(F)$ there are exactly two $F' \in \mathcal{M}$ having $E(F') = E(F) \setminus \{e\}$, namely $F/e$ and $F \setminus e$; (b) it is easy to distinguish them because $V(F \setminus e) \subseteq V(F/e)$. Next note that if we extend each $x \in V(F \setminus e)$ to a map $x': E \to T$ having $x'(e) = 0$, then $x' \in V(F)$. Typically $V(F)$ can be generated unambiguously from the $x'$, except in degenerate cases, where the structure of $V(F)$ can be determined from the contraction minors. For example, if $F \in \mathcal{F}_{GF(2)}$, the only ambiguity arises when for every choice of $e \in E(F)$, $V(F \setminus e)$ contains only the zero vector $(0, \ldots, 0)$. In this circumstance $V(F)$ could be either $\{(0, \ldots, 0)\}$ or $\{(0, \ldots, 0), (1, \ldots, 1)\}$, but the contraction minors $F/e$ immediately reveal which of the two possibilities holds. Note that if $|E(F)| = 1$, then $F$ cannot be reconstructed, since we cannot even recover $E(F)$.

Let $r(\mathcal{F})$ be the least integer $r$ such that every $F \in \mathcal{F}$ having $|E(F)| \geq r$ is reconstructible: $r(\mathcal{F}_m) = r(\mathcal{F}_\alpha) = 2$; $r(\mathcal{F}_G) = r(\mathcal{F}_P) = 3$; $r(\mathcal{F}_{GF(2)}) = 2$ and $r(\mathcal{F}_K) = 3$ for other $K$. We will say that $F \in \mathcal{F}$ is small if $|E(F)| < r(\mathcal{F})$.

**Proposition 3.** If $D_1$ and $D_2$ are distinct weak abstract dualities on $\mathcal{F}$, then $D_1(F) \neq D_2(F)$ for a small $F \in \mathcal{F}$.

The proof of Proposition 3 uses the following result. For $q \in \mathbb{Z}_+$ denote by $\mathcal{F}^q$ the subfamily $\{F \in \mathcal{F} | q \geq |E(F)|\}$ of $\mathcal{F}$.  

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Lemma 3.1. Suppose $D_1$ and $D_2$ are weak abstract dualities on $\mathcal{F}$ and for some positive integer $q$: (i) $D_1(F) = D_2(F)$ \( \forall F \in \mathcal{F}^q \); (ii) every $F \in \mathcal{F} \setminus \mathcal{F}^q$ is reconstructible. Then $D_1 = D_2$.

Proof. Suppose $\hat{F} \in \mathcal{F}$ has $|E(\hat{F})| = q + 1$. $D_1(\hat{F})$ is determined uniquely by $\mathcal{M} = \{D_1(\hat{F}) \setminus e \mid e \in E(\hat{F})\} \cup \{D_1(\hat{F})/e \mid e \in E(\hat{F})\}$.

By applying (III), then (i), and then (III) again to the members of $\mathcal{M}$, we find that $\mathcal{M} = \{D_2(\hat{F}) \setminus e \mid e \in E(\hat{F})\} \cup \{D_2(\hat{F})/e \mid e \in E(\hat{F})\}$, which determines $D_2(\hat{F})$ uniquely. Thus (i) holds for all $F \in \mathcal{F}^q + 1$, and, since $\mathcal{F} \setminus \mathcal{F}^q + 1 \subseteq \mathcal{F} \setminus \mathcal{F}^q$, so does (ii). By induction $D_1 = D_2$.

To prove Proposition 3, take $q$ in Lemma 3.1 to be $r(\mathcal{F}) - 1$.

Now to demonstrate that a particular family $\mathcal{F}$ has at most one abstract duality, it suffices to prove that

(4) all abstract dualities $D_1$ and $D_2$ on $\mathcal{F}$ agree on all small $F \in \mathcal{F}$.

In the combinatorial cases, $\mathcal{F}_m, \mathcal{F}_g, \mathcal{F}_c, \mathcal{F}_m$, this is facilitated by the modest size of $r(\mathcal{F})$ and by the following lemma. For any finite set $E$ let $\mathcal{F}(E)$ be the subfamily $\{F \in \mathcal{F} : E(F) = E\}$.

Lemma 4. Let $D$ be a weak abstract duality on $\mathcal{F}$, let $E$ be a finite set, and let $e^* \in E$ and $F^* \in \mathcal{F}(E \setminus \{e^*\})$ be fixed. Then the restriction of $D$ to $\mathcal{F}' = \{F \in \mathcal{F}(E) \mid F \setminus e^* = F^*\}$ is a bijection from $\mathcal{F}'$ to $\{F \in \mathcal{F}(E) \mid F/e^* = D(F^*)\}$.

In $\mathcal{F}_m$, for example, consideration of all matroids on two or fewer elements is enough, with Lemma 4, to prove that all weak abstract
dualities on $\mathcal{F}_m$ agree on all $F$ having $|E(F)| < 2$. That the argument uses only matroids on two or fewer elements leads to a stronger result.

Corollary 5. Let $\mathcal{F}$ be any subfamily of $\mathcal{F}_m$ that is closed under duality and under contraction and deletion. If $\mathcal{F}$ contains all matroids on two or fewer elements, then the unique weak abstract duality on $\mathcal{F}$ is the restriction of matroid duality to $\mathcal{F}$.

So, for example, each of the subfamilies consisting of planar graphic matroids, and matroids representable over a particular field, or all fields in some set, has a unique weak abstract duality.

It is in proving (4) for $\mathcal{F}_0$, $\mathcal{F}_w$, and $\mathcal{F}_K$, $K$ as in Theorem 2e, that (IV) is used. The proof of (4) for $\mathcal{F}_K$ is much longer and more difficult than for the combinatorial examples. This proof also reveals what happens for $\mathcal{F}_K$, where $K$ is an arbitrary field. This is summarized in

Theorem 6. Let $K$ be an arbitrary field.

(a) The orthogonality relation (1) is an abstract duality for $\mathcal{F}_K$.

(b) Any involutory automorphism $\varphi$ on $K$ induces naturally an involution $\phi$ on $\mathcal{F}_K$, and $\phi$ commutes with the contraction operation ($\sim$), the deletion operation ($\backslash$), and the orthogonality relation (1). Hence $\phi \circ \perp$ is an abstract duality for $\mathcal{F}_K$.

(c) Every abstract duality for $\mathcal{F}_K$ arises as in (b).

(d) If $K$ has no nontrivial involutory automorphism, e.g. if $K = \mathbb{Q}, \mathbb{R}$, or $GF(p^n)$, $p$ prime and $n$ odd, then (1) is the unique abstract duality for $\mathcal{F}_K$.
The involution \( \phi \) on \( \mathcal{F}_K \) induced by the involutory field automorphism \( \varphi \) arises by applying \( \varphi \) to each component of each vector. That is, for \( F = (E, V) \in \mathcal{F}_K \), \( \phi(F) = (E, V') \), where \( V' = \{ y : E \to K \mid \text{for some } x \in V, y(e) = \varphi(x(e)) \forall e \in E \} \).

Theorem 1 says that \( \mathcal{F}_m \) and \( \mathcal{F}_o \) have only one weak abstract duality. In the other examples, \( \mathcal{F}_o, \mathcal{F}_w, \mathcal{F}_K \) there are weak abstract dualities that arise from the standard duality by negating signs on fixed sets of elements. In the cases of \( \mathcal{F}_o \) and \( \mathcal{F}_w \) all of the weak abstract dualities arise in this way.

It should be noted that under (I) and (II), either half of (III) implies the other. We include both conditions (a) and (b) of (III) to emphasize symmetry. Theorem 1 and the combinatorial parts, (a) - (d), of Theorem 2 remain valid if (I) is relaxed to require only that \( D \) be one-to-one.

This investigation began with the narrower question of whether antimatroids (also called anti-exchange closures, convex geometries, or shelling structures) [6, 7, 10, 12] have an interesting duality relation. We can now make the question well-defined.

An antimatroid on the finite set \( E \) can be defined in terms of a closure operation \( \tau : 2^E \to 2^E \) with \( X \subseteq \tau(X) \) \( (V X \subseteq E) ; \tau(X) = \tau(X) \) \( (V X \subseteq E) ; x, y \notin \tau(X), x \in \tau(X \cup \{y\}) \Rightarrow y \notin \tau(X \cup \{x\}) \) \( (V X \subseteq E) ; x, y \in E \). Given \( E \) and \( \tau \) let \( V \) be the set of maps \( x : E \to T = \{-1, 0, +1\} \) such that \( \{ e \in E \mid x(e) = -1 \} \subseteq \tau(\{ e \in E \mid x(e) = +1 \}) \). Note that \( \tau \) can be recovered from \( V \). The family \( \mathcal{F}_a \) of all pairs
that arise in this way from antimatroids is closed under the contraction and deletion operations (1), which correspond to the operations that Korte and Lovasz [12] call deletion and trace, respectively (see [5]). Although antimatroids $F \in \mathcal{F}_a$ having $|E(F)| \geq 3$ are reconstructible from their simple minors, there is no abstract duality on $\mathcal{F}_a$. 
REFERENCES


