J': A New Triangulation of $\mathbb{R}^n$

by

Michael J. Todd*

*Research partially supported by a fellowship from the Alfred P. Sloan Foundation and by NSF Grant ENG82-15361.

Typist: Anne T. Kline
Abstract

This paper introduces a new triangulation of $\mathbb{R}^n$ and two families of related triangulations. Our interest is primarily in the use of such triangulations in piecewise-linear homotopy algorithms for solving systems of nonlinear equations, and we provide both theoretical and computational evidence of the efficiency of the new triangulations for this purpose. However, the triangulations we propose may also be of independent interest.
1. Introduction

This paper introduces a new triangulation of \( \mathbb{R}^n \) and two families of related triangulations. Our interest is primarily in the use of such triangulations in piecewise-linear homotopy algorithms for solving systems of nonlinear equations - see, e.g., Allgower and Georg [1], Eaves [5], and Todd [14,18]. However, the triangulations we propose may be of independent interest.

In three dimensions the new triangulation \( J' \) is similar to the \( A^*K_1 \) triangulation of van der Laan and Talman [8] and identical to a triangulation proposed by Buneman in a different context [4]. For dimensions greater than three it is very similar to the triangulation \( J_1 \). We will assume that the reader is familiar with Tucker's triangulation \( J_1 \) and Freudenthal's triangulation \( K_1 \) - see, e.g., [15]. In section 2 we define \( J' \) and prove it a triangulation of \( \mathbb{R}^n \), i.e., a locally finite collection of \( n \)-simplices covering \( \mathbb{R}^n \) such that any two intersect in a common face (perhaps empty). To do this we show that it can also be viewed as a polyhedral subdivision of \( \mathbb{R}^n \) generated by a family of hyperplanes.

Let \( H \) be a family of hyperplanes that is the union of a finite number of families of evenly-spaced parallel hyperplanes. Then \( H \) divides \( \mathbb{R}^n \) into polyhedral sets, and it is easy to see that the result is a polyhedral subdivision of \( \mathbb{R}^n \), i.e., a locally finite collection of \( n \)-polyhedra such that any two intersect in a common face. We say the polyhedral subdivision is generated by the family \( H \) of hyperplanes.

This technique also provides an easy proof that \( J_1 \) and \( K_1 \) are triangulations.
In section 3 we consider the special case when $n = 3$. Section 4 calculates various measures of the triangulation $J'$. While it does not subdivide a cube (or paralleloiped) into a small number of simplices, it appears superior to $K_1$ and dominates $J_1$ according to average directional (or surface) density.

Section 5 demonstrates how two families of triangulations, $J'_k(n)$ and $J''_k(n)$, of $\mathbb{R}^n$ are "induced" by $J'$. Various members of these families are suited to various piecewise-linear homotopy methods. Finally, in section 6 we give some computational experience with the new triangulation $J'_{n+1}(n+1)$ in a restart algorithm. A consistent improvement is observed.

Our notation is as follows. Subscripts of vectors denote coordinates while superscripts are used for sequences. The jth unit vector is denoted $e^j$ and $e$ is the vector of ones. We use $[a, b, ..., z]$ to denote the convex hull of the vectors $a, b, ..., z$.

2. The Triangulation $J'$

In this section we define the new triangulation $J'$, giving descriptions of each simplex both by its vertices and by its facets. We also prove that $J'$ is indeed a triangulation and state its pivot rules.

First we define $J'$ and its simplices via their vertices.

**Definition 2.1** The set of vertices of $J'$ is the set of vectors $v \in \mathbb{R}^n$ with each component an integer, such that there is not precisely one even component nor precisely one odd component. Each simplex $\sigma$ of $J'$ is of the form $\sigma = j'(v, \pi, s) = [v^0, ..., v^n]$, where $v$ is a vector each of whose
components is an even integer, \( \pi = (\pi(1), \ldots, \pi(n)) \) is a permutation of \( (1, \ldots, n) \) and \( s \) is a sign vector (each \( s_j = \pm 1 \)) with \( s_{\pi(1)} = s_{\pi(n)} = 1 \).

To define the vertices of \( \sigma \) it is convenient to let \( \tilde{e}^j \) denote \( s_{\pi(i)} e^{\pi(i)} \), where \( e^j \) is the jth unit vector. Thus \( \tilde{e}^1, \ldots, \tilde{e}^n \) are possibly permuted and reversed unit vectors. For \( n \geq 4 \), we have

\[
\begin{align*}
v^0 &= v ; \\
v^1 &= v^0 + 2\tilde{e}^1 ; \\
v^2 &= v^1 - \tilde{e}^1 + \tilde{e}^2 ; \\
v^j &= v^{j-1} + \tilde{e}^j, \quad 3 \leq j \leq n - 2 ; \\
v^{n-1} &= v^{n-2} + \tilde{e}^{n-1} - \tilde{e}^n ; \\
v^n &= v^{n-1} + 2\tilde{e}^n .
\end{align*}
\]

For \( n = 3 \), the formulae for \( j = 2 \) and \( j = n - 1 \) are combined to give

\[
v^2 = v^1 - \tilde{e}^1 + \tilde{e}^2 - \tilde{e}^3 .
\]

Note that the restrictions \( s_{\pi(1)} = s_{\pi(n)} = 1 \) are only present to give a one-to-one correspondence between simplices and their descriptions \( j'(v, \pi, s) \). Without this restriction \( v = v^0 \) could be replaced by \( v = v^1 \) with \( s_{\pi(1)} = -1 \), and \( s_{\pi(n)} = 1 \) could be replaced by \( s_{\pi(n)} = -1 \). There are other ways to resolve the non-uniqueness - we could insist that \( v_{\pi(1)} \)
be a multiple of 4 and that $v^n_{\pi(n)}$ be one more than a multiple of 4, for instance, but our choice is simpler if less symmetrical than other possibilities.

We remark that we could also insist that each component of $v$ be odd rather than even. The same simplex $\sigma$ is of the form $j'(v', \pi', -s+2e^{-1}+2e^n)$ where $v' = v^{n-1}$ and $\pi' = (\pi(n), \ldots, \pi(1))$. Thus $J'$ is invariant under permutations of coordinates, under reflections in coordinate hyperplanes and under translations by vectors with each component even or each component odd.

Note that each simplex of $J'$ is the union of four simplices of $J_1$. Indeed, $\sigma = j'(v, \pi, s) \in J'$ is the union of $j_1(v, \pi, s)$, $j_1(v, \pi, s-2e^n)$, $j_1(v+2e^{-1}, \pi, s-2e^{-1})$ and $j_1(v+2e^{-1}, \pi, s-2e^{-1}-2e^n)$.

Our first task is to obtain a facetal description of $J'$. To this end, let $\sigma = j'(v, \pi, s) = [v^0, \ldots, v^n]$. Let $V$ be the matrix with ith column \( (\begin{array}{l} 1 \\ \vdots \\ 1 \end{array}) \) for $0 \leq i \leq n$, and let $V_0$ be the matrix whose every column is \( (\begin{array}{l} 0 \\ \vdots \\ 0 \end{array}) \). Let $S$ be the diagonal matrix with diagonal entries $1, s_1, \ldots, s_n$, $P$ the permutation matrix \( (\begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \end{array}) \), where the ith column of $\hat{P}$ is $e^{\pi(i)}$, and

$$
Y = \begin{bmatrix}
1 & 1 & 1 & \ldots & 1 \\
2 & 1 & \ldots & 1 \\
1 & \ldots & 1 \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & & \ddots & \ddots & \ddots \\
\vdots & & & \ddots & \ddots & \ddots \\
1 & 1 \\
-1 & 1
\end{bmatrix}
$$

(2.2)
It is easy to see that

\[ V = V_0 + S \, P \, Y. \]  \hspace{1cm} (2.3)

Since each of \( S, P \) and \( Y \) is clearly nonsingular, this implies that \( V \) is also, since \( V - V_0 \) results from subtracting multiples of its zeroth row from each other row. Thus \( \sigma \) is indeed a simplex, i.e., its vertices are affinely independent.

To express an arbitrary vector \( x \in \mathbb{R}^n \) as an affine combination of \( v^0, \ldots, v^n \), we need to solve \( V_\lambda = \left( \begin{array}{c} 1 \\ x \end{array} \right) \); since \( e^T \lambda = 1 \) implies \( V_0 \lambda = \left( \begin{array}{c} 0 \\ v_0 \end{array} \right) \), writing \( z = \left( \begin{array}{c} 1 \\ x - v_0 \end{array} \right) \) we find \( (V - V_0) \lambda = z \) or, using (2.3)

\[ \lambda = Y^{-1} \, p^T \, S \, z \]  \hspace{1cm} (2.4)

It is easy to check that

\[ Y = \begin{bmatrix}
1 & -1/2 & -1/2 \\
1/2 & -1/2 \\
-1 & 1 & -1 \\
1/2 & -1/2 \\
1/2 & 1/2
\end{bmatrix} \]  \hspace{1cm} (2.5)
Hence we obtain

**Proposition 2.2** Given \( \sigma = j'(v, \pi, s) \) and \( x \in \mathbb{R}^n \), let

\[
 w_i = s_{\pi(i)}(x_{\pi(i)} - v_{\pi(i)}) \quad 1 \leq i \leq n.
\]

Then \( x \in \sigma \) iff

\[
 2 - w_2 \geq w_1 \geq w_2 \geq \cdots \geq w_{n-1} \geq w_n \geq - w_{n-1}.
\]

(2.6)

**Proof** Note that \( P^T \Sigma z = \begin{pmatrix} 1 \\ w \end{pmatrix} \) and that \( x \in \sigma \) iff \( \lambda \) in (2.4) is nonnegative. \( \Box \)

We call (2.6) a facetal description of \( \sigma \).

**Theorem 2.3** \( J' \) is a triangulation of \( \mathbb{R}^n \). In fact, \( J' \) is also the subdivision of \( \mathbb{R}^n \) generated by all hyperplanes of the form \( x_i \pm x_j \in 2\mathbb{Z} \).

**Proof** It is clear that \( J' \) is locally finite, i.e., that each point of \( \mathbb{R}^n \) has a neighborhood meeting only finitely many simplices of \( \mathbb{R}^n \). We complete the proof by showing that \( J' \) is indeed the simplicial subdivision claimed. First note that, by proposition 2.2, each simplex of \( J' \) is bounded by hyperplanes of the form \( x_i \pm x_j \in 2\mathbb{Z} \). Next we show that each simplex is a simple piece of the subdivision. If not, there is some hyperplane that cuts a simplex of \( J' \), and hence, that has two vertices of the simplex strictly on opposite sides. Without loss of generality, we can assume that the hyperplane is \( x_1 = x_2 \) and the simplex \( \sigma = j'(v, \pi, s) \), where \( i = \pi^{-1}(1) < \pi^{-1}(2) = j \). By considering the cases \( i = 1, j = 2, i = 1, 2 < j < n, i = 1, j = n, 2 < i < j < n, 2 < i < n-1, j = n \) and
i = n - 1, j = n we can easily show that all vertices of \( \sigma \) lie on the same side of the hyperplane.

Finally we must show that there are no other pieces of the subdivision. Thus we show that each \( x \in \mathbb{R}^n \) lies in some \( \sigma \in J' \). For each \( i \) let \( v_i \) be a closest even integer to \( x_i \) and choose a permutation \( \pi \) and a sign vector \( s \) so that

\[
1 \geq w_1 \geq ... \geq w_n \geq 0
\]

where \( w_i = s_{\pi(i)}(x_{\pi(i)} - v_{\pi(i)}) \). Since \( w_2 \leq 1 \), we have \( 2 - w_2 \geq 1 \geq w_1 \) and since \( w_{n-1} \geq 0 \) we have \( w_n \geq 0 \geq -w_{n-1} \). Thus (2.6) holds.

However we may have \( s_{\pi(1)} \) or \( s_{\pi(n)} \) equal to \(-1\). In the latter case, we may simply reset \( s_{\pi(n)} \) to \(+1\) so that \( w_n \) switches sign but \( w_{n-1} \geq w_n \geq -w_{n-1} \) still holds. If \( s_{\pi(1)} = -1 \) the reset \( s_{\pi(1)} \) to \(+1\) and decrease \( v_{\pi(1)} \) by 2. Then \( w_1 \) becomes \( 2 - w_1 \) so that the inequalities \( 2 - w_2 \geq w_1 \geq w_2 \) are still satisfied. After these changes (2.6) holds with \( s_{\pi(1)} = s_{\pi(n)} = 1 \), so that \( x \) belongs to \( J'(v, \pi, s) \) as desired. \( \square \)

To conclude this section we give the pivot rules of \( J' \). Suppose \( \tilde{\sigma} = j'(\overline{v}, \overline{\pi}, \overline{s}) \) contains all vertices of \( \sigma = j'(v, \pi, s) \) except \( v^i \), with \( \overline{\sigma} \neq \sigma \). Then we can obtain \( \overline{v}, \overline{\pi}, \overline{s} \) and the index \( j \) of the new vertex of \( \overline{\tilde{\sigma}} \) from the table below. As in definition 2.1, \( \tilde{e}^i \) denotes \( s_{\pi(i)}e_{\pi(i)} \).
<table>
<thead>
<tr>
<th>$i$</th>
<th>$s_{\pi(2)}$</th>
<th>$v$</th>
<th>$\pi$</th>
<th>$s$</th>
<th>$j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$+1$</td>
<td>$v + 2\tilde{e}^1$</td>
<td>$(\pi(2), \pi(1), \ldots, \pi(n))$</td>
<td>$s - 2\tilde{e}^1$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$-1$</td>
<td>$v + 2\tilde{e}^1 + 2\tilde{e}^2$</td>
<td>$(\pi(2), \pi(1), \ldots, \pi(n))$</td>
<td>$s - 2\tilde{e}^1 - 2\tilde{e}^2$</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$+1$</td>
<td>$v$</td>
<td>$(\pi(2), \pi(1), \ldots, \pi(n))$</td>
<td>$s$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$-1$</td>
<td>$v + 2\tilde{e}^2$</td>
<td>$(\pi(2), \pi(1), \ldots, \pi(n))$</td>
<td>$s - 2\tilde{e}^2$</td>
<td>0</td>
</tr>
<tr>
<td>$1 &lt; i &lt; n-1$</td>
<td>$v$</td>
<td>$(\pi(1), \ldots, \pi(i+1), \pi(i), \ldots, \pi(n))$</td>
<td>$s$</td>
<td>$i$</td>
<td></td>
</tr>
<tr>
<td>$i=n-1$</td>
<td>$+1$</td>
<td>$v$</td>
<td>$(\pi(1), \ldots, \pi(n-2), \pi(n), \pi(n-1))$</td>
<td>$s$</td>
<td>$n-1$</td>
</tr>
<tr>
<td></td>
<td>$-1$</td>
<td>$v$</td>
<td>$(\pi(1), \ldots, \pi(n-2), \pi(n), \pi(n-1))$</td>
<td>$s - 2\tilde{e}^{n-1}$</td>
<td>$n$</td>
</tr>
<tr>
<td>$i=n$</td>
<td>$+1$</td>
<td>$v$</td>
<td>$(\pi(1), \ldots, \pi(n-2), \pi(n), \pi(n-1))$</td>
<td>$s - 2\tilde{e}^n$</td>
<td>$n-1$</td>
</tr>
<tr>
<td></td>
<td>$-1$</td>
<td>$v$</td>
<td>$(\pi(1), \ldots, \pi(n-2), \pi(n), \pi(n-1))$</td>
<td>$s - 2\tilde{e}^{n-1} - 2\tilde{e}^n$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

Table 2.4
3. The 3-dimensional Case

When \( n = 3 \), each simplex of \( J' \) is of the form \( \sigma = j'(v, \pi, s) = [v^0, v^1, v^2, v^3] \) where

\[
\begin{align*}
    v^0 &= v \\
    v^1 &= v^0 + 2e^{\pi(1)} \\
    v^2 &= v^1 - e^{\pi(1)} + s_{\pi(2)}e^{\pi(2)} - e^{\pi(3)} \\
    v^3 &= v^2 + 2e^{\pi(3)}
\end{align*}
\]

In this section, we show that \( J' \) is similar to \( A^*_K \), where \( A^*_K \) is the matrix \((n+1 + \sqrt{n+1})I - ee^T\), the "optimal" linear transformation of the Freudenthal triangulation \( K_1 \); see van der Laan and Talman [8], Eaves [6], and [20]. Here similar means "obtainable by an orthogonal matrix and a scaling."

First note that, for any \( n \), \( A^*_K \) are similar. Indeed, \( A^*_K = ((n+1 - \sqrt{n+1})/(n+1 + \sqrt{n+1})) (I - 2ee^T/n) A^*_K \), and \( I - 2ee^T/n \) is easily seen to be orthogonal.

Thus it suffices to show that \( A^*_K \) and \( J' \) are similar when \( n = 3 \). In fact, we will show them to be identical. For \( n = 3 \),

\[
A^*_K = \begin{bmatrix}
1 & -1 & -1 \\
-1 & 1 & -1 \\
-1 & -1 & 1
\end{bmatrix}.
\]
We will denote the columns of $A^* a^1, a^2$ and $a^3$. Note first that the set of vertices of $A^*_1$ is \( \{v \in \mathbb{Z}^3 : \text{all components of } v \text{ are even or all are odd} \} = \{v \in \mathbb{Z}^3 : v \text{ does not have exactly 1 nor exactly } n - 1 = 2 \text{ odd components} \} \). Thus the two triangulations have the same vertices.

Let $\sigma = [v^0, v^1, v^2, v^3] \in A^*_1$, so that $v^0$ has all components odd or all even,

\[
v^1 = v^0 + a^i, \quad v^2 = v^1 + a^j, \quad v^3 = v^2 + a^k
\]

for some permutation \((i, j, k)\) of \((1, 2, 3)\). Then either $v^2$ or $v^3$ has all components even. In the first case,

\[
v^1 = v^3 + 2e^i, \quad v^2 = v^1 - e^i + e^j - e^k, \quad v^0 = v^2 + 2e^k,
\]

so $\sigma = [v^3, v^1, v^2, v^0] \in J'$. In the second case,

\[
v^0 = v^2 + 2e^k, \quad v^3 = v^0 - e^k - e^j - e^i, \quad v^1 = v^3 + 2e^i,
\]

so $\sigma = [v^2, v^0, v^3, v^1] \in J'$.

Conversely, let $\sigma = [v^0, v^1, v^2, v^3] \in J'$, so that $v^0$ has all components even,

\[
v^1 = v^0 + 2e^i, \quad v^2 = v^1 - e^i + s_j e^j - e^k, \quad v^3 = v^2 + 2e^k
\]

for some permutation \((i, j, k)\) of \((1, 2, 3)\) and some $s_j \in \{+1, -1\}$. 
Then if \( s_j = +1 \),

\[
v^1 = v^3 + a^i, v^2 = v^1 + a^j, v^0 = v^2 + a^k,
\]

so \( \sigma = [v^3, v^1, v^2, v^0] \in \mathbb{A}_-^1 \). On the other hand, if \( s_j = -1 \), we have

\[
v^3 = v^1 + a^k, v^0 = v^3 + a^j, v^2 = v^0 + a^i,
\]

so \( \sigma = [v^1, v^3, v^0, v^2] \in \mathbb{A}_+^1 \). This completes the proof that \( J' \) is identical to \( \mathbb{A}_-^1 \), and thus is similar to \( \mathbb{A}_+^1 \).

4. Measures for \( J' \)

In this section we compute several measures to evaluate the new triangulation \( J' \).

The first crude measure is the number of simplices used to triangulate the unit cube \([0,1]^n\). This is \( n! \) for \( J_1 \) and \( K_1 \), but there are triangulations with far fewer simplices in the unit cube - see Lee [10] and Sallee [13]. Unfortunately, the unit cube is not triangulated by \( J' \) whose generating hyperplanes are of the form \( x_i \pm x_i \in 2\mathbb{Z} \); nor is any cube for odd \( n \). However, we can find parallelopipeds that are triangulated by \( J' \) for each \( n \). One such is

\[
\mathbb{C}^n = \{ x \in \mathbb{R}^n : 0 \leq x_{2i-1} - x_{2i} \leq 2, 0 \leq x_{2i-1} + x_{2i} \leq 2, \\
i = 1, 2, \ldots, [n/2], \text{ and } 0 \leq x_{n-1} + x_n \leq 2 \text{ if } n \text{ is odd} \}
\]
Let $A_n$ be the $n \times n$ matrix given by

$$\begin{bmatrix}
1 & 1 \\
1 & -1 \\
& & \ddots & 1 \\
& & & & 1 & 1 \\
& & & & 1 & -1
\end{bmatrix}, \quad A_{2k+1} = 
\begin{bmatrix}
1 & 1 \\
1 & -1 \\
& & \ddots & 1 & 1 \\
& & & & 1 & -1 \\
& & & & & & 1 & 1
\end{bmatrix};$$

then $\mathcal{C}_n = \{x \in \mathbb{R}^n : 0 \leq y = A_n x \leq 2e\}$. Since $A_n$ has determinant $2^{\lfloor n/2 \rfloor}$ and $(y \in \mathbb{R}^n : 0 \leq y \leq 2e)$ has volume $2^n$, $\mathcal{C}_n$ has volume $2^{\lceil n/2 \rceil}$. Since each simplex of $J'$, as the union of four simplices of $J_1$, has volume $4/n!$ (this can easily be seen directly from (2.2), (2.3)), we obtain

**Theorem 4.1** There are parallelopipeds in $\mathbb{R}^n$ that are triangulated by $J'$ into $n! \cdot 2^{\lceil n/2 \rceil}/4$ simplices.

For $n \geq 4$, this measure is worse than $J_1$ and $K_1$. To me, this indicates the inadequacies of the measure - the problem is not the simplices in $J'$ but the lack of small parallelopipeds. Notice also that, for $n \geq 4$, the triangulation of $\mathcal{C}_n$ has vertices that are not vertices of $\mathcal{C}_n$.

While the simplices of $J'$ have volume four times those of $J_1$ or $K_1$, they share the small mesh size of these triangulations of the unit cube. The mesh size of a triangulation is the supremum of the diameters of its simplices, or the supremum of the lengths of its 1-simplices. In fact, we have
Proposition 4.2 The mesh size of \( J' \) is \( \max(2, \sqrt{n}) \).

Next we compute the average directional density of \( J' \) [16]. This is, roughly, the rate at which a random straight line meets facets of \( J' \) per unit length. Alternatively, Eaves and Yorke [7] have shown that it is the surface density, i.e. the surface area of simplices per unit volume, up to a scale factor. Since \( x \) lies in a facet of a simplex of \( J' \) iff \( x_i \pm x_j \) is an even integer for some \( i, j \), we obtain by the arguments of [16]:

Theorem 4.3 The directional density of \( J' \) in direction \( d \) is \( N(J', d) = \sum_{i<j} 1/2(\|d_i + d_j\| + \|d_i - d_j\|) \) and its average directional density is \( \binom{n}{2}/\sqrt{n} g_n \), where \( g_n = 2\Gamma(n/2)/(n-1)^{\frac{n}{2}} \Gamma(n-1)/2 \).

In a companion paper [20], we show that, of all triangulations \( AJ' \), with \( A \) a nonsingular linear transformation such that \( AJ' \) has the same mesh size as \( J' \), \( J' \) itself has the smallest average directional density.

5. Triangulations induced by \( J' \)

Note that \( J' \) refines the cubical subdivision of \( \mathbb{R}^n \), that is, the polyhedral subdivision generated by all hyperplanes of the form \( x_i \pm x_j = 0 \) (whose pieces are cones with faces of a cube centered at the origin as cross sections). This follows directly from theorem 2.3. An immediate implication is that \( J' \) can be used in the octahedral piecewise-linear homotopy algorithm of Wright [21].

However, \( J' \) does not refine the octahedral (or orthant) subdivision of \( \mathbb{R}^n \), since \( x_i = 0 \) is not among its generating hyperplanes. Thus it cannot be used in the cubical (or \( 2n \)-) algorithm of van der Laan and Talman [9].
and Reiser [12]. Even more apparently limiting is the fact that its \((n+1)\)-dimensional version does not also triangulate \(R^n \times [0,1]\), and thus it cannot directly be used in a restart method such as Merrill's [11].

In this section we show that a wealth of other triangulations are induced by \(J'\), so that these objections lose their force. Indeed, it follows from, e.g., theorems 3.1 and 7.1 of [19] that, if \(S\) is a triangulation of \(R^n\) generated by a family of hyperplanes, and if \(H\) is one of these hyperplanes, then \(T = \{ \tau \in H \} \) triangulates \(H\). We say \(T\) is induced by \(S\). If \(\alpha\) is an affine isomorphism between \(H\) and \(R^{n-1}\), then \(\alpha T\) is a triangulation of \(R^{n-1}\), and we also say \(\alpha T\) is induced by \(S\). We use these ideas to construct from \(J'\) two families of triangulations. In order to specify the dimension, we write \(J'(n)\) for the triangulation \(J'\) of \(R^n\).

**Definition 5.1** For \(1 \leq k \leq n + 1\), let \(J'_k(n)\) denote the polyhedral subdivision of \(R^n\) generated by the hyperplanes \(x_i \pm x_j \in Z\), \(1 \leq i < j \leq n\) and \(x_i \in Z\), \(k \leq i \leq n\), and let \(J''_k(n)\) denote that generated by the hyperplanes \(x_i \pm x_j \in Z\), \(1 \leq i < j < n\), \(x_i \in 2\mathbb{Z}\), \(1 \leq i < k\), and \(x_i \in Z\), \(k \leq i \leq n\). We write \(J''(n)\) for \(J''_{n+1}(n)\) (note that \(J'(n) = J'_{n+1}(n)\)). Also note that \(J'_1(n) = J''_1(n) = J_1(n)\).

In this section, we shall prove

**Theorem 5.2** \(J'_k(n)\) is a triangulation of \(R^n\) for \(n = 1, 2\) and \(1 \leq k \leq n\) and for \(n \geq 3\) and \(1 \leq k \leq n + 1\). \(J''_k(n)\) is a triangulation of \(R^n\) for \(n \geq 1\) and \(1 \leq k \leq n + 1\).

Let \(B_{kn}\), for \(n \geq 2\) and \(2 \leq k \leq n\), denote the \(n \times n\) matrix
Thus if \( T(n) \) is a subdivision of \( \mathbb{R}^n \) generated by hyperplanes containing \( x_{k-1} \pm x_k \in 2\mathbb{Z} \), then the nonsingular transformation \( x + y = B_{kn} x \) takes \( T(n) \) into a subdivision \( B_{kn} T(n) \), also generated by hyperplanes, and \( y_{k-1} \in \mathbb{Z}, y_k \in \mathbb{Z} \) are among these.

Next let \( S(n) \) be a hyperplane-generated subdivision of \( \mathbb{R}^n \) with \( x_k = 0 \) one of its generating hyperplanes, \( 1 \leq k \leq n \). Then \( S(n) \) induces
a subdivision $S'$ of $H = \{ x \in \mathbb{R}^n : x_k = 0 \}$. Let $\alpha$ be the affine isomorphism of $H$ and $\mathbb{R}^{n-1}$ defined by

$$
\alpha(x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n) = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n).
$$

Then we denote the subdivision $\alpha S'$ of $\mathbb{R}^{n-1}$ by $P_k S(n)$. Finally, let $Q_k T(n) = P_k B_{kn} T(n)$.

Note that, from the remarks above Definition 5.1, if $T(n)$, $S(n)$ are triangulations then so are $Q_k T(n)$ and $P_k S(n)$.

**Lemma 5.3**

(a) $Q_{k-1} J_k^i(n) = J_{k-2}^i(n-1)$ ($3 \leq k \leq n + 1$).

(b) $P_n J_k^i(n) = J_k^i(n-1)$ ($1 \leq k \leq n$).

(c) $Q_n J_k^i(n) = J_k^i(n-1)$ ($1 \leq k < n$).

(d) $Q_k J_k^i(n) = J_{k-1}^i(n-1)$ ($2 \leq k \leq n$).

(e) $Q_{k-1} J_k^u(n) = J_{k-2}^u(n-1)$ ($3 \leq k \leq n + 1$).

(f) $P_n J_k^u(n) = J_k^u(n-1)$ ($1 \leq k \leq n$).

(g) $Q_n J_k^u(n) = J_k^u(n-1)$ ($1 \leq k < n$).

(h) $Q_k J_k^u(n) = J_{k-1}^u(n-1)$ ($2 \leq k \leq n$).

(i) $P_{k-1} J_k^u(n) = J_{\min\{k, n\}}^u(n-1)$ ($2 \leq k \leq n + 1$).
Proof  In each case we merely need to check the generating hyperplanes. We will show (a) and (b) - the reader will have no difficulty in verifying (c)-(i).

(a) Consider \( y = B_k x, x = B_k^{-1} y \). Then the hyperplanes \( x_i \pm x_j \in 2\mathbb{Z} \), for \( \{i,j\} \cap \{k-2,k-1\} = \emptyset \) become \( y_i \pm y_j \in 2\mathbb{Z} \). The hyperplanes \( x_{k-2} \pm x_{k-1} \in 2\mathbb{Z} \) become \( y_{k-2} \in \mathbb{Z}, y_{k-1} \in \mathbb{Z} \). The hyperplanes \( x_i \pm x_{k-2} \in 2\mathbb{Z} \) and \( x_i \pm x_{k-1} \in 2\mathbb{Z} \), \( i \neq \{k-2,k-1\} \), become \( y_i \pm y_{k-2} \pm y_{k-1} \in 2\mathbb{Z} \). Finally, the hyperplanes \( x_i \in \mathbb{Z}, i \geq k \), become \( y_i \in \mathbb{Z}, i \geq k \). Now intersecting these hyperplanes with \( y_{k-1} = 0 \) and projecting down to \( \mathbb{R}^{n-1} \) gives the hyperplanes \( x_i \pm x_j \in 2\mathbb{Z} \), all \( i,j \) and \( x_i \in \mathbb{Z} \), \( i \geq k-2 \), as desired.

(b) Consider the effect of intersecting hyperplanes with \( x_n = 0 \) and then projecting down to \( \mathbb{R}^{n-1} \). The hyperplanes \( x_i \pm x_j \in 2\mathbb{Z} \), \( i < j < n \), remain the same, as do \( x_i \in \mathbb{Z}, k \leq i < n \). The hyperplanes \( x_i \pm x_n \in 2\mathbb{Z} \), \( i < n \), become \( x_i \in 2\mathbb{Z}, i < n \). Thus we have the generating family of \( J_k^n(n-1) \). \( \square \)

Proof of Theorem 5.2  Applying (a) inductively implies that each \( J_k^i(n) \) is a triangulation. Indeed, if we have established that all \( J_k^i(n) \) where \( n \geq m \) and \( n - k \leq \ell \) are triangulations, then (a) shows that all \( J_k^i(n) \) for \( n \geq m - 1 \) and \( n - k \leq \ell + 1 \) are triangulations. The case \( n = k = 1 \) is trivial. Similarly, (b) now shows that all \( J_k^n(n) \) are triangulations. \( \square \)

Note that parts (f) and (g) of the lemma show that \( J_1 \) induces only copies of itself. A similar analysis demonstrates that \( K_1 \) also induces only copies of itself.
Let us tabulate the symmetry properties of these triangulations. By invariance under permutations, we mean under permutations of \{1,\ldots,n\} that leave \{1,\ldots,k-1\} and \{k,\ldots,n\} invariant, and by invariance under reflection, we mean under transformations \(x_i \mapsto -x_i\) for any \(i\). Even translations are translations by vectors with all components even integers. Table 5.4 also notes which triangulations refine the cubical and octahedral (orthant) subdivisions of \(\mathbb{R}^n\) and which also triangulate \(\mathbb{R}^{n-1}x[0,1]\).

Among the new triangulations we have introduced, \(J'(n)\) seems most suitable for the octahedral algorithm and \(J''(n)\) for the cubical algorithm. We also recommend the restriction of \(J_{n+1}^{'}(n+1)\) to \(\mathbb{R}^nx[0,1]\) for use in a restart algorithm. There is, however, another candidate; we may use \(Q_{n+1,n+1}^{'}(n+1)\). This triangulation is generated by hyperplanes \(x_i + x_j \in 2\mathbb{Z}, i < j < n; x_i + x_n + x_{n+1} \in 2\mathbb{Z}, i < n;\) and \(x_n \in \mathbb{Z}, x_{n+1} \in \mathbb{Z}\). It appears at first sight (and based on its directional density) to be inferior to \(J_{n+1}^{'}(n+1)\). However, \(Q_{n+1,n+1}^{'}(n+1)\) has vertices with \(x_{n+1} = 1/2\). For example, consider its simplex \([0, 2e^1, e^{1+1/2}e^4+1/2e^5, e^1e^5, e^1+e^5-e^2-e^3, e^1+e^5-e^2+e^3]\) when \(n = 4\). In a restart algorithm we may choose the image of such a vertex arbitrarily. By making this choice appropriately, we may identify two vertices of several simplices, thus obtaining a new and simplified triangulation. For example, if we round \(x_{n+1}\) to the nearest odd integer and \(x_n\) to the nearest even integer when either (and hence both) is an odd multiple of \(1/2\), then the sample simplex above disappears. The details of this approach have not yet been worked out, but there remains the possibility that some such modification of \(Q_{n+1,n+1}^{'}(n+1)\) will be a reasonable choice for a restart method.
<table>
<thead>
<tr>
<th>Invariant under</th>
<th>( J'(n) )</th>
<th>( J'_k(n) ) for ( 1 &lt; k \leq n )</th>
<th>( J''(n) )</th>
<th>( J''_k(n) ) for ( 1 &lt; k \leq n )</th>
<th>( J_1(n) )</th>
<th>( k_1(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Even Permutations</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Even Reflections</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Even Translations</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Even Translations by e</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Refines Cubical Octahedral</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Triangulates ( R^{n-1}\times{0,1} )</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

Table 5.4
So far the various induced triangulations have been defined only by
their generating hyperplanes. We now describe them by their individual
simplices. We consider $J^l_k(n)$ or $J^u_k(n)$. Each simplex takes the following
form. As in Section 2, we let $v$ be a vector of $\mathbb{R}^n$ with each component
an even integer, $\pi$ be a permutation of $\{1, \ldots, n\}$ and $s$ a sign vector.
We require $s_{\pi(1)}$ to be 1 if $\pi(1) < k$, and $s_{\pi(n)}$ to be 1 if
$\pi(n) < k$ and we are considering $J^l_k(n)$. We let $\bar{e}^1 = s_{\pi(i)}e^{\pi(i)}$. Also,
set $(\alpha, \beta)$ to $(2, -1)$ if $\pi(1) < k$ and to $(1, 0)$ otherwise, and set
$(\gamma, \delta)$ to $(-1, 2)$ if $\pi(n) < k$ and we are considering $J^l_k(n)$ and to
$(0, 1)$ otherwise. Then the vertices of the corresponding simplex are

$$
v^0 = v;$$

$$
v^1 = v^0 + \alpha \bar{e}^1;$$

$$
v^2 = v^1 + \beta \bar{e}^1 + \bar{e}^2;$$

$$
v^j = v^{j-1} + \bar{e}^j, 3 \leq j \leq n - 2;$$

$$
v^{n-1} = v^{n-2} + \bar{e}^{n-1} + \gamma \bar{e}^n;$$

$$
v^n = v^{n-1} + \delta \bar{e}^n.$$

We leave the reader the verification of this description and the derivation
of appropriate pivot rules for a particular $J^l_k(n)$ or $J^u_k(n)$.

The following results are easy to derive.
Proposition 5.5 For \( k > 1 \), the mesh sizes of \( J^i_k(n) \) and \( J^u_k(n) \) are max\( \{2, \sqrt{n}\} \), while for \( k = 1 \) the figure is \( \sqrt{n} \).

Theorem 5.6 The directional density of \( J^i_k(n) \) in direction \( d \) is

\[
N(J^i_k(n), d) = \sum_{i \geq k} |d_i| + \sum_{i < j} 1/2(|d_i + d_j| + |d_i - d_j|)
\]

while for \( J^u_k(n) \) it is

\[
N(J^u_k(n), d) = \sum_{i < k} 1/2|d_i| + \sum_{i \geq k} |d_i| + \sum_{i < j} 1/2(|d_i + d_j| + |d_i - d_j|).
\]

The average directional densities are

\[
N(J^i_k(n)) = \left( (n+1-k) + \frac{n}{2} \sqrt{2} \right) g_n \quad \text{and}
\]

\[
N(J^u_k(n)) = \left( (n + 1/2 - k/2) + \frac{n}{2} \sqrt{2} \right) g_n
\]

where \( g_n \) is as in theorem 4.3.

6. Computational Experience

In this section we give the results of some numerical experimentation with the use of \( J^i_{n+1}(n+1) \) in a restart algorithm. Note that Broadie [3] has compared \( J^i \) to \( J^u_1 \) and \( K^i \) in the octahedral algorithm - his tests indicate that \( J^i \) almost always requires fewer function evaluations.

In all our runs we use the homotopy \( h(x,t) = tf(x) + (1-t) r(x) \) where \( r \) is the identity. Thus we employed the large pieces induced by the linearity of \( r \) as described in [17]. The code PLALGO [18] was used and required very little change from its program for the large pieces for \( J^i_1 \).
We compared Merrill's restart algorithm with large pieces based on 
\( J'_{n+1}(n+1) \) and \( J_1(n+1) \) and van der Laan and Talman's cubical (or 2n-) 
algorithm with the triangulation \( K'(n) \). We distinguish these cases below 
by writing \( J', J_1 \) or \( K' \) respectively.

For the economic equilibrium problems we chose an initial grid size of 
\( 1/(n+1) \) for problems in dimension \( n \). The equilibrium problems were 
converted to zero-finding problems as in [2]. We solved the pure trade 
examples of dimension 4, 7 and 9 (E1 - E3) and the examples with production 
of dimension 5 and 13 (EP1 and EP2) in Scarf with Hansen [14]; note that 
in all cases the number of prices (commodities) is one larger than the 
dimension. For these problems, we report the results of a run as \( p/q/r, \) 
where \( p \) linear programming pivots, \( q \) function evaluations, and \( r \) demand 
evaluations were required.

For E1 - E3, a refinement factor between restarts of .37 was used 
while the convergence test was \( \| f(x) \|_{\infty} \leq 10^{-12} \). For EP1 and EP2 and 
the other runs below, the default refinement factor of .5 was used. In EP1 
and EP2, the convergence tolerance was relaxed to \( 10^{-10} \). The remaining 
parameters in PLALGO had their default values - thus quasi-Newton accel-
eration was employed.

Our next test problem is Brown's almost linear function, defined by

\[
\begin{align*}
  f_1(x) &= \prod_{j=1}^{n} x_j - 1 \\
  f_j(x) &= \sum_{i=1}^{n} x_i + x_j - n - 1, \quad j > 2.
\end{align*}
\]
We solved this for $n = 10, 15$ and $20$, with starting point the origin and initial grid size $\delta = .5$. The convergence tolerance was $10^{-10}$ for $n = 10$ and $n = 15$, and $10^{-8}$ for $n = 20$. We report the results as $p/q$, with $p$ and $q$ as above.

Finally we considered Watson's test function, defined by

$$f_j(x) = x_j - \exp(\cos(\sum_{i=1}^{n} x_i)).$$

We solved this for $n = 1, 2, \ldots, 10$, with starting point the origin, initial grid size $\delta = .5$ and convergence tolerance $10^{-8}$. The results are reported similarly.

The results demonstrate a consistent advantage of $J'_n(n+1)$ over $J_1(n+1)$ in Merrill's algorithm, and (usually) a considerable advantage over the cubical algorithm with $K'(n)$. Other experimentation has shown that the first statement holds true over a variety of test problems, while the comparative advantages of Merrill's algorithm and the cubical algorithm can depend considerably on the problem type.
<table>
<thead>
<tr>
<th></th>
<th>E1</th>
<th>E2</th>
<th>E3</th>
<th>EP1</th>
<th>EP1</th>
</tr>
</thead>
<tbody>
<tr>
<td>J'</td>
<td>56/65/65</td>
<td>82/91/91</td>
<td>60/73/73</td>
<td>119/126/56</td>
<td>726/689/70</td>
</tr>
<tr>
<td>J1</td>
<td>60/68/68</td>
<td>104/112/112</td>
<td>70/84/84</td>
<td>124/132/53</td>
<td>842/802/96</td>
</tr>
</tbody>
</table>

Economic equilibrium problems

\[
\begin{align*}
n &= 10 \\ n &= 15 \\ n &= 20 \\
J' &= 93/92 \\ J_1 &= 117/116 \\ K' &= 146/153
\end{align*}
\]

Brown's almost-linear function

\[
\begin{align*}
n &= 1 \\ n &= 2 \\ n &= 3 \\ n &= 4 \\ n &= 5 \\ n &= 6 \\ n &= 7 \\ n &= 8 \\ n &= 9 \\ n &= 10 \\
\end{align*}
\]

Watson's test function

Table 6.1
References


