DISCRETE LINEAR DUALITY

by

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Abstract

We investigate specific instances of a linear duality model which generalizes orthogonality of subspaces and polarity of cones. The instances studied here are "integral duality" and "nonnegative integral duality", which are related to the problems of finding integral solutions and nonnegative integral solutions, respectively, for linear systems.

For each instance, the validity of properties analogous to results of Weyl, Minkowski and Farkas for cones is examined. We also characterize the constrained sets under each duality.
Introduction

In a preceding paper ([3]) we investigated a linear duality model, originally proposed in [11], which generalizes the notions of orthogonality of subspaces and polarity of cones. In this paper we examine specific instances of this model related to the problems of finding integral solutions and nonnegative integral solutions for linear systems. In particular, we investigate whether analogues to the theorems of Weyl, Minkowski and Farkas hold for these dualities. The closed (or constrained sets) for these models are also characterized. The presentation here closely follows that in [2].
1. The \((X,D)\)-duality model

Here, we briefly summarize the general theory of \((X,D)\)-duality. The reader is referred to [3] for a more detailed presentation.

Given a commutative ring \(R\) and subsets \(X\) and \(D\) of \(R\) satisfying \(\{0,1\} \subseteq X\), the \((X,D)\)-dual of any subset \(S\) of \(X^n\) is defined as

\[
S^* = \{x \in X^n : Sx \in D\},
\]

where \(Sx \in D\) denotes \(sx = (s_1 x_1 + \ldots + s_n x_n) \in D\), \(\forall s \in S\).

Sets of the form \(S^*\) for some (finite) \(S \subseteq X^n\) are said to be (finitely) constrained. A set \(S \subseteq X^n\) is closed if \(S^{**} = S\).

The following properties are derived easily from the above definitions (see [3] for proofs).

1.1 Proposition: Suppose \(S\) and \(T\) are subsets of \(X^n\). Then:

(a) \(S \subseteq T \Rightarrow S^* \subseteq T^*\);
(b) \((S \cup T)^* = S^* \cap T^*\);
(c) \(S \subseteq S^{**}\);
(d) \(S^* = S^{***}\);
(e) \(S\) is closed \(\iff\) \(S\) is constrained;
(f) \(S^{**}\) is the smallest closed set containing \(S\). \(\blacksquare\)

In [3] we show that it is appropriate to define the set generated by \(S \subseteq X^n\) as

\[
\sigma(S) = \{x \in X^n : x = y_1 a_1 + \ldots + y_m a_m, \text{ where } m \geq 1, a_1, \ldots, a_m \in S \text{ and } y \in (D^n)^*\},
\]
where \((D^m)^*\) is the \((X,D)\)-dual of \(D^m\). It is clear that
\(S \subseteq T \Rightarrow \sigma(S) \subseteq \sigma(T)\); also, \(S \subseteq \sigma(S)\), since the unit vectors
\(e_i = (0, \ldots, 1, \ldots, 0) \in (D^m)^*\), \(i = 1, \ldots, m\).

A set \(S \subseteq X^n\) is said to be (finitely) generated if
\(S = \sigma(T)\) for some (finite) \(T \subseteq X^n\).

The following properties are established in [3].

1.2 Proposition: Let \(S \subseteq X^n\).
(a) \(S\) is a generated set \(\iff\) \(\sigma(S) = S\);
(b) \(\sigma(S)\) is the smallest generated set containing \(S\);
(c) \(\sigma(S^*) = (\sigma(S))^* = S^*\);
(d) \(S\) is constrained \(\Rightarrow\) \(S\) is generated;
(e) If \(S = \{yA : y \in (D^m)^*\}\) then \(S^* = \{x \in X^n : Ax \in D^m\}\)
(i.e., \(S\) is finitely generated \(\Rightarrow\) \(S^*\) is finitely constrained).

The properties described so far are valid for any \((X,D)\)-duality. However, the classical theorems of Farkas, Weyl and Minkowski for convex cones and that of Fulkerson for blocking pairs of polyhedra suggest that further relationships between (finitely) generated and (finitely) constrained sets may hold for specific instances of \((X,D)\)-duality. Thus, we consider the following additional properties of an \((X,D)\)-duality.

1.3 Farkas property: For any \(A \in X^m \times X^n\) and any \(c \in X^n\),
exactly one of the following holds:
(a) \(\exists y \in (D^m)^*\) such that \(yA = c\);
(b) \(\exists x \in X^n\) such that \(Ax \in D^m\) and \(cx \not\in D\).
1.4 **Minkowski property:** Every finitely constrained set is finitely generated.

1.5 **Weyl property:** Every (nonempty) finitely generated set is finitely constrained.

1.6 **Fulkerson property:** Every finitely constrained set has a finitely constrained dual.

Several relationships can be established among these properties. The proofs of the following can be found in [3].

1.7 **Proposition:** If the Minkowski property holds for a specific \((X,D)\)-duality then the Fulkerson property holds for this duality.

1.8 **Theorem:** For a specific \((X,D)\)-duality, the following are equivalent:

(a) The Farkas property holds.

(b) Every finitely generated set is constrained.

(c) If \(S = \{x \in X^n : Ax \in D^m\}\), where \(A \in X^m \times X^n\), then \(S^* = \{yA : y \in (D^m)^*\}\). Thus, any finitely constrained set has a finitely generated dual.

1.9 **Theorem:** For a specific \((X,D)\)-duality, the following are equivalent:

(a) The Weyl property holds.

(b) The Farkas and Minkowski properties hold.

(c) The Farkas and Fulkerson properties hold.

Given a specific \((X,D)\)-duality model we are interested in:

i) determining whether the properties 1.3-6 hold;
ii) characterizing the class of all subsets of $X^n$ which are constrained (or closed) with respect to this duality.

In the next sections these questions are examined for the following instances of $(X,D)$-duality: $X = \mathbb{Q}, D = \mathbb{Z}$ ("integral duality"); $X = \mathbb{Z}, D = \mathbb{Z}_+$ ("nonnegative integral duality in $\mathbb{Z}$"); and $X = \mathbb{Q}, D = \mathbb{Z}_+$ ("nonnegative integral duality in $\mathbb{Q}$").

We close this section by briefly reviewing the above questions for the well-known instances of "subspace duality" and "cone duality". The reader is referred to [3] for more details.

For subspace duality (i.e., $X = \mathbb{R}, D = \{0\}$) it is easy to check (see [3]) that the classes of generated, finitely generated, constrained and finitely constrained sets are all the same, namely the class of all (nonempty) subspaces of $\mathbb{R}^n$. Hence the four properties 1.3-6 hold trivially.

For cone duality ($X = \mathbb{R}, D = \mathbb{R}_+$), it is well-known that these properties are valid (see [10], for example). However, not all generated sets (i.e., convex cones) are finitely generated or constrained. It can be shown that the constrained sets under cone duality are precisely the convex cones which are topologically closed (see [10, p. 121]).

We remark that the above properties of subspace and cone dualities remain unchanged if we replace the field of the real numbers by the field of the rational numbers. That is, the dualities determined by $X = \mathbb{Q}, D = \{0\}$ and $X = \mathbb{Q}, D = \mathbb{Q}_+$, respectively, have the same properties as above and will also be referred to as subspace and cone dualities. Throughout the
remaining sections of this paper we will deal only with vector spaces and cones of rational numbers.

2. Integral duality

We term integral duality the (X,D)-duality obtained by taking \( R = X = \mathbb{Q} \) and \( D = \mathbb{Z} \).

In this case:

\[
(D^n)^* = \{ x \in \mathbb{Q}^m : xy \in \mathbb{Z}, \forall y \in \mathbb{Z}^m \} = \mathbb{Z}^m.
\]

Hence, the generated sets here are exactly those subsets of \( \mathbb{Q}^n \) which are closed under integral linear combinations, that is, modules over the ring of integers \( \mathbb{Z} \) (or \( \mathbb{Z} \)-modules for short).

Given \( S \subseteq \mathbb{Q}^n \), the set \( M(S) \) of all finite integral linear combinations of elements of \( S \) is called the \( \mathbb{Z} \)-module generated by \( S \). If \( S \) is finite, \( M(S) \) is finitely generated. Hence, finitely generated \( \mathbb{Z} \)-modules are sets of the form

\[
M = \{ yA : y \in \mathbb{Z}^m \}, \quad \text{where} \quad A \in \mathbb{Q}^m \times \mathbb{N}.
\]

Not all \( \mathbb{Z} \)-modules are finitely generated. The set of the dyadic rationals \( D = \{ m/2^n : m, n \in \mathbb{Z} \} \) is an example of a \( \mathbb{Z} \)-module which is not finitely generated. Vector subspaces of \( \mathbb{Q}^n \) (except for \( \{0\} \)) constitute another important example of such \( \mathbb{Z} \)-modules.

The \( \mathbb{Z} \)-dual of a set \( S \subseteq \mathbb{Q}^n \) is the set \( S^\# \) defined by

\[
S^\# = \{ x \in \mathbb{Q}^n : Sx \in \mathbb{Z} \}. \quad \text{Recall that by Proposition 1.2(d),} \quad S^\# \text{ is a generated set, i.e. a \( \mathbb{Z} \)-module, for every} \quad S \subseteq \mathbb{Q}^n. \text{ \( \mathbb{Z} \)-modules of the form} \quad M = S^\# \text{ for some} \quad S \subseteq \mathbb{Q}^n \text{ are called constrained}.
\]
Z-modules. If, in addition, S is finite, M is said to be
finitely constrained. Hence, finitely constrained Z-modules are
sets of the form \( \{ x \in Q^n : A x \in Z^m \} \), where \( A \in Q^m \times n \).

The general duality properties reviewed in Section 1 provide
immediately the following:

2.1 Proposition: Let \( S, T \) denote subsets of \( Q^n \).

(a) \( S \subseteq T \Rightarrow S^\# \subseteq T^\# \).

(b) \( (S \cup T)^\# = S^\# \cap T^\# \).

(c) \( S \subseteq S^{###} \).

(d) \( S^\# = S^{####} \).

(e) \( S = S^{##} \Leftrightarrow S \) is a constrained Z-module.

(f) \( S^{##} \) is the smallest constrained Z-module containing \( S \).

(g) If \( S \) generates the Z-module \( M \), then \( S^\# = M^\# \).

(h) If \( S = \{ yA : y \in Z^m \} \), where \( A \in Q^m \times n \), then
\( S^\# = \{ x \in Q^n : A x \in Z^m \} \)

(i) If \( S \) and \( T \) are Z-modules then \( (S + T)^\# = S^\# \cap T^\# \).

Proposition 2.1(h) provides an expression for the Z-dual of a
finitely generated Z-module. It is also of interest to obtain an
expression for the Z-dual of Z-modules which are the sum of a
finitely generated Z-module and a subspace (as we will see, in
Theorem 2.9 below, these are exactly the constrained sets for
Z-duality).

2.2 Proposition: Let \( S = \{ yA + zB : y \in Q^m, z \in Z^p \} \), where
\( A \in Q^m \times n \) and \( B \in Q^p \times n \). Then
\( S^\# = \{ x \in Q^n : A x = 0, B x \in Z^p \} \).
Proof: \( S = T + U \), where \( T = \{ yA: y \in \mathbb{Q}^n \} \) and \( U = \{ zB: z \in \mathbb{Z}^p \} \). By Proposition 2.1(h), \( U^\# = \{ x \in \mathbb{Q}^n: Bx \in \mathbb{Z}^p \} \). Now,

\[
T^\# = \{ x \in \mathbb{Q}^n: (yA)x \in \mathbb{Z}, \forall y \in \mathbb{Q}^m \} \\
= \{ x \in \mathbb{Q}^n: y(Ax) \in \mathbb{Z}, \forall y \in \mathbb{Q}^m \} \\
= \{ x \in \mathbb{Q}^n: Ax = 0 \}.
\]

Hence, by Proposition 2.1(i),

\[
S^\# = T^\# \cap U^\# = \{ x \in \mathbb{Q}^n: Ax = 0, Bx \in \mathbb{Z}^p \}. \quad \blacksquare
\]

Next we shall examine the properties of finitely generated and finitely constrained \( \mathbb{Z} \)-modules. As seen in Section 1, the study of these properties for subspaces was greatly simplified by the fact that the properties of being generated, finitely generated, constrained and finitely constrained are all equivalent. For \( \mathbb{Z} \)-modules, as in the cone case, this is not true. We have already seen that the set \( D \) of the dyadic rationals provides an example of a \( \mathbb{Z} \)-module which is not finitely generated. \( D \) also gives an example of a \( \mathbb{Z} \)-module which is not constrained (just observe that \( D^\# = \{ x \in \mathbb{Q}: mx/2^n \in \mathbb{Z}, \forall m, n \in \mathbb{Z} \} = \{ 0 \} \), which implies \( D^{#\#} = \mathbb{Q} \neq D \)). It is also easy to see that not all constrained \( \mathbb{Z} \)-modules are finitely constrained. Let \( S = \{ x \in \mathbb{Q}^n: Bx = 0 \} \), where \( B \in \mathbb{Q}^m \times \mathbb{Q}^n \), be a subspace of \( \mathbb{Q}^n \) of dimension less than \( n \). Then \( S \) is constrained (since \( S = \{ yB: y \in \mathbb{Q}^m \}^\# \)) but not finitely constrained, since the restrictions \( Bx = 0 \) cannot be replaced by a finite number of restrictions of the type \( ax \in \mathbb{Z} \) (see [2] for more details).
Actually, this example shows that the Weyl and Minkowski properties, as described in Section 1, cannot hold for \( \mathbb{Z} \)-duality. Indeed, \( \{0\} \) is an example of a finitely generated \( \mathbb{Z} \)-module which is not finitely constrained. Also, \( \mathbb{Q}^n \) is a finitely constrained \( \mathbb{Z} \)-module (note that \( \mathbb{Q}^n = \{x \in \mathbb{Q}^n: 0x \in \mathbb{Z}\} \) which is not finitely generated. The Fulkerson property also fails: \( \mathbb{Q}^n \) is finitely constrained but its dual \( \{0\} \) is not.

The difference between the duality properties of \( \mathbb{Z} \)-modules and cones results from the fact that subspaces behave differently with respect to these two dualities. Subspaces can be viewed both as finitely generated and finitely constrained cones. However, subspaces are neither finitely generated nor finitely constrained \( \mathbb{Z} \)-modules (unless their dimension is 0 or \( n \), respectively). In order to obtain an equivalence property between finitely generated and finitely constrained \( \mathbb{Z} \)-modules, we have to treat their span and lineality explicitly (the lineality of a \( \mathbb{Z} \)-module \( M \subset \mathbb{Q}^n \) is the largest subspace contained in \( M \); i.e., it is the set of all vectors \( x \in \mathbb{Q}^n \) such that \( \lambda x \in M, \forall \lambda \in \mathbb{Q} \)). That is, we must deal with sets of the form \( \{yA + zB: y \in \mathbb{Q}^m, z \in \mathbb{Z}^p\} \) on the one hand and \( \{x \in \mathbb{Q}^n: Cx = 0, Dx \in \mathbb{Z}^m\} \) on the other. We shall show that in fact these two forms are equivalent, which means that Weyl-, Farkas-, Minkowski- and Fulkerson-type properties are valid for these "extended" concepts of finitely generated and finitely constrained sets.

To prove the equivalence between the extended notions of finitely generated and finitely constrained \( \mathbb{Z} \)-modules, we
introduce the concept of unimodular elimination, a refinement of Gaussian elimination which preserves integrality.

A matrix $P \in \mathbb{Z}^m \times m$ is said to be unimodular if $\det P = \pm 1$. Observe that if $P$ is unimodular, then $P^{-1}$ exists and has integral entries. Hence, for every $y \in \mathbb{Q}^m$ we have $y \in \mathbb{Z}^m \iff yP \in \mathbb{Z}^m$.

Now, let $A \in \mathbb{Q}^m \times n$, let $P \in \mathbb{Z}^m \times m$ be unimodular and define $B = PA$. Then, using the fact that $y \in \mathbb{Z}^m \iff yP \in \mathbb{Z}^m$, we have

$\{yA: y \in \mathbb{Z}^m\} = \{(yP)A: y \in \mathbb{Z}^m\} = \{yB: y \in \mathbb{Z}^m\}$.

Thus, the module generated by the rows of a matrix $A$ does not change when $A$ is premultiplied by a unimodular matrix.

This fact can be used to row-reduce a rational matrix to a triangular form without altering the $\mathbb{Z}$-module which it generates. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

be a rational matrix. Let $L$ be an integer such that $La_{11}$ and $La_{21}$ are integers. Let $g = \gcd(La_{11}, La_{21})$. It follows from the Euclidean algorithm (see [1]) that there are integers $p$ and $q$ such that $pLa_{11} + qLa_{21} = g$. Consider the matrix

$$P = \begin{bmatrix} p & q \\ -La_{21}/g & La_{11}/g \end{bmatrix}.$$  

$P$ has integral entries, its determinant is 1 (i.e., $P$ is unimodular) and $PA$ is of the form
By repeatedly applying this process, an arbitrary rational matrix can be row-reduced to an upper triangular matrix which generates the same \( Z \)-module. We call this process unimodular elimination.

By combining Gaussian and unimodular elimination we obtain the following:

2.3 **Theorem:** Let \( M = \{ yA + zB : y \in Q^m, z \in Z^p \} \) where \( A \in Q^m \times n \) and \( B \in Q^p \times n \). Let \( r = \text{rank}(A) \) and \( s = \text{rank}(A_B) - \text{rank}(A) \). Assume that the first \( r + s \) columns of \( [A_B] \) are linearly independent. Then there are matrices \( A' \in Q^r \times n \) and \( B' \in Q^s \times n \) such that \( M = \{ yA' + zB' : y \in Q^r, z \in Z^s \} \), with

\[
\begin{bmatrix}
A' \\
B'
\end{bmatrix} = \begin{bmatrix}
A_1 & A_2 \\
0 & B_1 & B_2
\end{bmatrix},
\]

where \( A_1 \in Q^r \times r \) and \( B_1 \in Q^s \times s \) are upper triangular matrices with nonzero diagonal entries.

**Proof:** By using Gaussian elimination, we can row reduce \( [A_B] \) to a matrix of the form

\[
\begin{bmatrix}
A_1 & A_2 \\
0 & 0 & 0
\end{bmatrix},
\]

where \( A_1 \in Q^r \times r \) is upper triangular.

This is equivalent to premultiplying \( [A_B] \) by a matrix of the form
\[
\begin{bmatrix}
P_1 & 0 \\
0 & I_p
\end{bmatrix},
\]

where \(P_1 \in \mathbb{Q}^{m \times m}\) is a nonsingular matrix and \(I_p\) denotes the \(p \times p\) identity matrix.

Now, by using unimodular elimination, we can find a unimodular matrix \(Q\) such that \(QS\) is of the form

\[
\begin{bmatrix}
B_1 & B_2 \\
0 & 0
\end{bmatrix},
\]

where \(B_1 \in \mathbb{Q}^{s \times s}\) is upper triangular.

Hence, premultiplying \([A]_B\) by the product

\[
\begin{bmatrix}
I_m & 0 \\
0 & Q
\end{bmatrix} \begin{bmatrix}
P_1 & 0 \\
P_2 & I_p
\end{bmatrix} = \begin{bmatrix}
P_1 & 0 \\
QP_2 & Q
\end{bmatrix}
\]

yields a matrix of the form:

\[
\begin{bmatrix}
A_1 & A_2 \\
0 & 0 \\
0 & B_1 \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
A' \\
0 \\
0 \\
B'
\end{bmatrix}.
\]

The correspondence

\([y,z] \leftrightarrow [y',z']\]

\[
\begin{bmatrix}
P_1 & 0 \\
QP_2 & Q
\end{bmatrix}
\]

defines a bijection on \(\mathbb{Q}^m \times \mathbb{Z}^p\), since:

\(z' \in \mathbb{Z}^p \leftrightarrow z'Q \in \mathbb{Z}^p \leftrightarrow z \in \mathbb{Z}^p\).
Thus:

\[ x \in M \iff \exists y \in \mathbb{Q}^m, \exists z \in \mathbb{Z}^p \text{ such that } [y, z] \begin{bmatrix} A \\ B \end{bmatrix} = x \]

\[ \iff \exists y' \in \mathbb{Q}^m, \exists z' \in \mathbb{Z}^p \] such that \[ [y', z'] \begin{bmatrix} \frac{P_1}{Q P_2} & 0 \\ Q & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = x \]

\[ \iff \exists y' \in \mathbb{Q}^m, \exists z' \in \mathbb{Z}^p \text{ such that } [y', z'] \begin{bmatrix} A' \\ -B' \end{bmatrix} = x \]

\[ \iff \exists y'' \in \mathbb{Q}^r, \exists z'' \in \mathbb{Z}^s \text{ such that } [y'', z''] \begin{bmatrix} A' \\ -B' \end{bmatrix} = x. \]

We can now state and prove the following:

2.4 Theorem: Let \( S = \{yA + zB : y \in \mathbb{Q}^m, z \in \mathbb{Z}^p\} \), where \( A \in \mathbb{Q}^m \times n \) and \( B \in \mathbb{Q}^p \times n \). Then there are integers \( r, s \) and matrices \( C \in \mathbb{Q}^r \times n \) and \( D \in \mathbb{Q}^s \times n \) such that

\[ S = \{x \in \mathbb{Q}^n : Cx = 0, Dx \in \mathbb{Z}^p\}. \]

(Note that by Proposition 2.2 this implies that \( S \) is constrained.)

Proof: By Theorem 2.3 we can assume, without loss of generality, that

\[ \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & & & \\ 0 & B_1 & & & \\ & & \ddots & & \\ & & & 0 & B_2 \end{bmatrix}, \]

where \( A_1 \in \mathbb{Q}^m \times m \) and \( B_1 \in \mathbb{Q}^p \times p \) are upper triangular matrices having nonzero diagonal entries.
Hence, using Gaussian elimination, we can find a matrix $Q$ such that

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} I_m & 0 & 0 \\ 0 & I_p & 0 \end{pmatrix}. $$

Therefore

$$S = \{x: x = yA + zB, y \in Q^m, z \in Z^p\}$$

$$= \{x: xQ = (yA + zB)Q, y \in Q^m, z \in Z^p\}$$

$$= \{x: xQ = (y, z, 0), y \in Q^m, z \in Z^p\}$$

$$= \{x: xQ^j \in Z, \text{ for } m + 1 \leq j \leq m + p \text{ and } xQ^j = 0, \text{ for } j > m + p\}$$

(where $Q^j$ is the $j$th column of $Q$)

$$= \{x: Cx \in Z, Dx = 0\}, \text{ where } C = [Q^{m+1}, \ldots, Q^{m+p}]^t \text{ and } D = [Q^{m+p+1}, \ldots, Q^n]^t.$$ 

Theorem 2.4 is a Weyl-type property. By an analysis similar to that used to establish the results 1.7-1.9 (see [3]) corresponding Farkas-, Minkowski- and Fulkerson-type properties result. Hence, we have

2.5 Corollary:

(a) For any $A \in Q^m \times n$, $B \in Q^p \times n$ and $C \in Q^n$ exactly one of the following is true.

(i) $\exists y \in Q^m$, $z \in Z^p$ such that $yA + zB = C$;

(ii) $\exists x \in Q^n$ such that $Ax = 0$, $Bx \in Z^p$ and $Cx \notin Z$.

(b) If $S = \{x \in Q^n: Ax = 0, Bx \in Z^p\}$, where $A \in Q^m \times n$ and $B \in Q^p \times n$, then $S^{\#} = \{yA + zB: y \in Q^m, z \in Z^p\}$. 
(c) If \( S = \{ x \in \mathbb{Q}^n : Ax = 0, Bx \in \mathbb{Z}^p \} \), then there are integers \( r \) and \( s \) and matrices \( C \in \mathbb{Q}^{r \times n} \) and \( D \in \mathbb{Q}^{s \times n} \) such that \( S = \{ yC + zD : y \in \mathbb{Q}^r, z \in \mathbb{Z}^s \} \).

Now, let us again consider the Weyl, Minkowski, Farkas and Fulkerson properties for \( \mathbb{Z} \)-duality, in the strict sense defined in Section 1. Recall that the Weyl, Minkowski and Fulkerson properties fail. However, the validity of the extended versions of these properties (derived in Theorem 2.4 and Corollary 2.5) are sufficient to show that the Farkas property still holds in the original sense. Indeed, taking \( A \) vacuous (i.e., \( m = 0 \)) in 2.5(a) yields:

2.6 Corollary: The Farkas property holds for \( \mathbb{Z} \)-duality. That is, for any \( B \in \mathbb{Q}^{p \times n} \) and \( c \in \mathbb{Q}^n \), exactly one of the following is true:

(i) \( \exists y \in \mathbb{Z}^p \) such that \( yB = c \);

(ii) \( \exists x \in \mathbb{Q}^n \) such that \( Bx \in \mathbb{Z}^p \) and \( cx \notin \mathbb{Z} \).

We now focus on the problem of characterizing constrained (or closed) \( \mathbb{Z} \)-modules, i.e., \( \mathbb{Z} \)-modules with the property that \( s^{##} = s \).

We will show that the \( \mathbb{Z} \)-modules which are constrained are precisely the ones which are the sum of a subspace and a finitely generated \( \mathbb{Z} \)-module, i.e., modules of the form \( S = \{ yA + zB : y \in \mathbb{Q}^m, z \in \mathbb{Z}^p \} \) or, equivalently, of the form \( \{ x \in \mathbb{Q}^n : Cx = 0, Dx \in \mathbb{Z}^p \} \). Observe that, in this aspect, \( \mathbb{Z} \)-duality differs from cone duality, since not all constrained cones can be expressed as a sum of a subspace and a finitely generated cone (recall that a
cone is constrained if and only if it is topologically closed; circular cones are examples of cones which are topologically closed -- hence constrained -- but which cannot be expressed as a sum of a subspace and a finitely generated cone).

We will use the fact that a submodule of a finitely generated \( \mathbb{Z} \)-module is again finitely generated. This is a well-known theorem in algebra. However, we present here an elementary proof of this result (see [1, p. 496] for another simple proof).

First we need the following:

2.7 Lemma: Let \( M \subset \mathbb{Q}^n \) be a \( \mathbb{Z} \)-module. Then \( M \) is finitely generated if and only if there is \( k \in \mathbb{Z} \) such that \( kM \subset \mathbb{Z}^n \).

Proof:

a) \( \Rightarrow \) Let \( M = \{ yA : y \in \mathbb{Z}^m \} \), where \( A \in \mathbb{Q}^m \times \mathbb{Z}^n \). Since all entries in \( A \) are rational numbers, we can choose \( k \in \mathbb{Z} \) such that \( kA \in \mathbb{Z}^m \times \mathbb{Z}^n \). Hence \( kM = \{ y(kA) : y \in \mathbb{Z}^m \} \subset \mathbb{Z}^n \).

b) \( \Leftarrow \) Suppose that \( N = kM \subset \mathbb{Z}^n \). Let \( S \) be the subspace generated by \( N \). Let \( a_1, \ldots, a_r \) denote elements of \( N \) such that \( \{a_1, \ldots, a_r\} \) constitute a basis of \( S \).

For any \( x \in N \), there exist rationals \( \alpha_1, \ldots, \alpha_r \) (uniquely determined) such that \( x = \sum_{i=1}^{r} \alpha_i a_i \).

Define \( T = \{ x \in N : 0 \leq \alpha_i \leq 1, \forall i \} \). (Observe that \( T \supset \{a_1, \ldots, a_r\} \)).

Since \( N \subset \mathbb{Z}^n \), it is clear that \( T \) is finite. Let \( T = \{t_1, \ldots, t_s\} \).

Clearly, \( \{ \sum_{i=1}^{s} y_i t_i : y_i \in \mathbb{Z} \} \subset N \), since \( N \) is closed under integer linear combinations.
Now, let \( x \in \mathbb{N} \). We can express \( x \) as:

\[
x = \sum_{i=1}^{n} \alpha_i a_i = \sum_{i=1}^{n} \alpha_i' a_i + \sum_{i=1}^{n} (\alpha_i - \alpha_i') a_i,
\]

which shows that every \( x \in \mathbb{N} \) is an integer combination of elements of \( T \). Hence, \( N \subseteq \{ \sum_{i=1}^{s} y_i t_i : y_i \in \mathbb{Z} \} \).

Therefore, we proved that \( N = \{ \sum_{i=1}^{s} y_i t_i : y_i \in \mathbb{Z} \} \), which shows that \( N \) (and hence \( M \)) is finitely generated.\( \blacksquare \)

2.8 Proposition: Let \( M \subseteq \mathbb{Q}^n \) be a finitely generated \( \mathbb{Z} \)-module. Then any submodule of \( M \) is also finitely generated.

Proof: Let \( N \) be a submodule of \( M \). By Lemma 2.7, there exists \( k \in \mathbb{Z} \) such that \( kN \subseteq kM \subseteq \mathbb{Z}^n \). But, again by Lemma 2.7, this implies that \( N \) is finitely generated.\( \blacksquare \)

Now, we are ready for the main result of this section.

2.9 Theorem: Let \( M \subseteq \mathbb{Q}^n \). Then \( M \) is a constrained \( \mathbb{Z} \)-module if and only if there exist matrices \( A \in \mathbb{Q}^{m \times n} \) and \( B \in \mathbb{Q}^{p \times n} \) such that \( M = \{ yA + zB : y \in \mathbb{Q}^m, z \in \mathbb{Z}^p \} \).

Proof: The "if" part was established in Theorem 2.4. To show the converse, suppose that \( M \) is a constrained \( \mathbb{Z} \)-module. That is, \( \exists S \subseteq \mathbb{Q}^n \) such that \( M = S^\# = \{ x \in \mathbb{Q}^n : Sx \in \mathbb{Z} \} \).

Let \( m \) be the dimension of the subspace spanned by \( S \). We consider two cases:

(a) \( m = n \)

Since \( S \) is full dimensional, we can choose \( n \) linearly independent elements \( s_1, \ldots, s_n \in S \). Let \( S_0 \) be the matrix which
has rows \( s_1, \ldots, s_n \) and let \( M_0 = (s_1, \ldots, s_n)^\# = \{ x \in \mathbb{Q}^n : S_0 x \in \mathbb{Z}^n \} \).

Then:

\[
x \in M_0 \iff \\
S_0 x \in \mathbb{Z}^n \iff \\
\exists y \in \mathbb{Z}^n \text{ s.t. } y = S_0 x \iff \\
\exists y \in \mathbb{Z}^n \text{ s.t. } x = (S_0)^{-1} y.
\]

Thus, \( M_0 = \left( (S_0)^{-1} y : y \in \mathbb{Z}^n \right) \), which shows that \( M_0 \) is finitely generated.

But since \( \{ s_1, \ldots, s_n \} \subseteq S \) we have \( M \subseteq M_0 \). Hence, Proposition 2.8 implies that \( M \) is finitely generated.

(b) \( m < n \)

Let \( T = [t_1 t_2 \cdots t_m t_{m+1} \cdots t_n] \in \mathbb{Q}^n \times n \) be a matrix with columns \( t_1, \ldots, t_n \), where \( \{ t_{m+1}, \ldots, t_n \} \) is a basis of the subspace orthogonal to \( S \) and \( \{ t_1, \ldots, t_n \} \) is a basis of \( \mathbb{Q}^n \).

Let \( ST \) denote the set of all vectors of the form \( sT \), where \( s \in S \). Since the last \( n - m \) columns of \( T \) are orthogonal to \( S \), the last \( n - m \) components of all elements of \( ST \) are zero. We will represent this by writing

\[
ST = [S' \mid 0], \text{ where } S' \subseteq \mathbb{Q}^m.
\]

(Actually, "post-multiplying" \( S \) by \( T \) is equivalent to doing "column operations" in \( S \) to reduce the last \( n - m \) "columns" to 0).
Define $N = T^{-1}M = \{T^{-1}m: m \in M\}$. Then:

$$N = \{y \in \mathbb{Q}^n: Ty \in M\} = \{y \in \mathbb{Q}^n: STy \in \mathbb{Z}\} = \{y \in \mathbb{Q}^n: [S' \ 0]y \in \mathbb{Z}\} = \{(y_1, y_2): S'y_1 \in \mathbb{Z} \text{ and } y_2 \in \mathbb{Q}^{n-m}\} = \{(y_1, 0): S'y_1 \in \mathbb{Z}\} + \{(0, y_2): y_2 \in \mathbb{Q}^{n-m}\}.$$

Since rank $S' = m$, by part (a) we conclude that $\{(y_1, 0): S'y_1 \in \mathbb{Z}\}$ is finitely generated (say by the columns of $[A]$, where $A \in \mathbb{Q}^{m \times p}$).

Thus:

$$N = \left\{ \left[ \begin{array}{c} A \\ 0 \end{array} \right] w + \left[ \begin{array}{c} 0 \\ I \end{array} \right] y_2: w \in \mathbb{Z}^p, y_2 \in \mathbb{Q}^{n-m} \right\},$$

and hence

$$M = TN = \left\{ T \left[ \begin{array}{c} A \\ 0 \end{array} \right] w + T \left[ \begin{array}{c} 0 \\ I \end{array} \right] y_2: w \in \mathbb{Z}^p, y_2 \in \mathbb{Q}^{n-m} \right\},$$

which is the desired result. $lacksquare$

In [2] we also give an alternative proof of Theorem 2.9, based on geometrical properties of $\mathbb{Z}^n$. The above approach, which is fundamentally based on Proposition 2.8, has, however, the advantage of being applicable to more general dualities, namely dualities in which $X = \mathbb{Q}$ and $D$ is any sub-ring of $\mathbb{Q}$.

Note, first, that unimodular elimination is still appropriate for D-modules. That is, given a unimodular matrix $P \in \mathbb{Z}^{m \times m}$, we have $\{yA: y \in D^m\} = \{y(PA): y \in D^m\}$, for any $A \in \mathbb{Q}^{m \times n}$ (this follows from the fact that, for $P$ unimodular,
\( y \in D^m \iff yP \in D^m \). Therefore, an analogue of Theorem 2.4 remains true for this case; i.e., sets of the form \( S = \{yA + zB: y \in \mathbb{Q}^m, z \in D^P\} \) are constrained under \((\mathbb{Q}, D)\)-duality whenever \( D \) is a sub-ring of \( \mathbb{Q} \).

The basic result used to prove the converse was Proposition 2.8. We show that a similar result holds for modules over any subring of the rationals. First, we show:

2.10 Lemma: Let \( D \) be a subring of \( \mathbb{Q} \). Suppose that \( p/q \in D \), where \( p \) and \( q \) are relatively prime integers. Then \( 1/q \in D \).

Proof: Since \( \gcd(p, q) = 1 \), there are integers \( a \) and \( b \) such that \( ap + bq = 1 \).

Since \( D \) is closed under addition and multiplication, we have

\[
a \left( \frac{p}{q} \right) + b \in D.
\]

But

\[
a \left( \frac{p}{q} \right) + b = \frac{ap + bq}{q} = \frac{1}{q},
\]

which shows that \( 1/q \in D \), as desired. ■

2.11 Proposition: Let \( D \) be a subring of \( \mathbb{Q} \) and let \( M \subset \mathbb{Q}^n \) be a finitely generated \( D \)-module. Then every submodule of \( M \) is finitely generated.

Proof: Let \( M = \{yA: y \in D^m\} \), where \( A \in \mathbb{Q}^m \times \mathbb{n} \). Choose \( k \in \mathbb{Z} \) such that \( kA \subset \mathbb{Z}^m \times \mathbb{n} \). Then \( kM \subset D^n \). Let \( N \) be a submodule of \( M \) and let \( N' = kN \). Consider the \( \mathbb{Z} \)-module \( N' \cap \mathbb{Z}^n \). By Proposition 2.8, \( N' \cap \mathbb{Z}^n \) is a finitely generated \( \mathbb{Z} \)-module. So,
assume that $a_1, \ldots, a_\xi$ generate $N' \cap \mathbb{Z}^n$. Let
\[ x = \left( \frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n} \right) \in N', \]
where $p_i$ and $q_i$ are relatively prime integers for each $i$. Let $z = (q_1q_2\cdots q_n)x$. Since $N'$ is closed under integer combinations and all components of $z$ are integral, we have $z \in N' \cap \mathbb{Z}^n$. Therefore, there exist integers $w_i, i = 1, \ldots, \xi$ such that $(q_1\cdots q_n)x = \sum_{i=1}^\xi w_i a_i$. But since $N' \subseteq D^n$, Lemma 2.10 implies $1/q_i \in D, i = 1, \ldots, n$. Therefore, every $x \in N'$ can be written as
\[ x = \sum_{i=1}^\xi \left( \frac{w_i}{q_1\cdots q_n} \right) a_i, \]
where $w_i/q_1\cdots q_n \in D, i = 1, \ldots, \xi$. Therefore $a_1, \ldots, a_\xi$ generate $N'$ as a $D$-module. Thus $N'$ (and hence $N$) is a finitely generated $D$-module.\[ ]

Proposition 2.11 implies that the proof of Theorem 2.9 remains valid with $\mathbb{Z}$ replaced by any subring $D$ of $Q$. Hence, we have the following:

2.12 Theorem: Let $D$ be a subring of $Q$. Then $S \subseteq Q^n$ is constrained under $(Q,D)$-duality if and only if there exist matrices $A \in Q^m \times n$ and $B \in Q^p \times n$ such that $S = \{yA + zB : y \in Q^m, z \in D^p\}$.\[ ]

We end this section by remarking that Theorem 2.9 can be stated in an even more general context. Actually, the results
concerning unimodular elimination remain true whenever D is a principal ideal domain and X is its field of quotients (see [12], for example). On the other hand, principal ideal domains are special cases of Noetherian rings, which have the property that submodules of finitely generated modules are again finitely generated (see [8], Section 18). Hence the proofs of both implications in Theorem 2.9 proceed as before, allowing us to state the following:

2.13 Theorem: Let D be a principal ideal domain and X its field of quotients. Then \( S \subseteq X^n \) is constrained under \((X,D)\)-duality if and only if there exist matrices \( A \in X^m \times X^n \) and \( B \in X^p \times X^n \) such that \( S = \{yA + zB : y \in X^m, z \in D^p\}\).

3. Nonnegative integral duality in \( \mathbb{Z} \)

As seen in section 1, the \((X,D)\)-duality model is relevant to the study of the problem of deciding whether there is a vector \( y \in (D^m)^* \) such that \( yA = c \). If \( X \) and \( D \) are chosen in such a way that \( (D^m)^* = \mathbb{Z}_+^m \), the duality thus obtained is useful to the study of the problem of existence of nonnegative integral solutions to a linear system. In this section we investigate one such model: the case \( X = \mathbb{Z}, D = \mathbb{Z}_+ ((\mathbb{Z},\mathbb{Z}_+)-duality \) or simply \( \mathbb{Z}_+-duality\)). In the next section we examine the case \( X = \mathbb{Q}, D = \mathbb{Z}_+ \).

The problem of deciding whether a linear system has a nonnegative integral solution (i.e., the feasibility problem for integer programming) is "hard", in the sense that it is
NP-complete and, hence, no polynomial algorithm is known to solve it. Our duality models will reflect this fact through the failure of the Farkas property.

First, we note that if \( X = \mathbb{Z} \) and \( D = \mathbb{Z}_+ \) we have:

\[
(D^m)^* = \{x \in \mathbb{Z}^m : xy \in \mathbb{Z}, \forall y \in \mathbb{Z}_+^m \} = \mathbb{Z}_+^m.
\]

Hence, the generated sets here are the subsets of \( \mathbb{Z}^n \) which are closed under nonnegative integral linear combinations, which we will call \( \mathbb{Z}_+ \)-modules. Given \( S \subseteq \mathbb{Z}^n \), the \( \mathbb{Z}_+ \)-module generated by \( S \) is the set \( \sigma(S) \) of all nonnegative integral combinations of elements of \( S \). Finitely generated \( \mathbb{Z}_+ \)-modules are those of the form \( \sigma(S) \) for some finite \( S \), i.e., sets of the form \( \{yA : y \in \mathbb{Z}_+^m\} \), where \( A \in \mathbb{Z}^m \times n \).

The \( \mathbb{Z}_+ \)-dual of a set \( S \subseteq \mathbb{Z}^n \) is the set \( S^* \) defined by \( S^* = \{x \in \mathbb{Z}^n : Sx \in \mathbb{Z}_+ \} \); i.e., \( S^* \) is the set of all vectors in \( \mathbb{Z}^n \) which make a nonnegative inner product with each element of \( S \) (observe that the integrality of the inner product is guaranteed by the fact that all sets involved are contained in \( \mathbb{Z}^n \)). In other words, \( S^* \) is the set of all integer-valued points in the cone \( S^+ \equiv \{x \in \mathbb{Q}^n : Sx \geq 0\} \) polar to \( S \). A set \( S \subseteq \mathbb{Z}^n \) is said to be (finitely) constrained if \( S = T^* \) for some (finite) \( T \subseteq \mathbb{Z}^n \). Note that finitely \( \mathbb{Z}_+ \)-constrained sets are of the form \( \{x \in \mathbb{Z}^n : Ax \geq 0\} \) for some \( A \in \mathbb{Z}^m \times n \). Also recall that Proposition 1.2(d) implies that every \( \mathbb{Z}_+ \)-constrained set is a \( \mathbb{Z}_+ \)-module.
From Propositions 1.1 and 1.2 it immediately follows that:

3.1 **Proposition:** Let \( S, T \) denote subsets of \( \mathbb{Z}^n \). Then:

(a) \( S \subseteq T \Rightarrow S^* \subseteq T^* \).
(b) \( (S \cup T)^* = S^* \cap T^* \).
(c) \( S \subseteq S^{**} \).
(d) \( S^* = S^{**} \).
(e) \( S = S^{**} \iff S \) is a constrained \( \mathbb{Z}_+ \)-module.
(f) \( S^{**} \) is the smallest constrained \( \mathbb{Z}_+ \)-module containing \( S \).
(g) If \( T \) generates the \( \mathbb{Z}_+ \)-module \( S \), then \( T^* = S^* \).
(h) If \( S = \{yA : y \in \mathbb{Z}_+^m\} \), for some \( A \in \mathbb{Z}_+^m \times \mathbb{Z}_+^n \), then
\[
S^* = \{x \in \mathbb{Z}_+^n : Ax \in \mathbb{Z}_+^m\}.
\]

In the definition of the \( \mathbb{Z}_+ \)-dual of a set \( S \subseteq \mathbb{Z}^n \) we observed that \( S^* \) is the set of all integer-valued points in the cone \( S^+ \) polar to \( S \). The next proposition emphasizes the relationship between constrained sets under \( \mathbb{Z}_+ \)-duality and constrained cones.

3.2 **Proposition:** Let \( S \subseteq \mathbb{Z}^n \). \( S \) is (finitely) constrained under \( \mathbb{Z}_+ \)-duality if and only if \( S = \mathbb{Z}^n \cap K \) for some (finitely) constrained cone \( K \subseteq \mathbb{Q}^n \).

**Proof:** First, suppose that \( S = T^* \) is a \( \mathbb{Z}_+ \)-constrained set. Then \( S = \{x \in \mathbb{Z}^n : Tx \geq 0\} = T^+ \cap \mathbb{Z}^n \), where \( T^+ = \{x \in \mathbb{Q}^n : Tx \geq 0\} \) is a constrained cone. Also, it is clear that if \( S \) is finitely constrained (i.e., \(|T| \) is finite), then \( T^+ \) is also finitely constrained.
Now, assume that $S = \mathbb{Z}^n \cap K$, where $K = \{x \in \mathbb{Q}^n : Tx \geq 0\}$ for some $T \subseteq \mathbb{Q}^n$. By properly scaling each element of $T$ we can assume, without loss of generality, that $T \subseteq \mathbb{Z}^n$. Thus, $S = \{x \in \mathbb{Z}^n : Tx \geq 0\}$, which shows that $S$ is $\mathbb{Z}_+^n$-constrained. If, in particular, $K$ is finitely constrained, it is clear that $S$ is also finitely constrained. □

The previous proposition related constrained $\mathbb{Z}_+^n$-modules to constrained cones. Now, we establish the relationship between $\mathbb{Z}_+^n$-modules and cones from the point of view of generation.

3.3 Proposition: Let $K \subseteq \mathbb{Q}^n$ be a cone, let $S = K \cap \mathbb{Z}^n$ and let $T \subseteq \mathbb{Z}^n$. If the $\mathbb{Z}_+^n$-module generated by $T$ is $S$, then the cone generated by $T$ is $K$.

Proof: Since $T \subseteq K$ it suffices to show that every $x \in K$ can be expressed as a nonnegative combination of elements of $T$. So, let $x \in K$. Since $K \subseteq \mathbb{Q}^n$, there is some $\lambda > 0$ such that $\lambda x \in \mathbb{Z}^n$, which implies $\lambda x \in S$. But $T$ generates $S$ as a $\mathbb{Z}_+^n$-module. Thus, $\lambda x$ can be expressed as a nonnegative integral combination of elements of $T$. This implies that $x$ is a nonnegative combination of elements of $T$, as desired. □

We can now improve the characterization of the $\mathbb{Z}_+^n$-constrained sets given in Proposition 3.2.

3.4 Proposition: Let $S \subseteq \mathbb{Z}^n$ and let $K \subseteq \mathbb{Q}^n$ be the cone generated by $S$. Then $S$ is a (finitely) constrained $\mathbb{Z}_+^n$-module if and only if (i) $K$ is a (finitely) constrained cone and (ii) $S = K \cap \mathbb{Z}^n$. 
Proof: By Proposition 3.2, (i) and (ii) immediately imply that $S$ is $\mathbb{Z}_+$-constrained (finitely constrained if $K$ is).

Now, suppose that $S$ is (finitely) $\mathbb{Z}_+$-constrained. By Proposition 3.2, $S = K' \cap \mathbb{Z}^n$ for some (finitely) constrained cone $K'$. But by Proposition 3.3, $K'$ is the cone generated by $S$, which shows that $K = K'$ and hence that (i) and (ii) hold.$\blacksquare$

Propositions 3.3 and 3.4 together with the theorems of Weyl and Minkowski for cones, imply an immediate relationship between finitely generated and finitely constrained $\mathbb{Z}_+$-modules. If a $\mathbb{Z}_+$-module $S$ is constrained but not finitely constrained, then it cannot be finitely generated (since this would imply that the cone generated by $S$ is finitely generated and hence finitely constrained, by Weyl's theorem). Another way of stating the same property is that if a finitely generated $\mathbb{Z}_+$-module is constrained, then it must be finitely constrained.

We are now ready to discuss the validity of the Farkas, Weyl and Minkowski properties for $\mathbb{Z}_+$-duality. Recall that, by Proposition 1.8(b), the Farkas property holds for $\mathbb{Z}_+$-duality if and only if every finitely constrained $\mathbb{Z}_+$-module is constrained. Let $S = \{yA : y \in \mathbb{Z}_+^m\}$, where $A \in \mathbb{Z}_+^m \times \mathbb{N}$, be a finitely generated $\mathbb{Z}_+$-module. It is easily shown that the cone generated by $S$ is $K = \{yA : y \in \mathbb{Q}_+^m\}$, which is constrained by the Farkas property for cones. Hence, Proposition 3.4 implies that $S$ is a constrained $\mathbb{Z}_+$-module if and only if $S$ contains all integral points in $K$. This is not always the case. For example, let $S = \{2y_1 + 3y_2 : y_1, y_2 \in \mathbb{Z}_+\} \subset \mathbb{Z}_+^1$. The cone generated by $S$ is
K = \mathbb{Q}_+.\ It is easy to see that 1 \not\in S;\ thus, S does not contain all integral points in K.

Therefore, the Farkas property (and hence the Weyl property) fails for \mathbb{Z}_+\-duality. That is, given \( A \in \mathbb{Z}^m \times \mathbb{Z}^n \) and \( c \in \mathbb{Z}^n \), it is not necessarily true that exactly one of the following alternatives holds:

(i) \( \exists y \in \mathbb{Z}_+^m \) such that \( yA = c \);

(ii) \( \exists x \in \mathbb{Z}_+^n \) such that \( Ax \geq 0 \) and \( cx < 0 \).

The Minkowski property, on the other hand, does hold for \mathbb{Z}_+\-duality (see Trotter [11]). It is an immediate consequence of the following result of Hilbert [9] (see Giles and Pulleyblank [7] for a simple proof).

3.5 Theorem (Hilbert's finite basis theorem [9]): Let \( K = \{x \in \mathbb{Q}^n : Ax \geq 0\} \), where \( A \in \mathbb{Q}^m \times \mathbb{Q}^n \), be a finitely constrained cone. Let \( S = K \cap \mathbb{Z}^n \). Then there is a finite set \( H \subset S \) such that \( H \) generates \( S \) as a \( \mathbb{Z}_+ \)-module.

3.6 Corollary: The Minkowski property (and hence the Fulkerson property) holds for \( \mathbb{Z}_+ \)-duality.

Motivated by Theorem 3.5, we say that a set \( H = \{h_1, \cdots, h_k\} \subset \mathbb{Z}^n \) is a Hilbert basis if every integral vector in the cone generated by \( H \) can be expressed as a nonnegative integral combination of \( h_1, \cdots, h_k \). Observe that Proposition 3.4 immediately implies the following result.

3.7 Proposition: A finitely generated \( \mathbb{Z}_+ \)-module \( S = \{yA: y \in \mathbb{Z}_+^m\} \) is constrained if and only if the set of row-vectors of
A constitutes a Hilbert basis (in this case, we say that A is a Hilbert matrix).

A rational linear system \((Ax \leq b)\) is totally dual integral (see Edmonds and Giles [6]) provided the linear programming problem \((\min yb: yA = c, y \geq 0)\) has an integral optimal solution for each \(c\) for which an optimal solution exists. Thus the Hilbert basis requirement of Proposition 3.7 is equivalent to the requirement that the system \((Ax \leq 0)\) be totally dual integral (see also Cook [5]). That is, homogeneous totally dual integral systems with integral coefficients correspond precisely to Hilbert bases which, in turn, correspond (by 3.7) precisely to those finitely generated \(\mathbb{Z}_+\)-modules for which the Weyl property holds.

We remark further that the failure of the Farkas property for \(\mathbb{Z}_+\)-duality can be seen as another indication that Integer Programming is "hard". By the same token, we may also expect that finitely generated \(\mathbb{Z}_+\)-modules which are also constrained (and in this case they must be finitely constrained, as pointed out earlier) should be associated with instances of integer programming problems which are "easy" to solve. The following theorem of Chandrasekaran [4] shows that this is indeed the case.

3.8 Theorem (Chandrasekaran [4]):

There is a polynomial time algorithm that, given a linear system \(yA = c\), where \(A \in \mathbb{Z}_+^{m \times n}\) and \(c\) is an integral vector in the cone generated by the rows of \(A\), either finds a nonnegative integral solution \(y\) or shows that \(A\) is not a
Hilbert matrix (i.e., that the \( \mathbb{Z}_+ \)-module generated by the rows of \( A \) is not constrained).

4. **Nonnegative integral duality in \( \mathbb{Q} \)

Now, we investigate another \( \mathbb{Z}_+ \)-duality model, which differs from \((\mathbb{Z}, \mathbb{Z}_+)\)-duality only in that \( X \) is taken as \( \mathbb{Q} \) instead of \( \mathbb{Z} \). This means that we are allowing rational data, instead of all-integral data. In this section we compare the properties of these two dualities.

We start by observing that for \( X = \mathbb{Q} \) and \( D = \mathbb{Z}_+ \) we still have

\[
(D^m)^* = \{ x \in \mathbb{Q}^n : xy \in \mathbb{Z}_+, \forall y \in \mathbb{Z}_m \} = \mathbb{Z}_m^m.
\]

Hence, as in \((\mathbb{Z}, \mathbb{Z}_+)\)-duality, the generated sets for \((\mathbb{Q}, \mathbb{Z}_+)\)-duality are those sets (now contained in \( \mathbb{Q}^n \)) which are closed under nonnegative integral combinations (we will continue to call such sets \( \mathbb{Z}_+ \)-modules).

Recall that, under \((\mathbb{Z}, \mathbb{Z}_+)\)-duality, the dual \( S^* \) of a set \( S \subseteq \mathbb{Z}^n \) is given by \( S^* = S^+ \cap \mathbb{Z}^n \), where \( S^+ \) is the cone polar to \( S \). Under \((\mathbb{Q}, \mathbb{Z}_+)\)-duality, the dual of a set \( S \subseteq \mathbb{Q}^n \) is given by

\[
S^* = \{ x \in \mathbb{Q}^n : Sx \in \mathbb{Z}_+ \} = \{ x \in \mathbb{Q}^n : Sx \in \mathbb{Z}_+ \} \cap \{ x \in \mathbb{Q}^n : Sx \geq 0 \} = S^\# \cap S^+,
\]

where \( S^\# \) and \( S^+ \) denote, respectively, the \( \mathbb{Z} \)-dual and the polar cone of \( S \).
Therefore, every constrained set under \((\mathcal{Q}, \mathbb{Z}_+)^\ast\)-duality is the intersection of a constrained \(\mathbb{Z}\)-module and a constrained cone. A natural question to ask at this point is whether the converse is true; i.e., is the intersection of a constrained \(\mathbb{Z}\)-module \(M^\#\) and a constrained cone \(K^+\) always constrained under \((\mathcal{Q}, \mathbb{Z}_+)^\ast\)-duality? The answer is no. For example, let \(M^\# = \mathbb{Z}\) and \(K^+ = \mathcal{Q}\). Then \(M^\#\) is a constrained \(\mathbb{Z}_+\)-module, \(K^+\) is a constrained cone, but \(M^\# \cap K^+ = \mathbb{Z}\) is not \((\mathcal{Q}, \mathbb{Z}_+)^\ast\)-constrained, since \(\mathbb{Z}^\ast = \{x \in \mathcal{Q} : xy \in \mathbb{Z}_+, \forall y \in \mathbb{Z}\} = \{0\}\) and hence \(\mathbb{Z}^{**} = \{0\}^\ast = \mathcal{Q} \neq \mathbb{Z}\).

To obtain conditions under which \(M^\# \cap K^+\) is \((\mathcal{Q}, \mathbb{Z}_+)^\ast\)-constrained we first establish some notation. For any \(T \subset \mathcal{Q}^n\) we denote by \(S(T), K(T)\) and \(M(T)\), respectively, the subspace, the cone and the \(\mathbb{Z}\)-module generated by \(T\). If \(T\) is closed under addition we denote by \(s(T)\) the \textbf{lineality space} of \(T\), i.e., the set of all \(x \in \mathcal{Q}^n\) such that \(\lambda x \in T\) for every \(\lambda \in \mathcal{Q}\). It is easy to see that for every \(T \subset \mathcal{Q}^n\) we have \(s(T^+) = s(T^\#) = s(T^\ast) = T^\perp\), where \(T^\perp\) is the subspace orthogonal to \(T\).

4.1 \textbf{Lemma}: Let \(M \subset \mathcal{Q}^n\) be a \(\mathbb{Z}\) module and let \(K \subset \mathcal{Q}^n\) be a cone. Assume that \(M\) and \(K\) generate the same subspace \(S\). Then

a) \(K(M \cap K) = K;\)

b) \(M(M \cap K) = M.\)

\textbf{Proof}: (a) Clearly, \(K(M \cap K) \subset K\). To prove the reverse inclusion, let \(u \in K\). Since \(S(K) = S(M)\), we have \(u \in S(M)\). Hence, there are elements \(m_1, \ldots, m_k \in M\) and rationals
\( \lambda_1, \ldots, \lambda_k \) such that \( u = \sum_{i=1}^k \lambda_i m_i \). Let \( \lambda \) be a positive integer such that \( \lambda \lambda_i \in \mathbb{Z}, i = 1, \ldots, k \). Then \( \lambda u \in M \cap K \), which implies that \( u \in K(M \cap K) \). Thus, \( K \subseteq K(M \cap K) \).

(b) Clearly, \( M(M \cap K) \subseteq M \). Now, select elements \( m_1, \ldots, m_k \in M \) such that they form a basis for the subspace \( S \) generated by \( M \). Let \( T = \{ x \in M : x = \sum_{i=1}^k \alpha_i m_i, \text{ with } 0 \leq \alpha_i \leq 1 \} \). Each element \( m \in M \) can be expressed as

\[
m = \sum_{i=1}^k \alpha_i m_i = \sum_{i=1}^k \alpha_i m_i + \left[ \sum_{i=1}^k (\alpha_i - 1) \alpha_i m_i \right],
\]

which is an integral combination of the elements \( m_1, \ldots, m_k, (\sum_{i=1}^k (\alpha_i - 1) \alpha_i m_i) \), all of which are in \( T \). Thus, \( T \) generates \( M \). Also, \( T \) is bounded, since for any \( x \in T \) we have

\[
||x|| = || \sum_{i=1}^k \alpha_i m_i || \leq \sum_{i=1}^k ||m_i|| = \delta.
\]

Choose any relatively interior element \( u \) of \( K \). As in part (a), there is a positive integer \( \lambda_1 \) such that \( \lambda_1 u \in M \cap K \). Since \( \lambda_1 u \) is again in the relative interior of \( K \), we can select a positive integer \( \lambda_2 \) such that the intersection of the sphere of center \( m_0 = \lambda_2 \lambda_1 u \) and radius \( \delta \) with \( S \) is contained in \( K \). Since \( T \subseteq S \) and \( ||t|| \leq \delta \) for all \( t \in T \), this implies that \( m_0 + T = \{m_0 + t : t \in T\} \subseteq K \). In fact, \( m_0 + T \subseteq M \cap K \), since \( m_0 \in M \) and \( T \subseteq M \). Finally notice that each \( t \in T \) is the difference \( (m_0 + t) - m_0 \) of two elements of \( M_0 + T \). Hence \( M(m_0 + T) \subseteq T \) and we have

\[
M(M \cap K) \subseteq M(m_0 + T) \subseteq M(T) = M. \quad \blacksquare
\]
Using Lemma 4.1 we can now establish necessary and sufficient conditions under which the \( \mathbb{Z} \)-module \( M \) and the cone \( K \) satisfy 
\[(M \cap K)^* = M^\# \cap K^+.\]

4.2 Theorem: Let \( M \subseteq \mathbb{Q}^n \) and \( K \subseteq \mathbb{Q}^n \) denote a \( \mathbb{Z} \)-module and a cone, respectively. Then \((M \cap K)^* = M^\# \cap K^+\) if and only if \( S(M) = S(K) \).

Proof: First, assume that \( M \) and \( K \) satisfy \( S(M) = S(K) \). By Lemma 4.1 we have \( K(M \cap K) = K \) and \( M(M \cap K) = M \). Therefore,
\[(M \cap K)^* = (M \cap K)^\# \cap (M \cap K)^+ = (M(M \cap K))^\# \cap (K(M \cap K))^+ = M^\# \cap K^+.\]

Now, suppose that \((M \cap K)^* = M^\# \cap K^+\). By considering the lineality spaces of the sets on each side of this equality we have:
\[s(M^\#) \cap s(K^+) = s((M \cap K)^*), \text{ which is equivalent to } M^\perp \cap K^\perp = (M \cap K)^\perp.\]

Using the fact that for any sets \( T_1, T_2 \subseteq \mathbb{Q}^n \) we have \( T_1 \cap T_2 = (S(T_1) + S(T_2))^\perp \), this last equality implies that \( S(M) + S(K) = S(M \cap K) \), which shows that \( S(M) = S(K) = S(M \cap K) \).

We can now obtain characterizations for constrained sets under \((\mathbb{Q}, \mathbb{Z}_+)^\)-duality.

4.3 Proposition: Let \( T \subseteq \mathbb{Q}^n \). Then \( T \) is \((\mathbb{Q}, \mathbb{Z}_+)^\)-constrained if and only if there exist a constrained \( \mathbb{Z} \)-module \( M \) and a
constrained cone \( K \) such that (i) \( s(M) = s(K) \) and (ii) \( T = M \cap K \).

**Proof:**

\( \Rightarrow \) Assume \( T = U^* \) for some \( U \subset Q^n \). Let \( M = U^\# \) and \( K = U^+ \). Clearly, \( T = U^\# \cap U^+ = M \cap K \). So, (ii) holds. Also, we have \( s(M) = s(U^\#) = U^\perp \) and \( s(K) = s(U^+) = U^\perp \). Thus, (i) also holds.

\( \Leftarrow \) Assume \( T = M \cap K \), where \( M \subset Q^n \) is a \( Z \)-module and \( K \subset Q^n \) is a cone. Suppose that \( s(M) = s(K) \). Since \( M = M^{#\#} \) and \( K = K^{++} \), this last condition implies that \( (M^\#)^\perp = (K^+)^\perp \), which in turn implies that \( S(M^\#) = S(K^+) \). Now, by Proposition 4.2, we have:

\[
(M^\# \cap K^+)^* = (M^\#)^* \cap (K^+) = M \cap K = T.
\]

Hence \( T \) is \( (Q,Z_+) \)-constrained. \( \blacksquare \)

The next proposition improves the characterization given in Proposition 4.3 by showing that \( M \) and \( K \) can be chosen as the \( Z \)-module and the cone generated by \( T \), respectively.

**4.4 Proposition:** Let \( T \subset Q^n \). Then \( T \) is \( (Q,Z_+) \)-constrained if and only if (i) \( M(T) \) is a constrained \( Z \)-module; (ii) \( K(T) \) is a constrained cone; (iii) \( s(M(T)) = s(K(T)) \) and (iv) \( T = M(T) \cap K(T) \).

**Proof:** The "if" part is immediately implied by Proposition 4.3.

Now, suppose that \( T \) is \( (Q,Z_+) \)-constrained. By Proposition 4.3, there exist a constrained \( Z \)-module \( M \) and a constrained
cone $K$ satisfying $T = M \cap K$ and $s(M) = s(K)$. Define $M' = M \cap S(T)$ and $K' = K \cap S(T)$. We have:

$$M' \cap T' = M \cap S(T) \cap K \cap S(T) = T \cap S(T) = T.$$  

Also, since $M$ and $K$ have the same lineality space, we have $s(M') = s(K')$. Moreover, $M'$ and $K'$ are a constrained $\mathbb{Z}$-module and a constrained cone, respectively, since $M'$ is the intersection of two constrained $\mathbb{Z}$-modules and $K'$ is the intersection of two constrained cones (recall that subspaces are special cases of constrained cones and $\mathbb{Z}$-modules).

Hence, if we show that $M' = M(T)$ and $K' = K(T)$ the proof will be complete. It is clear that $S(M') \subseteq S(T)$ and $S(K') \subseteq S(T)$. Since $T = M' \cap K'$ we also have $S(T) \subseteq S(M')$ and $S(T) \subseteq S(K')$. Therefore $S(M') = S(K') = S(T)$ and Lemma 4.1 implies:

$$M(T) = M(M' \cap K') = M'$$

and

$$K(T) = K(M' \cap K') = K'.$$

Applying Proposition 4.4 to the special case where $S$ is a finitely generated $\mathbb{Z}_+^n$-module yields the following (a pointed cone has lineality \{0\}):  

4.5 Proposition: Let $S = \{yA : y \in \mathbb{Z}_+^m\}$, where $A \in \mathbb{Q}^m \times \mathbb{n}$, be a finitely generated $\mathbb{Z}_+^n$-module. Then $S$ is finitely constrained if and only if:

(i) the cone $\{yA : y \in \mathbb{Q}_+^m\}$ is pointed;
(ii) $S = \{yA : y \in \mathbb{Q}_+^m\} \cap \{yA : y \in \mathbb{Z}_+^m\}$ (i.e., $S$ contains all points in the intersection of the $\mathbb{Z}$-module and the cone generated by the rows of $A$).

Proof: This follows from Proposition 4.4 by observing that $M(S) = \{yA : y \in \mathbb{Z}_+^m\}$ is a finitely generated $\mathbb{Z}$-module and hence has lineality space $\{0\}$. Thus, the condition $s(M(S)) = s(K(S))$ holds if and only if $K(S)$ is pointed. □

Proposition 4.5 establishes the conditions under which a finitely generated $\mathbb{Z}_+^m$-module $S = \{yA : y \in \mathbb{Z}_+^m\}$, where $A \in \mathbb{Q}_+^{m \times n}$, is constrained. First of all, the cone $K(S) = \{yA : y \geq 0\}$ must be pointed. Also $S$ must contain all points in $\{yA : y \geq 0\} \cap \{yA : y \in \mathbb{Z}_+^m\}$. Neither of these properties holds in general, as the following examples show:

a) Let $S = \mathbb{Z} = \{(1y_1 + (-1)y_2 : y_1, y_2 \in \mathbb{Z}_+)\}$. Then $S$ is a finitely generated $\mathbb{Z}_+^m$-module. Since $K(S) = \mathbb{Q}_+$ is not pointed, $S$ cannot be constrained.

b) Let $S = \{2y_1 + 3y_2 : y_1, y_2 \in \mathbb{Z}_+\} = \{0, 2, 3, 4, \ldots\}$. In this case $K(S) = \mathbb{Q}_+$ is a pointed cone. But $K(S) \cap M(S) = \mathbb{Q}_+ \cap \mathbb{Z} = \mathbb{Z}_+ \neq S$. Thus, $S$ is not constrained.

Therefore, the Farkas and Weyl properties fail again for $(\mathbb{Q}, \mathbb{Z}_+)$-duality, as they did for $(\mathbb{Z}, \mathbb{Z}_+)$-duality.

In contrast to $(\mathbb{Z}, \mathbb{Z}_+)$-duality, however, $(\mathbb{Q}, \mathbb{Z}_+)$-duality does not have the Minkowski and Fulkerson properties. To see this, consider $S = \{(1,0)^* = (x = (x_1, x_2) : x_1 \in \mathbb{Z}_+)\}$. Although $S$ is finitely constrained, it is not finitely generated as a $\mathbb{Z}_+^m$-module, since it contains the subspace $\{(0, x_2) : x_2 \in \mathbb{Q}\}$. Observe also
that \( S^* = \{(x_1, x_2) : x_1 \in \mathbb{Z}_+, x_2 = 0\} \). It is clear that the restriction \( x_2 = 0 \) cannot be represented by a finite number of restrictions of the form \( a_i x \in \mathbb{Z}_+ \). Thus \( S^* \) is not finitely constrained.

This example shows that the reason for the failure of the Minkowski property here is the same as that observed for \( \mathbb{Z} \)-duality: when the subspace \( S \) generated by the rows of the matrix \( A \in \mathbb{Q}^{m \times n} \) has dimension less than \( n \), the finitely constrained set \( \{x \in \mathbb{Q}^n : Ax \in \mathbb{Z}_+^m\} \) contains the subspace orthogonal to \( S \), which is neither finitely generated as a \( \mathbb{Z}_+ \)-module nor as a \( \mathbb{Z} \)-module. As done for \( \mathbb{Z} \)-duality, we can describe any finitely constrained \( \mathbb{Z}_+ \)-module as the sum of a subspace and a finitely generated \( \mathbb{Z}_+ \)-module.

4.6 **Theorem:** Let \( A \in \mathbb{Q}^{m \times n} \) and let \( T = \{x \in \mathbb{Q}^n : Ax \in \mathbb{Z}_+^m\} \) be a finitely constrained \( \mathbb{Z}_+ \)-module. Then there exist integers \( r \) and \( h \) and matrices \( C \in \mathbb{Q}^{r \times n} \) and \( H \in \mathbb{Q}^{h \times n} \) such that \( T = \{yC + wH : y \in \mathbb{Q}^r, w \in \mathbb{Z}_+^h\} \) (i.e., \( T \) is the sum of a subspace and a finitely generated \( \mathbb{Z}_+ \)-module).

**Proof:** By Corollary 2.5(c) there exist matrices \( C \in \mathbb{Q}^{r \times n} \) and \( D \in \mathbb{Q}^{s \times n} \) such that \( \{x : Ax \in \mathbb{Z}_+^m\} = \{yC + zD : y \in \mathbb{Q}^r, z \in \mathbb{Z}_+^s\} \). Thus, we have:

\[
T = \{yC + zD : y \in \mathbb{Q}^r, z \in \mathbb{Z}_+^s, (yC + zD)A^t \in \mathbb{Z}_+^m\}.
\]

Since \( CA^t = 0 \) (because the subspace \( \{yC : y \in \mathbb{Q}^r\} \) must be orthogonal to each row of \( A \)), we have:
\[ T = \{ y \in \mathbb{Q}^r, z \in \mathbb{Z}^s, \, z(\mathbf{D}^t) \in \mathbb{Z}_+^m \} \]
\[ = \{ y \in \mathbb{Q}^r \} + \{ z \in \mathbb{Z}^s, \, z(\mathbf{D}^t) \in \mathbb{Z}_+^m \} \]
\[ = \{ y \in \mathbb{Q}^r \} + \{ z \in \mathbb{Z}^s, \, z(\mathbf{D}^t) \geq 0 \}, \]
this last step being justified by the fact that \( (\mathbf{D}^t) \) must be an integral matrix, since the rows of \( \mathbf{D} \) are in the \( \mathbb{Z} \)-dual of the \( \mathbb{Z} \)-module generated by the rows of \( \mathbf{A} \).

By observing that \( \{ z \in \mathbb{Z}^s : z(\mathbf{D}^t) \geq 0 \} \) is the set of integral points in a cone, we conclude, by Theorem 3.5, that there exists a matrix \( \mathbf{H'} \in \mathbb{Z}^h \times \mathbb{Z}^s \) such that \( \{ z \in \mathbb{Z}^s : z(\mathbf{D}^t) \geq 0 \} = \{ \mathbf{wH'} : \mathbf{w} \in \mathbb{Z}^h_+ \} \). Hence, defining \( \mathbf{H} = \mathbf{H'} \mathbf{D} \), we obtain:
\[ T = \{ y \in \mathbb{Q}^r \} + \{ \mathbf{wH} : \mathbf{w} \in \mathbb{Z}^h_+ \} \]
\[ = \{ y \in \mathbb{Q}^r \} + \{ \mathbf{wH} : \mathbf{w} \in \mathbb{Z}^h_+ \}, \]
which is the desired result.\( \blacksquare \)

We remark that the proof of Theorem 4.6 demonstrates further that sets of the form \( T = \{ x \in \mathbb{Q}^n : \mathbf{A}x \in \mathbb{Z}_+^m, \, \mathbf{B}x = 0 \} \) can also be represented as the sum of a subspace and a finitely generated \( \mathbb{Z}_+ \)-module.
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