SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NEW YORK

TECHNICAL REPORT NO. 635

September 1984

A GEOMETRIC APPROACH TO CONSTRUCTING MARTINGALE MEASURES: THE FINITE CASE

by

Murad S. Taqqu[1] and Walter Willinger

[1]Research supported by the National Science Foundation grant ECS-84-08524 at Cornell University.

Typist: Anne T. Kline

A GEOMETRIC APPROACH TO CONSTRUCTING MARTINGALE MEASURES: THE FINITE CASE

Murad S. Taqqu[1] and Walter Willinger

School of Operations Research
and Industrial Engineering
Cornell University
Ithaca, NY 14853

Abstract: Consider a d-dimensional stochastic process $X = (X_t: t = 0,1,\ldots,T)$ on a finite probability space $(\Omega,\mathcal{F},P)$ and let $\mathcal{M}(X)$ denote the set of all probability measures on $(\Omega,\mathcal{F})$ that make X a martingale. We provide a necessary and sufficient condition for $\mathcal{M}(X)$ to be non-empty and we characterize the extreme points of $\mathcal{M}(X)$. The conditions are stated in terms of X and have a geometric interpretation.

Keywords: martingales, change of measure, extremal measures, convex hulls, security markets.

[1]Research supported by the National Science Foundation grant ECS-84-08524 at Cornell University.

1. Introduction

Let $X = (X_t: t = 0,1,\ldots,T)$, $T < \infty$, be a fixed d-dimensional stochastic process, with $d \geq 1$, on a finite probability space (Ω,F,P). We shall assume without loss of generality that $P[\{\omega\}] > 0$ for all $\omega \in \Omega$. Consequently, if Q is any other probability measure on (Ω,F), we have $Q \ll P$, that is, Q is absolutely continuous with respect P. We have $Q \sim P$, that is, Q is equivalent to P if and only if $Q[\{\omega\}] > 0$ for all $\omega \in \Omega$. Let $N = |\Omega| < \infty$ and, for $0 \leq t \leq T$, let P_t denote the smallest partition of Ω corresponding to the σ-algebra $F_t = \sigma(X_0,X_1,\ldots,X_t)$. For convenience, we take $F_0 = \{\emptyset,\Omega)$ and $F_T = F = 2^{\Omega}$ to be the σ-algebra containing all subsets of Ω.

A martingale measure associated with X is a probability measure Q on (Ω,F) such that the stochastic process X is a $(Q,(F_t))$ - martingale, i.e. $E_Q[X_{t+1}|F_t] = X_t$ a.s. Q for all $t \in \{0,1,\ldots,T-1\}$, where $E_Q[\cdot]$ denotes the mathematical expectation with respect to Q. Let $M(X)$ denote the set of all martingale measures associated with X. Since $M(X)$ is a convex set, we can define $M_e(X)$, the set of all extreme points of $M(X)$.

The purpose of this paper is to provide a necessary and sufficient condition for $M(X) \neq \emptyset$ and to characterize the probability measures that belong to $M_e(X)$. Our conditions on the vector-valued stochastic process X are geometric in nature. The results extend to processes X with arbitrary state space. (Details about this extension will appear elsewhere.)

We proceed as follows. Let Y be a fixed random variable with values in R^d such that $\sigma(Y) = F$. In Section 2, we show that the set $\{Q: E_Q[Y] = 0,\ Q \sim P\}$ is non-empty iff 0 belongs to the relative

interior of the convex hull of $\{Y(\omega): \omega \in \Omega\}$. A related result is obtained when the probability measure Q on (Ω,F) is not necessarily assumed to be equivalent to P. Section 3 characterizes all probability measures on (Ω,F) that are extreme points of the set $\{Q: E_Q[Y] = 0\}$. We show that for Q to be an extreme point, one must have $\dim\{\text{span}\{Y(\omega_1),\ldots,Y(\omega_n)\}\} = n-1$, where $\omega_1,\ldots,\omega_n$ are the distinct points of Ω to which Q assigns positive probability. Finally, in Section 4, we consider the d-dimensional stochastic process X and answer our questions about $M(X)$ and $M_e(X)$ by applying the previous results to the increments of the process X. Applications to the theory of finite security markets can be found in Taqqu and Willinger (1984).

2. A geometric condition for a change of measure

Let Y be a d-dimensional random variable $(d \geq 1)$ on (Ω,F,P) with the property that $\sigma(Y) = F$. Recall that P gives positive mass to all N points of Ω and that a probability measure Q on (Ω,F) is equivalent to P if it also gives positive probability to all N points of Ω. We shall find a necessary and sufficient condition for the existence of an equivalent change of measure that results in a probability measure Q for which $E_Q[Y] = 0$.

To gain insight, let us first consider the one-dimensional and two-dimensional cases. In the case $d = 1$, it is very easy to prove the following result:

$$\exists\ Q \sim P \text{ such that } E_Q[Y] = 0 \iff \begin{cases} Y(\omega) \geq 0 \ \forall\ \omega \in \Omega \Rightarrow Y \equiv 0 \\ Y(\omega) \leq 0 \ \forall\ \omega \in \Omega \Rightarrow Y \equiv 0. \end{cases} \tag{2.1}$$

This simply means that in order for such a Q to exist, the random variable Y has to assume positive and negative values (except when $Y \equiv 0$), so that probability mass can be "redistributed" in such a way that the "center of gravity" coincides with the origin. In the two-dimensional case, one can easily construct examples that show that condition (2.1) does not guarantee anymore the existence of a Q with the desired properties. However, these examples turn out not to satisfy either condition (I) or condition (II) of Lemma 2.1 below.

The convex hull of the set $\{Y(\omega): \omega \in \Omega\} \subset R^d$ is $C = \mathrm{conv}(\{Y(\omega): \omega \in \Omega\}) = \{x \in R^d: x = \sum_{\omega\in\Omega} \lambda_\omega Y(\omega), \sum_{\omega\in\Omega} \lambda_\omega = 1, \lambda_\omega \geq 0\}$ and its affine hull is $\mathrm{aff}(C) = \{x \in R^d: x = \sum_{\omega\in\Omega} \lambda_\omega Y(\omega), \sum_{\omega\in\Omega} \lambda_\omega = 1\}$. When dealing with convex sets C, the concept of interior is replaced by the more convenient concept of relative interior, reflecting the fact that C, regarded as a subset of R^d, does not have an interior when $\dim(C) < d$. The relative interior of C, denoted $\mathrm{ri}(C)$, is the interior of C when C is viewed as a subset of $\mathrm{aff}(C)$. Formally, $\mathrm{ri}(C) = \{x \in \mathrm{aff}(C): \exists \epsilon > 0$ such that $(x + \epsilon B) \cap (\mathrm{aff}(C)) \subset C\}$ where $B = \{x \in R^d: |x| \leq 1\}$ is the Euclidean unit ball in R^d. Let $\mathrm{cl}(C)$ denote the closure of C and note that $\mathrm{cl}(C) = \mathrm{cl}(\mathrm{ri}(C))$.

<u>Lemma 2.1</u>. The following two statements are equivalent.

(I) $0 \in \mathrm{ri}(C)$.

(II) There is no hyperplane that separates C and the origin 0 properly, i.e. there exists no $\alpha \in R^d$ such that $\alpha \cdot Y(\omega) \geq 0 \ \forall \omega \in \Omega$ but $\alpha \cdot Y \not\equiv 0$.

<u>Proof</u>. Let C_1 and C_2 be two non-empty convex sets in R^d. Then $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) = \emptyset$ iff there exists a hyperplane separating C_1 and C_2 properly (see Rockafellar (1970, p. 96)). Setting $C_1 = C$ and $C_2 = \{0\}$ concludes the proof. □

We also need

Lemma 2.2. Let A be a given $d \times N$ matrix. Then exactly one of the following two alternatives hold:

either (a) $A \cdot x = 0$ has a solution $x \in R^N$ with $x > 0$

or (b) there is a vector $y \in R^d$ with $y \cdot A \geq 0$, $y \cdot A \neq 0$.

Proof. Let $e = (1,1,\ldots,1) \in R^N$. Then we have (after scaling, if necessary),

(a) $\Leftrightarrow \exists u \in R^N$ such that $A \cdot u = A \cdot (-e)$, $u \geq 0$ (set $u = x - e$)
and

(b) $\Leftrightarrow \exists v \in R^d$ such that $v \cdot A \leq 0$ and $v \cdot (A \cdot (-e)) > 0$

(set $v = -y$).

Setting $b = A \cdot (-e)$, $z = u$ and $w = v$ we get

(a) $\Leftrightarrow$ (A): $\exists z \in R^N$, $z \geq 0$ such that $A \cdot z = b$

(b) $\Leftrightarrow$ (B): $\exists w \in R^d$ such that $w \cdot A \leq 0$ and $w \cdot b > 0$.

But Farkas Lemma (see Murty (1976, p. 175)) states that exactly one of the two alternatives (A) or (B) hold. □

Theorem 2.1. There exists a probability measure Q on $(\Omega, \mathcal{F})$ that is equivalent to P and for which $E_Q[Y] = 0$ if and only if statement (I) (or equivalently (II)) of Lemma 1.1 is satisfied.

Proof. The sufficiency is obvious. To prove necessity, define the $d \times N$ matrix A by

$$A = \begin{bmatrix} Y_1(\omega_1) & Y_1(\omega_2) & \cdots & Y_1(\omega_N) \\ Y_2(\omega_1) & Y_2(\omega_2) & \cdots & Y_2(\omega_N) \\ \vdots & \vdots & & \vdots \\ Y_d(\omega_1) & Y_d(\omega_2) & \cdots & Y_d(\omega_N) \end{bmatrix}$$

and suppose that statement (II) of Lemma 1.1 holds. Then alternative (b) of Lemma 1.2 cannot hold which implies that alternative (a) must hold. Thus, the equation $A \cdot x = 0$ has a <u>strictly positive</u> solution $x^* \in R^N$. Now set $Q[\{\omega_i\}] = (1/\sum_{j=1}^{N} x_j^*)x_i^*$ $(i = 1,\ldots,N)$. Clearly, Q is a probability measure on (Ω,F), $Q[\{\omega_i\}] > 0$ for $i = 1,\ldots,N$, and

$$\begin{aligned} E_Q[Y] &= \sum_{i=1}^{N} Y(\omega_i)Q[\{\omega_i\}] \\ &= \sum_{i=1}^{N} Y(\omega_i)(1/\sum_{j=1}^{N} x_j^*)x_i^* \\ &= (1/\sum_{j=1}^{N} x_j^*)A \cdot x^* \\ &= 0. \end{aligned}$$

Thus Q has the desired properties. □

The following is a modification of Theorem 1.1, where we allow a change of measure that results in a probability measure Q which may assign zero probability to some $\omega \in \Omega$. Thus, it provides a necessary and sufficient condition for the set $\{Q: E_Q[Y] = 0\}$ to be non-empty.

<u>Theorem 2.2</u>. There exists a probability measure Q on (Ω,F) that is absolutely continuous with respect to P and for which $E_Q[Y] = 0$ if and only if (I') (or equivalently (II')) is satisfied, where

(I') $0 \in cl(C)$.

(II') There is no hyperplane that separates C and the origin 0 strongly, i.e. there exists no $\alpha \in R^d$ such that $\alpha \cdot Y(\omega) > 0 \forall \omega \in \Omega$.

Proof. (I') is a formal statement of $Q << P$, $E_Q(Y) = 0$, and its equivalence to (II') follows from Rockafellar (1970, p. 96). □

3. Extremality and uniqueness of Q

Let Y be as in Section 2. Theorem 2.2 states conditions for the set $\{Q: E_Q[Y] = 0\}$ to be non-empty. We now wish to characterize those probability measures Q which are extreme points of the convex set $\{Q: E_Q[Y] = 0\}$. We shall use the following lemma which is a particular case of a result of Jacod (1979, Thm. (11.6)).

Lemma 3.1. Let Q be an extreme point of the set $\{Q: E_Q[Y] = 0\}$. If Z is an F-measurable real-valued random variable with the properties that $E_Q[Z] = 0$ and $E_Q[ZY] = 0$, then $Z = 0$.

Theorem 3.1. Fix $1 \leq n \leq N$ and let Q be a probability measure on (Ω, F) that puts positive mass on n points of Ω, say $\omega_1, \omega_2, \ldots, \omega_n$, and puts zero mass on the remaining $N - n$ points of Ω. Suppose Q is such that $E_Q[Y] = 0$. Then the following three statements are equivalent:

(a) Q is an extreme point of the set $\{Q: E_Q[Y] = 0\}$.

(b) The system

$$(S) \quad \sum_{i=1}^{n} Y(\omega_i)\cdot q_i = 0, \quad \sum_{i=1}^{n} q_i = 1, \quad q_i > 0 \quad \text{for} \quad i = 1,\ldots,n$$

has the unique solution $q_i = Q[\{\omega_i\}]$ $(i = 1,\ldots,n)$.

(c) $\dim\{\mathrm{span}\{Y(\omega_1),\ldots,Y(\omega_n)\}\} = n - 1$.

<u>Remark</u>. Note that different Q's satisfying (a) may put positive mass on different sets of n points of Ω and zero mass on the remaining ones. However, a Q which satisfies (a) is unique if it puts positive mass on all the N points of the sample space Ω, i.e. when $Q \sim P$, because in this case the system (S) has the unique solution $q_i = Q[\{\omega_i\}]$ $(i = 1,\ldots,N)$. Therefore, we get

<u>Corollary 3.1</u>. If $Q \sim P$ and Q is an extreme point of the set $\{Q: E_Q[Y] = 0\}$, then Q is unique.

<u>Proof of Theorem 3.1</u>. We first show that (a) => (b): Suppose, on the contrary, that there is a second solution $q_i' = Q'[\{\omega_i\}]$ $(i = 1,\ldots,n)$ for the system (S) with $Q'[\{\omega_i\}] = 0$ for $i = n + 1,\ldots,N$. Define the random variable Z' by

$$Z'(\omega_i) = \begin{cases} Q'[\{\omega_i\}]/Q[\{\omega_i\}] & \text{for} \quad i = 1,\ldots,n \\ \\ 0 & \text{for} \quad i = n + 1,\ldots,N \end{cases}$$

and set $Z = Z' - 1$. Then, we get

$$E_Q[Z] = E_Q[Z'] - 1 = 0,$$

and

$$E_Q[ZY] = E_Q[Z'Y] - E_Q[Y] = E_{Q'}[Y] - E_Q[Y] = 0.$$

By Lemma 3.1, this implies $Z = 0$, i.e. $Z' = 1$ or $q_i = q_i'$ for $i = 1,\ldots,n$. Thus (S) has a unique solution.

We now show that (b) => (a): Again, we proceed by assuming, on the contrary that $Q = \lambda Q' + (1-\lambda)Q''$ for some $\lambda \in (0,1)$ and some Q', Q'' with $E_{Q'}[Y] = E_{Q''}[Y] = 0$. Set $Q_1 = 1/2(Q + Q')$. Clearly, $Q'[\{\omega_i\}] = Q''[\{\omega_i\}] = 0$ for $i = n + 1,\ldots,N$ and hence $Q_1[\{\omega_i\}] > 0$ for $i = 1,\ldots,n$ and $Q_1[\{\omega_i\}] = 0$ for $i = n + 1,\ldots,N$. Moreover,

$$\sum_{i=1}^{n} Q_1[\{\omega_i\}] = 1/2 \sum_{i=1}^{n} Q[\{\omega_i\}] + 1/2 \sum_{i=1}^{n} Q'[\{\omega_i\}] = 1$$

and

$$\sum_{i=1}^{n} Y(\omega_i)Q_1[\{\omega_i\}] = 1/2\{\sum_{i=1}^{n} Y(\omega_i)Q[\{\omega_i\}] + \sum_{i=1}^{n} Y(\omega_i)Q'[\{\omega_i\}]\} = 0.$$

This shows that $\{Q_1[\{\omega_i\}]: i = 1,\ldots,n\}$ also satisfies (S), and therefore, $Q' = Q$ since we assume that (S) has a unique solution. Similarly, if $Q_1 = 1/2(Q + Q'')$, we get $Q'' = Q$. Thus Q is an extreme point of $\{Q: E_Q[Y] = 0\}$.

Finally we prove (b) <=> (c): This equivalence is a well-known result in convex analysis (see Rockafellar (1970, p. 7)). Indeed, (S) has a unique solution iff the vectors $Y(\omega_1),\ldots,Y(\omega_n)$ are affinely independent. But by definition, this is the case iff the vectors $Y(\omega_2) - Y(\omega_1),\ldots,Y(\omega_n) - Y(\omega_1)$ are linearly independent, i.e. $\dim\{\mathrm{span}\{Y(\omega_1),\ldots,Y(\omega_n)\}\} = n - 1$. □

4. Martingale measures associated with the process X

Let $X = (X_t: t = 0,1,\ldots,T)$ be the d-dimensional stochastic process on $(\Omega,\mathcal{F},P)$ with associated filtration $(\mathcal{F}_t: t = 0,1,\ldots T)$, introduced in Section 1. Note that the probability measure P can be "decomposed" into an initial distribution (which is a degenerate distribution since $\mathcal{F}_0$ is trivial) and successive transition probabilities. Moreover, the transition probabilities from time t to $t + 1$ $(t = 0,1,\ldots,T - 1)$ can be taken to be the conditional distribution of $X_{t+1} - X_t$ given $\mathcal{F}_t$.

By viewing the probability measures of the previous sections as transition probabilities and piecing them together so as to obtain a probability measure for the stochastic process X, we can apply the results of the preceeding section to the problems at hand. First, we want to obtain a necessary and sufficient condition for $\mathcal{M}(X) \neq \emptyset$, that is, for the existence of a change of measure which would make X a martingale. Secondly, we want to characterize the probability measures that belong to $\mathcal{M}_e(X)$.

<u>Theorem 4.1</u>. The following three statements are equivalent:

(i) There exists $Q \in M(X)$ with $Q \sim P$.

(ii) For all $t \in \{0,1,\ldots,T-1\}$ and all $A \in \mathcal{P}_t$,
$0 \in \mathrm{ri}(\mathrm{conv}(\{X_{t+1}(\omega) - X_t(\omega): \omega \in A\}))$.

(iii) For all $t \in \{0,1,\ldots,T-1\}$ and all $A \in \mathcal{P}_t$ there exists no hyperplane that separates $\mathrm{conv}(\{X_{t+1}(\omega) - X_t(\omega): \omega \in A\})$ and 0 properly, i.e. there exists no $\alpha \in R^d$ such that $\alpha\cdot(X_{t+1}(\omega) - X_t(\omega)) \geq 0 \ \forall\omega \in A$ but $\alpha\cdot(X_{t+1} - X_t) \not\equiv 0$.

<u>Proof</u>. Apply Theorem 2.1 to each transition probability. □

We shall now consider measures Q that do not necessarily put positive mass on all points of Ω.

<u>Theorem 4.2</u>. Consider the three statements:

(i) There exists $Q \in M(X)$

(ii) For all $t \in \{0,1,\ldots,T-1\}$ and all $A \in \mathcal{P}_t$, with $Q[A] > 0$,
$0 \in \mathrm{cl}(\mathrm{conv}(\{X_{t+1}(\omega) - X_t(\omega): \omega \in A\}))$.

(iii) For all $t \in \{0,1,\ldots,T-1\}$ and all $A \in \mathcal{P}_t$ with $Q[A] > 0$, there exists no hyperplane that separates $\mathrm{conv}(\{X_{t+1}(\omega) - X_t(\omega): \omega \in A\})$ and 0 strongly, i.e. there is no $\alpha \in R^d$ such that $\alpha\cdot(X_{t+1}(\omega) - X_t(\omega)) > 0$ for all $\omega \in A$.

Then the following implications hold:

(i) => (ii) <=> (iii)

Proof. Let $Q \in M(X)$. By applying Theorem 2.2 to each $A \in \mathcal{P}_t$ ($t \in \{0,1,\ldots,T-1\}$) with $Q[A] > 0$, we get that (i) implies (ii). The equivalence between (ii) and (iii) follows from Theorem 2.2. □

In order to see that statement (ii) does not guarantee the existence of a martingale measure, consider the following example of a two-step stochastic process $X = (X_0, X_1, X_2)$ involving dyadic branching: $\mathcal{P}_0 = \{\Omega\}$, $\mathcal{P}_1 = \{B,C\}$ where $B = \{\omega_1,\omega_2\}$ and $C = \{\omega_3,\omega_4\}$, and $\mathcal{P}_2 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}$. Suppose that statement (ii) holds with Q such that $Q[B] = 0$ and hence $Q[C] = 1$. Now, Theorem 2.2, applied to the partition set $A = \Omega$ at $t = 0$ may yield a measure Q' with $Q'[B] > 0$ and $Q'[C] > 0$. Since we supposed $Q[B] = 0$, statement (ii) does not guarantee that $0 \in \mathrm{cl}(\mathrm{conv}(\{X_2(\omega) - X_1(\omega): \omega \in B\}))$ and therefore, Q' may not end up being a martingale measure.

We shall now provide a necessary and sufficient condition for a martingale measure to exist in terms of conditions of the type $0 \in \mathrm{cl}(\mathrm{conv}(\{X_{t+1}(\omega) - X_t(\omega): \omega \in A\}))$. The condition will be formulated in terms of an algorithm which labels both the partition sets $A \in \mathcal{P}_t$ ($t \in \{0,1,\ldots,T-1\}$) and the transitions leading from $B \in \mathcal{P}_t$ to $A \in \mathcal{P}_{t+1}$, where $A \subset B$ and $0 \leq t < T$.

The labeling algorithm

1. Labeling at $t = 0$.

The unique partition set is $A = \Omega$. Let $A_1, A_2, \ldots, A_n$ be an enumeration of the elements of $\mathcal{P}_1$ and let $X_1(\omega_i)$ denote the value of X_1 on A_i ($i = 1,2,\ldots,n$).

- If $0 \notin \mathrm{cl}(\mathrm{conv}(\{X_1(\omega) - X_0(\omega): \omega \in A\}))$, give A the label (-).
- If $0 \in \mathrm{cl}(\mathrm{conv}(\{X_1(\omega) - X_0(\omega): \omega \in A\}))$, give A the label (+).

Moreover, by Theorem 2.2, the system of equations

$$\sum_{i=1}^{n} \lambda_i(X_1(\omega_i) - X_0(\omega_i)) = 0, \quad \sum_{i=1}^{n} \lambda_i = 1, \ \lambda_i \geq 0 \ (i = 1,\ldots,n)$$

has a solution $(\overline{\lambda}_i)_{i=1,\ldots,n}$. Label the transition $A \to A_i$ by (1) or (0) according to whether $\overline{\lambda}_i > 0$ or $\overline{\lambda}_i = 0$ $(i = 1,\ldots,n)$.

2. <u>Labeling at t, where $t \in \{1,2,\ldots,T-1\}$</u>.

Label the partition set $A \in \mathcal{P}_t$ only if there is some $B \in \mathcal{P}_{t-1}$ with $A \subset B$ such that the transition $B \to A$ is labeled (1). Let $A_1, A_2, \ldots, A_n$ denote the elements of $\mathcal{P}_{t+1}$ for which $A_i \subset A$ and let $X_{t+1}(\omega_i)$ denote the value of X_{t+1} on A_i $(i = 1,2,\ldots,n)$.

- If $0 \notin \mathrm{cl}(\mathrm{conv}(\{X_{t+1}(\omega) - X_t(\omega): \omega \in A\}))$, give A the label (-).
- If $0 \in \mathrm{cl}(\mathrm{conv}(\{X_{t+1}(\omega) - X_t(\omega): \omega \in A\}))$, give A the label (+).

Moreover, by Theorem 2.2, the system of equations

$$\sum_{i=1}^{n} \lambda_i(X_{t+1}(\omega_i) - X_t(\omega_i)) = 0, \quad \sum_{i=1}^{n} \lambda_i = 1, \ \lambda_i \geq 0 \ (i = 1,\ldots,n)$$

has a solution $(\overline{\lambda}_i)_{i=1,\ldots,n}$. Label the transition $A \to A_i$ by (1) or (0) according to whether $\overline{\lambda}_i > 0$ or $\overline{\lambda}_i = 0$ $(i = 1,\ldots,n)$.

<u>Theorem 4.3</u>. $\mathcal{M}(X) \neq \emptyset$ (i.e. $\exists Q \in \mathcal{M}(X)$ with $Q << P$) if and only if there exists a way of labeling the partition sets (via the labeling algorithm described above) so that Ω is labeled (+) and all other partition sets are either labeled (+) or not labeled at all.

Thus, $\mathcal{M}(X) = \emptyset$ if and only if some partition set has to be labeled (-), that is iff there is no condition of the type "0 belongs to the corresponding closure" for that partition set. Note that in general, the system of equations at hand, discussed in the algorithm, may have different solutions so that there may be different ways of labeling the transitions between partition sets and therefore different ways of labeling the partition sets themselves. Hence, to ensure that $\mathcal{M}(X) = \emptyset$, it is necessary to verify that some partition set is labeled (-) and that no alternate labeling for that partition set is possible.

The next theorem characterizes the extreme points of the set $\mathcal{M}(X)$.

<u>Theorem 4.4</u>. Suppose $Q \in \mathcal{M}(X)$.
Then $Q \in \mathcal{M}_e(X)$ if and only if the $(Q,(\mathcal{F}_t))$ - martingale X satisfies the following two conditions:

(i) X_0 is a vector in R^d which is $\mathcal{F}_0$-measurable.

(ii) For each $0 \leq t < T$ and each set $A \in \mathcal{P}_t$ with $Q[A] > 0$, the conditional distribution of the increment $X_{t+1} - X_t$ given A has mean zero, and puts positive mass on exactly $n_{t+1,A}$ points, where
$n_{t+1,A} = \dim\{\mathrm{span}\{X_{t+1}(\omega) - X_t(\omega): \omega \in A,\ Q[\{\omega\}] > 0\}\} + 1.$

Proof. We may assume that Q is "decomposed" into an initial distribution and successive transition probabilities, characterized by the conditional distribution of $X_{t+1} - X_t$ given $A \in \mathcal{P}_t$ $(t = 0,1,\ldots,T-1)$. Note that Q is an extreme point of $M(X)$ if and only if each single transition probability is extreme in the sense of section 2. Now apply Theorem 3.1 to each transition. □

Corollary 4.1. If $Q \sim P$ and $Q \in M_e(X)$ then Q is unique.

Proof. In the proof of the preceeding theorem, apply Corollary 3.1 instead to Theorem 3.1. □

REFERENCES

Jacod, J. (1979) Calcul Stochastique et Problèmes de Martingles, Lecture notes in Math. 714 (Springer-Verlag, Berlin, Heidelberg, New York).

Murty, K.G. (1976), Linear and Combinatorial Programming, (Wiley, New York).

Rockafellar, R.T. (1970), Convex Analysis (Princeton Univ. Press).

Taqqu, M.S. and Willinger, W. (1984), The analysis of finite security markets using martingales. Operations Research Technical Report No. 636, Cornell University.