TAG SYSTEMS: A COMBINATORIAL ABSTRACTION OF INTEGRAL DEPENDENCE

by

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A set of rational vectors is said to be integrally dependent if one of the vectors can be written as an integral combination of the other vectors in the set. In this thesis we consider a natural combinatorial abstraction of the notion of integral dependence of vectors; we term the general form of this abstract combinatorial structure a tag system.

Three equivalent axiomatizations of tag systems are investigated. The first is in terms of the system's circuits. Each circuit of a tag system has a tag set which is a nonempty subset of the circuit. The circuits are then required to be "minimal with respect to these tag sets," that is, if one circuit is contained in another, then the two related tag sets must be disjoint. Any element in the tag set of a circuit is said to be (minimally) generated by the remaining elements of the circuit. A characterization of tag systems is also given in terms of its minimal generating sets. Finally, an axiomatization is described in terms of spanning sets, those sets which generate (not necessarily minimally) all elements of the tag system.

The concept of duality is studied and it is shown that dual pairs of tag systems are equivalent to direct sums of blocking
pairs of clutters. This leads to a painting theorem for tag systems which has as a special case the integral form of Farkas' Lemma; that is, given a rational $m \times n$ matrix $A$ and rational $m$-vector $b$, exactly one of the following holds:

(I) There exists an integral $n$-vector $x$ such that $Ax = b$.
(II) There exists a rational $m$-vector $y$ such that $yA$ is integral but $yb$ is not.

Another particular instance of the painting theorem is the "usual" Farkas Lemma (in which "integral" is replaced by "nonnegative" in (I) and (II) above). The proofs of these two results highlight the similarity between nonnegative and integral dependence systems. Another instance of this painting theorem is a theorem of the alternative for systems of equations in min-algebra. Methods similar to those employed in the above cases are used to prove this instance of the theorem.

Tag systems arise naturally from graphs in several ways. In one development, the stable sets of an arbitrary graph are the independent sets of the related tag system. In addition, it is proved that this system is dual to a tag system which represents integral dependence. Using this correspondence it is seen that finding the smallest cardinality spanning independent set of an integrally representable tag system is NP-Complete.
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CHAPTER 1

BACKGROUND

In Combinatorial Optimization, one is generally concerned with choosing a "best" object from a finite collection of possibilities. Usually, this finite set has an underlying structure which provides a framework for a rule or algorithm for selecting the desired object. An extremely well-studied topic in the field of combinatorics is matroid theory; matroids arise as a natural combinatorial abstraction of linear dependence (see Section 1.2 below). In this thesis, we present and study a general combinatorial structure which has as a particularization an abstraction of the algebraic relation of integral dependence; this abstraction is in the same spirit as the matroid abstraction of linear dependence. Other abstractions of algebraic dependence relations have been investigated in several works including [8], [9] and [21]. In the present chapter, we provide some background material on certain well-known combinatorial structures which will be important in our later development. Our introduction here will be brief; for further details we refer the reader to the following standard references: for graph theory [28]; for matroid theory [36]; for oriented matroids [4]; for greedoids [24].

1.1 Graphs

A graph, G, is a pair (N,A), where N is the node set, \{1,2,\ldots,n\} and A is the arc set, where A is a family of (unordered)
pairs of nodes. We also call the nodes vertices and the arcs edges. If we require the arcs to be ordered pairs of nodes, then the graph is directed. For ease of notation \{i,j\} will mean the undirected edge between nodes i and j, and (i,j) will denote the directed edge from node i to node j. If there is an arc \{i,j\}, then we say i and j are adjacent nodes; if in addition there is an arc \{i,k\}, then the two arcs are adjacent (at node i). Further, \{i,j\} is said to be incident upon nodes i and j. A vertex which is not adjacent to any other node is isolated. A set of vertices, no two of which are adjacent is a stable set or an independent set (of vertices). A set of edges, no two of which are adjacent is called a matching.

A (simple) path is a collection of arcs \{e_1,\ldots,e_p\}, where \(e_1 = \{i_1,i_2\}, e_2 = \{i_2,i_3\},\ldots,e_p = \{i_p,i_{p+1}\}\) and \(i_j \neq i_k\) for \(j \neq k\). If each edge \(e_k\) is directed from \(i_k\) to \(i_{k+1}\), then the path is called a dipath. If \(P = \{e_1,\ldots,e_p\}\) is a path and edge \(e_{p+1} = \{i_{p+1},i_1\}\) is an edge not in \(P\), or \(P = \emptyset\) and \(e_{p+1} = \{i,i\}\) for some \(i \in N\), then \(P \cup \{e_{p+1}\}\) is called a cycle or a circuit; a directed cycle or directed circuit is similarly defined. If there exists a path between each pair of distinct vertices of \(G\), we say \(G\) is connected. Vertices \(v\) and \(w\) are connected if there is a path between them.

For set \(F \subseteq A\), let \(G \setminus F\) denote the graph obtained by deleting all edges of \(F\); i.e., \(G \setminus F = (N,A \setminus F)\). Now suppose \(F \subseteq A\). Then, if there exist vertices \(v\) and \(w \in N\) which are connected in \(G \setminus U\) for every \(U \not\subseteq F\), but are not connected in \(G \setminus F\), then \(F\) is called a (minimal) cut set of \(G\).
1.1.1 Example: Let $G = (N,A)$ be given by

![Graph Diagram]

Then, for example, nodes 1 and 2 are adjacent, edges $a$ and $b$ are adjacent and edge $a$ is incident upon nodes 1 and 2. Note 6 is isolated and $G' = (N\setminus\{6\},A)$ is connected. $\{a,b,c\}$ is a path and $\{a,b,c,d\}$ is a cycle. $\{c,d\}$ is a cut set, $\{1,3,6\}$ is a stable set and $\{a,c\}$ is a matching.

1.2 Matroids

Matroids are a combinatorial abstraction of linear dependence (see 1.2.5 below). We indicate here three equivalent axiomatizations of matroids. Let $E$ be a finite set, called the ground set.

1.2.1 Circuit Axioms: A collection $C$ of subsets of $E$ is the set of circuits of a matroid on $E$ if and only if:

(C0) $X \in C \Rightarrow X \neq \emptyset$;

(C1) if $X \neq Y$ and $X \in C$, $Y \in C$, then $X \not\subset Y$;

(C2) if $X,Y$ are distinct members of $C$ and $x \in X \cap Y$, $y \in Y \setminus X$, then there exists $Z \in C$ with $y \in Z \subseteq (X \cup Y)\setminus\{x\}$. 
1.2.2 Independent Set Axioms: A collection \( I \) of subsets of \( E \) is the family of independent sets of a matroid on \( E \) if and only if:

\[(I0) \ \emptyset \in I;\]
\[(I1) \ \text{if} \ X \in I \ \text{and} \ Y \subseteq X, \ \text{then} \ Y \in I;\]
\[(I2) \ \text{for any} \ A \subseteq E, \ \text{all maximal independent subsets of} \ A \ \text{have the same cardinality.}\]

1.2.3 Closure Axioms: A function \( \sigma: 2^E \to 2^E \) is the closure operator of a matroid on \( E \) if and only if for all \( X, Y \subseteq E \) and \( x, y \in E, \)

\[(S1) X \subseteq \sigma X;\]
\[(S2) Y \subseteq X \Rightarrow \sigma Y \subseteq \sigma X;\]
\[(S3) \sigma X = \sigma \sigma X;\]
\[(S4) \text{if} \ y \notin \sigma X, \ y \in \sigma (X \cup \{x\}) \ \text{then} \ x \in \sigma (X \cup \{y\}).\]

The reader is referred to [36] for the proofs that 1.2.1, 1.2.2 and 1.2.3 are equivalent. We will use \( M = (E, C) \) to denote the matroid \( M \) defined in terms of its circuits and \( M = (E, I) \) when \( M \) has been specified by its independent sets.

1.2.4 For matroid \( M = (E, C) \), a cocircuit is a nonempty subset of \( E \), say \( C^* \), which is minimal with respect to the property that \( |C \cap C^*| \neq 1 \) for all \( C \in C. \) ("Minimality" means that the indicated condition holds for \( C^* \) but that for any proper nonempty subset of \( C^* \), this condition fails.) Call the collection of all such cocircuits \( C^*.\)
It is well-known that \( M^* = (E, C^*) \) is a matroid, called the dual of \( M = (E, C) \) and that \( M = M^{**} \).

We let \( \mathbb{Q} \) denote the set of rational numbers and \( \mathbb{Q}^{m \times n} \) the set of all \( m \times n \) rational matrices. Though many of the results of this thesis remain valid in a more general setting, we restrict attention to the rationals throughout, in order to allow accurate computational statements.

1.2.5 Given \( A \in \mathbb{Q}^{m \times n} \), let \( E = \{1, 2, \ldots, n\} \) and let \( C \) be the collection of nonempty subsets of \( E \) which index minimal linearly dependent sets of columns of \( A \). Then \( M = (E, C) \) is a matroid (see [36]). If, for a given matroid \( M = (E, C) \), there is a matrix \( A \in \mathbb{Q}^{m \times n} \) for which this correspondence exists, then \( M \) is termed representable over \( \mathbb{Q} \) or \( \mathbb{Q} \)-matric. Note that for this example the independent sets of \( M \) are the subsets of \( E \) which index linearly independent column sets for matrix \( A \), and for \( X \subseteq E \), \( \sigma(X) \) corresponds to the columns of \( A \) spanned by those columns whose indices comprise \( X \).

1.2.6 Define the support of vector \( x \) to be \( \{j: x_j \neq 0\} \). Given a rational subspace \( S \), an elementary vector of \( S \) is a nonzero vector which has minimal support in \( S \). The frame of \( S \) is the collection of all elementary vectors of \( S \).

It is easily verified that the dual of the matroid indicated in 1.2.5 is defined with \( C^* \) as the collection of supports of elementary vectors in the rowspace of \( A \) (see [36]). It is well-known that this dual matroid \( M^* = (E, C^*) \) is also representable over the rationals.
1.2.7 Another important dual pair of matroids is defined on a given graph \( G = (N,A) \). Let \( E = A \) and define \( M = (E,C) \) by \( C \in C \) if and only if \( C \) indexes a circuit of \( G \). Then the dual of \( M \) is given by \( M^* = (E,C^*) \) with \( C^* \in C^* \) if and only if \( C^* \) indexes a (minimal) cutset of \( G \). These are called \textit{graphic} and \textit{cographic} matroids, respectively. See [36] for the proof that these are in fact a dual pair of matroids.

1.2.8 A dual pair of matroids is \textit{orientable} (see [2]) if there exists a partition of each \( C \in C \) into \( C_+ \) and \( C_- \) and of each \( C^* \in C^* \) into \( C^*_+ \) and \( C^*_- \) such that

\[
C \cap C^* \neq \emptyset \implies \begin{cases} 
(C_+ \cap C^*_+) \cup (C_- \cap C^*_-) \neq \emptyset \\
(C_+ \cap C^*_-) \cup (C_- \cap C^*_+) \neq \emptyset.
\end{cases}
\]

This last condition is known as the \textit{orthogonality axiom}.

The matroid defined in 1.2.5 and its dual are seen to be orientable as follows. For \( C \in C \), there corresponds an \( x \in \mathbb{Q}^n \) such that \( x \neq 0 \) and \( Ax = 0 \), with the support of \( x \) equal to \( C \). Put \( j \in C_+ \) if and only if \( x_j > 0 \) and \( j \in C_- \) if and only if \( x_j < 0 \). Similarly, for \( C^* \in C^* \), there corresponds a \( y \in \mathbb{Q}^m \) such that \( yA = w \) and the support of \( w \) equals \( C^* \). As above, put \( j \in C^*_+ \) if and only if \( w_j > 0 \) and \( j \in C^*_- \) if and only if \( w_j < 0 \). The reader can easily verify that these partitioning rules satisfy the orthogonality axiom.
1.3 Greedoids

Greedoids are a generalization of matroids recently proposed and developed by B. Korte and L. Lovász (see [24],[25],[26],[27]). Let $E$ be a finite set and $F$ collection of subsets of $E$, called feasible sets. Then the greedoid $(E,F)$ is defined by the following axioms:

(H0) $\emptyset \in F$;

(H1) $\forall x \in F, \exists x \in X$ such that $X \setminus \{x\} \in F$;

(H2) if $X,Y \in F$ and $|X| = |Y|+1$, then $\exists x \in X \setminus Y$ such that $Y \cup \{x\} \in F$.

The reader can easily verify that if (H1) is replaced by (H1)' $\forall x \in F, X \setminus \{x\} \in F \forall x \in X$, then these axioms are equivalent to the independent set axioms of a matroid (1.2.2). Thus, the manner in which greedoids generalize matroids is apparent. For $A \subseteq E$, any maximal feasible set of $A$ is called a base of $A$.

One can also axiomatize greedoids in terms of a "simple language" $L$ over ground set $E$. Call the elements of $E$ letters, then a simple language is a collection of finite sequences (ordered) of elements of the ground set, called words, with no letter repeated in any word. For words $\alpha$ and $\beta$, we denote by $\alpha \cdot \beta$ the concatenation of $\alpha$ and $\beta$, that is, the sequence that is obtained by placing $\beta$ after $\alpha$. Then, a greedoid on finite set $E$ is a simple language $(E,L)$ such that:
(G0) $\emptyset \in L$;

(G1) if $\alpha \in L$ and $\alpha = \beta \cdot \gamma$ then $\beta \in L$;

(G2) if $\alpha, \beta \in L$ and $|\alpha| > |\beta|$, then $\exists$ a letter $x \in \alpha$

\[ \text{such that } \beta \cdot (x) \in L. \]

Shelling structures (see below) constitute an important particular example of greedoids. Let $E$ be a finite set and let, for each $e \in E$, a set system $H_e \subseteq 2^{E \setminus \{e\}}$ be given. The sets in $H_e$ are termed alternative precedences for $e$. Let

\[ L = \{x_1 \ldots x_k | \forall 1 \leq i \leq k, x_i \in E \text{ and } \exists U \in H_{x_i} \]

\[ \text{such that } U \subseteq \{x_1, \ldots, x_{i-1}\} \}. \]

We assume, in addition, that every element of $E$ appears in some word of $L$ (i.e., $L$ is normal). Then, $(E, L)$ is called a shelling structure. See [27] for a proof that $(E, L)$ is a greedoid.

Now suppose that $(E, F)$ is a shelling structure and $T \subseteq E$. Define $F: T = \{A \cap T: A \in F\}$. A set $T \subseteq E$ is called free if $F: T = 2^T$. A set $C$ is called a circuit if it is minimal non-free, i.e., if $C$ is not free but every proper subset of $C$ is free.

For $A \in F$, let $\Gamma(A) = \{x \in E \setminus A: A \cup \{x\} \in F\}$, the set of continuations of $A$. It is easy to verify (see [27]) that when $(E, F)$ is a shelling structure, $C$ is a circuit of $(E, F)$ and $A$ is a base of $E \setminus C$, then $|C \setminus \Gamma(A)| = 1$. This element of $C \setminus \Gamma(A)$ is called the root of $C$ (denoted $R(C)$). The following result, which we shall require in 2.7 is established in [27].
Lemma: Let \((E,F)\) be a shelling structure and \(A \subseteq E\). Then \(A \in F\) if and only if for each circuit \(C\), \(A \cap C\) is not the root of \(C\).

1.4 Overview of Thesis

The remainder of this thesis is devoted to the definition and study of tag systems, a combinatorial structure that abstracts integral linear dependence. Chapter two is concerned with the foundations of this analysis while Chapter three investigates the relationship between different "representable" dependence systems.

In Chapter two, tag systems are defined in terms of dependent sets, generating sets and a span function, and these three axiomatizations are shown to be equivalent. Duality is introduced and then pursued along the same lines used by Minty in his study of matroids (see [30]) and then by Bland in his analysis of oriented matroids (see [2]); that is, a definition is given for a dual system and then a painting theorem is presented and shown to characterize the duality. We also investigate certain combinatorial operations on tag systems, including analogues for the matroid minor operations, contraction and deletion. The precise relationship between tag systems and matroids is studied in detail; in particular, we investigate certain combinations of additional axiomatic properties required for a dual pair of tag systems in order to define a dual pair of matroids. Greedoids, when considered as a generalization of matroids, are shown to be distinct from tag systems, and yet, when looked at in a different way, provide a wealth of new examples of tag systems. Finally, it is shown that tag systems and direct sums of blocking pairs of clutters are equivalent combinatorial structures.
In Chapter three we illustrate fundamental similarities in the underlying structure of nonnegative rational, integral and min-algebraic dependence systems. In each case it is shown that the structures give rise to a dual pair of tag systems. Further, it is shown that the painting theorem (see Theorem 2.4.4 in Chapter two) is equivalent to the "theorem of the alternative" in each of these instances, thus producing a combinatorial statement of an algebraic result. Finally, we investigate various ways to define tag systems on graphs. Several of these examples are drawn from applications of the above work performed on the vertex-edge incidence matrix of a graph (see 3.5). We also show that the stable sets of an arbitrary graph are closely related to the independent sets of an integrally representable tag system. This implies that in order to achieve the level of generality suggested by integral dependence, we must also encompass the (excessive) generality of the stable sets of a graph and hence we cannot expect strong algorithmic results for general tag systems.
CHAPTER 2
FOUNDATIONS

2.1 Introduction

Theorems of the Alternative constitute an important class of results studied in Mathematical Programming. Perhaps the most widely known is Farkas' Lemma (see for example [7]). This theorem is important, in part, because it leads to basic results in Linear Programming duality theory. Below, we discuss three types of Theorems of the Alternative.

The first theorem, known as Fredholm's Theorem, characterizes the existence of solutions of rational equality systems, see e.g. [15].

2.1.1 Given a matrix $A \in \mathbb{Q}^{m \times n}$ and vector $b \in \mathbb{Q}^{m}$, exactly one of the following holds:

(P1) $\exists x \in \mathbb{Q}^{n}$ such that $Ax = b$,

(P2) $\exists y \in \mathbb{Q}^{m}$ such that $yA = 0$, but $yb \neq 0$.

Next, for nonnegative solutions of rational equality systems, we have the well-known Farkas Lemma [7]:

2.1.2 Given a matrix $A \in \mathbb{Q}^{m \times n}$ and vector $b \in \mathbb{Q}^{m}$, exactly one of the following holds:

(P1) $\exists x \in \mathbb{Q}^{n}$ such that $x \geq 0$ and $Ax = b$,

(P2) $\exists y \in \mathbb{Q}^{m}$ such that $yA \geq 0$, but $yb < 0$.
And finally, for integral solutions of rational equality systems:

2.1.3 Given a matrix $A \in \mathbb{Q}^{m \times n}$ and vector $b \in \mathbb{Q}^m$, exactly one of the following holds:

(P1) \exists x \in \mathbb{Z}^n$ such that $Ax = b,$

(P2) \exists y \in \mathbb{Q}^m$ such that $yA \in \mathbb{Z}^n$, but $yb \notin \mathbb{Z}$.

(See [33], for example).

A common technique in the area of Combinatorial Optimization is to abstract the notion of dependence from an algebraic structure and then to work with the resulting combinatorial framework. In the case of subspaces, one obtains matroids by abstracting the notion of linear dependence. Here, the following "painting" theorem due to Minty [30] is central to the development of matroid duality theory.

2.1.4 Given a matroid $M = (E,C)$ and a painting (partition) of $E$ into Red ($R$), Blue ($B$) and White ($W$), with $|R| = 1$, exactly one of the following holds:

(P1) \exists C \in C$ with $R \subseteq C \subseteq R \cup B,$

(P2) \exists C^* \in C$* with $R \subseteq C^* \subseteq R \cup W.$

Applying this theorem to the two $\mathbb{Q}$-matric matroids, (recall definition 1.2.5) on matrix $[A|-b]$, painting $A$ Blue and $(-b)$ Red, gives Theorem 2.1.1. Noticing that the key element in a proof of the fact that these matroids are duals is Gaussian Elimination leads to the conclusion that the abstraction of linear dependence and an
elimination scheme are sufficient to obtain the Theorem of the Alternative for subspaces.

Working with convex cones, abstracting the notion of signed linear dependence results in oriented matroids as discussed in 1.2.8. The following theorem is due independently to Bland and Las Vergnas (see for example [5]).

2.1.5 Given a dual pair of orientable matroids, \( M = (E, C) \) and \( M^* = (E, C^*) \), fix an orientation of \( C \) and \( C^* \) and distinguish some \( i \in E \). Then, for any painting of \( E \) into Red (R), Blue (B) and White (W) with \( i \in R \), exactly one of the following must hold:

(P1) \( \exists C \subseteq C, i \in C \subseteq R \cup B \) with the elements of \( R \cap C \) uniformly oriented.

(P2) \( \exists C^* \subseteq C^*, i \in C^* \subseteq R \cup W \) with the elements of \( R \cap C^* \) uniformly oriented.

Applying this theorem to the two matric matroids on matrix \( [A_i - b] \), painting all of the columns Red and distinguishing the \((-b)\) column, yields Theorem 2.1.2. Thus, using only the notion of signed linear dependence and Gaussian Elimination we can show that the matric matroids are orientable and thus obtain the important Theorem of the Alternative for convex cones.

Hence, in the sense discussed above, Theorems 2.1.1 and 2.1.2 are a result of purely combinatorial arguments. This thesis will follow the approach of Minty [30], and later of Bland [2], to investigate a combinatorial framework that underlies Theorem 2.1.3, the Theorem of the Alternative for integral modules. In Section two of this chapter,
an axiomatic description of this new structure, tag systems, is presented and several examples that play an important role in Chapter three are introduced. Section three defines the concepts of independent and spanning sets in tag systems and thus produces two additional distinct, but equivalent, axiomatizations of the structure. The concept of a base of a tag system is also investigated and contrasted to the corresponding concept for matroids. Duality is studied in Section four, where, following Minty's approach [30], a painting theorem is presented which later is shown to imply Theorem 2.1.3. Section five discusses minors of tag systems and Sections six and seven relate tag systems to two other combinatorial structures, matroids and greedoids.

In Section eight we study the correspondence (pointed out to us by Professor R.G. Bland) between dual pairs of tag systems and families of blocking pairs of clutters, called here blocking systems. It is shown that there is a one to one correspondence between these two types of structures and thus that one can view the class of tag systems as the class of direct sums of blocking pairs of clutters. Hence the reader might find it instructive to read Section eight before the remainder of this chapter. Using this approach, the results presented on duality of tag systems in 2.4.1 through 2.4.6 and the development of minors in 2.5.1 through 2.5.4 follow as direct consequences of analogous results for blocking systems (see [14] and [31]). This relationship advises us not to expect strong algorithmic results for arbitrary tag systems while it also suggests ways to restrict general tag systems in order to obtain
algorithmically tractable systems. One possible restriction is tagged circuit exchange (see 2.6.4). However, the results of 3.5 illustrate that requiring only one member of a dual pair of tag systems to satisfy tagged circuit exchange is insufficient to produce algorithmically tractable systems; and further it is shown in 2.6.12 that if both systems of the dual pair satisfy tagged circuit exchange then in fact they are dual matroids. Hence if we desire that the additional requirements both enable strong algorithmic results and be symmetric in the dual systems, then we must use a weaker version of circuit exchange. This task of finding appropriate additional axioms which will impose enough structure to produce algorithmically tractable systems while also retaining sufficient generality is an important open problem which is discussed further in Section eight.

2.2 Dependence

This section will present an axiomatization of an abstraction of integral dependence. Before studying integral dependence, we review some definitions regarding linear dependence.

A set of rational m-dimensional vectors (shortened to m-vectors) \( \{a_1, \ldots, a_n\} \) are \underline{linearly dependent} if one element of the set can be written as a rational combination of the remaining elements of the set; more precisely, if there exists an \( i, 1 \leq i \leq n \), and rational \( x_j, j \neq i \), such that \( a_i = \sum_{j \neq i} a_j x_j \). Clearly, this is equivalent to the condition that a nontrivial rational combination of the elements of the set produces zero; i.e., there exist \( x_j \in \mathbb{Q} \), not all zero, such that \( \sum_{j=1}^{n} a_j x_j = 0 \). We say that a linearly dependent
set is a minimal linearly dependent set if no proper subset of it is also linearly dependent.

In one standard development, the circuits of a matric matroid are defined to be minimal linearly dependent sets of vectors (see 1.2.5). We will demonstrate here why there is a need to refine this definition in order to model integrally dependent sets of rational vectors.

2.2.1 A set of rational m-vectors \( \{a_1, \ldots, a_n\} \) is integrally dependent if some element, \( a_i \), can be written as an integral combination of the remaining vectors, \( \sum_{j \neq i} a_j x_j, \ x_j \in \mathbb{Z} \).

Notice that this definition implies the existence of integers, \( z_j \), such that \( \sum_{j=1}^{n} a_j z_j = 0 \) (let \( z_i = -1, z_j = x_j \) for \( j \neq i \)). However, by definition 2.2.1, the set \( \{2,3\} \) is not integrally dependent, whereas there does exist a nontrivial integer combination of 2 and 3 which produces zero, e.g., \( 2(6) + 3(-4) = 0 \). The important thing to note here is that integral dependence implies the existence of a unit multiplier in the integral combination which gives zero.

To illuminate further the differences between linear (rational) and integral dependence, we consider the following example.

2.2.2 Example: Let \( A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 6 & 10 \\ 1 & 2 & 3 & 5 \end{bmatrix} \). For ease of notation, we will refer to the column vectors of \( A \) by their indices; for example, we refer to the vector \( \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \) by its column index, 1.

One can easily enumerate the linearly dependent sets of columns of \( A \): \( \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \ldots \).
\{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}. Thus the minimal linearly dependent sets are: \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}. Now let us investigate the integrally dependent sets. First notice that this concept is not "symmetric". Obviously \(2 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 2 \end{bmatrix}\); however, there is no integer \(z\) such that \(\begin{bmatrix} 2 \\ 4 \\ 2 \\ 1 \end{bmatrix} z = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\). Thus, we must note explicitly which elements in an integrally dependent set appear with unit weight. Next, it is important to note that different elements might have unit weight in different representations. As an example, \(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \langle 2 \rangle + \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} \langle 10 \rangle - \begin{bmatrix} 5 \\ 6 \\ 5 \end{bmatrix} = 0\) would put unit weights on columns three and four, and \(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \langle 2 \rangle + \begin{bmatrix} 6 \\ 3 \end{bmatrix} \langle 10 \rangle - \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 0\) has unit multipliers on the first and fourth columns. Continuing in this manner yields the following list of integrally dependent sets with the elements that have unit weight in some representation indicated with a bar: \{1, \bar{2}\}, \{1,\bar{3}\}, \{1,\bar{4}\}, \{1,2,\bar{3}\}, \{1,\bar{2},\bar{4}\}, \{1,\bar{3},4\}, \{2,\bar{3},4\}, \{1,\bar{2},\bar{3},4\}.

We now address the issue of minimality. No subset of \{1,2\} has an integral dependency, whereas there are subsets of \{1,2,3\} which are integrally dependent. However, no subset of \{1,2,3\} has a dependency relation with unit multiplier on column 1. In order to record this type of minimality, we will distinguish any element, \(i\), of dependent set, \(C\), which appears with unit weight in a dependency relation for \(C\), but does not appear with unit multiplier in any dependency relation for any proper subset of \(C\). Thus we would "tag" 1 in \{1,2,3\} but we would not tag 2 or 3. This yields the following list with the tagged elements "hatted": \{1,\hat{2}\}, \{1,\hat{3}\}, \{1,\hat{4}\}, \{\hat{1},2,3\}, \{\hat{1},2,\hat{4}\}, \{\hat{1},\hat{3},4\}, \{\hat{2},\hat{3},\hat{4}\}. Note that if a set has none of its elements
tagged then we do not consider it; for example, we discarded set \{1,2,3,4\} above. This list then gives the integrally dependent sets which are "minimal" with respect to at least one element, with these special elements "tagged". We now incorporate the above ideas into an axiomatic model for abstract integral dependence.

2.2.3 A tag system is a triple \( T = (E, C, \tau) \) with \( E \) a finite set called the ground set of elements, and \( C \subseteq 2^E \), the set of circuits and \( \tau: C \to 2^E \) the tag function with \( \tau(C) \) called the tag set of \( C \), satisfying:

\[
\begin{align*}
(TC1) \quad & \emptyset \neq \tau(C) \subseteq C; \\
(TC2) \quad & C_1 \nsubseteq C_2 \Rightarrow \tau(C_1) \cap \tau(C_2) = \emptyset.
\end{align*}
\]

As the above example of integrally dependent sets suggests, we call a triple satisfying (TC1) and (TC2) tag-minimal, since in this setting the circuits are minimal with respect to their tagged elements. In Example 2.2.2 above, recall that we listed each dependent set once and tagged any elements which appeared with unit multiplier in some representation; this gave a concise list of the dependent sets. In the same manner, the requirement that \( \tau \) be a function forces a concise description of the tag system.

We now briefly discuss some examples of structures which are modeled by tag systems.

2.2.4 Integral Dependence: For a matrix \( A \in \mathbb{Q}^{m \times n} \), the notion of integral dependence discussed in Example 2.2.2 defines circuits and
their tags which satisfy 2.2.3. This structure is discussed further in Chapter three.

2.2.5 Matroids: Given a matroid, \( M = (E, C) \), define a triple \( T = (E, C, \tau) \) by stipulating that \( \tau(C) = C \) for all \( C \in C \). Since the circuits of a matroid are minimal, the triple \( T \) is clearly a tag system. These systems are the topic of Section five of this chapter.

2.2.6 Vertex Adjacency: Given a graph, \( G = (N, A) \), define a triple \( T = (N, A, \tau) \) where \( \tau(C) = C \) for all \( C \in C \). Clearly \( \tau \) is a function and (TC1) is satisfied. (TC2) follows since with every circuit of cardinality two, it is impossible to have comparable circuits. This system is studied in Chapter three.

**Example:** Let \( G = \begin{array}{ccc} 1 & 4 \\ 2 & & 3 \end{array} \). Then \( T = (E, C, \tau) \) is defined by \( E = \{1, 2, 3, 4\} \); \( C_1 = \{1, 2\}, \tau(C_1) = \{1, 2\}; C_2 = \{1, 3\}, \tau(C_2) = \{1, 3\}; C_3 = \{1, 4\}, \tau(C_3) = \{1, 4\}; C_4 = \{2, 3\}, \tau(C_4) = \{2, 3\} \) and \( C_5 = \{3, 4\}, \tau(C_5) = \{3, 4\} \).

2.2.7 Vertex Neighborhoods: Again given a graph, \( G = (N, A) \), define a triple \( T = (N, C, \tau) \). Here, for each \( i \in N \), we define \( N(i) = \{j : \{i, j\} \in A\} \), the neighborhood of \( A \). Then, \( C \in C \) if and only if \( C = \{i\} \cup N(i) \) for some \( i \in N \) and we define \( \tau(C) = \{j \in C : C = \{j\} \cup N(j)\} \). Clearly \( \tau \) is a function from \( C \) to \( 2^E \) which satisfies (TC1). (TC2) is clear since for each \( i \in N \), there will be exactly one circuit with \( i \) in its tag set. This system will also be studied in Chapter three.
Example: For $G$ given above in 2.2.6, $T = (E, C, \tau)$ is defined by $E = \{1, 2, 3, 4\}$ and $C_1 = \{1, 2, 3, 4\}$, $\tau(C_1) = \{1, 3\}$; $C_2 = \{1, 2, 3\}$, $\tau(C_2) = \{2\}$ and $C_3 = \{1, 3, 4\}$, $\tau(C_3) = \{4\}$.

These examples serve to illustrate the variety of combinatorial settings in which tag systems find application. In the third chapter of this thesis these examples and others will be studied in greater detail and the relationships between different types of tag systems will be investigated.

2.3 Generation

In this section we turn our attention to the independence structure underlying tag systems. We will find in this section that the concept of a base of a tag system is more general than the analogous notion of a base of a matroid. In one development of $Q$-matric matroids, given a matrix $A$, the maximal independent sets of the matroid are defined to be the minimal spanning sets of the subspace generated by the columns of $A$. This construction yields an axiomatization of the independent sets of a matroid which is equivalent to the definition in terms of the minimal dependent sets, i.e. the circuits (see 1.2.1 and 1.2.2). Here we discover that while the circuits of the tag system defined through integral dependence of the columns of $A$ do give rise to a collection of maximal independent sets, these sets do not coincide exactly with the minimal spanning sets of the integer module generated by the columns of $A$. Thus we will need to broaden the definition of a base and explicitly require it to be both maximally independent and
minimally spanning. First, we focus on linear independence and study how this differs from integral independence.

Given a set \( A = \{a_1, \ldots, a_n\} \) of rational \( m \)-vectors, we call a subset \( B \subseteq A \) spanning if every \( a_i \) can be written as a rational combination of vectors in \( B \). Further, \( B \) is a minimal spanning set if no proper subset of it is also spanning. A subset of \( A \) is linearly independent if it is not linearly dependent and is considered a maximal linearly independent set if no superset of it contained in \( A \) is still independent. It is a well-known fact of linear algebra that for subspace \( S \subseteq \mathbb{Q}^m \), a set is a maximal linearly independent subset of \( S \) if and only if it is a minimal spanning set of \( S \). We will see in this section that this result does not carry over to the integral setting.

We now define analogues of the above concepts for the integral case. Given a set \( A = \{a_1, \ldots, a_n\} \) of rational \( m \)-vectors, we call a subset \( B \subseteq A \) integrally independent if it is not integrally dependent (see Definition 2.2.1), and \( B \) is a maximal (integrally) independent set if no superset of \( B \) contained in \( A \) is still integrally independent. Notice that a linearly dependent set can be integrally independent, as illustrated by the set \( \{2,3\} \). We call \( B \) a generating set if there is an \( x \in \mathbb{A} \setminus \mathbb{B} \) which can be represented as an integral combination of the vectors in \( B \), i.e., if there exist \( z_j \in \mathbb{Z} \) such that \( x = \sum_{a_j \in B} a_j z_j \). \( B \) is called a minimal generating set with respect to \( x \) if \( B \) generates \( x \) as above, but no subset of \( B \) has this property. Finally, a subset \( B \) is a spanning set if it generates each element of
A \backslash B. At the end of this section we demonstrate that for an integer
module, minimal spanning sets are not equivalent to maximal independent
sets.

To clarify the definitions above, we illustrate with the following
example.

2.3.1 Example: Let $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 4 & 6 & 8 & 10 \\ 2 & 3 & 4 & 5 \end{bmatrix}$, using the notation of column
indices introduced in Example 2.2.2. Here we list the minimal generating
sets and those elements which each set generates: \{1\},3; \{1,2\},4;
\{1,4\},2; \{2,3\},1,4; \{2,4\},1,3 ; \{3,4\},1,2. Notice that \{1,2\} is
minimal with respect to generating 4 even though \{1\} alone generates 3.

The above discussion of integral dependence suggests that given a
tag system $T = (E, C, \tau)$, we define a subset $L$ of $E$ to be a minimal
generating set if $L = C \setminus \{j\}$ for some $C \in C$ with $j \in \tau(C)$. We
also define the generation function $\gamma(\cdot)$ by

$\gamma(L) = \{j \notin L: L \cup \{j\} = C \text{ for some } C \in C \text{ with } j \in \tau(C)\}$

for each minimal generating set $L$. The following theorem indicates
that we can now give an equivalent axiomatization of tag systems using
the notion of minimal generating sets.

2.3.2 Theorem: Suppose $(E, L, \gamma)$ is given, where $E$ is a finite set,
$L \subseteq 2^E$ and $\gamma: L \to 2^E$. Then $L$ defines the minimal generating sets
of a tag system with associated generation function $\gamma(\cdot)$ if and only if

(MG1) $\emptyset \neq \gamma(L) \subseteq E \setminus L$ for all $L \in L$;

(MG2) for $L_1, L_2 \in L$, $L_1 \not\subset L_2 \Rightarrow \gamma(L_1) \cap \gamma(L_2) = \emptyset$. 
Proof: First suppose that $T = (E, C, \tau)$ is a tag system on $E$ with $(E, L, \gamma)$ as defined above. Then clearly $\gamma$ is a function from $L$ to $2^E$ which satisfies (MG1). Suppose $L_1, L_2 \in L$ with $L_1 \not\subset L_2$ and $j \in \gamma(L_1) \cap \gamma(L_2)$. Then by definition there exist $C_1, C_2 \in C$ with $j \in \tau(C_1) \cap \tau(C_2)$ and $C_1 = L_1 \cup \{j\} \not\subset L_2 \cup \{j\} = C_2$ which contradicts the tag-minimality of $T$. Thus (MG2) is satisfied.

Now suppose $(E, L, \gamma)$ is given satisfying (MG1) and (MG2). Then, define $T = (E, C, \tau)$ by

(i) $C \in C$ if and only if $C = L \cup \{j\}$ for some $L \in L$ with $j \in \gamma(L)$;

(ii) $\forall C \in C, \tau(C) = \{j \in C : C \setminus \{j\} = L \text{ for some } L \in L$ with $j \in \gamma(L)\}$.

Note that it is clear that $\tau$ is a function from $C$ to $2^E$ which satisfies (TC1). Assume there exist $C_1, C_2 \in C$ with $C_1 \not\subset C_2$ and $j \in \tau(C_1) \cap \tau(C_2)$. Then (ii) implies that there exist $L_1, L_2 \in L$ with $j \in \gamma(L_1) \cap \gamma(L_2)$ and $L_1 \cup \{j\} = C_1 \not\subset C_2 = L_2 \cup \{j\}$ which implies $L_1 \not\subset L_2$ thus contradicting (MG2). So (TC2) is satisfied and $T$ is a tag system. It is now straightforward to verify that the minimal generating sets of $T$ are in fact $L$ with generation function $\gamma(\cdot)$ and hence the proof is complete. $\Box$

In the discussion of linear (rational) and integral independence earlier, we defined the notion of a spanning set. However, in the development thus far, we have not yet axiomatized this concept of a set which can generate its complement relative to the ground set.

Given a subset of the ground set, $S \subseteq E$, we define the span
of \( S \), \( \sigma(S) \), to be the set of all elements which \( S \) generates, including itself. More precisely, for \( S \subseteq E \), let
\[
\sigma(S) = S \cup \{ x : \exists C \subseteq C \text{ with } C \subseteq S \cup \{ x \} \text{ and } x \in \tau(C) \}\text{ (or equivalently } S \cup \{ x : \exists L \subseteq L \text{ with } L \subseteq S \text{ and } x \in \gamma(L) \}).
\]

2.3.3 Theorem: A function \( \sigma : 2^E \rightarrow 2^E \) is the span function of a tag system if and only if it satisfies:

1. \( S \subseteq \sigma(S) \) for all \( S \subseteq E \);
2. \( S_1 \subseteq S_2 \Rightarrow \sigma(S_1) \subseteq \sigma(S_2) \).

**Proof:** First suppose \( \sigma : 2^E \rightarrow 2^E \) is a function which satisfies (S1) and (S2). Define \( T = (E, C, \tau) \) as follows: where \( C \subseteq E \), \( C \in C \) if and only if \( \{ i \in C : i \in \sigma(C \setminus \{ i \}) \text{ but } \forall k \in C \setminus \{ i \}, i \notin \sigma(C \setminus \{ i, k \}) \} \neq \emptyset \); and for \( C \in C \), \( \tau(C) = \{ i \in C : i \in \sigma(C \setminus \{ i \}) \text{ but } \forall k \in C \setminus \{ i \}, i \notin \sigma(C \setminus \{ i, k \}) \} \). Then clearly \( \tau \) is a function from \( C \) to \( 2^E \) which satisfies (TC1).

Suppose \( C_1, C_2 \in C \) with \( C_1 \neq C_2 \) and \( i \in \tau(C_1) \cap \tau(C_2) \). Then, let \( k \in C_2 \setminus C_1 \). Since \( i \in \tau(C_1) \), we must have \( i \in \sigma(C_1 \setminus \{ i \}) \); similarly, \( i \in \tau(C_2) \) and \( k \in C_2 \) imply \( i \notin \sigma(C_2 \setminus \{ i, k \}) \). But then \( C_1 \setminus \{ i \} \subseteq C_2 \setminus \{ i, k \} \) and \( \sigma(C_1 \setminus \{ i \}) \neq \sigma(C_2 \setminus \{ i, k \}) \) which contradicts (S2). Thus (TC2) is satisfied and \( T \) is a tag system. It is easy to verify that the span function for this system is \( \sigma \).

Now suppose \( T = (E, C, \tau) \) is a tag system and define
\[
\sigma(S) = S \cup \{ x : \exists C \subseteq S \cup \{ x \}, x \in \tau(C) \} \text{ for each } S \subseteq E.
\]
Then clearly \( S \subseteq \sigma(S) \), satisfying (S1). Suppose \( S_1 \subseteq S_2 \) and let \( x \in \sigma(S_1) \). If \( x \in S_2 \) then we have \( x \in \sigma(S_2) \). Alternatively, suppose \( x \notin S_2 \). Then
there is some \( C_1 \in C \) with \( C_1 \subseteq S \cup \{x\} \) and \( x \in \tau(C_1) \). But, then \( C_1 \subseteq S_2 \cup \{x\} \) and so \( x \in \sigma(S_2) \). Hence, (S2) is satisfied. \( \square \)

We will sometimes denote a tag system by \((E, \sigma)\) when it is convenient to do so.

Recall that given a tag system, a subset of the ground set is called independent if it contains no circuit. We now define a set \( S \) to be spanning if \( \sigma(S) = E \) and set \( B \) to be a base if it is both spanning and independent. We see below that, as in the subspace setting, a base here is in fact a maximal independent set and a minimal spanning set.

### 2.3.4 Proposition: If \( B \) is a base of tag system \( T = (E, C, \tau) = (E, \sigma) \), then it is a maximal independent set.

**Proof:** For any \( j \not\in B \), we know \( j \in \sigma(B) \setminus B \) which implies there exists \( C \in C \) with \( C \subseteq B \cup \{j\} \) and thus \( B \cup \{j\} \) is dependent. \( \square \)

### 2.3.5 Proposition: If \( B \) is a base of tag system \( T = (E, C, \tau) = (E, \sigma) \), then it is a minimal spanning set.

**Proof:** Since \( B \) is independent, we know there exists no \( C \in C \) with \( C \subseteq B \). Hence, for any \( j \in B \), there exists no \( C \in C \) with \( j \in \tau(C) \) and \( C \subseteq (B \setminus \{j\}) \cup \{j\} \). Thus, for any \( j \in B \), \( j \not\in \sigma(B \setminus \{j\}) \) and so \( B \) is a minimal spanning set. \( \square \)

These two propositions, 2.3.4 and 2.3.5, do not imply however, that every maximal independent set is a base, nor that every minimal spanning set is a base. The following example illustrates this.
2.3.6 Example: Let $E = \{1,2,3\}$ with $C_1 = \{1,2,3\}$, $\tau(C_1) = \{2,3\}$; $C_2 = \{1,2\}$, $\tau(C_2) = \{1\}$; $C_3 = \{1,3\}$, $\tau(C_3) = \{1\}$. Then the maximal independent sets are $L_1 = \{1\}$ and $L_2 = \{2,3\}$. The minimal spanning sets are $S_1 = \{1,2\}$, $S_2 = \{1,3\}$ and $S_3 = \{2,3\}$. Clearly the only base is $L_2 = S_3 = \{2,3\}$. □

2.4 Duality

In this section we define the dual of a tag system and, following Minty's development for matroids (see [30]), prove a painting theorem.

Given a tag system $T = (E, C, \tau)$, define for each $S \subseteq E$, the set

$$\tau'(S) = \{ j \in S : \text{there exists no } C \in C \text{ with } j \in \tau(C) \text{ such that } C \cap S = \{j\} \text{ and for all } k \in S \setminus \{j\}, \text{ there exists } C_k \in C \text{ with } j \in \tau(C_k) \text{ such that } C_k \cap S = \{j,k\} \}.$$

2.4.1 Define the dual of tag system $T = (E, C, \tau)$ to be $T^* = (E, C^*, \tau^*)$ on the same ground set $E$ with $C^* \subseteq 2^E$ and $\tau^*: C^* \to 2^E$ where $C^*$ is the collection of cocircuits such that for any $C^* \subseteq E$ we place $C^* \in C^*$ if and only if $\tau'(C^*) \neq \emptyset$ and we define $\tau^*(C^*) = \tau'(C^*)$.

Notice that for $C^* \in C^*$ and $C \in C$ we have $\tau(C) \cap \tau^*(C^*) \neq \emptyset$ implies $|C \cap C^*| \neq 1$ which is reminiscent of the duality condition for matroids (see 1.2.4).
2.4.2 Proposition: $T^*$ is a tag system.

Proof: Clearly $\emptyset \neq \tau^*(C^*) \subseteq C^*$ for all $C^* \in C^*$ and $\tau^*$ is a function. Suppose $C_1^*, C_2^* \in C^*$ with $C_1^* \subseteq C_2^*$ and $j \in \tau^*(C_1^*) \cap \tau^*(C_2^*)$, and let $k \in C_2^* \setminus C_1^*$. Then, by $j \in \tau^*(C_2^*)$ we know $j \in \tau'(C_2^*)$ and hence there exists $C_k \in C$ with $j \in \tau(C_k)$ such that $C_k \cap C_2^* = \{j, k\}$. However, this implies that $C_k \cap C^* = \{j\}$ which contradicts $j \in \tau^*(C_1^*)$.

Hence $(TC2)$ is satisfied and $T^*$ is a tag system. $\square$

The following example illustrates the dual of a tag system.

2.4.3 Example: Let $T = (E, C, \tau)$ be given by $E = \{1, 2, 3, 4\}$ with $C_1 = \{1, 2, 3\}, \tau(C_1) = \{1, 2, 3\}; C_2 = \{1, 2, 4\}, \tau(C_2) = \{4\}$ and $C_3 = \{3, 4\}, \tau(C_3) = \{4\}$. Then it is easily checked that the dual system, $T^* = (E, C^*, \tau^*)$, is defined by $C_1^* = \{1, 2\}, \tau^*(C_1^*) = \{1, 2\}; C_2^* = \{1, 3\}, \tau_2^*(C_2) = \{1, 3\}; C_3^* = \{2, 3\}, \tau^*(C_3^*) = \{2, 3\}; C_4^* = \{1, 3, 4\}, \tau^*(C_4^*) = \{4\}; C_5^* = \{2, 3, 4\}, \tau^*(C_5^*) = \{4\}.$

We now prove a painting theorem for a tag system and its dual.

2.4.4 Theorem: Given a tag system $T = (E, C, \tau)$ and its dual, $T^* = (E, C^*, \tau^*)$ and any painting of $E$ into Red (R), Blue (B) and White (W), with $|R| = 1$, exactly one of the following holds:

(P1) $\exists C \in C$ with $C \subseteq R \cup B$ and $R \subseteq \tau(C)$;

(P2) $\exists C^* \in C^*$ with $C^* \subseteq R \cup W$ and $R \subseteq \tau^*(C^*)$.

Proof: We first demonstrate that alternatives (P1) and (P2) cannot both hold. Assume that there exist both $C \subseteq R \cup B$ with $R \subseteq \tau(C)$
and \( C^* \subseteq R \cup W \) with \( R \subseteq \tau^*(C^*) \). Then \( R \subseteq \tau(C) \cap \tau^*(C^*) \) but \( C \cap C^* = R \) contradicting the definition of \( \tau^*(C^*) \). Hence at most one of \((P1)\) and \((P2)\) holds.

We now show that if \((P1)\) fails then \((P2)\) must hold. If \((P1)\) fails and there does not exist \( C \in C \) with \( R \subseteq \tau(C) \), then clearly \( R \in C^* \) with \( R = \tau^*(R) \) defines a cocircuit which satisfies \((P2)\). Thus, we assume that there is a \( C \in C \) with \( R \subseteq \tau(C) \). Since \((P1)\) fails, it is clear that every such circuit must intersect White.

Let \( S = \{ C \cap W : C \in C \text{ with } R \subseteq \tau(C) \} \). By the comment above, \( S \neq \emptyset \). Now we delete elements from \( S \) to obtain \( S^0 \) with the property

\[
(*) \quad \begin{cases} 
\text{For all } C \in C \text{ with } R \subseteq \tau(C), \ C \cap S^0 \neq \emptyset, \text{ and} \\ 
\text{for all } i \in S^0, \text{ there exists } C_i \in C \text{ with} \\ 
R \subseteq \tau(C_i) \text{ and } C_i \cap S^0 = \{ i \}. 
\end{cases}
\]

Now let \( C^* = S^0 \cup R \) and let \( \tau^*(C^*) = \tau'(C^*) \). We will show that \( R \subseteq \tau^*(C^*) \) and hence \( C^* \) satisfies \((P2)\).

Suppose that \( C \in C \) with \( R \subseteq \tau(C) \). Then, \( C \cap C^* = C \cap (S^0 \cup R) = (C \cap S^0) \cup R \) which, by construction, properly contains \( R \). Now, let \( k \in C^* \setminus R \) (which exists by construction). Then, there exists \( C_k \in C \) with \( R \subseteq \tau(C_k) \) such that \( C_k \cap C^* = \{ k \} \cup R \). Hence \( R \subseteq \tau^*(C^*) \) and the proof is complete. \( \square \)
The Matroid Painting Theorem actually characterizes dual pairs of matroids; the following proposition shows that the analogous result for tag systems is also true.

2.4.5 Proposition: If \( T = (E,C,\tau) \) and \( \hat{T} = (E,\hat{C},\hat{\tau}) \) are two tag systems which together satisfy the painting condition of Theorem 2.4.4, i.e., exactly one of the following holds for any painting of \( E = R \cup B \cup W \) with \(|R| = 1:\)

(P1) There exists \( C \subseteq R \cup B \) with \( R \subseteq \tau(C); \)
(P2) There exists \( \hat{C} \subseteq R \cup W \) with \( R \subseteq \hat{\tau}(\hat{C}), \)

then \( T^* = \hat{T}. \)

Proof: We must show that \( D \in \hat{C} \) if and only if \( D \in C^* \) and \( \hat{\tau}(D) = \tau^*(D) \) for all such \( D. \)

Suppose \( D \in \hat{C} \) with tag set \( \hat{\tau}(D). \) We will show that \( j \in \hat{\tau}(D) \) if and only if \( j \in \tau'(D) \) and hence \( D \in C^* \) with \( \tau^*(D) = \hat{\tau}(D). \)

Suppose \( j \in \hat{\tau}(D) \) and \( C \in C \) with \( j \in \tau(C) \) such that \( C \cap D = \{j\}. \) We arrive at a contradiction through painting. Paint \( \{j\} \) Red, \( C \setminus \{j\} \) Blue, \( D \setminus \{j\} \) White and \( E \setminus (D \cup C) \) arbitrarily Blue and White. Then \( C \subseteq R \cup B \) with \( R \subseteq \tau(C) \) and \( D \subseteq R \cup W \) with \( R \subseteq \hat{\tau}(D), \)
contradicting the "exactly one" clause of the painting stipulation.

Now suppose \( k \in D \setminus \{j\}. \) Paint \( \{j\} \) Red, \( D \setminus \{j,k\} \) White and \( (E \setminus D) \cup \{k\} \) Blue. If part (P2) of the painting condition held, there would be a \( \hat{C} \) with \( j \in \hat{\tau}(C) \) and \( \hat{C} \subseteq D \setminus \{k\}. \) This would contradict the tag-minimality of \( T, \) so part (P2) must fail. Hence part (P1) of the painting condition must hold. Thus, there exists \( C \in C \) with \( j \in \tau(C) \) and \( C \subseteq R \cup B = (E \setminus D) \cup \{j,k\}. \) So, \( \{j\} \subseteq C \cap D \subseteq \{j,k\}. \) Since we
know by the above argument that \(|D \cap C| \neq 1\) (since \(j \in \hat{\tau}(C) \cap \hat{\tau}(D)\)), \(D \cap C\) must equal \(\{j, k\}\). Hence \(j \in \hat{\tau}(D)\) implies \(j \in \hat{\tau}'(D)\), and since \(\hat{T}\) is a tag system, no \(\hat{\tau}(D)\) is empty, so we have \(D \in C^*\) with \(\hat{\tau}(D) \subseteq \tau^*(D)\).

Now suppose \(j \in D \setminus \hat{\tau}(D)\), but \(j \in \tau'(D)\). Paint \(\{j\}\) Red, \(D \setminus \{j\}\) White and \(E \setminus D\) Blue. Painting condition (P2) would imply the existence of \(\hat{C} \subseteq \hat{\tau}(\hat{C})\) with \(j \in \hat{\tau}(\hat{C})\) such that \(\hat{C} \subseteq D\). Since \(\hat{T}\) is a tag system, \(j \in \hat{\tau}(\hat{C})\setminus \hat{\tau}(D)\) implies \(\hat{C} \neq D\) and hence there exists \(k \in D \setminus \hat{C}\). By \(j \in \tau'(D)\), there exists \(C_k \subseteq C\) with \(j \in \tau(C_k)\) and \(C_k \cap D = \{j, k\}\). This implies \(C_k \cap \hat{C} = \{j\}\) and so \(j \notin \tau'(\hat{C})\). But, by the first part of this proof, we know \(j \in \hat{\tau}(\hat{C})\) implies \(j \in \tau'(\hat{C})\) and hence we have a contradiction. Thus part (P2) must fail and so part (P1) implies that there exists \(C \subseteq C\) with \(j \in \tau(C)\) and \(C \subseteq (E \setminus D) \cup \{j\}\). However, then \(C \cap D = \{j\}\) contradicting \(j \in \tau'(D)\). Hence, there is no such \(j \in \tau'(D)\setminus \hat{\tau}(D)\) and we conclude that \(j \in \tau'(D)\) if and only if \(j \in \hat{\tau}(D)\). Thus, \(D \in \hat{C}\) implies \(D \in C^*\) with \(\tau^*(D) = \hat{\tau}(D)\).

Now suppose \(D \in C^*\) with tag set \(\tau^*(D)\). We show that the existence of \(D \in \hat{C}\) with tag set \(\tau^*(D) = \hat{\tau}(D)\) is required by the painting theorem. Let \(j\) be chosen arbitrarily from \(\tau^*(D)\). Paint \(\{j\}\) Red, \(D \setminus \{j\}\) White and \(E \setminus D\) Blue. Condition (P1) would imply the existence of \(C \subseteq C\) with \(j \in \tau(C)\) and \(C \subseteq (E \setminus D) \cup \{j\}\). However, then \(C \cap D = \{j\}\), contradicting \(j \in \tau^*(D)\). Thus, part (P1) fails and condition (P2) implies the existence of \(\hat{C} \subseteq \hat{C}\) with \(j \in \hat{\tau}(\hat{C})\) and \(\hat{C} \subseteq D\). If \(\hat{C} = D\), then we have \(D \in \hat{C}\) and we conclude from the first part of this proof that \(\hat{\tau}(D)\) must equal \(\tau^*(D)\). If \(\hat{C} \neq D\),
then we may choose $k \in D \setminus \hat{C}$. Thus, by $j \in \tau^*(D)$, there exists $C_k \in C$ with $j \in \tau(C_k)$ and $C_k \cap D = \{j, k\}$. However, then $C_k \cap \hat{C} = \{j\}$, contradicting $\hat{C} \in C^*$ with $\hat{\tau}(\hat{C}) = \tau^*(\hat{C})$ (given by the first part of this proof). Thus $D \in \hat{C}$ with $\hat{\tau}(D) = \tau^*(D)$ and the proof is complete. □

2.4.6 Corollary: $(T^*)^* = T$

Proof: If $T$ and $T^*$ satisfy Theorem 2.4.4, then after switching the Blue and White elements, $T^*$ and $T$ satisfy the theorem. Thus, by Proposition 2.4.5, $(T^*)^* = T$. □

Hence, as in the matroid case, tag systems are characterized by the painting theorem, and the dual of the dual of system $T$ is again $T$.

The next proposition provides an expression for the span function of the dual of a tag system in terms of the original span function.

2.4.7 Proposition: Suppose we are given dual tag systems $T = (E, C, \tau)$ and $T^* = (E, C^*, \tau^*)$ with respective span functions $\sigma$ and $\sigma^*$. Then, for all $S \subseteq E$, $\sigma^*(S) = S \cup \{x \notin S : x \notin \sigma((E \setminus S) \setminus \{x\})\} = \Sigma(S)$.

Proof: Given $S \subseteq E$, let $x \in \sigma^*(S) \setminus \Sigma(S)$. Then, by $x \in \sigma^*(S)$, there exists $C^* \in C^*$ with $C^* \subseteq S \cup \{x\}$ and $x \in \tau^*(C^*)$. Since $x \notin \Sigma(S)$, we know there exists $C \in C$ with $C \subseteq E \setminus S$ and $x \in \tau(C)$. However, clearly $C \cap C^* = \{x\} = \tau(C) \cap \tau^*(C^*)$, contradicting $x \in \tau^*(C^*)$. Now, let $x \in \Sigma(S) \setminus \sigma^*(S)$. Then, there exists no $C \in C$ with $C \subseteq E \setminus S$ and $x \in \tau(C)$, and there exists no $C^* \in C^*$ with $C^* \subseteq S \cup \{x\}$ and $x \in \tau^*(C^*)$. 
Now paint \{x\} Red, \(E \setminus S \setminus \{x\}\) Blue and \(S\) White. (P1) of Theorem 2.4.4 would imply the existence of \(C \in C\) with \(C \subseteq E \setminus S\) and \(x \in \tau(C)\) and (P2) would imply the existence of \(C^* \in C^*\) with \(C^* \subseteq S \cup \{x\}\) and \(x \in \tau^*(C^*)\). Thus \(x \in \Sigma(S) \setminus \sigma^*(S)\) implies neither (P1) nor (P2) holds, contradicting Theorem 2.4.4. Thus, \(\sigma^*(S) = \Sigma(S)\). □

Note that the painting theorem, 2.4.4, can be restated in terms of the span functions.

2.4.8 Proposition: Given dual tag systems \(T = (E, \sigma)\) and \(T^* = (E, \sigma)\), paint \(E\) Red, Blue and White with \(|R| = 1\). Then exactly one of the following holds:

(P1)' \(\exists S \subseteq E\) such that \(S \subseteq B\) and \(R \subseteq \sigma(S)\);
(P2)' \(\exists S^* \subseteq E\) such that \(S^* \subseteq W\) and \(R \subseteq \sigma^*(S^*)\). □

One fundamental result of matroid theory is that the complement of a base of a matroid is a base of the dual matroid. We now see that the same holds true for tag systems.

2.4.9 Proposition: \(B\) is a base of \(T = (E, C, \tau) = (E, \sigma)\) if and only if \(E \setminus B\) is a base of \(T^* = (E, C^*, \tau^*) = (E, \sigma^*)\).

Proof: Note that since \(E \setminus (E \setminus B) = B\) and \(T^* = T\), we need only prove that if \(B\) is a base of \(T\), then \(E \setminus B\) is a base of \(T^*\).

Let \(B\) be a base of \(T\); i.e., let \(B\) be an independent set with \(\sigma(B) = E\). Then Proposition 2.4.7 implies that
\[ \sigma^*(E \setminus B) = (E \setminus B) \cup \{x \in B : x \notin \sigma(B \setminus \{x\})\}. \]

Since \( B \) is an independent set, \( x \in B \) implies \( x \notin \sigma(B \setminus \{x\}) \) and thus \( \sigma^*(E \setminus B) = E \). Now suppose \( E \setminus B \) is dependent in \( T^* \); i.e., there exists \( C^* \in C^* \) with \( C^* \subseteq E \setminus B \). Let \( i \in \tau^*(C^*) \). Then we know that \( i \in \sigma(B) \setminus B \) and hence there exists \( C \in C \) with \( C \subseteq B \cup \{i\} \) and \( i \in \tau(C) \). However, then \( C \cap C^* = \{i\} = \tau(C) \cap \tau^*(C) \), contradicting \( i \in \tau^*(C^*) \). Hence \( E \setminus B \) is independent in \( T^* \) and the proof is complete. \( \square \)

2.5 Operations on Tag Systems

This section investigates the operations of deletion and contraction of an element of a tag system and the process of taking the intersection of two tag systems. An interesting result here is that, unlike in the matroid setting, the intersection of tag systems is again a tag system.

2.5.1 Let \( T = (E, C, \tau) \) be a tag system and let \( e \) be an element of \( E \). Then define \( T_e = (E \setminus \{e\}, C, \tau) \) by \( C \in C \) if and only if \( \overline{C} \in C \) with \( e \notin \overline{C} \) and define \( \tau(C) = \tau(C \setminus \{e\}) \). It is clear that this is a tag system, it is called the tag system obtained by deleting element \( e \).

2.5.2 Again given \( T \) and \( e \in E \), form the following subset of \( 2^{E \setminus \{e\}} \):
\[ C = \{C \setminus \{e\} : C \in C \text{ and } \{e\} \notin \tau(C)\}. \]
Now for each \( C \in C \), define \( \tau(C) = \tau(C \setminus \{e\}) \). Then, form a tag-minimal family by the rule: if \( C_1, C_2 \in C \) with \( C_1 \neq C_2 \) and \( j \in \tau(C_1) \cap \tau(C_2) \), then replace \( \tau(C_2) \)
by \( \tau(C_2) \setminus \{j\} \). After this process, first remove any \( C \) with \( \tau(C) = \emptyset \), and then if \( C_1, C_2 \in C \) with \( C_1 = C_2 \), replace \( \tau(C_1) \) by \( \tau(C_1) \cup \tau(C_2) \) and delete \( C_2 \). Finally, call the remaining family \( \hat{C} \) and define \( \hat{\tau} : \hat{C} \rightarrow 2^E \) by \( \hat{\tau}(\hat{C}) = \hat{\tau}(C) \). It is clear that \( (E \setminus \{e\}, \hat{C}, \hat{\tau}) \) is a tag system; it is called the tag system obtained by contracting element \( e \) and denoted \( \hat{T}_e \).

The following proposition proves that these two tag systems are related in the same manner as are their matroid analogues, i.e., that the operations of deletion and contraction are dual to one another.

2.5.3 Proposition: \( (T_e)^* = (\hat{T}^*)_e \).

Proof: We will prove that \( T_e \) and \( (\hat{T}^*)_e \) are duals by showing that they satisfy the tag system painting theorem (2.4.4) together. For ease of notation, let \( (\hat{T}^*)_e = (E \setminus \{e\}, \hat{C}, \hat{\tau}) \).

Let \( E \setminus \{e\} \) be painted Red, Blue and White with \( |R| = 1 \). Suppose that both parts (P1) and (P2) of Theorem 2.4.4 are satisfied. Then there exists \( \bar{C} \subseteq R \cup B \) with \( R \subseteq \tau(\bar{C}) \) and \( \bar{C} \subseteq R \cup W \) with \( R \subseteq \bar{\tau}(\bar{C}) \).

Thus, there exists \( C \subseteq C \) with \( e \notin C \) and \( C \subseteq R \cup B \), \( R \subseteq \tau(C) \) and also \( C^* \subseteq C^* \) with \( \tau(C^*) \neq \{e\} \), \( C^* \setminus \{e\} \subseteq R \cup W \) and \( R \subseteq \tau(C^*) \setminus \{e\} \). Now paint \( e \) White. Then, \( C \) satisfies (P1) of the painting conditions for \( T \) and \( C^* \) satisfies (P2) for \( T^* \) thus contradicting the fact that \( T \) and \( T^* \) are dual tag systems. Therefore, at most one of (P1) and (P2) holds for \( T_e \) and \( (\hat{T}^*)_e \).

Now assume (P1) of the theorem fails; i.e., there is no \( \bar{C} \subseteq R \cup B \) with \( R \subseteq \bar{\tau}(\bar{C}) \). This implies that there does not exist \( C \subseteq C \) with
e \not \in C, C \subseteq R \cup B \text{ and } R \subseteq \tau(C). \text{ Thus, if e is painted White, then (P1) fails for T, and hence (P2) must hold for } T^*. \text{ This implies there exists a } C^* \in C^* \text{ with } R \subseteq \tau^*(C^*) \text{ and } C^* \subseteq R \cup W. \text{ Since e was painted White and } R \subseteq \tau^*(C^*), \tau^*(C^*) \text{ cannot equal } \{e\}. \text{ Therefore, } C^* \setminus \{e\} \text{ is an element of } \hat{C} \text{ with } \tau(C^* \setminus \{e\}) = \tau^*(C^*) \setminus \{e\} \text{ (see Definition 2.5.2). Hence, there exists a } \hat{C} \in \hat{C}^* \text{ satisfying (P2) for } (T^*)_e. \quad \Box

2.5.4 Example: We demonstrate the operations of deletion and contraction on Example 2.2.2; the reader can verify the tabulations in Table 1. Recall that } E = \{1,2,3,4\}, \text{ let } e = 1.

Thus we find that the operation of deleting an element in the primal system is dual to that of contracting an element in the dual system.

2.5.5 Another operation from matroid theory which remains useful in the setting of tag systems is truncation. Given a tag system } T = (E,\sigma) \text{ and integer } k > 0, \text{ consider the function } \sigma^k, \text{ where } \sigma^k(S) = \sigma(S) \text{ for all } S \subseteq E \text{ with } |S| \leq k \text{ and } \sigma^k(S) = E \text{ for all } S \text{ with } |S| > k. \text{ This clearly defines another span function. Since in the tag system } \tau^k = (E,\sigma^k) \text{ only sets of cardinality less than or equal to } k \text{ can be independent, we call this the } k\text{-truncated tag system.}

Whereas it is well-known that the common independent sets of two matroids do not themselves necessarily constitute the independent sets of a matroid, the following discussion illustrates that this is always true for tag systems.
<table>
<thead>
<tr>
<th>$T^*$</th>
<th>$\hat{T}^*_e$</th>
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<th>$\hat{\tau}^*_e$</th>
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<tbody>
<tr>
<td>$(E, C^<em>, \tau^</em>)$</td>
<td>$(E \setminus {e}, C^<em>, \tau^</em>)$</td>
<td>$(E \setminus {e}, C^<em>, \tau^</em>)$</td>
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</tbody>
</table>

**TABLE 1**
2.5.6 Given tag systems $T^1 = (E, C^1, \tau^1)$ and $T^2 = (E, C^2, \tau^2)$, form $T = (E, C, \tau)$ as follows: Define $T = (E, C, \tau)$ by $C \in C$ with $j \in \tau(C)$ if and only if $C \in C^i$ with $j \in \tau^i(C)$ for $i = 1$ or 2. Then, form $T$ by making $T = (E, C, \tau)$ a tag system as discussed in 2.5.2 for $\hat{T}_e$.

2.5.7 Proposition: $T = (E, C, \tau)$ is the tag system whose independent sets are exactly those which are independent in both $T^1$ and $T^2$.

Proof: This is clear since a set is dependent in $T$ exactly when it is dependent in either $T^1$ or $T^2$. [\\]

This last result shows that $T$ is actually the tag system obtained by "intersecting" $T^1$ and $T^2$, and thus will be referred to as $T^1 \cap T^2$. Iterating this process yields the following.

2.5.8 Corollary: The common independent sets of $n$ tag systems $T^1, \ldots, T^n$ constitute the independent sets of a tag system, denoted $T^1 \cap \ldots \cap T^n$.

2.5.9 Now, given tag systems $T^1 = (E, C^1, \tau^1)$ and $T^2 = (E, C^2, \tau^2)$, form $T = (E, C, \tau)$ as follows: Define $T = (E, C, \tau)$ by $C \in C$ with $j \in \tau(C)$ if and only if there exists $C^1 \in C^1$ and $C^2 \in C^2$ with $j \in \tau^1(C^1) \cap \tau^2(C^2)$, $C^1 \subseteq C$ and $C^2 \subseteq C$. Then, form $T$ by making $T$ a tag system as discussed earlier.

2.5.10 Proposition: $T = (E, C, \tau)$, defined in 2.5.9, is the tag system whose independent sets are exactly those which are independent in either $T^1$ or $T^2$. 
Proof: This is clear since a set is dependent in $T$ exactly when it is dependent in both $T^1$ and $T^2$. □

This result shows that $T$ is actually the tag system obtained by forming the "union" of $T^1$ and $T^2$, and thus will be denoted $T^1 \cup T^2$. Note that by iterating this process an analogous result to Corollary 2.5.8 is obtained.

2.5.11 Theorem: Given tag systems $T_1 = (E,C_1,\tau_1)$ and $T_2 = (E,C_2,\tau_2)$ and their duals $T^*_1$ and $T^*_2$, respectively, $(T_1 \cup T_2)^* = T^*_1 \cap T^*_2$.

Proof: We will prove this by showing that $(T_1 \cup T_2)$ and $(T^*_1 \cap T^*_2)$ satisfy the painting theorem, 2.4.4, together. For ease of notation, let $T_1 \cup T_2 = (E,D_1,\omega_1)$ and $T^*_1 \cap T^*_2 = (E,D_2,\omega_2)$.

Paint $E$ Red, Blue and White with $|R| = 1$. Then (P1) of the painting condition holds if and only if there exists $D_1 \in D_1$ with $D_1 \subseteq R \cup B$ and $R \subseteq \omega_1(D_1)$. This is equivalent to $C_1 \subseteq C_1$, $C_2 \subseteq C_2$ with $C_1 \subseteq R \cup B$, $C_2 \subseteq R \cup B$ and $R \subseteq \tau_1(C_1) \cap \tau_2(C_2)$. Since Theorem 2.4.4 holds for $T_1,T^*_1$ and $T_2,T^*_2$, this is equivalent to the nonexistence of $C^*_1 \subseteq R \cup W$ with $R \subseteq \tau^*_1(C^*_1)$ and the nonexistence of $C^*_2 \subseteq R \cup W$ with $R \subseteq \tau^*_2(C^*_2)$. Thus, there does not exist $D_2 \in D_2$ with $D_2 \subseteq R \cup W$ and $R \subseteq \omega_2(D_2)$ and hence (P1) holds if and only if (P2) fails. □

2.6 Matroids

Example 2.2.5 illustrates that a matroid naturally gives rise to a tag system. This section discusses this correspondence in more
detail. We show how certain properties of circuits and tag sets are related to a tag system's being a matroid.

2.6.1 **Proposition:** Given tag system $T = (E, C, \tau) = (E, \sigma)$, if $\tau(C) = C$ for all $C \in C$, then $\sigma$ satisfies the following property for all $x, y \in E$ and $X \subseteq E$:

$$(*) \, y \not\in \sigma(X) \text{ and } y \in \sigma(X \cup \{x\}) \Rightarrow x \in \sigma(X \cup \{y\}).$$

**Proof:** Let $X \subseteq E$ and $x, y \in E$ with $y \not\in \sigma(X)$ but $y \in \sigma(X \cup \{x\})$. Then, there exists no $C \in C$ with $C \subseteq X \cup \{y\}$ and $y \in \tau(C)$, but there does exist $C \in C$ with $C \subseteq X \cup \{x, y\}$ and $y \in \tau(C)$. $\tau(C) = C$ for all $C \in C$ thus implies $x \in \tau(C)$ and hence $x \in \sigma(X \cup \{y\})$. □

2.6.2 **Lemma:** Tag system $T = (E, C, \tau) = (E, \sigma)$ satisfies property $(*)$ above if and only if it satisfies the following property for all $C \in C$:

$$(**) \, i \in C \setminus \tau(C) \Rightarrow \exists C_1 \in C, C_1 \not\supseteq C \text{ with } i \in \tau(C_1).$$

**Proof:** Assume $T$ satisfies $(*)$ and let $C \in C$ with $i \in C \setminus \tau(C)$ and $j \in \tau(C)$. Then, $j \in \sigma(C \setminus \{j\})$ but $j \not\in \sigma(C \setminus \{i, j\})$ and so by $(*)$, $i \in \sigma(C \setminus \{i\})$. Thus there exists a circuit contained within $C$ with $i$ in its tag set, so $T$ satisfies $(**)$.

Now assume $T$ satisfies $(**)$, then $X \subseteq E$ and $x, y \in E$ such that $y \not\in \sigma(X)$ and $y \in \sigma(X \cup \{x\})$. Thus, there exists $C \in C$ with $x \in C \subseteq X \cup \{x, y\}$ with $y \in \tau(C)$. By $(**)$, either $x \in \tau(C)$ or there exists $C_1 \not\supseteq C$ with $x \in \tau(C_1)$. In either case, $x \in \sigma(X \cup \{y\})$ so $T$ satisfies $(*)$. □
2.6.3 Theorem: Given a dual pair of tag systems, \( T = (E, C, \tau) = (E, \sigma) \) and \( T^* = (E, C^*, \tau^*) = (E, \sigma^*) \), if both \( T \) and \( T^* \) satisfy (*), then \( \tau(C) = C \) for all \( C \in C \) and \( \tau^*(C^*) = C^* \) for all \( C^* \in C^* \).

Proof: Suppose there exists a circuit that is not equal to its tag set. Let \( C \) be a minimal such circuit (that is, for \( C \subset C \), \( \tau(C) = C \)), and let \( i \in C \setminus \tau(C) \). By 2.6.2, there exists \( C_1 \in C \) with \( C_1 \not\subseteq C \) and \( i \in \tau(C_1) = C_1 \). Let \( k \in \tau(C) \). Then, there exists \( C^* \in C^* \) with \( k \in \tau^*(C^*) \) and \( C^* \cap C = \{i, k\} \). Thus, since \( i \in C^* \), either \( i \in \tau^*(C^*) \) or there exists \( C_1^* \not\subseteq C^* \) with \( i \in \tau^*(C_1^*) \) (by 2.6.2). So, we have \( C_1^* \subseteq C^* \) with \( i \in \tau^*(C_1^*) \). However, then \( i \in C_1^* \cap C_1 \subseteq C_1 \cap C \subseteq C^* \cap C = \{i, k\} \) and \( i \in \tau^*(C_1^*) \cap \tau(C_1) \). Hence, \( k \in C_1 \cap C_1 \). But, \( k \in \tau(C) \) implies \( k \not\in C_1 = \tau(C_1) \) (otherwise it would contradict the tag-minimality of \( T \)). This is clearly a contradiction (\( k \not\in C_1 \) and \( k \in C_1 \cap C_1 \)). \( \square \)

2.6.4 A tag system \( T = (E, C, \tau) \) satisfies tag circuit exchange (TCE) if whenever \( C_1, C_2 \in C \) with \( i \in \tau(C_1) \cap C_2 \) and \( j \in \tau(C_2) \setminus C_1 \), there exists \( C_3 \in C \) with \( C_3 \subseteq (C_1 \cup C_2) \setminus \{i\} \) and \( j \in \tau(C_3) \).

Clearly the tag system obtained from a matroid satisfies (TCE).

2.6.5 Theorem: Given the tag system \( T = (E, C, \tau) = (E, \sigma) \), \( \sigma(S) = \sigma(\sigma(S)) \) for all \( S \subseteq E \) if and only if \( T \) satisfies (TCE).

Proof: First suppose \( T \) satisfies (TCE) and let \( S \subseteq E \). Then recall that \( \sigma(S) = S \cup \{x: \exists C \in C, C \subseteq S \cup \{x\} \text{ with } x \in \tau(C)\} \) and thus \( \sigma(\sigma(S)) = \sigma(S) \cup \{x: \exists C \in C, C \subseteq \sigma(S) \cup \{x\} \text{ with } x \in \tau(C)\} \).
Clearly \( \sigma(S) \subseteq \sigma(\sigma(S)) \) always. Suppose there exists \( x \in \sigma(\sigma(S)) \setminus \sigma(S) \) and note that this implies \( x \notin S \). Then, there exists \( C_1 \in C \) with \( C_1 \subseteq \sigma(S) \cup \{x\} \) and \( x \in \tau(C_1) \) and in addition, \( C_1 \) must contain some element, say \( y \), in \( \sigma(S) \setminus S \). This implies that there exists \( C_2 \in C \) with \( C_2 \subseteq S \cup \{y\} \) and \( y \in \tau(C_2) \). Now perform circuit exchange \((y \in \tau(C_2) \cap C_1 \text{ and } x \in \tau(C_1) \setminus C_2)\) to obtain \( C_3 \in C \) with \( x \in \tau(C_3) \) and \( C_3 \subseteq C_1 \cup C_2 \setminus \{y\} \). This \( C_3 \) is then contained in \((\sigma(S) \cup \{x\}) \setminus \{y\}\) and thus as before contains some \( z \in \sigma(S) \setminus (S \cup \{y\}) \). However, this implies the existence of \( C_4 \in C \) with \( z \in \tau(C_4) \) and \( C_4 \subseteq S \cup \{z\} \). Again we perform circuit exchange to obtain \( C_5 \in C \) with \( x \in \tau(C_5) \) and \( C_5 \subseteq (\sigma(S) \cup \{x\}) \setminus \{y, z\} \). Iterating this process results in a circuit \( C_n \in C \) with \( x \in \tau(C_n) \) and \( C_n \subseteq S \cup \{x\} \), contradicting \( x \notin \sigma(S) \).

Now suppose that \( \sigma(\sigma(S)) = \sigma(S) \) for all \( S \subseteq E \). Let \( C_1, C_2 \in C \) with \( i \in \tau(C_1) \cap C_2 \) and \( j \in \tau(C_2) \setminus C_1 \). Then, clearly \( i \in \sigma((C_1 \cup C_2) \setminus \{i, j\}) \). Also note that \( j \in \sigma((C_1 \cup C_2) \setminus \{j\}) \). However, \( i \in \sigma((C_1 \cup C_2) \setminus \{i, j\}) \) implies that \( (C_1 \cup C_2) \setminus \{j\} \subseteq \sigma((C_1 \cup C_2) \setminus \{i, j\}) \) and thus \( j \in \sigma((C_1 \cup C_2) \setminus \{i, j\}) \). Then \( \sigma(\sigma(S)) = \sigma(S) \) for all \( S \subseteq E \) implies that \( j \in \sigma((C_1 \cup C_2) \setminus \{i, j\}) \). Hence there exists \( C \subseteq C \) with \( j \in \tau(C) \) and \( C \subseteq (C_1 \cup C_2) \setminus \{i\} \) and so \( T \) satisfies (TCE).

2.6.6 Proposition: Given tag system \( T = (E, C, \tau) \). If \( T \) satisfies (TCE) then \( \overline{T}_e \) and \( \overline{T}_e \) also satisfy (TCE) for all \( e \in E \).
Proof: This is clear from the definitions for $\hat{T}_e$. For $\hat{T}_e$, suppose $\hat{C}_1, \hat{C}_2 \in \hat{C}$ with $i \in \hat{\tau}(\hat{C}_1) \cap \hat{C}_2$ and $j \in \hat{\tau}(\hat{C}_2) \setminus \hat{C}_1$. Then, $\hat{C}_1$ is derived from some $C_1 \setminus \{e\}$ with $\tau(C_1) \neq \{e\}$ and $\hat{\tau}(\hat{C}_1) \subseteq \tau(C_1) \setminus \{e\}$. Similarly, $\hat{C}_2$ comes from some $C_2 \setminus \{e\}$ with $\tau(C_2) \neq \{e\}$ and $\hat{\tau}(\hat{C}_2) \subseteq \tau(C_2) \setminus \{e\}$. Thus, $i \in \tau(C_1) \cap C_2$ and $j \in \tau(C_2) \setminus C_1$ and so by (TCE) for $T$, there exists $C_3 \in C$ with $j \in \tau(C_3)$ and $C_3 \subseteq (C_1 \cup C_2) \setminus \{i\}$. Clearly $\tau(C_3) \neq \{e\}$. Contracting $e$ and taking the tag-minimal such circuit (with respect to $j \in \tau(C_3)$) completes the proof. □

The remainder of this section highlights the relationship between matroids and tag systems. It is easily verified that the definition of the dual of a tag system (2.4.1) is equivalent to the definition of the matroid dual (1.2.4) given that the tag system is originally obtained from a matroid as in 2.2.5. Thus, the following is proved.

2.6.7 Proposition: Given matroid $M = (E, C)$ and its related tag system $T = (E, C, \tau)$, the dual matroid is $M^* = (E, C^*)$ if and only if the dual tag system is $T^* = (E, C^*, \tau^*)$. □

2.6.8 Theorem: Given tag system $T = (E, C, \tau)$ and its dual tag system $T^* = (E, C^*, \tau^*)$, if $\tau(C) = C$ for all $C \in C$ and $\tau^*(C^*) = C^*$ for all $C^* \in C^*$, then $M = (E, C)$ and $M^* = (E, C^*)$ are dual matroids.
Proof: This follows immediately from the observation that the given assumptions imply that \((E,C,C^*)\) satisfies Minty's self-dual axiomatization of dual matroids (see [30]). □

2.6.9 Corollary: Given tag system \(T = (E,C,\tau) = (E,\sigma)\) and its dual tag system \(T^* = (E,C^*,\tau^*) = (E,\sigma^*)\), if both \(T\) and \(T^*\) satisfy condition (*) of 2.6.1 then \(M = (E,C) = (E,\sigma)\) and \(M^* = (E,C^*) = (E,\sigma^*)\) are dual matroids.

Proof: This follows from combining 2.6.3 and 2.6.8. □

2.6.10 Corollary: Given dual tag systems \(T = (E,C,\tau)\) and \(T^* = (E,C^*,\tau^*)\) for which \(\tau(C) = C\) for all \(C \in C\) and \(\tau^*(C^*) = C^*\) for all \(C^* \in C^*\), then for any \(e \in E\), \(\hat{\tau}(\hat{C}) = \hat{C}\) for all \(\hat{C} \in \hat{C}\) and \(\hat{\tau}^*(\hat{C}^*) = \hat{C}^*\) for all \(\hat{C}^* \in \hat{C}^*\) when \(\hat{T}_e = (E \setminus \{e\},\hat{C},\hat{\tau})\) and \(\hat{T}^*_e = (E \setminus \{e\},\hat{C}^*,\hat{\tau}^*)\).

Proof: This follows immediately from the observations that tag system minors coincide with matroid minors when \(\tau(C) = C\) for all \(C \in C\) and that minors of matroids are matroids. □

2.6.11 Lemma: If \(T = (E,C,\tau)\) is a tag system which satisfies (TCE), and if \(C_1, C_2 \in C\) with \(C_1 \not\subset C_2\), then \(\tau(C_2) \subseteq C_1\).

Proof: Suppose the conclusion does not hold. Then, \(C_1, C_2 \in C\) with \(C_1 \not\subset C_2\) and there exists some \(j \in \tau(C_2) \setminus C_1\). Let \(i \in \tau(C_1)\), then since \(C_1 \not\subset C_2\), \(i \in \tau(C_1) \cap C_2\). Using circuit exchange, we obtain \(C_3 \in C\) with \(j \in \tau(C_3)\) and \(C_3 \subseteq (C_1 \cup C_2) \setminus \{i\} = C_2 \setminus \{i\}\). This contradicts the tag-minimality of \(T\) and hence the proof is complete. □
2.6.12 Theorem: Given tag system $T = (E, C, \tau)$ and its dual tag system $T^* = (E, C^*, \tau^*)$; if both systems satisfy tagged circuit exchange, then $M = (E, C)$ and $M^* = (E, C^*)$ are dual matroids.

Proof: We prove this by showing that $\tau(C) = C$ for all $C \in C$ or $\tau^*(C^*) = C^*$ for all $C^* \in C^*$. The result then follows from 2.6.7, since then either $T$ or $T^*$ itself comes from a matroid. The proof is by induction on $|E|$. Clearly the only tag systems on one element are matroids and hence the theorem is true when $|E| = 1$.

Now assume the result is true for all tag systems on a ground set of cardinality less than or equal to $n$, and suppose that $T = (E, C, \tau)$ and $T^* = (E, C^*, \tau^*)$ are dual tag systems on ground set $E$ with $|E| = n+1$, and that both systems satisfy (TCE). Let $C \in C$ and suppose $C \not\subseteq E$. Then, let $e \not\in C$ and consider $\overline{T_e}$, the tag system obtained by deleting element $e$. Since $e \not\in C$, we know that $C \in \overline{C}$ with $\tau(C) = \tau(C)$. Therefore, since $\overline{T_e}$ and $(T^*)_e$ are dual tag systems on $n$ elements, each of which must satisfy (TCE) (by 2.6.6), $C = \overline{\tau(C)} = \tau(C)$. Hence, it follows that for all $C \in C$ with $C \not\subseteq E$, $\tau(C) = C$.

Now suppose $C = E$. Clearly if $\tau(C) = C$ then the proof is completed, so assume $e \in C \setminus \tau(C)$. Then, either there is another $C_1 \in C$ with $e \in \tau(C_1)$ or there exists no circuit with $e$ in its tag set. Suppose first that $C_1 \in C$ with $e \in \tau(C_1)$. Then, since $C = E$, $C_1 \subseteq C$ and hence by 2.6.11, $\tau(C) \subseteq C_1$. As $e \in \tau(C_1) \setminus \tau(C)$, it follows that $C \not\subseteq C_1$ and hence $C_1 \not\subseteq C$. Thus, $C_1 \not\subseteq E$ and so
by the above argument, $\tau(C_1) = C_1$. Therefore, $\tau(C) \subseteq C_1 = \tau(C_1)$
thus contradicting the tag-minimality of $T$.

Now assume there exists no circuit containing $e$ in its tag set. Then it is clear that $\{e\} \in C^*$ with $\tau^*(\{e\}) = \{e\}$.
Hence, by 2.6.11, any $C^* \in C^*$ with $e \in C^*$ must have $\{e\} = \tau^*(C^*)$.
This, though, would contradict the tag-minimality of $T^*$ and thus, no other cocircuit contains $e$. Hence, no cocircuit equals the entire ground set $E$ and by the first part of this proof applied to $T^*$, we see that $\tau^*(C^*) = C^*$ for all $C^* \in C^*$. □

The following example illustrates that it is not enough for
a tag system to have all of its tag sets equal to the circuits and
for the dual to satisfy (TCE) in order to obtain dual matroids.

2.6.13 Example: Let $T = (E,C,\tau)$ be defined by $E = \{1,2,3,4\}$,
$C_1 = \{1,2\}$, $C_2 = \{2,3\}$, $C_3 = \{3,4\}$, $C_4 = \{1,4\}$ with $\tau(C) = C$
for all $C \in C$. Note that $M = (E,C)$ is not a matroid since it
does not satisfy circuit exchange (consider $C_1$ and $C_2$).

$T^* = (E,C^*,\tau^*)$ is given by $C^*_1 = \{1,2,4\}$, $\tau^*(C^*_1) = \{1\}$;
$C^*_2 = \{1,2,3\}$, $\tau^*(C^*_2) = \{2\}$; $C^*_3 = \{2,3,4\}$, $\tau^*(C^*_3) = \{3\}$; $C^*_4 = \{1,3,4\}$,
$\tau^*(C^*_4) = \{4\}$. Note that $T^*$ satisfies (TCE) vacuously and while
$(E,C^*)$ is a matroid, it is dual to itself rather than dual to
$M = (E,C)$.

2.6.14 Proposition: Given a dual pair of tag systems, $T = (E,\sigma)$
and $T^* = (E,\sigma^*)$, if $\sigma(\sigma(S)) = \sigma(S)$ and $\sigma^*(\sigma^*(S)) = \sigma^*(S)$ for
all $S \subseteq E$, then $M = (E,\sigma)$ and $M^* = (E,\sigma^*)$ constitute a dual
pair of matroids.
Proof: This follows by combining 2.6.5 and 2.6.12. □

2.7 Greedoids

As discussed in Chapter one, a recently developed generalization of matroids is given by greedoids. This section will investigate two distinct ways in which greedoids give rise to tag systems.

The reader will recall from the feasible set axiomatization of greedoids (1.3) that all of the independent set axioms for matroids (1.2.2) are satisfied except for (II), which requires that every subset of an independent set be independent. On the other hand, it is clear that the independent sets of a tag system have this property, hence the following result is proved.

2.7.1 Theorem: If \((E,F)\) defines both the feasible sets of a greedoid on ground set \(E\) and the independent sets of a tag system on \(E\), then it defines the independent sets of a matroid on \(E\). □

Thus, this informs us that from the independent set viewpoint, greedoids and tag systems are two distinct generalizations of matroids. If we concentrate instead on circuits, we obtain tag systems related to greedoids in a different way. Here we restrict our attention to shelling structures (see 1.3), as this appears to be the only setting in which circuits have been defined for greedoids.

If \((E,F)\) defines a shelling structure, let \(C\) be the family of circuits of this structure with \(\rho: C \to 2^E\) defined by \(\rho(C) = R(C)\), the root of circuit \(C\) for all \(C \in C\). It is shown in [27] that \((E,F)\) is determined by \((E,C,\rho)\).
2.7.2 Proposition: If \((E,C,\rho)\) defines a shelling structure, then it also defines a tag system.

Proof: First, it is clear that \(\rho\) is a function from \(C\) to \(2^E\) which satisfies (TC1). Since the circuits of a shelling structure are minimal, (TC2) is also satisfied. 

2.7.3 Theorem: If \(T = (E,C,\rho)\) is the tag system defined in 2.7.2 from shelling structure \((E,F)\), then the dual tag system \(T^* = (E,C^*,\tau^*)\) is defined by \(C^* \in C^*\) with \(j \in \tau^*(C^*)\) if and only if \(C^*\) is a minimal feasible set containing \(j\).

Proof: First notice that \(T^*\), so defined, is a tag system. Next, we need the following lemma.

2.7.4 Lemma: \(|\tau^*(C^*)| = 1\) for all \(C^* \in C^*\).

Proof: Suppose \(|\tau^*(C^*)| > 1\). Then, say, \(i,j \in \tau^*(C^*)\). By definition, no subset of \(C^*\) containing either \(i\) or \(j\) is feasible. This, though, contradicts the axiom which states that for all \(X \in F\), there exists \(x \in X\) such that \(X \setminus \{x\} \in F\), since any such \(X \setminus \{x\}\) must contain either \(i\) or \(j\). 

Now, continuing the proof of 2.7.3, suppose \(C^* \in C^*\) with \(j \in \tau^*(C^*)\). Then, we know that for all \(C \in C\) with \(\rho(C) = \{j\}\), \(C \cap C^* \neq \{j\}\). Further, let \(k \in C^* \setminus \{j\}\). Then, if there does not exist \(C \in C\) with \(\{j\} = \rho(C)\) and \(C \cap C^* = \{j,k\}\), \(C^* \setminus \{k\}\) intersects no circuit with root \(j\) on the subset \(\{j\}\). Hence \(C^* \setminus \{k\}\) is a feasible
set containing \( j \), contradicting our assumption that \( C^* \in C^* \) with \( \tau^*(C^*) \geq j \). Now, by 2.7.4, \( |\tau^*(C^*)| = 1 \) for all \( C^* \in C^* \), so the proof is complete. \( \square \)

Hence any shelling structure defines a dual pair of tag systems, thus providing an additional class of examples of tag systems.

2.8 Blocking Systems

This section will briefly investigate an intimate relation between systems of blocking clutters and dual pairs of tag systems. The relationship studied here was pointed out to us by Professor R.G. Bland. Given a finite set \( S \), a clutter on \( S \) is a family of noncomparable subsets of \( S \). Clearly, the circuits of matroid \( M = (E, C) \) form a clutter on \( E \).

2.8.1 Proposition: Given tag system \( T = (E, C, \tau) \) and \( e \in E \), the family \( C_e = \{C \in C \mid e \in \tau(C)\} \) is a clutter on \( E \setminus \{e\} \).

Proof: This is clear by the tag-minimality of \( T \). \( \square \)

Given clutter \( C \), call a set, \( C' \), a blocking circuit if \( C \cap C' \neq \emptyset \) for all \( C \in C \), and no subset of \( C' \) has no property. Define the blocker of \( C \) to be the collection, \( C' \), of all such \( C' \). It can be proved (see e.g. [14]) that \( C' \) is the blocker of \( C \) if and only if \( C \) is the blocker of \( C' \). We call \((C, C')\) a blocking pair.

2.8.2 Proposition: Given a dual pair of tag systems \( T = (E, C, \tau) \) and \( T^* = (E, C^*, \tau^*) \) and \( e \in E \), \( C_e \) and \( C^*_e \) are a blocking pair of clutters.
Proof: Suppose $C_e \in C$ and $C^* \in C^*$ and $C_e \cap C^* = \emptyset$. Then $C_e$ comes from $C = C_e \cup \{e\}$ with $C \in C$ and $e \in \tau(C)$ and similarly $C^* = C_e^* \cup \{e\}$ with $C^* \in C^*$ and $e \in \tau^*(C^*)$. But then $C \cap C^* = \{e\} = \tau(C) \cap \tau^*(C^*)$. This clearly contradicts the fact that $T$ and $T^*$ are dual tag systems. □

Note that if $E = \{1,2,\ldots,n\}$, then $\{C_i, C_i^*\}$ as stipulated in 2.8.1 is a blocking pair of clutters for each $i \in E$ (by 2.8.2) and hence any dual pair of tag systems gives rise to $n$ blocking pairs of clutters. The following illustrates that the converse is also true; that is, given $n$ blocking pairs of clutters, one can construct a unique pair of dual tag systems and hence there is a one to one correspondence between these two structures.

Suppose $E = \{1,2,\ldots,n\}$ and for each $i \in E$, $\{C_i, C_i^*\}$ is a blocking pair of clutters on $E \setminus \{i\}$. Then define $T = (E,C,\tau)$ by $C = \bigcup_{i=1}^n \{C \cup \{i\} : C \in C_i\}$ and $\tau(C) = \{i : (C \setminus \{i\}) \in C_i\}$ for each $C \in C$. Similarly, define $T^* = (E,C^*,\tau^*)$ by $C^* = \bigcup_{i=1}^n \{C^* \cup \{i\} : C^* \in C_i^*\}$ and $\tau^*(C^*) = \{i : (C^* \setminus \{i\}) \in C_i^*\}$ for each $C^* \in C^*$. It is clear that $T$ and $T^*$ are a dual pair of tag systems.

Using this correspondence, the painting theorem characterization of dual tag systems (2.4.5) is an immediate consequence of the following theorem ([14]):

2.8.3 Theorem: Given clutters $C$ and $C^*$ on ground set $E$, $C$ and $C^*$ are a blocking pair if and only if for any painting of $E$
into Blue and White, some member of $C$ is entirely Blue or some member of $C^*$ is entirely White, but not both. □

In Chapter three, we follow the pattern of defining two tag systems based on a dependence structure, proving that they are duals and thus arriving at Theorems 2.1.2 and 2.1.3 through the tag system painting theorem (2.4.4) (the same can clearly be done for Theorem 2.1.1). The correspondence discussed in this section illustrates that this approach is equivalent to defining a family of pairs of clutters, proving that each pair is a blocking pair and then proving the theorems through the blocking system painting theorem (2.8.3). Thus the reader should notice that all of the combinatorial abstractions that we achieve here through tag systems can equivalently be achieved by studying blocking systems.

It also becomes apparent from the vast generality of blocking systems that one should not expect strong algorithmic results for general tag systems. This is evidenced again in Chapter three where we define tag systems related to graph stable sets. However, the blocking system approach suggests a method for imposing further structure which could lead to special classes of tag systems which are algorithmically tractable. The extreme case in which the tag system under consideration is actually a matroid suggests an important direction for such refinement, namely the imposition of a relationship between the circuits of the different pairs of blocking families. One generalization of this is the stipulation that the circuits only satisfy an exchange axiom such as tagged circuit exchange. However,
the graphic tag systems of Chapter three illustrate that this is not a sufficient requirement for providing strong algorithmic results. It remains an important open problem to determine a subclass of tag systems which are more general than matroids and yet exhibit an algorithmically tractable structure.
CHAPTER 3
REPRESENTABILITY

3.1 Introduction

In this chapter, we define what is meant by a representable tag system and examine several examples of these. We also address the question of when a tag system is representable.

The general scheme adopted here for representable systems is drawn from [33] in which a general framework for duality is discussed (see also [6]). Let \( R \) be a commutative ring (see [1]), and let \( D \) be a subset of \( R \). For subset \( S \subseteq R^n \), consider the set \( S^* = \{ y \in R^n : yS \subseteq D \} \), where \( yS \subseteq D \) means \( \sum_{j=1}^{n} y_j s_j \in D \) for all \( s \in S \). We call \( S^* \) the \( D \)-dual of \( S \). Since the elements of \( S^* \) are required to satisfy constraints defined by elements of \( S \), we call \( S^* \) a constrained set.

In combinatorial optimization, we are generally concerned with finite relations and hence will only consider finitely constrained sets, i.e., those that are defined by a finite number of the constraints listed in \( yS \subseteq D \). Thus, we can write \( S \) as a matrix with columns from \( S \) without abuse of notation. Two well-known examples of finitely constrained sets arise for \( R = \mathbb{Q} \) with \( D = \{0\} \) and \( D = \mathbb{Q}_+ \). When \( D = \{0\} \), we obtain subspaces and with \( D = \mathbb{Q}_+ \), we arrive at polyhedral cones.
Notice that in these two examples, the concept of duality arises from considering sets generated by linear and non-negative combinations, respectively. Thus, we now consider sets that are generated by taking \((D^n)^*\)-combinations of a set of vectors in \(\mathbb{R}^n\). Note that for \(R = \mathbb{Q}\), \(D = \{0\}\) implies \(D^* = \mathbb{Q}\) and \(D = \mathbb{Q}_+\) implies \(D^* = \mathbb{Q}_+\) and hence we obtain the appropriate types of combinations in the above examples in this manner. To be precise, let \(\{t_1, \ldots, t_n\}\) be a subset of \(\mathbb{R}^m\) given by the columns of matrix \(T\). Consider the set \(\{y \in \mathbb{R}^m \mid Tx = y\}\) for some \(x \in (D^n)^*\), the set generated by \((D^n)^*\) combinations of the set \(\{t_1, \ldots, t_n\}\). We will discuss several examples, including those above, where a constrained set \(\{y \in \mathbb{R}^m : yS \in D\}\) and a generated set \(\{y \in \mathbb{R}^m : Tx = y, x \in (D^n)^*\}\) are related combinatorially and algebraically. Notice that we will only be considering cases where \((D^n)^* = (D^*)^n\). The next two sections of this chapter are concerned with the cases when \(R = \mathbb{Q}\) and \(D = D^* = \mathbb{Q}_+\) and \(D = D^* = \mathbb{Z}\).

3.1.1 Given a matrix \(A \in \mathbb{R}^{m \times n}\), we say that the columns of \(A\) are \(D^*\)-dependent with respect to column \(k\) if there exist \(x_j \in D^*\) such that \(A_k = \sum_{j \neq k} A_j x_j\); where \(A_i\) denotes column \(i\) of matrix \(A\). Form triple \(T = (E, C, \tau)\) on \(E = \{1, 2, \ldots, n\}\) with \(C \subseteq C\) and \(k \in \tau(C)\) if and only if \(C\) indexes a minimally \(D^*\)-dependent set of columns of \(A\) with respect to column \(k\); that is, \(C\) indexes a \(D^*\)-dependent set of columns of \(A\) with respect to column \(k\) and no proper subset of \(C\) has this property. It is clear that \(T\) is a tag system.
3.1.2 Now, given a vector \( x \in \mathbb{R}^n \), define the non-D support of \( x \), denoted \( ND(x) \) to be \( \{ j : x_j \neq 0 \} \). Given matrix \( A \in \mathbb{R}^{m \times n} \), form triple \( \widetilde{T} = (E, \tilde{C}, \tau) \) on \( E = \{1, 2, \ldots, n\} \) with \( \tilde{C} \subseteq \tilde{C} \) and \( k \in \tau(\tilde{C}) \) if and only if \( \tilde{C} \) indexes the non-D support of a vector in the rowspace of \( A \) and no proper subset of \( \tilde{C} \) containing \( k \) has this property. It is easily seen that \( \tau \) is a tag system.

One question often studied in relation to a particular \( R, D \) and \( D^* \) is whether they satisfy the following theorem, called here the Theorem of the Alternative for \( D \) and \( D^* \) over \( R \) (abbreviated to TOA(\( R; D, D^* \))).

3.1.3 TOA(\( R; D, D^* \)): For any \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \), exactly one of the following holds:

\[(P1) \exists \ x \in (D^n)^* \text{ such that } A x = b, \]
\[(P2) \exists \ y \in \mathbb{R}^m \text{ such that } y A \in D^n, \text{ but } y b \not\in D. \]

3.1.4 Proposition: If TOA(\( R; D, D^* \)) is satisfied then \( T \) and \( \tilde{T} \) as defined in 3.1.1 and 3.1.2, respectively, are dual tag systems.

Proof: Given \( A \in \mathbb{R}^{m \times n} \) and \( k, \ 1 \leq k \leq m \), define \( (A)_k \) to be the matrix obtained from \( A \) by deleting column \( A_k \). Then, by letting \( (A)_k \) play the role of matrix \( A \) in 3.1.3 and \( A_k \) the role of \( b \), it is easily checked that the assumption of this proposition is equivalent to the requirement that \( T \) and \( \tilde{T} \) satisfy the painting condition of Theorem 2.4.4. \( \square \)
Note that an equivalent approach for proving that these systems are duals is via the blocking system definitions in 2.8. Given matrix \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \), we can define a clutter \( C^1 \) to be the set of supports of vectors \( x \in (D^m)^* \) which satisfy \( Ax = b \) that have minimal support with respect to this property, and another clutter \( C^2 \), to be the set of non-D supports of vectors in the rowspace of \( A \), \( w = yA \), which satisfy \( yb \notin D \), where the non-D supports are minimal with respect to this property. Then, it is easy to verify that the proof, either using Theorem 3.1.3 and Proposition 3.1.4 or through the independent, direct methods discussed in the subsequent sections, that \( T \) and \( \bar{T} \) are dual tag systems is equivalent to the proof that \( C^1 \) and \( C^2 \) are blocking clutters (making the appropriate substitutions for \( A \) and \( b \) as in the proof of 3.1.4). Hence, when we later apply the tag system painting theorem to specific examples of \( T \) and \( \bar{T} \) to obtain a particular Theorem of the Alternative, this is equivalent to applying the blocking system painting theorem (2.8.1) to the corresponding \( C^1 \) and \( C^2 \) to obtain the same result.

Given a tag system \( T = (E, C, \tau) \), if there exist a commutative ring \( R \) and a \( D \subseteq R \) such that \( T \) is related to \( D^* \) as described in 3.1.1 for some \( A \in \mathbb{R}^{m \times n} \), i.e., if the circuits of \( T \) precisely correspond to the minimal \( D^* \)-dependent column sets of \( A \) and their tags specify those elements for which the dependence is minimal, then we say \( T \) is \textit{D*-representable} over \( R \). When \( R = \mathbb{Q} \) and \( D = \{0\} \), then \( D^* = \mathbb{Q} \) and the \( D^* \)-representable tag systems over \( \mathbb{Q} \) are exactly the matroids that are representable (in the sense of 1.2.5) over \( \mathbb{Q} \).
For $D = D^* = Q_+$ (the nonnegative rational numbers), we will obtain the tag systems discussed in Section three of this chapter and for $D = D^* = Z$, we obtain the integrally representable tag systems of Section two.

In the remainder of this section, we present some results that are related to the notion of representability.

3.1.5 Proposition: Suppose $R$ is a commutative ring and $D \subseteq R$ is closed under the multiplication of $R$. If $T = (E, C, \tau)$ is $D^*$-representable over $R$, then $T$ satisfies (TCE).

Proof: Suppose $T$ is $D^*$-representable over ring $R$. Then, there exists $A \in R^{m \times n}$ such that $C \subseteq C$ with $k \in \tau(C)$ if and only if there exist coefficients $x_j \in D^*$ with $A_k = \sum_{j \in C \setminus \{k\}} A_j x_j$ and in addition, $A_k$ cannot be written as a $D^*$-combination of the columns indexed by any proper subset of $C \setminus \{k\}$. Let $C_1, C_2 \subseteq C$ with $i \in \tau(C_1) \cap C_2$ and $j \in \tau(C_2) \setminus C_1$. Then, there exist $x_k \in D^*$ such that $A_i = \sum_{k \in C_1 \setminus \{i\}} A_k x_k$ and $y_\ell \in D^*$ such that $A_j = \sum_{\ell \in C_2 \setminus \{j\}} A_\ell y_\ell$.

Note that $y_1$ appears nontrivially in this representation of $A_j$ since $i \in C_2$. Then, through substitution,

$$A_j = \sum_{\ell \in (C_2 \setminus \{j\}) \setminus C_1} A_\ell y_\ell + \sum_{\ell \in (C_2 \cap C_1) \setminus \{i\}} A_\ell (y_\ell + x_\ell y_1) + \sum_{\ell \in C_1 \setminus C_2} A_\ell x_\ell y_1.$$

Hence, letting

$$w_\ell = \begin{cases} y_\ell & \text{if } \ell \in (C_2 \setminus \{j\}) \setminus C_1 \\ y_\ell + x_\ell y_i & \text{if } \ell \in (C_1 \cap C_2) \setminus \{i\} \\ x_\ell y_1 & \text{if } \ell \in C_1 \setminus C_2 \end{cases},$$

we have

$$A_j = \sum_{\ell \in (C_2 \setminus \{j\}) \setminus C_1} w_\ell + \sum_{\ell \in (C_1 \cap C_2) \setminus \{i\}} w_\ell (y_\ell + x_\ell y_1) + \sum_{\ell \in C_1 \setminus C_2} w_\ell x_\ell y_1.$$
we obtain a D*-representation of \( A_j \) using only columns indexed by \((C_1 \cup C_2)\setminus i\). Taking a minimal such representation provides the required \( C_3 \in \mathcal{C} \) with \( C_3 \subseteq (C_1 \cup C_2) \setminus i \) and \( j \in \tau(C_3) \). \( \Box \)

3.1.6 Example: The circuits and cocircuits of the following tag system both fail to satisfy (TCE) and hence neither \( T \) nor \( T^* \) is representable. Let \( E = \{1,2,3,4\} \) and \( T = (E, \mathcal{C}, \tau) \) be defined by \( C_1 = \{1,2,3,4\}, \tau(C_1) = \{1\}; C_2 = \{2,3\}, \tau(C_2) = \{2,3\}; C_3 = \{1,4\}, \tau(C_3) = \{4\}; \) and so \( C_1^* = \{1,2\}, \tau^*(C_1^*) = \{1\}; C_2^* = \{1,3\}, \tau^*(C_2^*) = \{1\}; C_3^* = \{1,4\}, \tau^*(C_3^*) = \{1,4\}; C_4^* = \{2,3\}, \tau^*(C_4^*) = \{2,3\}. \)

Note that \( 2 \in \tau(C_2) \cap C_1 \) and \( 1 \in \tau(C_1) \setminus C_2 \) but there is no \( C \in \mathcal{C} \) with \( C \subseteq (C_1 \cup C_2) \setminus \{2\} \) and \( 1 \in \tau(C) \). Also note that \( 1 \in \tau^*(C_1^*) \cap C_3^* \) and \( 4 \in \tau^*(C_3^*) \setminus C_1^* \) but there is no \( C^* \in \mathcal{C}^* \) with \( C^* \subseteq (C_1^* \cup C_3^*) \setminus \{1\} \) and \( 4 \in \tau^*(C^*) \). \( \Box \)

3.1.7 Corollary: If \( T \) and \( T^* \) are both representable, then they are dual matroids.

Proof: Proposition 3.1.5 implies that \( T \) and \( T^* \) both satisfy (TCE), thus the result follows from Proposition 2.6.12. \( \Box \)

A natural question concerns whether the converse of 3.1.5 is valid; that is, is every tag system which satisfies tagged circuit exchange D*-representable over some commutative ring \( R \)? This remains an open problem.
3.2 Integrally Representable Systems

In this section we discuss the tag systems that arise from consideration of \( R = Q \) and \( D = D^* = \mathbb{Z} \), as introduced in Section 3.1. Thus, we will study the relation of integral dependence among the columns of a rational matrix. First we will investigate an elimination scheme for systems of linear congruences. The reader is referred to [1] for background material on modulo arithmetic.

Given a system of linear congruences in one variable,

\[
x \equiv c_1 \pmod{m_1} \\
\vdots \\
x \equiv c_r \pmod{m_r},
\]

with all data integral, it is known, see for example Theorem 3.16 on page 62 of [29], that this system has a solution if and only if each pair of congruences

\[
x \equiv c_i \pmod{m_i} \\
x \equiv c_j \pmod{m_j}
\]

has a solution. The proof of Theorem 5.11 on pages 80-83 of [19] illustrates inductively how to use the pairwise solution to construct a global solution. Further, it is easy to show that the integrality of the data is not needed for this result to hold.
Recall that all solutions (not restricting attention to integral solutions) of the linear congruence \( ax \equiv b \pmod{m} \) are of the form \( x_0 + \left( \frac{m}{a} \right) t \) where \( x_0 \) is a particular solution and \( t \) is an integer. Hence, given a system

\[
a_1 x \equiv b_1 \pmod{m_1} \\
\vdots \\
a_r x \equiv b_r \pmod{m_r}
\]

with particular solution \( y_i \) for relation \( i \), one can form the equivalent system

\[
x \equiv y_1 \pmod{m_1/a_1} \\
\vdots \\
x \equiv y_r \pmod{m_r/a_r}.
\]

Thus, any system of linear congruences in one variable is consistent if and only if it is pairwise consistent.

Now suppose the following system is given in all integral data:

\[
(I) \quad a_{11} x_1 + \ldots + a_{1n} x_n \equiv b_1 \pmod{m_1} \\
\vdots \\
a_{r1} x_1 + \ldots + a_{rn} x_n \equiv b_r \pmod{m_r}
\]
Then form the "reduced" system:

\[(II) \quad (a) \quad a_{i1}x_1 + \ldots + a_{in}x_n \equiv b_i \pmod{m_i} \quad \text{for all } i \quad \text{with} \quad a_{i1} = 0 \]

\[(b) \quad a_{j1}(a_{j2}x_2 + \ldots + a_{jn}x_n) - a_{j1}(a_{i2}x_2 + \ldots + a_{in}x_n) \equiv a_{i1}b_j - a_{j1}b_i \pmod{g_{ij}} \quad \text{for all } i = 1, \ldots, r \]

and \( j > i \) such that \( a_{i1} \neq 0 \neq a_{j1} \),

where \( g_{ij} = \gcd(a_{i1}m_j, a_{j1}m_i) \) for all \( i = 1, \ldots, r \) and \( j > i \).

3.2.1 Theorem: If \( x = (x_1, \ldots, x_n) \) is a solution to system (I), then \( \overline{x} = (x_2, \ldots, x_n) \) solves (II) and conversely, if \( \overline{x} = (x_2, \ldots, x_n) \) is a solution to system (II) then there exists an \( x_1 \) such that \( x = (x_1, \overline{x}) \) solves (I).

Proof: First suppose \( x = (x_1, \ldots, x_n) \) solves system (I). Then clearly \( \overline{x} = (x_2, \ldots, x_n) \) solves each relation of \( \text{II}(a) \). Further, \( a_{i1}x_1 + \ldots + a_{in}x_n - b_i = m_i k_i \) for some \( k_i \in \mathbb{Z} \) and \( a_{j1}x_1 + \ldots + a_{jn}x_n - b_j = m_j k_j \) for some \( k_j \in \mathbb{Z} \) for each \( i \) and \( j \) with \( a_{i1} \neq 0 \neq a_{j1} \). Thus,

\[
\begin{align*}
& a_{i1}(a_{j1}x_1 + \ldots + a_{jn}x_n) - a_{j1}(a_{i1}x_1 + \ldots + a_{in}x_n) \\
& \quad = a_{i1}b_j + a_{i1}m_j k_j - a_{j1}b_i - a_{j1}m_i k_i \\
& \quad = (a_{i1}b_j - a_{j1}b_i) + a_{i1}m_j k_j - a_{j1}m_i k_i \\
& \quad = a_{i1}b_j - a_{j1}b_i + g_{ij} k' \quad \text{for some } k' \in \mathbb{Z},
\end{align*}
\]
where this last equation follows since the greatest common divisor of $a_{ji}m_i$ and $a_{ij}m_j$ is $g_{ij}$. Hence $\bar{x}$ solves each relation of II(b). Thus, $\bar{x}$ is a solution of (II).

Now suppose $\bar{x}$ solves system (II). We will show that there exists an $x_1^{(i,j)}$ which together with $\bar{x} = (x_2, \ldots, x_n)$ satisfies equations $i$ and $j$ of system (I) simultaneously. Hence we will have demonstrated that system (I) viewed as a system of linear congruences in one variable $(x_1)$ is pairwise consistent. Thus, by the comments at the beginning of this section it will follow that there exists $x_1$ such that $x = (x_1, \bar{x})$ satisfies system (I).

Notice that for a given $i$ and $j$, if $a_{il} = a_{jl} = 0$ then clearly $x_1^{(i,j)} = 0$ will suffice and further if $a_{il} = 0$ but $a_{jl} \neq 0$ then it is also easy to find an $x_1^{(i,j)}$ which will satisfy these equations simultaneously. Thus, we need only consider $i$ and $j$ with $a_{il} \neq 0 \neq a_{jl}$. Then we wish to find $x_1^{(i,j)}$ such that

$$a_{il}x_1^{(i,j)} = b_i - a_{i2}x_2 - \cdots - a_{in}x_n + m_ik_i$$

and

$$a_{jl}x_1^{(i,j)} = b_j - a_{j2}x_2 - \cdots - a_{jn}x_n + m_jk_j$$

for some integers $k_i$ and $k_j$. This is equivalent to finding integral $k_i$ and $k_j$ such that

$$a_{jl}(b_i - a_{i2}x_2 - \cdots - a_{in}x_n + m_ik_i)$$

\[= a_{il}(b_j - a_{j2}x_2 - \cdots - a_{jn}x_n + m_jk_j).\]
We know by the relation of II(b) which pairs \( i \) and \( j \) that

\[
a_{i1}(b_j - a_{j2}x_2 - \cdots - a_{jn}x_n)
= a_{j1}(b_i - a_{i2}x_2 - \cdots - a_{in}x_n) + g_{ij} \ell_{ij}
\]
for some \( \ell_{ij} \in \mathbb{Z} \).

Hence we need to show that

\[
a_{j1}(b_i - a_{i2}x_2 - \cdots - a_{in}x_n + m_ik_1)
= a_{j1}(b_i - a_{i2}x_2 - \cdots - a_{in}x_n) + g_{ij} \ell_{ij} + a_{ij}m_jk_j.
\]

Or, equivalently, we wish to determine integral \( k_i \) and \( k_j \) such that

\[
a_{j1}m_ik_i - a_{ij}m_jk_j = g_{ij} \ell_{ij}.
\]

This is clearly possible by the definition of \( g_{ij} \). Thus the proof is complete. \( \square \)

3.2.2 Given rational \( m \times n \) matrix \( A \), let \( E = \{1,2,\ldots,n\} \) and define \( T = (E,C,\tau) \) by \( C \in \mathcal{C} \) with \( k \in \tau(C) \) if and only if the \( k \)th column of \( A \), denoted \( A_k \), can be represented as an integral combination of the columns indexed by \( C \setminus \{k\} \), but not as an integral combination of the columns indexed by any proper subset of \( C \setminus \{k\} \). Note that this condition is equivalent to the stipulation
that there exist an integral vector whose support is $C$, $x_k = \pm 1$, $Ax = 0$ and with $C$ minimal with respect to these requirements.

See Example 2.2.2 for an illustration of such a system.

3.2.3 Again, given a rational $m \times n$ matrix $A$, we define a tag system on $E = \{1, 2, \ldots, n\}$. $\hat{\tau} = (E, \hat{C}, \hat{\tau})$ is given by $\hat{C} \in \hat{C}$ with $k \in \hat{\tau}(\hat{C})$ if and only if there exists a vector $w$ in the rowspace of $A$ with nonintegral support $\text{NZ}(w) = \{j : w_j \not\in \mathbb{Z}\} = \hat{C}$ and in addition no proper subset of $\hat{C}$ containing $k$ is the nonintegral support of a vector in the rowspace of $A$. It is easy to verify that this is a tag system. Note that Corollary 3.1.7 implies that this system will generally not be representable.

3.2.4 Let

$$A = \begin{bmatrix}
1 & 2 & 3 & 6 \\
1 & 0 & 1 & 2 \\
1 & 1 & 2 & 4
\end{bmatrix}.$$ 

Then the following vectors define $\hat{\tau} = (E, \hat{C}, \hat{\tau})$ on $E = \{1, 2, 3, 4\}$:

<table>
<thead>
<tr>
<th>$y$</th>
<th>$w = yA$</th>
<th>$\hat{C}$</th>
<th>$\hat{\tau}(\hat{C})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, \frac{1}{2}, 0)$</td>
<td>$(\frac{1}{2}, 0, \frac{1}{2}, 1)$</td>
<td>$(1, 3)$</td>
<td>$(1, 3)$</td>
</tr>
<tr>
<td>$(0, 0, \frac{1}{2})$</td>
<td>$(\frac{1}{2}, \frac{1}{2}, 1, 2)$</td>
<td>$(1, 2)$</td>
<td>$(1, 2)$</td>
</tr>
<tr>
<td>$(0, \frac{1}{2}, \frac{1}{2})$</td>
<td>$(1, \frac{1}{2}, \frac{3}{2}, 2)$</td>
<td>$(2, 3)$</td>
<td>$(2, 3)$</td>
</tr>
<tr>
<td>$(\frac{1}{3}, 0, \frac{1}{3})$</td>
<td>$(\frac{2}{3}, \frac{1}{3}, \frac{5}{3}, \frac{10}{3})$</td>
<td>$(1, 3, 4)$</td>
<td>$(4)$</td>
</tr>
<tr>
<td>$(0, \frac{2}{3}, \frac{1}{3})$</td>
<td>$(\frac{2}{3}, \frac{4}{3}, \frac{8}{3})$</td>
<td>$(2, 3, 4)$</td>
<td>$(4)$</td>
</tr>
</tbody>
</table>
3.2.5 Lemma: Suppose $A$ is a rational matrix and $i \in \text{NZ}(w)$ for some $w$ in the rowspace of $A$, $R(A)$. Then there exists a vector $v \in R(A)$ such that $i \in \text{NZ}(v) \subseteq \text{NZ}(w)$ and $\text{NZ}(v)$ is minimal with respect to $i \in \text{NZ}(v)$ and $v \in R(A)$.

Proof: Clearly since $w \in R(A)$ and $i \in \text{NZ}(w)$, there exists a vector $v \in R(A)$ for which $\text{NZ}(v)$ is minimal with respect to $v \in R(A)$, $i \in \text{NZ}(v)$ and $\text{NZ}(v) \subseteq \text{NZ}(w)$. Thus, $\text{NZ}(v)$ is also minimal simply with respect to $i \in \text{NZ}(v)$ and $v \in R(A)$, since otherwise there would exist a vector $u \in R(A)$ with $i \in \text{NZ}(u)$ and $\text{NZ}(u) \subseteq \text{NZ}(v)$ contradicting the choice of $v$. \qed

Now we show that the tag systems $T$ and $\bar{T}$ defined above in 3.2.3 and 3.2.4, respectively, are dual to one another.

3.2.6 Theorem: Given rational $m \times n$ matrix $A$, define $T$ and $\bar{T}$ as described in 3.2.3 and 3.2.4, respectively. Then $T = \bar{T}^*$ (and thus $T^* = \bar{T}$).

Proof: We will prove that $T$ and $\bar{T}$ are dual tag systems by illustrating that they satisfy the painting theorem (Theorem 2.4.4, see Proposition 2.4.5) together. Without loss of generality, assume $A$ is an integral matrix. Consider an arbitrary painting of $E$ into $R \cup B \cup W$ with $|R| = 1$.

First, suppose both (P1) and (P2) of the painting theorem are satisfied. Then there exists $C \in C$ with $C \subseteq R \cup B$ and $R \subseteq \tau(C)$,
or equivalently, an \( x \in \mathbb{Z}^n \) with \( x_{R'} = \pm 1 \), the support of \( x \) given by \( C \subseteq R \cup B \) and \( Ax = 0 \). Also, there exists \( \tilde{C} \subseteq \tilde{C} \) with \( \tilde{C} \subseteq R \cup W \) and \( R \subseteq \tilde{\tau}(\tilde{C}) \), that is, a \( y \in \mathbb{Q}^m \) with \( yA = w \), and \( R \subseteq \text{NZ}(w) = \tilde{C} \subseteq R \cup W \). Hence,

\[
0 = yAx = w \cdot x = \sum_{j \in B} w_j x_j + \sum_{j \in W} w_j x_j + w_{R'} = \sum_{j \in B} w_j x_j \pm w_{R'}.
\]

But, for all \( j \in B \), \( x_j \in \mathbb{Z} \) and \( w_j \in \mathbb{Z} \), and \( w_{R'} \notin \mathbb{Z} \). Thus, this is clearly impossible and so both (P1) and (P2) cannot hold simultaneously.

Now suppose (P1) fails, where \( |B| = k \ (k \geq 0) \), and call the Blue submatrix \( B \) and the Red column \( r \). Without loss of generality we assume \( B \) has full row rank.

First suppose that \( B \) has full column rank. Then, we can perform row operations (Gaussian Elimination) on \( B \) using only integral multipliers to derive \( B' \) where \( B' \) is in permuted diagonal form, that is, each row and column of \( B \) contains exactly one nonzero entry. Let \( P \) be the product of the elementary matrices used to derive \( B' \) and let \( Pr = r' \). Since (P1) fails, there exists an \( i, 1 \leq i \leq m \), such that the nonzero entry in row \( i \), say \( b'_{ij} \), does not divide \( r'_i \). Then, letting \( P_i \) denote the \( i \)th row of \( P \) and \( y = P_i/b'_{ij} \) yields \( yB \in \mathbb{Z}^k \) and \( yr \notin \mathbb{Z} \). Let \( \tilde{C}' \) correspond to the nonintegral support of \( yA \). Then Lemma 3.2.5 shows us that there exists a \( \tilde{C} \subseteq \tilde{C}' \) with \( R \subseteq \tilde{\tau}(\tilde{C}) \) and \( \tilde{C} \subseteq \tilde{C}' \subseteq R \cup W \). Hence (P2) is satisfied.
Now suppose $B$ does not have full column rank. If $p \in \mathbb{Q}^m$, $M \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$ then $Mx \equiv b \pmod{p}$ will denote $M^i x \equiv b_i \pmod{p_i}$ for $i = 1, \ldots, m$. Now form system

\[
\begin{bmatrix}
B \\
I
\end{bmatrix}
\begin{bmatrix}
x
\end{bmatrix}
\equiv
\begin{bmatrix}
r \\
0
\end{bmatrix} \pmod{1}.
\]

Then, we can perform integral row operations (premultiplying by matrix $P_1$),

\[
P_1 = \begin{bmatrix}
\bar{P}_1 & 0 \\
0 & 1
\end{bmatrix},
\]

to derive equivalent system

\[
\begin{bmatrix}
B' \\
I
\end{bmatrix}
\begin{bmatrix}
x
\end{bmatrix}
\equiv
\begin{bmatrix}
r' \\
0
\end{bmatrix} \pmod{1};
\]

where $B'$ has an $m \times m$ submatrix in permuted diagonal form. Without loss of generality, assume this matrix is diagonal and is in the first $m$ columns of $B'$, that is, $B' = [D; C]$, where $D$ is an integral diagonal matrix $= \begin{bmatrix}
b_{11}' & & & \\
& \ddots & & \\
& & b_{m}' & \\
& & & b_{mm}'
\end{bmatrix}$ and $C$ is an integral $m \times (k-m)$ matrix.

Now note that requiring $b_{i1}' x_1 = r_i' - b_{i,m+1}' x_{m+1} - \ldots - b_{ik}' x_k$ with $x_1$ integral is equivalent to requiring

\[
b_{i,m+1}' x_{m+1} + \ldots + b_{ik}' x_k \equiv r_i' \pmod{b_{ii}'}.\]

Hence we can substitute the $i^{th}$ equation of $B'x = r'$ into the $i^{th}$ relation of $Ix \equiv 0 \pmod{1}$ to obtain an equivalent system. Explicitly, let
\[ P_2 = \begin{bmatrix} I & 0 & 0 \\ I & -D & 0 \\ 0 & 0 & I \end{bmatrix} , \]

then

\[ P_2 \begin{bmatrix} B' \\ I \end{bmatrix} = P_2 \begin{bmatrix} D & C \\ I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} D & C \\ 0 & C \\ 0 & I \end{bmatrix} \]

and

\[ P_2 \begin{bmatrix} r' \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} r' \\ 0 \\ r' \end{bmatrix}. \]

Hence the equivalent system is

\[
\begin{bmatrix}
D & -C \\
0 & C \\
0 & 1
\end{bmatrix} \times \begin{bmatrix}
r' \\
0
\end{bmatrix} = \begin{bmatrix}
r' \\
r'
\end{bmatrix} \pmod{\Delta},
\]

where \( \Delta_i = b'_{ii} \) for \( 1 \leq i \leq m \) and 1 otherwise. Now we can apply Theorem 3.2.1 to

\[
\begin{bmatrix}
0 & C \\
0 & I
\end{bmatrix} \times \equiv \begin{bmatrix}
r'
\end{bmatrix} \pmod{\Delta}
\]
to derive an equivalent system \( Ox \equiv s' (\text{mod } \Delta') \). Note that this system will generally have a different number of constraints than the original system. Suppose

\[
Q \begin{bmatrix} 0 & C \\ 0 & I \end{bmatrix} = 0 \quad \text{and} \quad Q \begin{bmatrix} r' \\ 0 \end{bmatrix} = s'.
\]

Then notice that

\[
Q' = \begin{bmatrix} I & 0 \\ 0 & Q \end{bmatrix}
\]

will transform \( \begin{bmatrix} B' \\ 0 & C \\ 0 & I \end{bmatrix} \) to \( \begin{bmatrix} B' \\ 0 \end{bmatrix} \) and

\[
Q' \begin{bmatrix} r' \\ r' \\ 0 \end{bmatrix} = \begin{bmatrix} r' \\ s' \end{bmatrix}.
\]

Since (P1) fails, there exists an \( i \) such that \( s_i' \) is not divisible by \( \Delta_i' \). Suppose \( Q \) is the product of matrices \( Q = Q_k \ldots Q_m+1 \), where \( Q_j \) "eliminates" variable \( x_j \) from the system as described in Theorem 3.2.1. Define \( Q_m \) by the relation

\[
P_2 = \begin{bmatrix} I & 0 & 0 \\ I & -D & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ -D & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]
Then let $Q(\varepsilon) = Q_\varepsilon Q_{\varepsilon-1} \ldots Q_m$ for $m \leq \varepsilon \leq k$, and suppose

$Q(\varepsilon) \begin{bmatrix} B' & I \\ I & 0 \end{bmatrix} = \Lambda(\varepsilon)$, with $\Lambda(\varepsilon) = [\lambda_{ij}(\varepsilon)]$ and $Q(\varepsilon) \begin{bmatrix} r' \\ 0 \end{bmatrix} = s'(\varepsilon)$.

Let the $i^{th}$ relation of $\Lambda(\varepsilon)x \equiv s'(\varepsilon)$ be taken modulo $\Delta_i(\varepsilon)$. Then we shall require the following result.

3.2.7 Claim: Let $Q^h(\varepsilon)$ denote the $h^{th}$ row of $Q(\varepsilon)$. Then for $m \leq \varepsilon \leq k$ and $p \geq m+1$, $\Delta_h(\varepsilon)$ divides the $p^{th}$ entry in $Q^h(\varepsilon)$.

Proof: We shall prove this by induction on $\varepsilon$. Recall that

$Q_m = \begin{bmatrix} I & -D & 0 \\ 0 & 0 & I \end{bmatrix}$

and $\Delta_h(m) = b_{hh}^i$ for $1 \leq h \leq m$ and $\Delta_h(m) = 1$ for $m+1 \leq h \leq k$.

Clearly the result holds here (recall that

$D = \begin{bmatrix} b_{11}' & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & b_{mm}' \end{bmatrix}$).

Assume now that the result holds for $m, \ldots, \varepsilon-1$ and consider $\varepsilon$, where $m+1 \leq \varepsilon \leq k$. Clearly if the $h^{th}$ row of $\Lambda(\varepsilon)$ is of type II(a) in Theorem 3.2.1 (i.e., is also present in $\Lambda(\varepsilon-1)$), then the result is true. Thus suppose that $\Lambda^h(\varepsilon)$ is of type II(b) in Theorem 3.2.1.
Then there exist \( i \) and \( j \) such that

\[
\Lambda^h(\ell) = -\lambda_i,\ell(\ell-1)\Lambda^j(\ell-1) + \lambda_j,\ell(\ell-1)\Lambda^i(\ell-1),
\]

and hence

\[
Q^h_\ell = [0, \ldots, 0, -\lambda_i,\ell(\ell-1), 0, \ldots, 0, \lambda_j,\ell(\ell-1), 0, \ldots, 0].
\]

Thus,

\[
Q^h(\ell) = -\lambda_i,\ell(\ell-1)Q^j(\ell-1) + \lambda_j,\ell(\ell-1)Q^i(\ell-1)
\]

and

\[
\Delta_h(\ell) = \gcd(\lambda_i,\ell(\ell-1)\Delta_j(\ell-1), \lambda_j,\ell(\ell-1)\Delta_i(\ell-1)).
\]

Let \( p \geq m+1 \); then since \( \Delta_j(\ell-1) \) divides the \( p^{th} \) entry of \( Q^j(\ell-1) \) by the induction hypothesis and \( \lambda_i,\ell(\ell-1) \) clearly divides itself, we obtain that \( \Delta_h(\ell) \) divides the \( p^{th} \) entry of \( (-\lambda_i,\ell(\ell-1)Q^j(\ell-1)). \)

Similarly, since \( \Delta_i(\ell-1) \) divides the \( p^{th} \) entry of \( Q^i(\ell-1) \) and \( \lambda_j,\ell(\ell-1) \) divides itself, we see that \( \Delta_h(\ell) \) divides the \( p^{th} \) entry of \( Q^h(\ell) \). \( \square \)

Now we complete the proof of Theorem 3.2.6. Recall that at the end of the procedure we have an \( s'_i \) that is not divisible by \( \Delta_i \).

Then note that \( s'_i \) is the \((i+m)^{th}\) component of \( \begin{bmatrix} r'' \\ s' \end{bmatrix} \). Let
\[ y = \frac{(Q'P_2P_1)^{i+m}}{\Delta_i} = \frac{(Q'P_2)^{i+m}P_1}{\Delta_i}, \]

where we recall that \( Q' = \begin{bmatrix} I & 0 \\ 0 & Q \end{bmatrix} \). Then \( y \begin{bmatrix} B \\ I \end{bmatrix} = 0 \) and \( y \begin{bmatrix} r \\ 0 \end{bmatrix} = s_i'. \)

Now note that Claim 3.2.7 implies that

\[ \frac{(Q'P_2)^{i+m}}{\Delta_i} = [w_1, w_2] \]

where \( w_1 \in \mathbb{Q}^m \) and \( w_2 \) is integral. Hence

\[ y = [w_1, w_2]P_1 = [w_1, w_2] \begin{bmatrix} \bar{P}_1 & 0 \\ 0 & I \end{bmatrix} = [w_1 \bar{P}_1, w_2]. \]

Thus since \( y \begin{bmatrix} B \\ I \end{bmatrix} = 0 \), \( (w_1 \bar{P}_1)B = -w_2I \) is integral but \( (w_1 \bar{P}_1)r = y \begin{bmatrix} r \\ 0 \end{bmatrix} = s_i' \) is not integral. Letting \( \tilde{C}' \) correspond to the nonintegral support of \( (w_1 \bar{P}_1)A \) we see that the Theorem is proved as before. \( \square \)

We now interpret the painting theorem for this dual pair of tag systems. Given rational \( m \times n \) matrix \( A \) and rational \( m \)-vector \( b \), form the extended matrix \([A|b]\). Paint the columns of \( A \) Blue and the \( b \) column Red.
Then, alternative (P1) implies that there is a $C \in \mathcal{C}$ with $C \subseteq R \cup B$ and $R \subseteq \tau(C)$. Hence, there exists an integral $x$ with $x_{n+1} = \pm 1$ and $[A;b]x = 0$. Thus, there is an $x \in \mathbb{Z}^n$ such that $Ax = b$.

Alternative (P2) implies the existence of a $\tilde{C} \in \tilde{\mathcal{C}}$ with $\tilde{C} \subseteq R \cup W$ and $R \subseteq \tilde{\tau}(\tilde{C})$. Thus, there is a $y \in \mathbb{Q}^m$ such that $yA \in \mathbb{Z}$, but $yb \notin \mathbb{Z}$.

The reader will recognize this as Theorem 2.1.3, the integral form of Farkas' Lemma.

3.3 Nonnegative Rational Dependence

In this section we discuss the tag systems that arise from consideration of $R = \mathbb{Q}$ and $D = D^* = \mathbb{Q}_+$ as defined in Section one of this chapter. It is important to note the direct analogy between the development here and that of the previous section.

3.3.1 Given a rational $m \times n$ matrix $A$, let $E = \{1,2,\ldots,n\}$ and define $T = (E,C,\tau)$ by $C \in \mathcal{C}$ with $k \in \tau(C)$ if and only if column $A_k$ can be generated as a nonnegative rational combination of the columns indexed by $C \setminus \{k\}$, but not as a nonnegative rational combination of the columns indexed by any proper subset of $C \setminus \{k\}$.

Note that this condition is equivalent to the stipulation that there exist a rational vector $x$ whose support is $C$, $x_k = -1$ and $x_j \geq 0$ for $j \neq k$ with $Ax = 0$ and $C$ minimal with respect to these requirements. It is easy to verify that $T$ is a tag system.
3.3.2 Again, given a rational $m \times n$ matrix $A$, we define a tag system on $E = \{1, 2, \ldots, n\}$. $\bar{T} = (E, \bar{C}, \bar{\tau})$ is given by $\bar{C} \in \bar{C}$ with $k \in \bar{\tau}(\bar{C})$ if and only if there exists a vector $w$ in the rowspace of $A$ with negative support $NQ_+(w) = \{j : w_j < 0\} = \bar{C}$ and in addition, no proper subset of $\bar{C}$ containing $k$ is the negative support of a vector in the rowspace of $A$. It is easy to verify $\bar{T}$ is a tag system. As mentioned in Section two, Corollary 3.1.7 implies that this system will generally not be representable.

The following lemma is easily proven as in Section two.

3.3.3 Lemma: Suppose $A$ is a rational matrix and $i \in NQ_+(w)$ for some $w \in R(A)$. Then, there exists a vector $v \in R(A)$ such that $i \in NQ_+(v) \subseteq NQ_+(w)$ and $NQ_+(v)$ is minimal with respect to $i \in NQ_+(v)$ and $v \in R(A)$.

3.3.4 Theorem: Given rational $m \times n$ matrix $A$, define $T$ and $\bar{T}$ as in 3.3.1 and 3.3.2, respectively. Then $T = \bar{T}^*$.

Proof: As in Section 3.2, we prove that $T$ and $\bar{T}$ are dual tag systems by illustrating that they satisfy the painting condition of Theorem 2.4.4. Consider an arbitrary painting of $E$ into $R \cup B \cup W$ with $|R| = 1$.

Suppose both alternatives (P1) and (P2) of the painting theorem are satisfied. Then there exist both a $C \in \bar{C}$ with $C \subseteq R \cup B$ and $R \subseteq \bar{\tau}(C)$ and a $\bar{C} \in \bar{C}$ with $\bar{C} \subseteq R \cup W$ and $R \subseteq \bar{\tau}(\bar{C})$. Equivalently,
there exists a rational vector \( \mathbf{x} \) with \( x_R = -1 \) and \( x_j \geq 0 \) for \( j \notin R \), with the support of \( \mathbf{x} \) given by \( C \subseteq R \cup B \) and \( A\mathbf{x} = 0 \), and there exists a rational vector \( \mathbf{y} \) with \( yA = w \) and \( R \subseteq N(w) = \tilde{C} \subseteq R \cup W \). Hence,

\[
0 = yAx = w \cdot \mathbf{x} = \sum_{j \in B} w_j x_j + \sum_{j \in W} w_j x_j + w_R x_R = \sum_{j \in B} w_j x_j - w_R.
\]

But, for all \( j \in B \), \( w_j \geq 0 \) and \( x_j \geq 0 \) and \( w_R < 0 \) and hence the right hand side of this equation is strictly greater than zero. Thus, both (P1) and (P2) cannot hold.

Now suppose (P1) fails. Let \( B \) denote the Blue submatrix and \( r \) the Red column. Then without loss of generality let \( B \in \mathbb{Q}^{m \times k} \), \( k \geq 0 \), and let \( B \) have full row rank,

First suppose \( B \) also has full column rank. Then we can perform Gaussian Elimination to derive \( B'\mathbf{x} = r' \) where \( B' \) is in permuted diagonal form. Suppose \( PB = B' \) and \( Pr = r' \). Then since (P1) fails, there exists an \( i \), \( 1 \leq i \leq m \), such that the nonzero entry in row \( i \), say \( b_{ij}^i \), and \( r_i \) are of opposite sign. Thus letting \( y = p_i^i \) or \(-p_i^i \) yields \( yB \geq 0 \) and \( yr < 0 \). Let \( \tilde{C}' \) correspond to the negative support of \( yA \). Then Lemma 3.3.3 shows us that there exists a \( \tilde{C} \subseteq \tilde{C} \) with \( R \subseteq \tilde{T}(C) \) and \( \tilde{C} \subseteq \tilde{C}' \subseteq R \cup W \) and so (P2) is satisfied.

Now suppose \( B \) does not have full column rank. Then form the system

\[
\begin{pmatrix}
B \\
I
\end{pmatrix} \begin{pmatrix}
\mathbf{x} \\
r
\end{pmatrix} = \begin{pmatrix}
\geq \\
0
\end{pmatrix}.
\]

Then we can perform row operations (Gaussian
Elimination) by premultiplying by \( P_1 = \begin{bmatrix} \bar{P}_1 & 0 \\ 0 & I \end{bmatrix} \) to obtain equivalent system \[
\begin{bmatrix} B' \\ I \end{bmatrix} x \begin{bmatrix} r' \\ 0 \end{bmatrix}, \quad \text{where } B' \text{ has an } m \times m \text{ permuted identity submatrix. Without loss of generality assume this is a diagonal submatrix in the first } m \text{ columns, that is, } B' = [I_i^t C]. \text{ Then we can substitute the } i^{th} \text{ equation of } B'x = r' \text{ into the } i^{th} \text{ relation of } I_x \geq 0 \text{ as in the first phase of Fourier-Motzkin Elimination (see [7]). More explicitly, let}
\]
\[
P_2 = \begin{bmatrix} I & 0 & 0 \\ -I & I & 0 \\ 0 & 0 & I \end{bmatrix};
\]

then
\[
P_2 \begin{bmatrix} I & C \\ I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & C \\ 0 & -C \\ 0 & I \end{bmatrix} \text{ and } P_2 \begin{bmatrix} r' \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} r' \\ -r' \\ 0 \end{bmatrix}.
\]

Hence an equivalent system is \( B'x = r' \), \( \begin{bmatrix} 0 & -C' \\ 0 & I \end{bmatrix} x \geq \begin{bmatrix} -r' \\ 0 \end{bmatrix} \). Now we can perform the "reduction" phase of Fourier-Motzkin Elimination on \( \begin{bmatrix} 0 & -C' \\ 0 & I \end{bmatrix} x \geq \begin{bmatrix} -r' \\ 0 \end{bmatrix} \) to derive an equivalent system \( Ox \geq s' \). Let
\[
Q \begin{bmatrix} 0 & -C' \\ 0 & I \end{bmatrix} = 0 \text{ and } Q \begin{bmatrix} -r' \\ 0 \end{bmatrix} = s',
\]
and denote \( Q' = \begin{bmatrix} I & 0 \\ 0 & Q \end{bmatrix} \). Since (P1) fails, there is an \( i \) such that \( s^i_1 > 0 \). Note that \( s^i_1 \) is the \((i+m)\)th component of \( \begin{bmatrix} r' \\ s' \end{bmatrix} \)

and let \( y = -(Q'P_2p_1)^{i+m} \). Then,

\[
y = -(Q'P_2)^{i+m}p_1 = -(Q'P_2)^{i+m} \begin{bmatrix} \overline{p}_1 & 0 \\ 0 & I \end{bmatrix} = (w_1, w_2) \begin{bmatrix} \overline{p}_1 & 0 \\ 0 & I \end{bmatrix} = (w_1\overline{p}_1, w_2).
\]

Now for all \( p \geq m+1 \), the \( p \)th entry of \((Q'P_2)^{i+m}\) is nonnegative by construction (\( Q \geq 0 \) for Fourier-Motzkin Elimination) and hence \( w_2 \leq 0 \). Now since \( y \begin{bmatrix} B \\ I \end{bmatrix} = 0 \), we see that \( (w_1\overline{p}_1)B = -w_2I \geq 0 \) and \( (w_1\overline{p}_1)r = y \begin{bmatrix} r' \\ 0 \end{bmatrix} = -s^i_1 < 0 \). Thus, if we let \( \tilde{C} \) be the negative support of \((w_1\overline{p}_1)A\), then as before (P2) is satisfied and the proof is complete. \( \square \)

We now interpret the painting theorem for this dual pair of tag systems. Given rational \( m \times n \) matrix and rational \( m \)-vector \( b \), form the extended matrix \([A|b]\) and paint the columns of \( A \) Blue and the \( b \) column Red.

Then, alternative (P1) implies the existence of \( C \subseteq C \) with \( C \subseteq R \cup B \) and \( R \subseteq \tau(C) \). Hence, there exists a rational \( x \) with \( x_{n+1} = -1 \) and \( x_j \geq 0 \) for \( j < n+1 \) and \([A|b]x = 0 \). Thus, there exists \( x \geq 0 \) such that \( Ax = b \).

Alternative (P2) gives a \( \tilde{C} \subseteq \tilde{C} \) with \( \tilde{C} \subseteq R \cup W \) and \( R \subseteq \tilde{\tau}(\tilde{C}) \). Thus, there exists \( y \in Q^{m} \) such that \( yA \geq 0 \), but \( yb < 0 \).

The reader will recognize this as Theorem 2.1.2, the usual form of Farkas' Lemma.
3.4 Min-Algebraic Dependence

In this section we provide results analogous to those presented in Sections two and three, here using Hoffman's "min algebra" (see [21]). This algebra has proven useful in obtaining graph theoretic results and also in extending the notion of linear programming duality theory to new settings. While much is known about the min algebra, no elimination scheme for inductively solving systems of equations has been proposed. As such a scheme has played a crucial role in establishing the link between algebraic and combinatorial structures in the settings of the two previous sections, we now introduce such a procedure for the min algebra. The reader should note that while the full strength of this elimination scheme is not explicitly needed for the results produced in this section on tag systems, the procedure could prove useful in finding new solution techniques for min algebraic systems.

As in [21], the min algebra is defined by replacing ordinary addition by a minimum operator and ordinary multiplication by addition. Hence for two n-vectors $x$ and $y$, the min algebraic inner product $x \cdot y$ is calculated by the rule $x \cdot y = \min_{1 \leq j \leq n} \{x_j + y_j\}$. We now show that we can solve a system $Ax = b$ inductively by reducing the number of variables by one at each iteration.

3.4.1 Given system (1): $Ax = b$ where $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$, define system (2): $\bar{A}[\begin{pmatrix} 0 \\ x \end{pmatrix}] = b$ where $\Delta_1 = \max_{1 \leq i \leq m} \{b_i - a_{i1}\}$ and
\[ \bar{A} = \begin{bmatrix}
    a_{11} + \Delta_1 & \cdots & a_{1n} \\
    \vdots & & \vdots \\
    a_{m1} + \Delta_1 & \cdots & a_{mn}
\end{bmatrix} \quad \text{and} \]

\[ \bar{x} = (x_2, \ldots, x_n). \]

3.4.2 Theorem: Given \( A \in \mathbb{Z}^{m \times n} \) and \( b \in \mathbb{Z}^m \), if \( x \) solves system (1) above then \( \bar{x} \) solves system (2). Further, if \( \bar{x} \) solves system (2) and we define \( x_1 = \Delta_1 \), then \( x = (x_1, \bar{x}) \) solves system (1).

Proof: First suppose \( x \) solves system (1). Then we know that 
\[
\min_{1 \leq j \leq n} \{ a_{1j} + x_j \} = b_1 \quad \text{for all } i.
\]
Thus 
\[
\min_{1 \leq j \leq n} \{ a_{1j} + x_j \}, \min_{j \geq 2} \{ a_{ij} + x_j \} = b_i
\]
for all \( i \), so \( a_{11} + x_1 \geq b_1 \) for all \( i \). Hence \( x_1 \geq \max_{1 \leq i \leq m} (b_i - a_{11}) = \Delta_1 \)
so 
\[
\min_{1 \leq j \leq n} \{ a_{1j} + \Delta_1 \}, \min_{j \geq 2} \{ a_{ij} + x_j \} = b_i \quad \text{for all } i
\]
and thus \( \bar{x} \) solves (2).

Now assume \( \bar{x} \) solves (2). Thus, 
\[
\min_{1 \leq j \leq n} \{ a_{1j} + \Delta_1 \}, \min_{j \geq 2} \{ a_{ij} + x_j \} = b_1
\]
for all \( i \). Letting \( x_1 = \Delta_1 \), it is clear that \( x = (x_1, \bar{x}) \) solves (1). \( \Box \)

3.4.3 In fact, there is a very easy, efficient way to find a solution to \( Ax = b \) in the min algebra sense or to prove that the system is inconsistent. Define \( \Delta_j = \max_{1 \leq i \leq m} \{ b_i - a_{ij} \} \). Then form
\[
\bar{A} = \begin{bmatrix}
a_{11} + \Delta_1 & \cdots & a_{1n} + \Delta_n \\
\vdots & \ddots & \vdots \\
a_{m1} + \Delta_1 & \cdots & a_{mn} + \Delta_n 
\end{bmatrix}.
\]

Then, if \( \bar{A} \cdot 0 = b \), the vector \( \Delta = (\Delta_1, \ldots, \Delta_n) \) solves \( A x = b \), otherwise the system has no solution. The proof of this follows from inductive application of Theorem 3.4.2.

3.4.4 Example: Let

\[
A = \begin{bmatrix}
1 & 5 & 1 \\
0 & 4 & 2 \\
2 & 1 & 1
\end{bmatrix}
\text{ and } \quad b = \begin{bmatrix}
3 \\
2 \\
1
\end{bmatrix}.
\]

Then \( \Delta_1 = 2, \Delta_2 = 0 \) and \( \Delta_3 = 2 \) giving

\[
\bar{A} = \begin{bmatrix}
3 & 5 & 3 \\
2 & 4 & 4 \\
4 & 1 & 3
\end{bmatrix}.
\]

\( \bar{A} \cdot 0 = b \) and thus \( \Delta = (2,0,2) \) solves the original system. This is easily checked. \( \square \)

3.4.5 Example: Let

\[
A = \begin{bmatrix}
1 & 5 & 1 \\
1 & 4 & 2 \\
2 & 1 & 1
\end{bmatrix}
\text{ and } \quad b = \begin{bmatrix}
3 \\
2 \\
1
\end{bmatrix}.
\]
Then again $\Delta_1 = 2$, $\Delta_2 = 0$ and $\Delta_3 = 2$ yielding

$$
\bar{A} = \begin{bmatrix}
3 & 5 & 3 \\
3 & 4 & 4 \\
4 & 1 & 3
\end{bmatrix}.
$$

Here, however, $\bar{A} \cdot 0 = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} \neq b$ and hence the original system has no solution. Notice that in the original system, we have

\[
\min\{1 + x_1, 5 + x_2, 1 + x_3\} = 3, \quad \min\{1 + x_1, 4 + x_2, 2 + x_3\} = 2 \quad \text{and}
\]

\[
\min\{2 + x_1, 1 + x_2, 1 + x_3\} = 1 \quad \text{implying } x_1 \geq 2, \quad x_2 \geq 0, \quad x_3 \geq 2 \quad \text{but}
\]

for the second equation to hold, $\min\{1 + x_1, 4 + x_2, 2 + x_3\} = 2$, we must have $x_1 = 1, \quad x_2 = -2$ or $x_3 = 0$, a clear contradiction. □

3.4.6 Now, given an integral $m \times n$ matrix $A$, we define two tag systems on $E = \{1, 2, \ldots, n\}$. Let $T = (E, C, \tau)$ be defined by $C \in C$ with $k \in \tau(C)$ if and only if there is an integral vector $x$ such that $\min_{j \in C \setminus \{k\}} \{a_{ij} + x_j\} = a_{ik}$ for all $i$ and no proper subset of $C$ containing $k$ has this property. It is easy to verify that $T$ is a tag system.

3.4.7 Define $\tilde{T} = (E, \tilde{C}, \tilde{\tau})$ by $\tilde{C} \in \tilde{C}$ with $k \in \tilde{\tau}(\tilde{C})$ if and only if there exists a rational vector $y$ such that $\min_{i} (y_i + a_{ij}) \in \mathbb{Z}$ for all $j \notin \tilde{C}$ and $\min_{i} (y_i + a_{ij}) \notin \mathbb{Z}$ for all $j \in \tilde{C}$ and no proper subset of $\tilde{C}$ containing $k$ has this property.
3.4.8 Example: Let

\[ A = \begin{bmatrix} 1 & 5 & 3 \\ 0 & 4 & 2 \\ 2 & 1 & 1 \end{bmatrix}. \]

Then the reader can check that the following vectors define \( T \) and \( \bar{T} \) on \( A \).

\[
\begin{array}{cccccccc}
\text{x} & \text{(A)} & \text{C} & \text{\( \tau(C) \)} & \text{y} & \text{yA} & \text{\( \bar{C} \)} & \text{\( \bar{\tau(C)} \)} \\
\hline
\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} & \{1,2,3\} & \{3\} & \begin{pmatrix} 0, -\frac{1}{2}, 0 \end{pmatrix} & \begin{pmatrix} -\frac{1}{2}, 1, 1 \end{pmatrix} & \{1\} & \{1\} \\
\begin{pmatrix} 0, -3, -\frac{1}{2} \end{pmatrix} & \begin{pmatrix} -3, \frac{1}{2}, -1 \end{pmatrix} & \{2\} & \{2\} \\
\begin{pmatrix} 0, -\frac{1}{2}, 1 \end{pmatrix} & \begin{pmatrix} -\frac{1}{2}, 2, 1 \frac{1}{2} \end{pmatrix} & \{1,3\} & \{3\} \\
\begin{pmatrix} 0, 0, -\frac{1}{2} \end{pmatrix} & \begin{pmatrix} 0, 1, 1 \end{pmatrix} & \{2,3\} & \{3\} \\
\end{array}
\]

3.4.9 Theorem: Given matrix \( A \in \mathbb{Z}^{m \times n} \), define \( T \) and \( \bar{T} \) as in 3.4.6 and 3.4.7, respectively. Then \( T^* = \bar{T} \).

Proof: As before, we prove that \( T \) and \( \bar{T} \) are dual tag systems by illustrating that they satisfy the painting theorem together. Consider an arbitrary painting of \( E \) into \( R \cup B \cup W \) with \( |R| = 1 \).

Now suppose both (P1) and (P2) of the painting theorem are satisfied. Then there exists a set \( C, R \subseteq C \subseteq R \cup B \), which is minimal with respect to the following property: there exists an integral \( x \)
such that \( \min\{a_{ij} + x_j\} = a_{iR} \) for all \( i \). In addition, there exists a set \( \bar{C} \), \( R \subseteq \bar{C} \subseteq R \cup W \), which is minimal with respect to containing \( R \) and the requirement that there exist a rational vector \( y \) such that \( \min\{y_i + a_{ij}\} \notin \mathbb{Z} \) if and only if \( j \in \bar{C} \). These together imply that

\[
\min\{y_i + a_{iR}\} = \min\{\min\{a_{ij} + x_j\} + y_i\} = \min\{\min\{y_i + a_{ij}\} + x_j\}.
\]

However, by \( R \subseteq \bar{C} \), we know that \( \min\{y_i + a_{iR}\} \notin \mathbb{Z} \), but the final minimum on the right hand side of the above relation is taken over only integral values and hence is an integer. Clearly this is impossible and thus (P1) and (P2) cannot both hold.

Now assume that the first \( k-1 \) columns \( (k \geq 1) \) of \( A \) are painted Blue and the \( k \)th column in Red. Then suppose (P1) fails and form \( \bar{A} \) by subtracting the \( k \)th column of \( A \) from each of the columns of \( A \):

\[
\bar{A} = \begin{bmatrix}
a_{11} - a_{1k} & \cdots & 0 & \cdots & a_{1n} - a_{1k} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & a_{mk} - a_{mk} & 0 & \cdots & a_{mn} - a_{mk}
\end{bmatrix}.
\]

Let \( \bar{A} = [\bar{a}_{ij}] \) and let \( \bar{a}_{ij}, j = \min\{\bar{a}_{ij}\} \) for each \( j \in B \); that
is, \( i_j \) is the index of some row in which the minimum entry in
column \( j \) occurs. Note that for some \( j's, i_j \) might take on multiple
values.

Suppose that each \( i = 1, \ldots, m \) appears in \( \{i_j\} \). Then let
\( x_j = -\min_{i} \{a_{ij}\} \) for each \( j \in B \). This implies that for each \( i \),

\[
\min_{j \in B} \{a_{ij} - a_{ik} + x_j\} = \min_{j \in B} \{a_{ij} - a_{ik} - \min_{k} \{a_{kj} - a_{kk}\}\} \geq 0.
\]

Suppose for some \( i^* \), this latter expression is strictly greater
than zero. Then, \( \min_{j \in B} \{a_{ij} - a_{ik}\} \) is not equal to \( a_{i^*j} - a_{i^*k} \) for
any \( j \in B \) and hence \( a_{i^*j} \neq a_{i_j,j} \) for any \( j \in B \). This contra-
dicts \( i^* \in \{i_j\} \) and thus \( \min_{j \in B} \{a_{ij} - a_{ik} + x_j\} = 0 \) for all \( i \).

However, this implies \( \min_{j \in B} \{a_{ij} + x_j\} = a_{ik} \) for all \( i \) which implies
there exists a \( C \in C \) with \( \{k\} = R \subseteq \tau(C) \) and \( C \subseteq R \cup B \) contra-
dicting the failure of (P1).

Hence, since (P1) fails, there exists an \( i^* \notin \{i_j\} \). Then, let
\( \overline{y} \) be defined by \( \overline{y}_{i^*} = -\frac{1}{2} \) and \( \overline{y}_i = 0 \) for \( i \neq i^* \). Note that
\( \min_{i} \{\overline{y}_i + \overline{a}_{ij}\} = \min_{i \neq i^*} \{-\frac{1}{2} + \overline{a}_{i^*j}, \min_{i \neq i^*} \{\overline{a}_{ij}\}\} \). However, \( a_{ij} \in Z \) for
all \( i \) and \( j \) implies the same for \( \overline{a}_{ij} \), and so if \( \overline{a}_{ij} < \overline{a}_{k,j} \)
then \( \overline{a}_{ij} < \overline{a}_{i_j,j} - \frac{1}{2} \). Thus, since \( i^* \notin \{i_j\} \),

\[
\min_{i} \{\overline{y}_i + \overline{a}_{ij}\} = \min_{i \neq i^*} \{\overline{y}_i + \overline{a}_{ij}\} \in Z
\]

for all \( j \in B \). However, \( \min_{i} \{\overline{y}_i + \overline{a}_{ik}\} = \min_{i} \{\overline{y}_i\} = -\frac{1}{2} \notin Z \). Now,
let \( y_i = \overline{y_i} - a_{ik} \) for each \( i \). Then, for all \( j \in B \),

\[
\min_{i} \{ y_i + a_{ij} \} = \min_{i} \{ \overline{y_i} - a_{ik} + a_{ij} \} = \min_{i} \{ \overline{y_i} + \overline{a_{ij}} \} \in \mathbb{Z},
\]

but \( \min_{i} \{ y_i + a_{ik} \} = \min_{i} \{ \overline{y_i} + \overline{a_{ik}} \} \notin \mathbb{Z} \). Thus, there exists a \( \tilde{C} \in \tilde{C} \) with \( R \subseteq \tilde{\tau}(\tilde{C}) \) and \( \tilde{C} \subseteq R \cup W \) and hence (P2) is satisfied. □

We now interpret the painting theorem for this pair of dual tag systems. Given \( A \in \mathbb{Z}^{m \times n} \) and \( b \in \mathbb{Z}^m \), form \([A^t;b]\) and paint \( A \) Blue and \( b \) Red.

Property (P1) implies there exists an \( x \in \mathbb{Z}^n \) and a set \( J \subseteq \{1, \ldots, n\} \) such that \( \min_{j \in J} \{a_{ij} + x_j\} = b_i \) for all \( i \). Alternative (P2) guarantees the existence of a rational \( y \) such that \( \min_{1 \leq i \leq m} \{ y_i + a_{ij} \} \in \mathbb{Z} \), for all \( j \), but \( \min_{1 \leq i \leq m} \{ y_i + b_i \} \notin \mathbb{Z} \).

The reader will recognize this as a min algebra analogue of Theorem 2.1.3.

### 3.5 Tag Systems on Graphs

There are several distinct ways to define a tag system on a given graph. In this section we briefly discuss some of the methods and relate one particular class of tag systems to the integrally representable tag systems of 3.2.

Suppose \( G \) is a directed graph and define the **node-arc incidence matrix** of \( G, M \), by \( M = (m_{ij}) \) where
\[ m_{ij} = \begin{cases} 
-1 & \text{if arc } j = (i,k) \text{ for some } k \in N \\
1 & \text{if arc } j = (k,i) \text{ for some } k \in N \\
0 & \text{otherwise.} 
\end{cases} \]

Then the graphic matroid defined on \( G \) (see 1.2.7) is equivalent to the \( Q \)-matrix matroid defined on \( M \) (see 1.2.5 and [36]). For each underlying circuit of the graph, choose one direction "around the circuit" as the positive direction and label each arc of the circuit \( + \) if its orientation is consistent with this positive direction and \( - \) otherwise. Then, the matroid discussed above is orientable with the orientation of the elements of each matroid circuit given by the orientation in the equivalent graphic circuit.

In accordance with the definitions above, we will consider each circuit, \( C \), of a directed graph to be partitioned into its positively directed arcs, \( C_+ \), and its negatively directed arcs, \( C_- \).

Now suppose \( F \) is a cutset in \( G \); that is, as in 1.1, there exist \( u,v \in N \) such that \( u \) and \( v \) are connected in \( G \) and in \( G \setminus U \) for all \( U \not\supset F \) but \( u \) and \( v \) are not connected in \( G \setminus F \). Then there exist vertex sets \( X \) and \( \overline{X} \) with \( u \in X \) and \( v \in \overline{X} \), \( X \cap \overline{X} = \emptyset \), such that each arc of \( F \) is incident upon one vertex of \( X \) and one vertex of \( \overline{X} \). For \( f = (p,q) \in F \), when \( p \in X \) we denote \( f \in F_+ \) and when \( q \in X \) we denote \( f \in F_- \). Thus we will also consider each cutset to be partitioned into positively oriented arcs, \( F_+ \), and negatively oriented arcs, \( F_- \).
Now define $T = (A, C, \tau)$ in terms of nonnegative rational dependence of the columns of $M$. Then $C \in C$ with $\{k\} = \tau(C)$ if and only if the (undirected) edges underlying the arcs indexed by $C$ form a circuit with $C_+ = C \setminus \{k\}$ and $C_- = \{k\}$ or $C_+ = \{k\}$ and $C_- = C \setminus \{k\}$. Further, using the painting theorem, it is easy to verify that the dual of this system is given by $T^* = (A, C^*, \tau^*)$ where $C^* \in C^*$ with $k \in \tau^*(C^*)$ if and only if there is a cutset of $G$, $F$, such that $F_+ = C^*$ or $F_- = C^*$, and no proper subset of $C^*$ containing $k$ has this property.

For the remainder of this section we let $G = (N, A)$ be an undirected graph and we consider tag systems which are distinct from those that arise from dependency relations among the columns of the incidence matrix.

We define the edge-neighborhood of edge $a \in A$ by $N(a) = \{b \in A: a$ and $b$ are adjacent$\}$. Now define $T = (A, C, \tau)$ by $C \in C$ if and only if $C = \{a, b\}$ where $a$ and $b$ are adjacent edges and $\tau(C) = C$ for all $C \in C$. Define $T = (A, \tilde{C}, \tilde{\tau})$ by $\tilde{C} \in \tilde{C}$ if and only if $\tilde{C} = \{a\} \cup N(a)$ for some $a \in A$ and $\tilde{\tau}(\tilde{C}) = \{b \in \tilde{C}: \tilde{C} = \{b\} \cup N(b)\}$. The arguments indicated in 2.2.6 and 2.2.7 also prove that $T$ and $T$ are tag systems.

**3.5.1 Proposition:** $T$ and $T$, as defined above, comprise a dual pair of tag systems.
Proof: We prove this by illustrating that $T$ and $\tilde{T}$ satisfy the tag system painting theorem together. Suppose the arcs of $G$ are painted $R \cup B \cup W$ with $|R| = 1$. Then property (P1) of the painting theorem is equivalent to the Red edge being adjacent to a Blue edge, while condition (P2) is equivalent to the Red edge being adjacent only to White edges. Clearly exactly one of (P1) and (P2) must hold. □

3.5.2 Proposition: The bases of $T$ are exactly the maximal matchings of $G$.

Proof: Let $B$ be a base of $T$. Then since $B$ is independent, no two edges of $B$ are adjacent and hence $B$ is a matching. Further, $\sigma(B) = E$ implies that every edge of $E \setminus B$ is adjacent to some edge of $B$ and thus $B$ is a maximal matching.

Now suppose $B$ is a maximal matching of $G$. Then it is clear that $B$ is independent in $T$ and that $\sigma(B) = E$ and hence $B$ is a base of $T$. □

Now note that we can repeat the above development for tag systems defined by adjacent vertices and vertex neighborhoods as discussed in 2.2.6 and 2.2.7. The proof that the analogous $T$ and $\tilde{T}$ are duals follows exactly as the proof of 3.5.1. In addition, it is clear that the bases of $T$ are exactly the maximal stable sets of $G$. Any tag system whose circuits correspond precisely to the
neighborhood sets of vertices of a graph will be called graphic. Its dual will be called cographic. Note that these are not equivalent to the graphic and cographic matroids. We now provide characterizations of graphic and cographic tag systems.

3.5.3 Proposition: Cographic tag systems are exactly those tag systems, $T = (E,C,\tau)$, which have the property that $|C| = 2$ and $\tau(C) = C$ for every $C \in C$.

Proof: First, it is clear that every cographic system satisfies this property. Now suppose $T = (E,C,\tau)$ has $|C| = 2$ and $\tau(C) = C$ for every $C \in C$. Then define $G = (N,A)$ by $N = E$ and for each $i \in N$, the neighbors of $i$ are given by $N(i) = \{j \in N : \{i,j\} \in C\}$. Then, it is clear that graph $G$ is well-defined and it is easily verified that $T$ is precisely the cographic tag system defined on $G$. $\square$

3.5.4 Proposition: Graphic tag systems are exactly those tag systems, $T = (E,C,\tau)$, which have the property that for each $i \in E$, there exists exactly one $C \in C$ with $i \in \tau(C)$, say $C_i$, and if $i \in C_j$, then $j \in C_i$.

Proof: It is clear that every graphic tag system satisfies this condition. Now assume $T = (E,C,\tau)$ satisfies the assumption. Then define graph $G = (N,A)$ by $N = E$ and for each $i \in N$, $N(i) = \{j : j \in C_i\}$. Again it is clear that $G$ is well-defined and that $T$ is the graphic tag system defined on $G$. $\square$
In the remainder of this section we define an existence problem for finding a base of an integral dependence system and relate this to graphic tag systems in order to show that the original problem is NP-Complete. (See [17] for background material on computational complexity.) This result indicates that an abstraction of integral dependence must encompass the graphic tag systems defined above. Thus any such combinatorial abstraction must also model the stable sets of arbitrary graphs and hence render unlikely the possibility for general algorithmic tractability.

3.5.5 $\mathbf{ZBASE}$: Given finite subset $S \subseteq \mathbf{Z}$ and integer $k > 0$, does there exist a subset $B \subseteq S$ so that $B$ is independent, $B$ integrally generates $S$ and $|B| \leq k$?

We prove that $\mathbf{ZBASE}$ is NP-Complete by providing a polynomial reduction of $\mathbf{STABLESET}$ to $\mathbf{ZBASE}$. Recall the $\mathbf{STABLESET}$ existence problem:

3.5.6 $\mathbf{STABLESET}$: Given graph $G = (N,A)$ with $|N| = n$, and integer $k > 0$, is there a stable set of $G$ of cardinality greater than or equal to $k$?

Suppose graph $G = (N,A)$ with $|N| = n$ and integer $k > 0$ are given. We will define a construction which assigns integers to the nodes of $G$ in such a way that the graphic tag system defined on $G$ is exactly the integrally representable tag system defined on
the \(1 \times n\) matrix whose entries are the integral node assignments. Then, letting \(k = n - \lambda\), \(M = \mathbb{Z}\) and

\[ S = \{z: z \text{ is assigned to a node of } G\}. \]

and solving \(Z\text{BASE}\) would be equivalent to finding a base of the graphic tag system of cardinality less than or equal to \(n - \lambda\). Thus, its complement would be a base of the cographic tag system, and hence a stable set of \(G\), of cardinality greater than or equal to \(\lambda\). So, any polynomial algorithm for \(Z\text{BASE}\) could be applied to result in a polynomial algorithm for \(Z\text{ABLESET}\). Furthermore, \(Z\text{BASE}\) is in the class \(NP\) of decision problems, since the Euclidean Algorithm can be used to perform greatest common divisor calculations for the integers in any subset of \(S\) (see [17]). Thus \(Z\text{BASE}\) is \(NP\)-Complete.

The construction is inductively applied to a general graph \(G\). First note that if \(G = (N,A)\) with \(N = \{v\}\), then \(Z\text{ABLESET}\) is clearly an easy problem. Hence we will assume \(|N| \geq 2\). Further, since \(Z\text{ABLESET}\) is \(NP\)-Complete for connected graphs, it will suffice to do the construction for connected graphs. Hence we assume throughout that \(G\) is connected.

Suppose \(G = (N,A)\) with \(N = \{v,w\}\). Then, since \(G\) is connected, assigning each of \(v\) and \(w\) the integer \(\pm 1\) will provide the correct tag system. We now illustrate how to start with a correctly labeled graph and add a new edge or a new edge with a new vertex. For ease of notation, we let the integer assigned to node \(i\) be denoted \(z_i\) and we use \(p_v, p_{ij}\), etc. to denote a prime number
which has not yet been used in the labeling of $G$. The procedures follow: Let $\pi$ initially be $1$.

3.5.7 Procedure 1: Add vertex $w$ adjacent to vertex $v$:

1. Assign $z_w = p_v z_v \pi$.
2. For each $j \in N(v) \cup \{w\}$ and for each $i \in N(j)$, multiply each index of $N \cup \{w\}\{i,j\}$ by $p_{ij}$ and replace $\pi \leftarrow p_{vw} \pi^2$ if $|N| > 2$ and replace $\pi \leftarrow p_{jv} p_{vw} \pi$ if $N = \{v,j\}$.

3.5.8 Procedure 2: Add edge $\{v,w\}$:

1. For each $j \in N(v)$ multiply each index in $N\{v,j\}$ by $p_{vj}$.
2. For each $i \in N(w)$ multiply each index in $N\{w,i\}$ by $p_{wj}$.
3. Multiply each index in $N\{v,w\}$ by $p_{vw}$ and replace $\pi \leftarrow p_{vw} \pi$.

3.5.9 Theorem: (A) If $G = (N,A)$ is labeled correctly, then $G' = (N \cup \{w\}, A \cup \{v,w\})$ is labeled correctly after Procedure 1.

(B) If $G = (N,A)$ is labeled correctly, then $G' = (N, A \cup \{v,w\})$ is labeled correctly after Procedure 2.

Proof: In both parts of the proof, we let $z_j$ denote the index on node $j$ prior to the procedure and $z'_j$ the index after the procedure. Note that at any given iteration, $\pi$ has a unique prime factor for each edge $\{i,j\}$ of the graph; we will denote this factor by $p_{ij}^*$.

(A) Let $j \in N$. Then note that $\gcd_{i \in N(j)} \{z_i\}$ divides $z_j$ since there exist $\lambda_i \in \mathbb{Z}$ such that $z_j = \sum_{i \in N(j)} \lambda_i z_i$. Further, it is not possible through Procedure 1 to multiply all of the indices of
the vertices in $N(j)$ by a prime without also multiplying $z_j'$ by that prime. Hence $\gcd \{z_i'\}_{i \in N(j)}$ also divides $z_j'$ and so we can write $z_j'$ as an integral combination of the $z_i'$ for $i \in N(j)$.

Now it is clear inductively that for any $i \in N(j)$, $p_{ij}^*$ divides each index of $N \setminus \{i,j\}$ and hence since $p_{ij}^*$ divides $\pi$ we also have that $p_{ij}^*$ divides $z_w'$. Thus, $p_{ij}^*$ divides the greatest common divisor of the indices $N \cup \{w\} \setminus \{i,j\}$ but does not divide $z_i'$ or $z_j'$. Hence $z_i'$ must be included in any integral representation of $z_j'$. So, the only minimal integral representation of $z_j'$ is in terms of all of the indices of $z_j'$'s neighbors.

Now consider vertex $w$. Clearly $z_w'$ is an integral multiple of $z_v'$ and since $z_v' = z_v$, it is an integral multiple of $z_v'$. Further $p_{vw}^*$ divides each $z_k'$ for $k \in N \setminus \{v,w\}$ so $N(w) = \{v\}$ gives a unique minimal integral representation of $z_w'$.

(B) Let $j \in N \setminus \{v,w\}$.

The argument for the fact that $z_j'$ can be written as an integral combination of $z_i'$, $i \in N(j)$, is the same here as in part A.

Let $i \in N(j)$. Then it is clear that $p_{ij}^*$ divides $z_k'$ for each $k \in N \setminus \{i,j\}$ but does not divide $z_i'$ nor $z_j'$. Hence again $N(j)$ provides the unique minimal representation of $z_j'$.

Now consider vertex $v$. The argument for $w$ is symmetric, so it will not be discussed separately. Let $\sigma_1$ denote the product of the primes $p_{vj}$ for $j \in N(v)$ and $\sigma_2$ denote the product of the
primes $p_{wi}$ for $i \in N(w)$ introduced in this iteration of Procedure 2. Then it is easy to verify that for $k \in N(v) \setminus N(w)$, $z'_k = \frac{\sigma_1}{p_{vk}} \frac{\sigma_2}{p_{wk}} p_{vw} z_k$ and for $k \in N(v) \cap N(w)$, $z'_k = \frac{\sigma_1}{p_{vk}} \frac{\sigma_2}{p_{wk}} p_{vw} z_k$. Now let $\sigma_2$ denote the product of the primes $p_{wi}$ for $i \in N(w) \setminus N(v)$. Then, $\gcd(z'_k: k \in N(v)) = \sigma_2 p_{vw} \cdot \gcd(z'_k: k \in N(v))$. Further, since $p_{vw}$ has not been introduced prior to this iteration we know that $p_{vw}$ does not divide $z'_w$. Hence,

$$\gcd(z'_w, z'_k: k \in N(v)) = \gcd(z'_w, \sigma_2 \gcd(z'_k: k \in N(v)).$$

Now note that $z'_v = \sigma_2 z_v$ and recall that by definition $\sigma_2$ divides $\sigma_2$ and by the induction hypothesis $\gcd(z'_k: k \in N(v))$ divides $z_v$. Hence the greatest common divisor of the indices of $v$'s new neighborhood ($\{w\} \cup N(v)$) divides the index on node $v$. So, $z'_v$ can be written as an integral combination of the indices of the nodes in $N(v) \cup \{w\}$. Finally, note that for any $j \in N(v)$, $p^*_{jv}$ divides all of the indices of $N \setminus \{j, v\}$ but not $z'_j$ or $z'_v$ and hence $N(v)$ gives a unique minimal integral representation of $z'_v$. \[ 3.5.10 \quad \text{Example: Suppose} \]

$$G = \begin{array}{ccc}
\text{d} & \text{a} & \text{c} \\
& \text{b} & \\
\end{array}$$

Then we "grow" $G$ as follows:
3.5.11 Theorem: The construction described above using Procedure 1 of 3.5.9 and Procedure 2 of 3.5.8 to assign integers to the nodes of a general graph requires a polynomial (in $|N| + |A|$) number of multiplications and the numbers produced remain polynomially "small."

Proof: First note that each execution of Procedure 1 or 2 introduces at most $O(|N|^2)$ primes and hence at most $O(|N|^3)$ multiplications. Further, exactly $|N| - 2 + (|A| - 1 - (|N| - 2)) = |A| - 1$
iterations are required, since we begin the procedure with a single edge and then execute Procedure 1 $|N| - 2$ times and Procedure 2 $|A| - 1 - (|N| - 2)$ times. Hence in total at most $O(|A||N|^3)$ multiplications are performed and at most $O(|A||N|^2)$ primes are needed. It is a well-known result in Number Theory (see [20]) that the $n^{th}$ prime is less than or equal to $4^n$. Hence the largest number required to reach $|A||N|^2$ distinct primes is on the order of magnitude of $4|A||N|^2$. Since we never multiply more than $O(|A||N|^2)$ of these together, each number written remains less than or equal to $O(4|A|^2|N|^4)$. Thus we need at most approximately $|A|^2|N|^4 \log 4$ bits to write the largest number used in the construction. □
BIBLIOGRAPHY


